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► **To cite this version:**

Przemyslaw Wojtaszczyk. Stability of l_1 minimisation in compressed sensing. SPARS'09 - Signal Processing with Adaptive Sparse Structured Representations, Inria Rennes - Bretagne Atlantique, Apr 2009, Saint Malo, France. inria-00369340

HAL Id: inria-00369340

<https://inria.hal.science/inria-00369340>

Submitted on 19 Mar 2009

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Stability of l_1 minimisation in compressed sensing

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Abstract

We discuss known results (c.f. [16, 6]) about stability of l_1 minimisation (denoted Δ_1) with respect to the measurement error and how those results depend on the measurement matrix Φ . Then we produce a large class of measurement matrices Φ for which we can apply results from [16] so we have estimate

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + k^{-1/2}\sigma_k^1(x)). \quad (1)$$

We conclude with a modification of l_1 minimisation which gives (1) for most random measurement matrices considered in compressed sensing literature. We also discuss stability of instance optimality in probability.

1 General description of compressed sensing

Let us start by explaining the general setup of compressed sensing. Given $N \gg n$ we look for $n \times N$ a matrix Φ such that vector $y = \Phi x \in \mathbb{R}^n$ preserves information about $x \in \mathbb{R}^N$. We need a decoder (generally nonlinear) $\Delta : \mathbb{R}^n \rightarrow \mathbb{R}^N$ such that $\Delta(\Phi x)$ looks like x . We require a k so $\Delta(\Phi x) = x$ for x any k -sparse vector. We want Δ to be numerically friendly and k big. This leads to requiring that Φ has RIP(k, δ).

Definition 1.1 ([3]) *Matrix Φ has RIP(k, δ), $0 < \delta < 1$ if*

$$(1 - \delta)\|x\|_2 \leq \|\Phi x\|_2 \leq (1 + \delta)\|x\|_2$$

for every k sparse vector $x \in \mathbb{R}^N$.

The largest possible k is $\sim n/\log(N/n)$.

All matrices which perform well for this maximal range of k are random. In this paper we consider only random matrices $\Phi(\omega) = (\phi_{i,j}(\omega))$ where $\phi_{i,j}$'s are independent, i.i.d. subgaussian random variables e.g. Gaussian, Bernoulli. We normalize $\mathbb{E}|\phi_{i,j}|^2 = 1/n$ so columns Φ_j of Φ have typically norm one. Let us call them standard matrices.

Given $0 < \delta < 1$ standard $\Phi(\omega)$ satisfies RIP(k, δ) for $k = \lfloor c_1(\delta)n/\log(N/n) \rfloor$ with probability $\geq 1 - e^{-c_2(\delta)n}$ where $c_1, c_2 > 0$. This is well known, see e.g. [7, 3, 1, 11]. The important point is that RIP is not practically verifiable even for moderately large N and k .

There are two main approaches to finding Δ for the above matrices. One approach, introduced by (E. Candes, D. Donoho et. al. see e.g. [3, 7]) is l_1 minimization Δ_1 i.e.

$$\Delta_1(y) = \text{Argmin}\{\|z\|_1 : \Phi(z) = y\}. \quad (2)$$

Another uses greedy algorithm. We start with an algorithm AL which for $y \in \mathbb{R}^n$ and vectors $(\Phi_j)_{j=1}^N$ gives a subset $\Lambda \subset \{1, \dots, N\}$ with $\#\Lambda \leq k$ and

$\sum_{j \in \Lambda} a_j \Phi_j$. We set $\Delta_{AL}(y) = \sum_{j=1}^N a_j e_j$. This approach was proposed with AL=OMP by A.Gilbert, J. Tropp et al. see e.g. [15]. Some variants of OMP were used by D.Needell, J.Tropp, R.Vershynin et al. see [13, 14, 5].

Instance optimality originate in the work of A.Gilbert, M.Strauss and coauthors and was formally introduces in [4]. Define $\sigma_k^p(x) = \inf\{\|x - z\|_p : z \text{ is } k \text{ sparse}\}$. We would like to have

$$\|x - \Delta(\Phi(x))\|_2 \leq C\sigma_k^2(x) \text{ for all } x \in \mathbb{R}^N. \quad (3)$$

Cohen Dahmen DeVore [4] showed this is impossible. So a pair random measurement matrix $\Phi(\omega)$ and decoder $\Delta(\omega)$ is *instance optimal in probability* (for k with constant C) if for every $x \in \mathbb{R}^N$ there exists a set $\Omega(x)$ of probability very close to 1 such that for $\omega \in \Omega(x)$ we have $\|x - \Delta(\Phi(x))\|_2 \leq C\sigma_k^2(x)$. This is possible – it was shown [4] that there exists such a decoder but it was totally impractical. In [16], [6] and [5] instance optimality in probability was shown for more practical decoders.

