Rigid Shape Interpolation Using Normal Equations
William Baxter, Pascal Barla, Ken-Ichi Anjyo

To cite this version:

HAL Id: inria-00360532
https://inria.hal.science/inria-00360532
Submitted on 9 Jun 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract

In this paper we provide a new compact formulation of rigid shape interpolation in terms of normal equations, and propose several enhancements to previous techniques. Specifically, we propose 1) a way to improve mesh independence, making the interpolation result less influenced by variations in tessellation, 2) a faster way to make the interpolation symmetric, and 3) simple modifications to enable controllable interpolation. Finally we also identify 4) a failure mode related to large rotations that is easily triggered in practical use, and we present a solution for this as well.

Keywords: npr, interpolation, morphing, in-betweening, least-squares, Poisson problem

1 Introduction

Two-dimensional morphing and shape interpolation algorithms have many practical applications. Algorithms for generating 2D morphs are commonly found in commercial video editing software such as Adobe After Effects as well as packages for creating web-oriented or cartoon animations, such as Adobe Flash or Toon Boom. Animated two dimensional shapes also have applications for real-time simulation of large crowds [Kavan et al. 2008].

Recently, a variety of new algorithms that preserve rigidity [Alexa et al. 2000; Sheffer and Kraevoy 2004; Fu et al. 2005; Xu et al. 2005; Sumner et al. 2005] have been introduced. These rigidity-preserving methods generate interpolations which have a very natural, almost physical, appearance, and do so in a very efficient manner, making them suitable for interactive applications. Rigidity preserving interpolation works well and is a very practical way to generate high-quality shape interpolations. However there are a number of limitations in practice which we address here:

- **Mesh independence:** To the extent possible, the results should be independent of the tessellation of the underlying mesh used. This property is lacking in [Gotsman and Surazhsky 2001; Surazhsky and Gotsman 2001; Alexa et al. 2000].
- **Symmetry:** It is often desirable for an interpolation to be symmetric in its arguments. That is, the vertex paths generated interpolating from shape A to shape B should be the same as from B to A. One solution is presented in [Alexa et al. 2000], but it increases the computation load of the algorithm tremendously.
- **Controllability:** once the interpolation has been computed, it is desirable to be able to tweak it in some manner if it does not generate exactly the animation desired. Controllability was not addressed in the previous work on rigid interpolation ([Alexa et al. 2000; Xu et al. 2005]).
- **Rotational consistency:** For large rotations (>180 degrees), which are common in practical 2D animations, rotational inconsistencies can occur causing severe artifacts. This affects both [Alexa et al. 2000] and [Xu et al. 2005].

We provide a solution to each of these issues in this paper. Furthermore, in support of our explanation of these solutions, we provide an alternate presentation of the mathematics behind rigid interpolation in terms of normal equations. We believe the normal equations offer a more intuitive formulation, and also make the path to implementation straightforward. Our derivation follows that of [Alexa et al. 2000] most closely, since we find it to be the most accessible and since, similar to that work, we are also primarily interested in 2D. The technique of [Xu et al. 2005] is closely related, however, and we discuss this connection in Section 3.2.

2 Previous Work

The term “morphing” was introduced by [Beier and Neely 1992] to describe a combination of image-space warping with cross-dissolve between pixels. Many other image-space morphing algorithms were introduced subsequently with many refinements, a good example being that of [Lee et al. 1995]. A more complete survey of image-space morphing can be found in [Wolberg 1998]. One problem with image-space morphing algorithms is that they have
difficulty handling large relative rotations in an image, resulting in
unnaturally distorted geometry, self-intersections, and/or local ori-
entation reversals ("flips"). It is also difficult to deal with images
where foreground and background should behave differently.

Object-space morphing is an alternative which addresses some of
those issues. A variety of techniques have been presented that op-
erate on both 2D and 3D geometry, e.g. [Gregory et al. 1998;
Lee et al. 1999; Kanai et al. 2000]. A good, though slightly dated,
survey can be found in [Alexa 2002]. Some works have ad-
vanced the state of the art of object space-morphing by guaranteeing
intersection-free solutions to the vertex path problem [Gotsman and
Surazhsky 2001; Surazhsky and Gotsman 2001].

