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Numerical approximation of Backward Stochastic Differential Equations with Jumps

Antoine Lejay* Ernesto Mordecki† Soledad Torres‡

Abstract. In this paper we propose a numerical method to approximate the solution of a Backward Stochastic Differential Equations with Jumps (BSDEJ). This method is based on the construction of a discrete BSDEJ driven by a complete system of three orthogonal discrete time-space martingales, the first a random walk converging to a Brownian motion; the second, another random walk, independent of the first one, converging to a Poisson process. The solution of this discrete BSDEJ is shown to weakly converge to the solution of the continuous time BSDEJ. An application to partial integro-differential equations is given.
Keywords: Backward SDEs with jumps, Skorokhod topology, Poisson Process, Monte Carlo method.

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*Projet TOSCA (INRIA and Institut Élie Cartan, UMR CNRS, Nancy-Université, CNRS, INRIA), IECN, BP 239, 54506 Vandœuvre-lès-Nancy CEDEX, France; email: `Antoine.Lejay@iecn.u-nancy.fr`

†Centro de Matemática, Facultad de Ciencias, Universidad de la República. Iguá 4225, 11400, Montevideo, Uruguay; email: `mordecki@cmat.edu.uy`. This author acknowledges support of Proyecto Fondo Clemente Estable 10101.

‡DEUV-CIMFAV, Universidad de Valparaíso, Casilla 5030 Valparaíso, Chile; email: `soledad.torres@uv.cl`. Partially supported by Laboratorio de Análisis Estocástico, PBCT-ACT13 and Fondecyt Grant N.1070919.

1 Introduction

Consider a stochastic process $\{Y_t, Z_t, U_t \colon 0 \le t \le T\}$ that is the solution of a Backward Stochastic Differential Equation with Jumps (in short BSDEJ) of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_{(t,T]\times\mathbb{R}} U_s(x)\tilde{N}(ds, dx), \qquad 0 \le t \le T. \quad (1.1)$$

Here $\{B_t \colon 0 \le t \le T\}$ is a one dimensional standard Brownian motion; $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dt, dx)$ is a compensated Poisson random measure defined in $[0, T] \times (\mathbb{R} \setminus \{0\})$. Both processes are defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, and, as usual in this framework, we assume that they are independent. The *terminal condition* ξ is a square integrable and $\mathcal{F}_T$ measurable random variable (see condition (B) in section 2); the *coefficient* f is a continuous, bounded and Lipschitz function $f \colon [0, T]\times\mathbb{R}^3 \to \mathbb{R}$ (see conditions (A1) and (A2) in section 2). The problem of solving a BSDEJ given the terminal condition ξ, the coefficient f, and the driving processes B and $\tilde{N}$, defined on a stochastic basis $\mathcal{B}$, is to find three adapted processes $\{Y_t, Z_t, U_t \colon 0 \le t \le T\}$ such that (1.1) holds.

In this paper –once the existence and uniqueness of the solution of a BSDEJ as described above is known– we propose a numerical method that approximates the solution of the equation (1.1) in the case when $\tilde{N}$ is a compensated Poisson process.

Nonlinear backward stochastic differential equations of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \le t \le T,$$

were first introduced by E. Pardoux and S. Peng [14] in 1990. Under some Lipschitz conditions on the generator f, the authors stated the first existence and uniqueness result. Later on, the same authors in 1992 developed the BSDE theory relaxing the hypotheses that ensure the existence and uniqueness on this type of equations in [15]. Many subsequent efforts have been made in order to relax further the assumptions on the coefficient $f(s, y, z)$, and many applications in mathematical finance have been proposed.

The backward stochastic differential equations with jumps theory begins with an existence result obtained by S. Tang and X. Li [11]. The authors

stated such a theorem when the generator satisfies some Lipschitz condition. Another relevant contribution on BSDEJ is the paper by R. Situ [17]. In [2], G. Barles, R. Buckdahn and E. Pardoux consider a BSDEJ when the driving noises are a Brownian motion and an independent Poisson random measure. They show the existence and uniqueness of the solution, and in addition, they establish a link with a partial integro-differential equation (in short PIDE).

A relevant problem in the theory of BSDEs is to propose implementable numerical methods to approximate the solution of such equations. Several efforts have been made in this direction as well. For example, in the Markovian case, J. Douglas, J. Ma and P. Protter [9] proposed a numerical method for a class of forward-backward SDEs, based on a four step scheme developed by J. Ma, P. Protter and J. Yong [13]. On the other hand, D. Chevance [6] proposed a numerical method for BSDEs. In [18], J. Zhang proposed a numerical scheme for a class of backward stochastic differential equations with possible path-dependent terminal values. See also [4, 10], among others, where numerical methods for decoupled forward-backward differential equations are proposed, and [3, 1, 12] for backward differential equations.