From practical point of view it is unrealistic to expect that we can get $\Phi(x)$ exactly; generally we should expect some measurement error. So we apply our decoder to $\Phi(x) + r$ for some $r \in \mathbb{R}^n$ and expect the result to be close to x . For ℓ_1 minimisation results of this type were proved first in [16] for Gaussian measurement matrix and later in [6] for standard matrices.

2 Stability of ℓ_1 minimization

Our arguments are largely geometric. On \mathbb{R}^n and \mathbb{R}^N we will consider the following well known norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$. The unit balls in those norms will be denoted \mathcal{B}_2 , \mathcal{B}_1 and \mathcal{B}_∞ respectively with the dimension of the space added as superscript if needed. We will also use on \mathbb{R}^n the following family of norms: $\|x\|_{J(\alpha)} = \max(\|x\|_2, \alpha\|x\|_\infty)$ for $\alpha > 1$. The unit ball in this norm will be denoted $\mathcal{B}_{J(\alpha)}$. If $\alpha = \sqrt{N}$ this norm will be denoted by $\|\cdot\|_J$ and the ball \mathcal{B}_J . By \mathcal{S} we will denote the euclidean unit sphere.

In [16] and in [6] we introduced the following new geometric properties of the measurement matrix

Definition 2.1 A matrix Φ has $LQ(\mu, k)$ property if for every vector $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^N$ such that $\Phi x = y$ and $\|x\|_1 \leq \mu^{-1}\sqrt{k}\|y\|_2$. Equivalently $\Phi(\mathcal{B}_1^N)$ (which is the convex hull of $(\pm\Phi_j)_{j=1}^N$) contains $(\mu/\sqrt{k})\mathcal{B}_2$.

In [16] it was proved that for some $\mu > 0$ and $c > 0$ and $\delta > 0$ the Gaussian random measurement matrix satisfies $LQ(\mu, k)$ and $RIP(k, \delta)$ with $k = \lfloor cn/\log(N/n) \rfloor$ with overwhelming probability.

Definition 2.2 A matrix Φ has $J(\mu, k)$ property if for every vector $y \in \mathbb{R}^n$ there is $x \in \mathbb{R}^N$ such that $\Phi x = y$ and $\|x\|_1 \leq \mu^{-1}\sqrt{k} \max(\|y\|_2, \sqrt{n/k}\|y\|_\infty)$. Equivalently $\Phi(\mathcal{B}_1^N)$ (which is the convex hull of $(\pm\Phi_j)_{j=1}^N$) contains $(\mu/\sqrt{k})\mathcal{B}_{J(\sqrt{n/k})}$.

In [9] (see also [6]) it was proved that any standard measurement matrix for some $c, \mu, \delta > 0$ has $J(\mu, k)$ and $RIP(k, \delta)$ with $k = \lfloor cn/\log(N/n) \rfloor$ with overwhelming probability.

Using those concepts we proved

Theorem 2.3 ([16]) If Φ satisfies $RIP(k, \delta)$ and $LQ(\mu, k)$ then there exists C such that for every $x \in \mathbb{R}^N$ and $r \in \mathbb{R}^n$

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (4)$$

and

$$\|\Delta_1(\Phi(x) + r) - x\|_2 \leq C(\|r\|_2 + \sigma_k^2(x) + \|\Phi(x|S^c)\|_2) \quad (5)$$

where S is a k element set for which $\sigma_k^2(x) = \|x|S^c\|_2$.

Theorem 2.4 ([6]) If Φ satisfies $RIP(k, \delta)$ and $J(\mu, k)$ then there exists C such that for every $x \in \mathbb{R}^N$ and $r \in \mathbb{R}^n$

$$\|\Delta(\Phi(x) + r) - x\|_2 \leq C(\|r\|_J + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (6)$$

and

$$\|\Delta(\Phi(x) + r) - x\|_2 \leq C(\|r\|_J + \sigma_k^2(x) + \|\Phi(x|S^c)\|_J) \quad (7)$$

where S is a k element set for which $\sigma_k^2(x) = \|x|S^c\|_2$.

This in particular implies that Δ_1 is instance optimal in probability for all standard matrices. But it also shows stability of this decoder with respect to the measurement error r .