Recently a great deal of attention has focused on methods which
attempt to preserve rigidity, or local differential quantities [Alexa
et al. 2000; Sheffer and Kraevoy 2004; Xu et al. 2005; Fu et al.
2005; Sumner et al. 2005]. These mesh-based object-space algo-
rithms solve a vertex optimization problem to find intermediate
shapes that introduce as little local shear and stretch as possible.
The work of [Schaefer et al. 2006] is unique in that it introduces
an image-space rigidity-preserving deformation method, though it
suffers from several of the same problems of other image space
methods. However, in general, these newer rigidity-preserving, dif-
ferential coordinate methods handle rotations very naturally, pro-
duce low distortion, and maintain a low runtime cost. Nevertheless
there are some issues that remain with these methods, as mentioned
in the introduction, and addressing these issues is the focus of our
work.

Recently we became aware of work by Choi and Szymczak [2003]
by way of [Fu et al. 2005] who also tackled the rotation consistency
problem (called “rotation coherence” in their work) in a manner
very similar to what we propose here. Their work also provides
a detailed proof that the procedure we describe will succeed when-
ever a consistent rotation exists. We offer a small improvement over
their method by taking an additional step to ensure that the rotation
assignment chosen is unique and minimal. On the other hand their
method is slightly more succinct than ours, made so by the elimi-
nation of the special boundary handling step.

3 Background

3.1 “As-rigid-as-possible” interpolation

Alexa et al. [2000] presents rigidity-preserving interpolation as a
quadratic minimization problem. The problem is as follows: given
two meshes \( P \) and \( Q \) composed of corresponding triangles \( P_T \) and
\( Q_T \) for \( T = 1 \ldots M \), find an interpolation that preserves rigid-
ity. The procedure proposed is the following. For each triangle
pair, compute the Jacobian of the affine transformation that maps
one triangle to the other, \( A_T \) (Fig. 2), and interpolate these in-

dividual matrices independently to get \( A_T(t) \), with \( A_T(0) = I \)
and \( A_T(1) = A_T \). This is done using a polar decomposition,
and linearly interpolating rotation and shear components indepen-
dently. Specifically, if the polar decomposition of \( A_T \) is \( R(\alpha)S \),
then \( A_T(t) = R(t\alpha)(1-t)I + (S) \), where \( \alpha \) is a rotation an-
gle extracted from the orthogonal part of the polar decomposition.
Finally, the key idea is to find intermediate vertex trajectories \( v_i(t) \)
by minimizing a quadratic error function:

\[
E(t) = \sum_{T=1}^{M} \| B_T(t) - A_T(t) \|^2
\]

where \( \| \cdot \| \) is the Frobenius norm and \( B_T(t) \) are the transfor-
mations that relate the initial vertices of \( P_T \) to unknown vertices \( u_i(t) \).

As explained in [Alexa et al. 2000], by taking the derivative of (1),
setting it to zero, and rearranging all the terms, a linear system of
the form \( V(t) = -H^{-1}G(t) \) can be obtained where \( V(t) \) is a vector
containing all the unknowns, and \( H \) is a constant matrix in-
dependent of \( t \). Thus straightforward, the steps required to get
from (1) to the final linear system using direct differentiation are
laborious and yield little insight. Consequently, the intermediate
steps are omitted in [Alexa et al. 2000]. However, by expressing
the problem instead in terms of least squares normal equations, in
Section 4 we are able to provide a complete and succinct end-to-
end derivation that reveals more clearly the structure of the linear
system.

One key aspect of the differential coordinate formulations like
[Alexa et al. 2000; Xu et al. 2005; Sumner et al. 2005] is that
they result in linear systems for which the left hand side (LHS,
here \( H \)) is independent of \( t \). Thus the interpolation problem can
be solved very efficiently by LU-decomposing the LHS once for
the whole interpolation, and finding solutions for each given \( t \)
by quick LU back-substitution. This advantage is lost, however, when
symmetrizing the cost function as proposed in [Alexa et al. 2000].
In Section 5 we show that a symmetric cost can be achieved with
little additional per-t cost.