In the present work we propose to approximate the solution of a BSDEJ driven by a Brownian Motion and an independent compensated Poisson process, through the solution of a discrete backward equation, following the approach proposed for BSDE by Briand, Delyon and Mémin in [5]. The algorithm to compute this approximation is simple.

The rest of the paper is organized as follows. In section 2 we present the problem, propose an approximation process and a simple algorithm to compute it, and present the main convergence result of the paper. Section 3 is devoted to the proof of the previous main result, in the adequate topology. In section 4 we present an application of the main result to a decoupled system of a stochastic differential equation and a backward stochastic differential equation, resulting in a numerical approximation of the solution of an associated partial integro-differential equation.

2 Main result

In order to propose an implementable numerical scheme we consider that the Poisson random measure is simply generated by the jumps of a Poisson process For the sake of simplicity of notation we work in the time interval

$[0,1]$. We then consider a Poisson process $\{N_t\colon 0 \le t \le 1\}$ with intensity λ and jump epochs $\{\tau_k\colon k = 0, 1, \dots\}$. The random measure is then

$$\tilde{N}(dt, dx) = \sum_{k=1}^{N_1} \delta_{(\tau_k,1)}(dt, dx) - \lambda dt \delta_1(dx),$$

where δ_a denotes the Dirac delta at the point a. We also denote $\tilde{N}_t = N_t - \lambda t$. As a consequence, the unknown function $U_t(x)$ that in principle depends on the jump magnitude x becomes $U_t = U_t(1)$, and our BSDEJ in (1.1) becomes

$$\begin{aligned} Y_t &= \xi + \int_t^1 f(s, Y_s, Z_s, U_s) ds - \int_t^1 Z_s dB_s - \sum_{i=N_t+1}^{N_1} U_{T_i} + \lambda \int_t^1 U_s ds, \\ &= \xi + \int_t^1 f(s, Y_s, Z_s, U_s) ds - \int_t^1 Z_s dB_s - \int_{(t,1]} U_s d\tilde{N}_s, \end{aligned} \tag{2.2}$$

for $0 \le t \le 1$. In order to ensure existence and uniqueness of the solution of this equation we consider the following conditions on the coefficient:

(A1) the function f is continuous and bounded.

(A2) the function f is uniformly Lipschitz with respect to variables (y, z, u), with Lipschitz constant K, i.e.

$$|f(s, y_1, z_1, u_1) - f(s, y_2, z_2, u_2)| \le K\big(|y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2|\big).$$

In what respects the terminal condition, we assume:

(B) the random variable ξ is $\mathcal{F}_T$ measurable and $\mathbb{E}(\xi^2) < \infty$.

2.1 Discrete time BSDE with jumps

We propose to approximate the solution of the BSDEJ in (2.2) by the solution of a *discrete* backward stochastic differential equation with jumps in a discrete stochastic basis with a filtration generated by two independent, centered random walks. In order to obtain the representation property a third martingale is considered. The convergence of this approximation relies on the fact that the first random walk converges to the driving Brownian motion, and the second to the driving compensated Poisson process. Let us define these two random walks.

For $n \in \mathbb{N}$ we introduce the first random walk $\{W_k^n \colon k = 0, \dots, n\}$ by

$$W_0^n = 0, \qquad W_k^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \epsilon_i^n \qquad (k = 1, \dots, n), \tag{2.3}$$

where $\epsilon_1^n, \dots, \epsilon_n^n$ are independent symmetric Bernoulli random variables:

$$\mathbb{P}\big(\epsilon_k^n = 1\big) = \mathbb{P}\big(\epsilon_k^n = -1\big) = 1/2, \qquad (k = 1, \dots, n).$$

The second random walk $\{\tilde{N}_k^n \colon k = 0, \dots, n\}$ is non symmetric, and given by

$$\tilde{N}_0^n = 0, \qquad \tilde{N}_k^n = \sum_{i=1}^{k} \eta_i^n \qquad (k = 1, \dots, n), \tag{2.4}$$

where $\eta_1^n, \dots, \eta_n^n$ are independent and identically distributed random variables with probabilities, for each k, given by

$$\mathbb{P}(\eta_k^n = \kappa_n - 1) = 1 - \mathbb{P}(\eta_k^n = \kappa_n) = \kappa_n \qquad (k = 1, \dots, n), \tag{2.5}$$

where $\kappa_n = e^{-\lambda/n}$. We assume that both sequences $\epsilon_1^n, \dots, \epsilon_n^n$ and $\eta_1^n, \dots, \eta_n^n$ are defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (that can be enlarged if necessary), and that they are mutually independent. The (discrete) filtration in the probability space is $\mathbb{F}^n = \{\mathcal{F}_k^n \colon k = 0, \dots, n\}$ with $\mathcal{F}_0^n = \{\Omega, \emptyset\}$ and $\mathcal{F}_k^n = \sigma\{\epsilon_1^n, \dots, \epsilon_k^n, \eta_1^n, \dots, \eta_k^n\}$ for $k = 1, \dots, n$.