Let us note that (4) and (6) build upon and improve results from [2].

Actually Theorem 2.4 was not stated in [6]. We were interested in instance optimality and did not want to get into abstract formulations. However all the arguments needed to the proof of Theorem 2.4 are in [6] (see also [16]). If we compare Theorems 2.3 and 2.4 we see that when $r = 0$ they have the same conclusions. The only difference is in how the measurement error influences the estimate. Since $\|x\|_2 \leq \|x\|_J$ the estimates in Theorem 2.3 are better. Later on we will comment on this in more detail. Unfortunately LQ in the maximal range of k was shown in [16] only for Gaussian standard measurement matrix and for matrix whose columns are independently drawn from \mathcal{S} (but this is quite close to Gaussian). LQ clearly fails for Bernoulli random matrix. This is the main drawback of Theorem 2.3.

3 Random matrices satisfying LQ

Now we want to produce a large family of measurement matrices which satisfy $\text{LQ}(\mu, k)$ for $k = \lfloor cn \log(N/n) \rfloor$. First we need a unitary matrix in \mathbb{R}^n . In his fundamental paper [8] B.S. Kashin formulated the following

Theorem 3.1 *There exists a constant $c > 0$ such that with overwhelming probability a unitary matrix U satisfies*

$$c\|x\|_2 \leq \frac{1}{2\sqrt{n}}(\|U(x)\|_1 + \|x\|_1) \leq \|x\|_2 \quad (8)$$

for all $x \in \mathbb{R}^n$.

He provided only a sketch of the proof, and stated only the existence of a matrix satisfying (8) but his proof easily gives that it holds with overwhelming probability on U . By duality we get that if a unitary matrix U satisfies (8) then $\text{conv}(U^*(\frac{1}{\sqrt{n}}B_\infty) \cup \frac{1}{\sqrt{n}}B_\infty) \supset cB_2$. To see this note that if this is not

the case then there exists x with $\|x\|_2 = 1$ such that $|\langle x, U^*z \rangle| < c\sqrt{n}$ and $|\langle x, z \rangle| < c\sqrt{n}$ for all $z \in B_\infty$. Thus we obtain $\|x\|_1 < c\sqrt{n}$ and $\|Ux\|_1 < c\sqrt{n}$ what contradicts (8). Since $B_J \supset \frac{1}{\sqrt{n}}B_\infty$ we get

Corollary 3.2 *With overwhelming probability on U we have $\text{conv} U(B_J) \cup B_J \supset cB_2$.*

We fix one such unitary matrix. Now we define a random matrix $\Phi(\omega)$ as follows. We take two symmetric, subgaussian random variables η and τ such that $\mathbb{E}|\eta|^2 = \mathbb{E}|\tau|^2 = 1$. Let $\eta_{j,i}$ and $\tau_{j,i}$ denote independent copies of η (resp. τ) and also $\eta_{j,i}$'s are independent from $\tau_{i,j}$'s. Our matrix Φ has columns ϕ_j where $\phi_j = n^{-1/2}(\eta_{j,1}, \dots, \eta_{j,n})$ for $j < N/2$ and $\phi_j = n^{-1/2}U(\tau_{j,1}, \dots, \tau_{j,n})$ for $j \geq N/2$. From [6] we know that with overwhelming probability $\text{conv}(\phi_j)_{j < N/2} \supset cB_J$ and $\text{conv}(\phi_j)_{j \geq N/2} \supset cU(B_J)$ so $\text{conv}(\phi_j)_{j=1}^N \supset c'B_2$. So we need to show that the matrix Φ satisfies RIP. Actually we will show the appropriate concentration inequality. Let us denote $\Phi_1 = (\phi_j)_{j=1}^{N/2}$ and $\Phi_2 = (\phi_j)_{j \geq N/2}$. We know (see e.g. [6]) that each of those matrices satisfies the concentration estimate: for each $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for each x

$$\mathbb{P}(|\|\Phi_s(x)\|^2 - \|x\|^2| > \epsilon\|x\|^2) \leq 2e^{-nc(\epsilon)}. \quad (9)$$

Thus (see [1]) matrices Φ_1 and Φ_2 satisfy $\text{RIP}(k, \delta)$. Now given $x \in \mathbb{R}^N$ we write $x = x_1 + x_2$ where x_1 equals x on coordinates $< N/2$ and x_2 equals x on coordinates $\geq N/2$. Note that

$$\begin{aligned} \|\Phi(x)\|^2 &= \|\Phi_1(x_1) + \Phi_2(x_2)\|^2 \\ &= \|\Phi_1(x_1)\|^2 + \|\Phi_2(x_2)\|^2 + 2\langle \Phi_1(x_1), \Phi_2(x_2) \rangle. \end{aligned} \quad (10)$$