3.2 Comparison with Poisson interpolation

Xu et al. [2005] suggest an alternate way to generate rigidity-
 preserving interpolations, by solving a Poisson problem on a do-
main mesh. This method is, in fact, nearly identical to that of
Alexa et al. [2000]. The gradients taken by Xu et al. are those
of the coordinate functions: \( \nabla x(x, y), \nabla y(x, y) \). By definition,
these gradients stacked together form the Jacobian. Xu et al. ac-
knowledge this, however, they claim that their solution differs from
that of Alexa et al. in that, in contrast to the Frobenius norm, their
formulation “directly minimizes least squares differences between
[gradient] vectors using the \( L^2 \) norm”. However, that is precisely
what the Frobenius norm gives as well. There is one difference
between the techniques, however: weighting. Relying on [Tong
et al. 2003], Xu et al. note that their discrete Poisson formulation is
equivalent to finding \( S \) that minimizes \( \sum_T a_T \| \nabla S - G \|^2 \) (Xu et
al. Eq. 10), where \( a_T \) is the area of triangle \( P_T \). In our notation,
\( B_T(t) = \nabla S \) and \( A_T(t) = G \), so this is precisely (1) with an addi-
tional weighting factor of \( a_T \). If we introduce this weighting in (1)
then, for 2D problems, as-rigid-as-possible interpolation becomes
identical to Poisson shape interpolation. Without the weighting, the
results of Alexa et al. can be seen to depend significantly on the
tessellation (see Figure 3).

Xu et al. made a significant contribution in the form of providing
a more rigorous and general mathematical foundation for rigid in-
terpolation by relating it to the Poisson problem. However, given
that solving the Poisson problem involves second order differential

![Figure 2: Triangle mapping notation: The Jacobian, \( A_T \), of
the mapping between two triangles. We define \( A_T(t) \) to be the
desired interpolated Jacobian at parameter \( t \), and \( B_T(t) \) to be that
actually obtained by the interpolation.](image-url)
as in the original method. The notation just presented enables writing the equation more explicitly. As noted in Section 3.2, for consistency we also weight the equations by triangle areas. The final weighted least squares problem becomes:

\[ E(t) = \sum_{T=1}^{M} a_T \| P_T^T V T - A_T(t) \|^2, \quad (3) \]

where \( a_T \) is the area of triangle \( P_T \). Rewriting each expression in terms of a global \( N \times 2 \) matrix of unknowns, \( V \), we obtain:

\[ E(t) = \sum_{T=1}^{M} a_T \| P_T V - A_T(t) \|^2, \quad (4) \]

where for a triangle index \( T \) with vertices \( \{i, j, k\} \), \( P_T \) is a sparse \( 2 \times N \) matrix with non-zero entries only in columns \( i, j, k \):

\[
P_T = \begin{bmatrix}
\ldots & [P_T^{i1}] & \ldots \\
\ldots & [P_T^{j1}] & \ldots \\
\ldots & [P_T^{k1}] & \ldots 
\end{bmatrix}
\]

With this, we can now express the problem in terms of normal equations. We wish to find a minimum residual solution to \( P_TV = \hat{A} \) with a given weighting. The normal equations that express that are:

\[
(P_TW^T)V = \hat{P}^T \hat{W} \hat{A}
\]

where

\[
P = \begin{bmatrix} P^T_1 & \ldots & P^T_M \end{bmatrix}^T
\]

\[
\hat{A} = \begin{bmatrix} A^T_1 & \ldots & A^T_M \end{bmatrix}^T
\]

\[
\hat{W} = \text{diag}(\begin{bmatrix} a_1 & \ldots & a_M \end{bmatrix})
\]

In terms of implementation, the core routine consists of assembling the global \( N \times N \) matrix \( P^T WP \) and \( N \times 2 \) matrix \( P^T W \hat{A} \). These are precisely the \( -H \) and \( G \) matrices of Alexa et al., respectively. It can be seen that these matrices have the form \( \sum_T a_T P_T^T P_T \) and \( \sum_T a_T P_T^T k_T \), in other words a sum of very sparse matrix multiplies. Thus implementation consists of just computing small dense products of the form \( a_T (P_T^*)^T (P_T) \) and \( a_T (P_T^*)^T A_T \) and then scattering the computed elements to their appropriate destinations in the global LHS and RHS matrices, which are sparse. This is identical to the “global assembly” stage of a finite element method.