In this discrete stochastic basis, given an $\mathcal{F}_{k+1}^n$ measurable random variable y_{k+1}, to represent the martingale difference $m_{k+1} := y_{k+1} - \mathbb{E}\big(y_{k+1} \mid \mathcal{F}_k^n\big)$ we need a set of three strongly orthogonal martingales. This is a motivation to introduce a third martingale increments sequence $\{\mu_k^n = \epsilon_k^n \eta_k^n \colon k = 0, \dots, n\}$. In this context there exist unique $\mathcal{F}_k^n$ measurable random variables z_k, u_k, v_k such that

$$m_{k+1} = y_{k+1} - \mathbb{E}\big(y_{k+1} \mid \mathcal{F}_k^n\big) = \frac{1}{\sqrt{n}} z_k \epsilon_{k+1}^n + u_k \eta_{k+1}^n + v_k \mu_{k+1}^n, \tag{2.6}$$

that can be computed as

$$z_k = \mathbb{E}\big(\sqrt{n} y_{k+1} \epsilon_{k+1}^n \mid \mathcal{F}_k^n\big) = \sqrt{n}\, \mathbb{E}\big(y_{k+1} \epsilon_{k+1}^n \mid \mathcal{F}_k^n\big), \tag{2.7}$$

$$u_k = \frac{\mathbb{E}\big(y_{k+1} \eta_{k+1}^n \mid \mathcal{F}_k^n\big)}{\mathbb{E}\big((\eta_{k+1}^n)^2 \mid \mathcal{F}_k^n\big)} = \frac{1}{\kappa_n(1 - \kappa_n)}\, \mathbb{E}\big(y_{k+1} \eta_{k+1}^n \mid \mathcal{F}_k^n\big), \tag{2.8}$$

$$v_k = \frac{\mathbb{E}\big(y_{k+1} \mu_{k+1}^n \mid \mathcal{F}_k^n\big)}{\mathbb{E}\big((\mu_{k+1}^n)^2 \mid \mathcal{F}_k^n\big)} = \frac{1}{\kappa_n(1 - \kappa_n)}\, \mathbb{E}\big(y_{k+1} \mu_{k+1}^n \mid \mathcal{F}_k^n\big).$$

(Observe that the martingales are orthogonal but not orthonormal, from this fact the necessity of the normalization.) With this information we proceed to formulate the discrete BSDEJ. Consider then a square integrable and $\mathcal{F}_n^n$ measurable final condition ξ^n, the same coefficient f as in equation (2.2), and denote $h = 1/n$. Consider the discrete time backward stochastic differential equation with terminal condition $y_n^n := y_{t_n^n}^n = \xi^n$, and for $k = n-1, \dots, 0$ given by

$$y_k^n = \xi^n + \sum_{i=k}^{n-1} hf(t_i^n, y_i^n, z_i^n, u_i^n) - \sum_{i=k}^{n-1} \left(\sqrt{h} z_i^n \epsilon_{i+1}^n + u_i^n \eta_{i+1}^n + v_i^n \mu_{i+1}^n \right), \quad (2.9)$$

where $y_i^n := y_{t_i^n}^n$, and $t_i^n = i/n$ $(i = 0, \dots, n)$. A solution to (2.9) is an $\mathbb{F}^n$-adapted sequence $\{y_k^n, z_k^n, u_k^n, v_k^n \colon k = 0, \dots, n\}$ such that $y_n^n = \xi^n$ and (2.9) holds.

Equation (2.9) is equivalent to a backward recursive system, beginning by $y_n^n = \xi^n$, followed by

$$\begin{aligned} y_k^n &= y_{k+1}^n + hf(t_k^n, y_k^n, z_k^n, u_k^n) - \sqrt{h} z_k^n \epsilon_{k+1}^n - u_k^n \eta_{k+1}^n - v_k^n \mu_{k+1}^n \\ &= y_{k+1}^n + hf(t_k^n, y_k^n, z_k^n, u_k^n) - m_{k+1}^n, \end{aligned} \quad (2.10)$$

for $k = n-1, \dots, 0$. In view of the representation property (2.6), this last equation (2.10) is equivalent to

$$y_k^n = \mathbb{E}(y_{k+1}^n \mid \mathcal{F}_k^n) + hf(t_k^n, y_k^n, z_k^n, u_k^n). \quad (2.11)$$

The solution can be computed by the following simple algorithm:

(I) Set $(y_n^n, z_n^n, u_n^n, v_n^n) = (\xi^n, 0, 0, 0)$.

(II) For k from $n-1$ down to 0, compute

$$\mathbb{E}(y_{k+1}^n \mid \mathcal{F}_k^n),$$

Compute z_k^n and u_k^n as in (2.7) and (2.8) and solve (2.11) to find y_k^n, using a fixed point principle.