Since Φ_1 and Φ_2 are independent for a fixed $b = \Phi_2(x_2)$ we have

$$\begin{aligned} \mathbb{P}(|\langle \Phi_1(x_1), b \rangle| > \frac{\epsilon}{3}\|b\| \cdot \|x_1\|) \\ = \mathbb{P}\left(\left|\sum_i \sum_j^{N/2} x_j b_i \eta_{i,j}\right| > \frac{\epsilon}{3}\|b\| \cdot \|x_1\|\right). \end{aligned}$$

Since $\eta_{i,j}$ are independent subgaussian variables and $\sum_i \sum_j^{N/2} |x_j|^2 |b_i|^2 = \|b\|^2 \|x_1\|^2$ we can continue as

$$\leq 2e^{-nc(\epsilon/3)} \quad (11)$$

From (9) for $s = 2$ we see that $\|b\| \sim \|x_2\|$ with big probability so from (11), (11) and (9) we get

$$\mathbb{P}\left(\left|\|\Phi(x)\|^2 - \|x\|^2\right| > \epsilon\|x\|^2\right) \leq 6e^{-nc(\epsilon/3)}.$$

So we proved a concentration inequality and thus we infer (see [1]) that Φ satisfies RIP(k, δ) for $\delta > 0$ with $k = \lfloor c(\delta)n/\log(n/N) \rfloor$.

4 Norm $\|\cdot\|_J$

In Theorem 2.4 the norm $\|\cdot\|_J$ appears as a measure which estimates the influence of the measurement error on the accuracy of the recovery. Now we want to make some remarks on this norm. First we observe that

$$\|x\|_J \leq \sqrt{\log N}\|x\|_2 \quad (12)$$

so Theorem 2.4 implies for any standard measurement matrix the error estimate

$$\|\Delta(\Phi(x)+r)-x\|_2 \leq C(\sqrt{\log N}\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}}) \quad (13)$$

and also shows that for each $x \in \mathbb{R}^N$ with big probability on the draw of Φ we have

$$\|\Delta(\Phi(x)+r)-x\|_2 \leq C(\sqrt{\log N}\|r\|_2 + \sigma_k^2(x)). \quad (14)$$

To see (14) we must use that with the big probability on the draw of Φ we have $\|\Phi(x|S^c)\|_J \leq \sigma_k^2(x)$. But this was already proved in [6] Lemma 5.5. The point is that for a random vector $z = \sum_j z_j \eta_j$ where η_j 's are independent symmetric random variables with the same subgaussian distribution it is very unlikely that $\|z\|_2 < \|z\|_J$.

But we can make the same observation about the error. If the error is as above e.g. random Gaussian it is exponentially unlikely that $\|z\|_2 < \|z\|_J$. The same is true if our error is uniformly drawn from \mathcal{S} , σ is the normalized surface measure on \mathcal{S} . The following is well known (see e.g. [12, p.5])

Lemma 4.1 For $0 \leq \alpha \leq 1$

$$\sigma(\{x \in \mathcal{S} : x_1 \geq \alpha\}) \leq \sqrt{\frac{\pi}{2}} e^{-\frac{\alpha^2 n}{2}}. \quad (15)$$

Note that $\{x \in \mathcal{S} : x_1 \geq \alpha\} \subset \{x \in \mathcal{S} : x_1 \geq \sin \alpha\}$.

Let us denote $I_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta$. From classical formulas for σ we get

$$\sigma(\{x \in \mathcal{S} : x_1 \geq \sin \alpha\}) = I_n^{-1} \int_{\alpha}^{\pi/2} \cos^n \theta d\theta.$$

Since $\cos \theta \leq e^{-\theta^2/2}$ for $0 \leq \theta \leq \pi/2$ we get

$$\begin{aligned} \int_{\alpha}^{\pi/2} \cos^n \theta d\theta &= \frac{1}{\sqrt{n}} \int_{\alpha\sqrt{n}}^{\sqrt{n}\pi/2} \cos^n\left(\frac{t}{\sqrt{n}}\right) dt \\ &\leq \frac{1}{\sqrt{n}} \int_{\alpha\sqrt{n}}^{\sqrt{n}\pi/2} e^{-t^2/2} dt \end{aligned}$$

and substituting $u = t - \alpha\sqrt{n}$ we can continue

$$\leq \frac{1}{\sqrt{n}} e^{-\alpha^2 n/2} \int_0^{\infty} e^{-u^2/2} du = \sqrt{\frac{\pi}{2n}} e^{-\alpha^2 n/2}.$$

To estimate I_n from below we integrate by parts to get $I_n = \frac{n-2}{n-1} I_{n-2} \geq \sqrt{\frac{n-3}{n-1}} I_{n-2}$. From this we get $\sqrt{n} I_n \geq 1$.