5 Inexpensive symmetric interpolation

One desirable property of an interpolation technique is symmetry. By this it is meant that interpolation \( P \rightarrow Q \) generates the exact same interpolation as \( Q \rightarrow P \) under the substitution \( t \leftarrow (1 - t) \). [Alexa et al. 2000] presents a modification to the cost equation that makes interpolation symmetric:

\[ E(t) = (1 - t) \tilde{E}(t) + t \tilde{E}(1 - t); \]

however, modifying the cost in this way requires re-solving the linear system for each new \( t \) value rather than reusing the fixed, pre-computed LU decomposition. By formulating as a least squares problem it is easy to see that one can symmetrize the cost function simply by including both sets of constraint equations in a single matrix. I.e. we wish to find a minimum-residual solution to:

\[
\begin{bmatrix}
\hat{P} \\
\hat{P}
\end{bmatrix}
\begin{bmatrix}
V \\
\hat{A}
\end{bmatrix}
 =
\begin{bmatrix}
\hat{A} \\
\hat{A}
\end{bmatrix}
\]
### Table 1: Computation Time

<table>
<thead>
<tr>
<th>Package</th>
<th>Size</th>
<th>Factor (µsec)</th>
<th>Solve (µsec)</th>
<th>Factor/Solve</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAPACK⁷</td>
<td>328×328</td>
<td>10506</td>
<td>646.56</td>
<td>16.2</td>
</tr>
<tr>
<td>TAUCS²</td>
<td>328×328</td>
<td>3249</td>
<td>165.68</td>
<td>19.6</td>
</tr>
<tr>
<td>SuperLU³</td>
<td>328×328</td>
<td>3005</td>
<td>160.85</td>
<td>18.7</td>
</tr>
<tr>
<td>UMFPACK⁴</td>
<td>328×328</td>
<td>3022</td>
<td>133.59</td>
<td>22.6</td>
</tr>
<tr>
<td>LAPACK</td>
<td>164×164</td>
<td>2264</td>
<td>65.45</td>
<td>34.6</td>
</tr>
<tr>
<td>TAUCS</td>
<td>164×164</td>
<td>1110</td>
<td>72.25</td>
<td>15.4</td>
</tr>
<tr>
<td>SuperLU</td>
<td>164×164</td>
<td>638</td>
<td>39.66</td>
<td>16.1</td>
</tr>
<tr>
<td>UMFPACK</td>
<td>164×164</td>
<td>1185</td>
<td>38.88</td>
<td>30.5</td>
</tr>
</tbody>
</table>

Table 1: **Computation Time:** Time required to factor and solve two sparse systems of equations using different numerical packages. The 328×328 matrix has 3228 nonzero elements; 164×164 has 892. LAPACK is a dense solver; the others are direct, sparse solvers. (Measurements made on a Pentium IV XEON 3.60Ghz processor.) 1. http://www.netlib.org/lapack/ 2. http://www.tau.ac.il/~stoledo/taucs/ 3. [Demmel et al. 1999] 4. [Davis 2004]

in the weighted least-squares sense. This still leads to a system matrix of the exact same size and sparsity, but creates a symmetric interpolation that needs be factored only once for all \( t \). This is very significant as can be seen in Table 1, since factoring requires around 15-30 times as much computation as back-substitution.

The key difference is that [Alexa et al. 2000] interpolate the two costs. The approach just explained is equivalent to simply adding the two costs together, as \( E(t) = E(t') + E(1-t) \). See Figure 4 for a comparison of interpolations with and without the symmetric cost.