This algorithm shows the existence and uniqueness of the solution of the discrete BSDEJ defined in (2.9), as it is always possible to solve it using the fixed point principle for n large enough, such that $Kh = K/n < 1$. Furthermore, the construction and the Lipschitz property (A1) shows that

$$\mathbb{E}\big(\sup_{0 \le k \le n} (y_k^n)^2 \big) < \infty. \quad (2.12)$$

In what respects $\{v_k^n\}$ it is shown in section 3 that this sequence vanishes, so, in principle, it is not relevant to compute it.

Remark 1 (Computing conditional expectations). Our scheme is very simple to implement. However, as for any scheme that solves BSDEs numerically, it requires to compute conditional expectations. Various methods have then been proposed. In our case, for a function $\Phi\colon \mathbb{R}^{2k+2} \to \mathbb{R}$ we use the formula

$$\begin{aligned}
&\mathbb{E}[\Phi(\epsilon_1^n, \dots, \epsilon_{k+1}^n, \eta_1^n, \dots, \eta_{k+1}^n) \,|\, \mathcal{F}_k^n] \\
&\qquad = \frac{\kappa_n}{2}\Phi(\epsilon_1^n, \dots, \epsilon_k^n, 1, \eta_1^n, \dots, \eta_k^n, \kappa_n - 1) \\
&\qquad + \frac{\kappa_n}{2}\Phi(\epsilon_1^n, \dots, \epsilon_k^n, -1, \eta_1^n, \dots, \eta_k^n, \kappa_n - 1) \\
&\qquad + \frac{1-\kappa_n}{2}\Phi(\epsilon_1^n, \dots, \epsilon_k^n, 1, \eta_1^n, \dots, \eta_k^n, \kappa_n) \\
&\qquad + \frac{1-\kappa_n}{2}\Phi(\epsilon_1^n, \dots, \epsilon_k^n, -1, \eta_1^n, \dots, \eta_k^n, \kappa_n). \qquad (2.13)
\end{aligned}$$

Remark 2. In principle our proposal can be implemented for a d-dimensional Brownian motion and a Poisson process with a jump measure of the form $\Pi(dz) = \sum_{i=1,\dots,\ell} \pi_i \mu_{x_i}(dz)$. In this case, there exist a set of $2^{d+\ell} - 1$ martingales with the representation property (2.6). It should be taken into account that the computation of conditional expectation in (II) involves the sum of $2^{d+\ell}$ terms.

2.2 The main convergence result

Consider the continuous time version $\{Y_t^n, Z_t^n, U_t^n, V_t^n \colon 0 \le t \le 1\}$ of the solution $\{y_i^n, z_i^n, u_i^n, v_i^n \colon i = 0, \dots, n\}$ of the discrete equation (2.10) defined by

$$Y_t^n = y_{[tn]}^n, \quad Z_t^n = z_{[tn]}^n, \quad U_t^n = u_{[tn]}^n, \quad V_t^n = v_{[tn]}^n, \qquad (2.14)$$

for $t \in [0,1]$. Note that Y_t^n, Z_t^n, U_t^n and V_t^n are measurable w.r.t. the σ-algebra $\mathcal{F}_{[nt]}^n$ when $t \in [0,1]$. The discrete BSDEJ in (2.9), denoting $c_n(t) = [nt]/n$ $(0 \le t \le 1)$, can be written as $Y_1^n = \xi^n$ and

$$\begin{aligned}
Y_t^n = \xi^n + \int_{(t,1]} f(s, Y_{s-}^n, Z_{s-}^n, U_{s-}^n) dc_n(s) - \int_{(t,1]} Z_{s-}^n dW_s^n \\
- \int_{(t,1]} U_{s-}^n d\tilde{N}_s^n - \int_{(t,1]} V_{s-}^n d\tilde{M}_s^n
\end{aligned}$$

for $t \in [0,1]$, and $\tilde{M}^n_s = W^n_s \times \tilde{N}^n_s$. With our notations (2.10) becomes

$$Y^n_{t^n_i} = Y^n_{t^n_{i+1}} + \frac{1}{n} f\big(t^n_i, Y^n_{t^n_i}, Z^n_{t^n_i}, U^n_{t^n_i}\big) - \frac{1}{\sqrt{n}} Z^n_{t^n_i} \epsilon^n_{i+1} - U^n_{t^n_i} \eta^n_{i+1} - V^n_{t^n_i} \mu^n_{i+1}, \quad (2.15)$$

In Section 2.1 and Remark 1, we gave a numerical scheme to compute the solution $\{Y^n_{t^n_i}, Z^n_{t^n_i}, U^n_{t^n_i} \colon 0 \le t \le 1\}$.

Theorem 1. *Under the assumptions* (A1), (A2), (B) *and* (H), *the set of processes* $(\xi^n, Y^n, \int_0^\cdot Z^n_s ds, \int_0^\cdot U^n_s ds)$ *converges in the Skorokhod topology, and in probability, towards the solution* $(\xi, Y, \int_0^\cdot Z_s ds, \int_0^\cdot U_s ds)$ *of the BSDEJ* (1.1).