Now if $\|z\|_J > \|z\|_2$ then for some coordinate s we have $|x_s| > \frac{1}{\sqrt{\log N}}$. The probability that this happens is at most

$$2n\sigma\left\{x \in \mathcal{S} : x_1 > \frac{1}{\sqrt{\log N}}\right\} \leq 2n\sqrt{\frac{\pi}{2}} e^{-\frac{n}{2\log N}}.$$

5 New algorithm

Now we want to suggest a modified ℓ_1 minimization algorithm which will have stability properties in ℓ_2 norm. The main point in our improvement is that we do not change the measurement matrix as it may be prescribed by other considerations. So suppose we have a fixed measurement matrix $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^n$ satisfying RIP(k, δ) for $k = \lfloor cn/\log n/N \rfloor$ and some $\delta > 0$. Given a signal $x \in \mathbb{R}^N$ we observe $y = \Phi(x)$ or $y = \Phi(x) + r$. Our algorithm works as follows:

1. We fix at random $n \times N$ Gaussian matrix $\Psi = \left(\frac{1}{\sqrt{n}} \gamma_{i,j}\right)_{i=1, j=1}^{n, N}$.

- We consider $n \times 2N$ matrix $\Gamma = (\Phi, \Psi)$ and solve the ℓ_1 minimization problem

$$\bar{z} = \text{Argmin } z \{z \in \mathbb{R}^{2N} : \Gamma(z) = y\}$$

- We define the decoder as $\Delta(y) = \bar{z}|_{\{1, \dots, N\}}$

If the matrix Φ is not fixed but is a random standard measurement matrix we proceed exactly the same but we must make sure that Ψ and Φ are independent.

The main observation is that with big probability on the draw of Ψ (or with big probability on the joint draw of Φ and Ψ) the matrix Γ satisfies RIP and LQ for some constants and $k = \lfloor cn/\log N/n \rfloor$. Since $\|x - \Delta(y)\| \leq \|x - \bar{z}\|$ for every norm we are considering we get:

- $\|x - \Delta(y)\|_2 \leq C(\|r\|_2 + \frac{\sigma_k^1(x)}{\sqrt{k}})$
- If Φ is random then for every $x \in \mathbb{R}^N$ with big probability we get

$$\|x - \Delta(x)\|_2 \leq C(\|r\|_2 + \sigma_k^2(x)).$$

This is an immediate consequence of Theorem 2.3 once we show that the matrix Γ with big probability satisfies RIP and LQ for some constants and $k = \lfloor cn/\log N/n \rfloor$. But Γ has LQ because Ψ has. We will show that Γ has RIP($k, \frac{4}{5}\delta$). The argument for this is standard so we will provide only a sketch.

- We take k columns from Γ . If all are columns of Φ or Ψ then we are done, so assume we have $k > s > 0$ such columns $(\phi_j)_{j \in A}$ from matrix Φ and $k - s$ columns $(\psi_j)_{j \in B}$ from Ψ . There are $\sum_{s=1}^{k-1} \binom{N}{s} \binom{N}{k-s} < N^{k-1}$ such possibilities.

- Let $X = \text{span}(\phi_j)_{j \in A}$ and $Z = \text{span}(\psi_j)_{j \in B}$. In order to show RIP it suffices to show that $|\langle x, z \rangle| \leq \frac{\delta}{10} \|x\| \|z\|$ for $x \in X$ and $z \in Z$.

- It is sufficient to show this for x and z from η nets in the unit balls (with appropriate η . Since (see e.g. [10, Ch. 15 Prop. 1.3.]) the unit ball in d dimensional space has an η net of cardinality not exceeding $(6/\eta)^d$ applying this to X and Z we see that we must to consider $(6/\eta)^k$ pairs x, z .

- A standard estimate shows that $\mathbb{P}(|\langle x, z \rangle| \geq \lambda \|x\| \|z\|) \leq 4e^{-\lambda^2 n/2}$. This is true both when Φ is fixed and when Φ is random.

- We put those estimates together (see e.g. [1]) and infer that we can find k of the desired magnitude.

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