### 6 Constraints and control

One limitation of the techniques of [Alexa et al. 2000] and [Xu et al. 2005] is that they offer no means for control. Given start and end configurations, the math inexorably leads to a unique set of vertex trajectories. There is no recourse if these trajectories are not satisfactory to the animator. We present some simple and direct methods to influence the interpolation while still attempting to preserve rigidity. These take the form of hard or soft linear constraints, which are easily expressed using the normal equations formulation.

For hard linear constraints there are two main options: substitution and Lagrange multipliers. Substitution is optimal in that it actually reduces the number of degrees of freedom in the system, however it requires that you express your equations exclusively in terms of these remaining degrees of freedom. For this reason, Lagrange multipliers are generally considered to be easier to work with in terms of modular software design. To enforce some additional linear constraints, \( CV = D \), using Lagrange multipliers one forms the augmented system:

\[
\begin{bmatrix}
  P^T WP & C^T \\
  C & 0 \\
\end{bmatrix} \mathbf{v} = \begin{bmatrix}
  P^T W A \\
  D \\
\end{bmatrix} \quad (5)
\]

Some simple possible constraints are

- \( C = [\ldots 0 1 \ldots ] \) enables the constraining of a single vertex location.
- \( C = [1/N 1/N \ldots ] \) constrains the mean of all the vertices.
- \( C = [\ldots 1 \ldots 1 \ldots ] \) to enforce that the difference between two vertices have a particular value.
Soft constraints are also easy to incorporate. For these we need a
correction weight, which is easy using the weighted least squares form-
ulation. For this, constraints are expressed by adding additional rows to \( P \). To impose soft constraints \( F \mathbb{V} = G \):

\[
\begin{bmatrix}
P^T \\
F^T
\end{bmatrix}
W
\begin{bmatrix}
P \\
F
\end{bmatrix}
V
=
\begin{bmatrix}
P^T \\
F^T
\end{bmatrix}
W
\begin{bmatrix}
A \\
G
\end{bmatrix}
(6)
\]

The strength of the penalty forces can be altered by changing the corresponding weights along the diagonal of \( W \).

These two simple modifications allow us to exercise arbitrary con-
trol over individual vertices or collections of vertices while letting the as-rigid-as-possible technique determine the remaining trajec-
tories. Please see Figure 5 and the supplemental video for demonstra-
tions.

7 Making rotation consistent

In a significant number of practical situations, orientation may be
interpolated incorrectly by current rigid interpolation approaches such as [Alexa et al. 2000; Xu et al. 2005]. The difficulty stems from the inherent ambiguity of extracting a rotation angle from a rotation matrix. The resulting angle is not unique: \( \alpha + 2k\pi \) for \( k \in \mathbb{Z} \). The result of this can most clearly be seen in Figures 6 and 7. Looking at the per-triangle rotations, large discontinuities can be seen where the rotation direction changes abruptly because of naïvely choosing the smallest magnitude rotation for each individual triangle.

Some means is needed to make these rotations consistent. Inconsis-
tent rotations can occur anywhere in a shape, either in the interior or on the boundary. To prove this, consider a single triangulated
shape and a copy of it rotated \( 180 - \epsilon \) degrees. Now rotate any
triangle of the copy by \( 2\epsilon \) degrees. This will introduce a rotational
discontinuity in the neighborhood of that triangle. In fact, Figure 1 results
from a closely related scenario: a copy of the turtle has been rotated
by exactly 180 degrees to create the target shape. Due to numerical
imprecision, however, triangles end up with rotations of \( 180 \pm \epsilon \). The situation also arises in less degenerate situations as well (Figures 6, 7). In particular, any interpolation where the rota-
tion required is greater than 180 degrees is problematic, since, ab-
sent other information, angle extraction techniques typically return
an answer on the \([-180, 180]\) domain.

Our approach is to start from the boundaries and work inward. We
wish to find places where the angle extracted from polar decom-
positions have large discontinuities between adjacent triangles (we
use 180 degrees as a threshold), and add or subtract multiples of
360 degrees as needed to get all jumps below 180 degrees. Walk-
ing first around the boundary gives us a linear sequence of angles
which is quick and easy to process.