3 Proofs

Our theorem is inspired in the main result of [5]. The proof follows, when possible, the main steps of the proof of this result. The main difference appears due to the fact that the underlying representation theorem for the simple symmetric Bernoulli random walk does not take place in our case, beeing necessary to consider a complete system of orthogonal martingales in order to have this representation property, as we have seen in equation (2.10). The idea is then to consider for both the discrete and the continuous time equations the approximations provided by the Picard's method.

In the continuous case denote $Y^{\infty,0} = Z^{\infty,0} = U^{\infty,0} = 0$ and define $\{Y^{\infty,p+1}, Z^{\infty,p+1}, U^{\infty,p+1} \colon 0 \le t \le 1\}$ inductively as the solution of the backward differential equation

$$Y^{\infty,p+1}_t = \xi + \int_t^1 f(s, Y^{\infty,p}_s, Z^{\infty,p}_s, U^{\infty,p}_s) ds - \int_t^1 Z^{\infty,p+1}_s dB_s - \int_{(t,1]} U^{\infty,p+1}_s d\tilde{N}_s$$

In the discrete case, given n denote $y^{n,0}_k = z^{n,0}_k = u^{n,0}_k = v^{n,0}_k = 0$ for $k = 0, \dots, n$ and define $\{y^{n,p+1}_k, z^{n,p+1}_k, u^{n,p+1}_k, v^{n,p+1}_k \colon k = 0, \dots, n\}$ inductively as the solution of the backward difference equation with final condition $y^{n,p+1}_n = \xi^n$ and backwards recursion defined by

$$\begin{aligned} y^{n,p+1}_k &= y^{n,p+1}_{k+1} + hf(t^n_k, y^{n,p}_k, z^{n,p}_k, u^{n,p}_k) - \sqrt{h} z^{n,p+1}_k \epsilon^n_{k+1} - u^{n,p+1}_k \eta^n_{k+1} - v^{n,p+1}_k \mu^n_{k+1} \\ &= y^{n,p+1}_{k+1} + hf(t^n_k, y^{n,p}_k, z^{n,p}_k, u^{n,p}_k) - m^{n,p+1}_{k+1}, \end{aligned} \quad (3.16)$$

If we consider the continuous time versions of the discrete Picard approximations as defined in (2.14), our method of proof relies on the decompositions

$$\begin{aligned} Y^n - Y &= (Y^n - Y^{n,p}) + (Y^{n,p} - Y^{\infty,p}) + (Y^{\infty,p} - Y) \\ Z^n - Z &= (Z^n - Z^{n,p}) + (Z^{n,p} - Z^{\infty,p}) + (Z^{\infty,p} - Z) \\ U^n - U &= (U^n - U^{n,p}) + (U^{n,p} - U^{\infty,p}) + (U^{\infty,p} - U) \\ V^n &= (V^n - V^{n,p}) + V^{n,p}. \end{aligned}$$

We prove the convergence of the discrete solution as $n \to \infty$ to the solution of (1.1), by proving the uniform convergence in the the Picard iteration principle, as well as the convergence of the approximation of this solution given by this iteration principle at each step when the time step is refined.

3.1 Convergence of the Picard's iteration procedure in the discrete case

The next lemma evaluates the convergence rate of the Picard approximation sequence $(y^{n,p}, z^{n,p}, u^{n,p})$ to (y^n, z^n, u^n) in the discrete scheme, and shows this rate is uniform in the time step $1/n$. It also provides the convergence of the discrete auxiliar sequence $(v_k^{n,p})$ to zero.

We denote $(y, z, u) = \{(y_k, z_k, u_k) \colon k = 0, \dots, n\}$ (with or without superscript n) and, for $\gamma > 1$ introduce the norms

$$\begin{aligned} \|(y^n, z^n, u^n)\|_\gamma^2 &:= \mathbb{E}\left(\sup_{0\le k\le n} \gamma^{k/n} |y_k^n|^2\right) + \frac{1}{n}\mathbb{E}\left(\sum_{k=0}^{n-1} \gamma^{k/n} |z_k^n|^2\right) \\ &\quad + \mathbb{E}\left(\kappa_n(1-\kappa_n)\sum_{k=0}^{n-1} \gamma^{k/n} |u_k^n|^2\right), \\ \|v^n\|_\gamma^2 &:= \mathbb{E}\left(\kappa_n(1-\kappa_n)\sum_{k=0}^{n-1} \gamma^{k/n} |v_k^n|^2\right). \end{aligned}$$

Lemma 1. *There exists $\gamma > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and $p \in \mathbb{N}^*$,*

$$\begin{aligned} &\left\|\left(y^{n,p+1} - y^{n,p}, z^{n,p+1} - z^{n,p}, u^{n,p+1} - u^{n,p}\right)\right\|_\gamma^2 + \left\|v^{n,p+1} - v_p^n\right\|_\gamma^2 \\ &\qquad \le \frac{1}{4}\left\|\left(y^{n,p} - y^{n,p-1}, z^{n,p} - z^{n,p-1}, u^{n,p} - u^{n,p-1}\right)\right\|_\gamma^2. \end{aligned}$$

Proof. We introduce some notation. As n remains fixed during the main part of the proof, it will be omitted in the notations whenever possible. We denote

$$\delta x_k^{p+1} = x_k^{n,p+1} - x_k^{n,p}$$

for a quantity $x = y, z, u, v, m$. With this notation observe that

$$\delta y_k^{p+1} = \delta y_{k+1}^{p+1} + \frac{1}{n}\delta f_k^p - \delta m_{k+1}^{p+1}, \tag{3.17}$$

where m is given in (2.10). If we denote

$$\delta f_k^p = f(t_k^n, y_k^{n,p}, z_k^{n,p}, u_k^{n,p}) - f(t_k^n, y_k^{n,p-1}, z_k^{n,p-1}, u_k^{n,p-1}).$$

Now, for some $\beta > 1$ we develop the quantity

$$\beta^n(\delta y_n^{p+1})^2 - \beta^k(\delta y_k^{p+1})^2 = -\beta^k(\delta y_k^{p+1})^2$$

through a discrete (time dependent) Itô Formula (compare with [16, VII §9]).

$$\begin{aligned}
-\beta^k(\delta y_k^{p+1})^2 &= \sum_{i=k}^{n-1}\left(\beta^{i+1}(\delta y_{i+1}^{p+1})^2 - \beta^i(\delta y_i^{p+1})^2\right) \\
&= (\beta-1)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + \sum_{i=k}^{n-1}\beta^{i+1}\left((\delta y_{i+1}^{p+1})^2 - \beta^i(\delta y_i^{p+1})^2\right) \\
&= (\beta-1)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}(\delta y_{i+1}^{p+1} - \delta y_i^{p+1}) \\
&\quad + \beta\sum_{i=k}^{n-1}\beta^i(\delta y_{i+1}^{p+1} - \delta y_i^{p+1})^2
\end{aligned} \tag{3.18}$$

From (3.17) follows, that

$$(\delta y_{i+1}^{p+1} - \delta y_i^{p+1})^2 \geq \frac{1}{2}\left(\delta m_{i+1}^{p+1}\right)^2 - \frac{1}{n^2}\left(\delta f_i^p\right)^2.$$

Changing signs, using the previous inequality and (3.17) again, from (3.18)

we obtain

$$\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\leq (1-\beta)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\left(\frac{1}{n}\delta f_{i+1}^p - \delta m_{i+1}^{p+1}\right)$$
$$+\frac{\beta}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2. \tag{3.19}$$

We now use the inequality, for $\lambda > 0$,

$$\left(\delta y_i^{p+1}\right)\left(\frac{2\beta}{n}\delta f_{i+1}^p\right) \leq \lambda\left(\delta y_i^{p+1}\right)^2 + \frac{2\beta^2}{\lambda n^2}\left(\delta f_{i+1}^p\right)^2$$

in the second term of (3.19) to obtain

$$\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\leq (1+\lambda-\beta)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + \frac{\beta+4\lambda^{-1}\beta^2}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2$$
$$-2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}$$

We now assume that $1+\lambda-\beta < 0$ and denote $B = \beta + 4\lambda^{-1}\beta^2$ to obtain

$$\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\leq \frac{B}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2 - 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}. \tag{3.20}$$

Formula (3.20) is our first main inequality. ¿From it we obtain the following two results. First, as the last summand is a martingale, taking expectations with $k = 0$, we obtain

$$\frac{\beta}{2}\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2 \leq \frac{B}{n^2}\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2. \tag{3.21}$$

Second, taking supremum over $k = 0, \dots, n$ we have

$$\sup_{0\le k\le n} \beta^k (\delta y_k^{p+1})^2 \le \frac{B}{n^2} \sum_{i=0}^{n-1} \beta^i \left(\delta f_{i+1}^p\right)^2 + 4\beta \sup_{0\le k\le n} \left| \sum_{i=0}^{k} \beta^i \delta y_i^{p+1} \delta m_{i+1}^{p+1} \right| \quad (3.22)$$

To obtain a convenient bound in the last term of (3.21), we use Davis (see [16, VII §3]) and afterwards Hölder inequalities. With F an universal constant, we obtain

$$\begin{aligned} 4\beta \mathbb{E} \sup_{0\le k\le n} \left| \sum_{i=0}^{k} \beta^i \delta y_i^{p+1} \delta m_{i+1}^{p+1} \right| &\le F\, \mathbb{E} \sqrt{\sum_{i=0}^{n} \beta^{2i} \left(\delta y_i^{p+1}\right)^2 \left(\delta m_{i+1}^{p+1}\right)^2} \\ &\le 4\beta F\, \mathbb{E} \sqrt{\sup_{0\le k\le n} \beta^k (\delta y_k^{p+1})^2 \sum_{i=0}^{n} \beta^i \left(\delta m_{i+1}^{p+1}\right)^2} \\ &\le \frac{\beta}{2} \mathbb{E} \sup_{0\le k\le n} \beta^k (\delta y_k^{p+1})^2 + 32\beta F^2 \mathbb{E} \sum_{i=0}^{n} \beta^i \left(\delta m_{i+1}^{p+1}\right)^2 . \end{aligned}$$