Removing discontinuities in this sequence of boundary rotations is
in fact a classic problem. It is precisely the problem faced when ex-
tracting the phase from a discrete Fourier transform. Most libraries
that provide a 1D FFT function also provide some sort of phase
unwrap() function which does exactly what we need. We put our
rotations into an array and just pass it to such a function. The basic
algorithm is \( O(N) \) and just involves scanning through the values
one-by-one looking for discontinuities, and raising or lowering the
tail end as needed to eliminate discontinuities greater than 180 de-

gatees.

Once we obtain the new boundary rotations via the FFT phase
unwrap() procedure, we treat these rotations as ground truth and
proceed to propagate these “good” rotations inward in a breadth
first manner.

Specifically, we start by pushing all boundary triangles on a stack,
tagging them all as VISITED. We then iteratively pop them off one
by one. For each triangle popped off, we examine its neighbors, and
any which are VISITED are ignored. For the others we adjust the
rotations of any that are inconsistent with the popped triangle, set
their VISITED flags and push each of them back on the stack. The
process continues until the stack is empty. The corrected results in
Figures 1, 6 and 7 were generated in this way.

After this is complete, the rotations should be consistent (if a con-
sistent assignment is possible); however, the rotation may not be
minimal, as can be seen in Figure 8. Furthermore the presence or
lack of excess rotation depends upon the choice of initial triangle.
This is true of [Choi and Szymczak 2003] as well. Such depend-
ence is undesirable and should be eliminated. The extra rotation
itself may or may not be desired by the user; however, we believe
the minimal rotation is likely to be the one desired in the vast majority of cases, and thus makes the best default.

We use a very simple approach to eliminate the excess rotation. We simply take the area-weighted average rotation over all the triangles and add or subtract multiples of 360 until the average is as close to zero as possible. For the area of triangle \( \{i, j, k\} \) we use the average of triangle areas \( \{p_i, p_j, p_k\} \) and \( \{q_i, q_j, q_k\} \).

8 Discussion and future work

We have presented both a new mathematical formulation and analysis of rigidity preserving interpolation. We pointed out the equivalence of Alexa et al.’s least squares solution and Xu et al.’s Poisson formulation. We additionally identified several areas in which the methods could be improved and offered solutions providing: tessellation-independence (adapted from Xu et al.), inexpensive symmetric cost, direct vertex control of results, and finally we fixed a significant issue with the handling of large rotations.

There is still room for improvement, however. In particular, with our constraint technique, the underlying ideal rotations \( A_T \) do not “know” about the constraints we have imposed, and thus unnatural behavior will result if the constraints are not somewhat consistent with the underlying rotations. Thus there is a limit to the amount of control offered by this technique.

In the future we are interested in pursuing a least-squares formulation of 3D mesh interpolation using the same connection between least-squares and the Poisson equation exploited here for 2D. With the recent popularity of Poisson-based techniques, it also seems likely that many other recent works could take advantage of the simpler, but mathematically equivalent, weighted least squares formulation. Finally, we are currently pursuing ideas for how to provide more flexible rigidity-preserving animation controls.

Acknowledgments: This work was supported in part by the Japan Science and Technology Agency, CREST project.

A Linear least squares and normal equations

Since it is the basis of our reformulation, we present here a brief review some fundamental aspects of linear least squares. More information can be found in [Pighin and Lewis 2007].

Given a set of over- or under-constrained linear equations in unknowns, \( u \), we may express them in matrix form as \( A u = b \), where \( A \) is non-square. We can find a solution that minimizes \( \|Au - b\|^2 = \sum_i \|\{A_i\} u - [b_i]\|^2 \) via the normal equations: \( A^T A u = A^T b \). \( A^T A \) is square, so can be inverted if non-singular or decomposed to find \( u \). In weighted least squares, we wish to minimize \( \sum_i w_i \|\{A_i\} u - [b_i]\|^2 \). This can also be expressed in matrix form as \( A^T W A u = A^T W b \), where \( W \) is a diagonal matrix with \( W_{ii} = w_i \). Note that \( u \) and \( b \) need not be simple vectors—they may be matrices as well.

References