Taking expectation in (3.22), using the previous result, and finally (3.21) we obtain, that

$$(1-\beta/2)\mathbb{E} \sup_{0\le k\le n} \beta^k (\delta y_k^{p+1})^2 \le \frac{B}{n^2} \left(1 + 64\beta F^2\right) \sum_{i=0}^{n-1} \beta^i \left(\delta f_{i+1}^p\right)^2 . \quad (3.23)$$

Combining (3.21) and (3.23) we arrive to

$$\mathbb{E} \sup_{0\le k\le n} \beta^k (\delta y_k^{p+1})^2 + \mathbb{E} \sum_{i=k}^{n-1} \beta^i \left(\delta m_{i+1}^{p+1}\right)^2 \le \frac{C}{n^2} \mathbb{E} \sum_{i=k}^{n-1} \beta^i \left(\delta f_{i+1}^p\right)^2 , \quad (3.24)$$

with $C = 2B(1 + 1/\beta + 64\beta F^2)/(1-\beta/2)$.

Using now the Lipschitz property (A1) we see that there exists a constant $\tilde{K}$ such that

$$|\delta f_i^p|^2 \le \tilde{K} \left((\delta y_i^p)^2 + (\delta z_i^p)^2 + n\kappa_n (1-\kappa_n)(\delta u_i^p)^2 \right) . \quad (3.25)$$

Equations (3.24) and (3.25) give

$$\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2+\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\le\frac{C\tilde{K}}{n}\Bigg(\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^p)^2+\frac{1}{n}\mathbb{E}\sum_{k=0}^{n-1}\beta^k\left(\delta z_k^p\right)^2$$
$$+\kappa_n(1-\kappa_n)\mathbb{E}\sum_{k=0}^{n-1}\beta^k\left(\delta u_k^p\right)^2\Bigg).$$

It remains to choose properly β and λ as a function of n. For some $\gamma>1$, set $\beta=\gamma^{1/n}$ and $\lambda=\beta/n$. The condition $1+\lambda<\beta$ is then equivalent to $\gamma>(1-1/n)^{-n}$. Thus, if $\gamma>e$, for n large enough this choice is suitable with our assumptions on β and λ, and C/n remains bounded as $n\to\infty$.

Computing $\mathbb{E}\left(\delta m_{i+1}^{p+1}\right)^2$, we conclude the proof. □

Just like in the continuous case, we can use the Cauchy criterion and the preceding lemma to get the following result.

Proposition 1. *Following the notations of* (2.15) *and* (3.16),

$$\mathbb{E}\left(\sup_{0\le t\le 1}\left|Y_t^{n,p}-Y_t^n\right|^2+\int_0^1\left|Z_t^{n,p}-Z_t^n\right|^2dt+\lambda\int_0^1\left|U_t^{n,p}-U_t^n\right|^2dt\right)$$

converges to 0 *uniformly in* n *as* $p\to\infty$.

3.2 Approximation of Brownian motion and Poisson process

In order to establish convergence in probability, we consider that all the processes are defined on the same probability space.

Lemma 2. *(I) Let us consider a compensated Poisson process* $\widetilde{N}_t=N_t-\lambda t$, *where* N *is a Poisson process of intensity* λ. *For any integer* n, *one can find a family of realizations of independent random variables* $(\eta_i^n)_{i=1,\dots,n}$ *such that (i) the distributions of the* η_s^n*'s are given by* (2.5). *(ii) For* $\widetilde{N}_t^n=\sum_{i=1}^{\lfloor t/h\rfloor-1}\eta_i^n$, *then* $\widetilde{N}^n$ *is a martingale and there exists a family* (ϕ^n) *of one-to-one random time changes from* $[0,T]$ *to* $[0,T]$ *such that* $\sup_{t\in[0,T]}|\phi^n(t)-t|\xrightarrow[n\to\infty]{}0$ *almost*

surely and $\sup_{t\in[0,T]}|\widetilde{N}_t - \widetilde{N}^n_{\phi^n(t)}| \xrightarrow[n\to\infty]{} 0$ *in probability. In other words,* $\widetilde{N}^n$ *converges in probability to* $\widetilde{N}$ *in the* J_1*-Skorokhod topology.*

(II) Let W *be a Brownian motion. Then there exists a family of realizations independent random variables* ϵ^n_k *such that* $\mathbb{P}(\epsilon^n_k = 1) = \mathbb{P}(\epsilon^n_k = -1) = 1/2$ *and* $W^n = h\sum_{i=1}^{\lfloor t/h\rfloor -1}\epsilon^n_i$ *converges uniformly in probability to* W*.*

(III) The couple $(W^n, \widetilde{N}^n)$ *converges in the* J_1*-Skorokhod topology in probability to* $(W, \widetilde{N})$*.*

Proof. (I) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which N is defined. Denote by $\{\tau_i\}_{i=1,\dots,\ell-1}$ the time jumps de N. To simplify the notations, we set $\tau_0 = 0$ and $\tau_\ell = T$. Let A^n the the event

$$A^n = \left\{\begin{array}{l}\text{there is at most one jump of } N\\ \text{on any interval } [kh, h(k+1)) \text{ for } k = 0,\dots,n\end{array}\right\}.$$

We denote by $\overline{A}^n$ the complementary event of A^n. Let $\phi^n(t,\omega)$ be the random piecewise linear functions on $[0,T]\times\Omega$ defined by

$$\phi^n(t,\omega) = \begin{cases}\phi^n(\tau_i,\omega) + \frac{t-\tau_i(\omega)}{\tau_{i+1}(\omega)-\tau_i(\omega)}(\pi_n(\tau_{i+1}(\omega)) - \pi_n(\tau_i(\omega)))\\ \quad\text{when } \omega\in A^n \text{ and } t\in[\tau_i(\omega),\tau_{i+1}(\omega)],\\ t \text{ when } \omega\in\overline{A}^n\end{cases}$$

where $\pi_n(t) = h(\lfloor t/h\rfloor + 1)$. Let us remark that $\phi^n(\tau_i) = \pi_n(\tau_i)$ for $i = 0,\dots,\ell$.

Let us note that for $t\in[\tau_i,\tau_{i+1})$,

$$|\phi^n(t) - t| \le |\phi^n(\tau_i) - \tau_i| + \left|(t-\tau_i)\left(\frac{\pi_n(\tau_{i+1}) - \pi_n(\tau_i)}{\tau_{i+1}-\tau_i} - 1\right)\right|$$

on the event A^n. Hence, one obtains easily that

$$\sup_{t\in[0,T]}|\phi^n(t) - t| \le 2T\ell \sup_{i=0,\dots,\ell-1}\left|\frac{\pi_n(\tau_{i+1}) - \pi_n(\tau_i)}{\tau_{i+1}-\tau_i} - 1\right|,$$

so that we get easily the uniform convergence of $\phi^n(t)$ to t almost surely and in $\mathrm{L}^1(\mathbb{P})$ (since $\mathbb{E}(\ell) < +\infty$).

Let us set for $k = 1,\dots,n$,

$$\eta^n_k = \begin{cases}\kappa & \text{if } N_{kh} - N_{(k-1)h} \ge 1,\\ \kappa - 1 & \text{if } N_{kh} - N_{(k-1)h} = 0.\end{cases}$$

¿From the properties of the Poisson process, we see that the η_k^n's are independent and distributed as in (2.5). In addition, we set $\widetilde{N}_t = \sum_{k=1}^{\lfloor t/h \rfloor} \eta_k^n$ with the convention that $\widetilde{N}_t = 0$ on $[0, h)$.

It is then easily proved that on A^n, for $t \in [\tau_i, \tau_{i+1}]$,

$$|\widetilde{N}_t - \widetilde{N}_{\tau_i} - \widetilde{N}^n_{\phi^n(t)} + \widetilde{N}^n_{\phi^n(\tau_i)}| \leq |\lambda(t - \tau_i) + (\kappa - 1)\lfloor(\phi^n(t) - \phi^n(\tau_i))/h\rfloor|.$$

On the other hand, if $\tau_i \in [kh, (k+1)h)$ and on A^n,

$$\widetilde{N}_{\phi^n(\tau_i)} - \widetilde{N}_{\phi^n(\tau_i -)} = \kappa - 1 + \widetilde{N}_{\tau_i} - \widetilde{N}_{\tau_i -}.$$

For $t \in [\tau_i, \tau_{i+1})$,

$$\begin{aligned}|\widetilde{N}_t - \widetilde{N}_{\phi^n(t)}| &\leq |\widetilde{N}_t - \widetilde{N}_{\tau_i} - \widetilde{N}_{\phi(t)} + \widetilde{N}_{\phi(\tau_i)}| \\ &\quad + \sum_{i \text{ s.t. } \tau_i \leq t,\ i \geq 1} |\widetilde{N}_{\tau_i} - \widetilde{N}_{\tau_i -} - \widetilde{N}_{\phi(\tau_i)} + \widetilde{N}_{\phi(\tau_i -)}| \\ &\quad + \sum_{i \text{ s.t. } \tau_i \leq t,\ i \geq 1} |\widetilde{N}_{\tau_i -} - \widetilde{N}_{\tau_{i-1}} - \widetilde{N}_{\phi(\tau_i)-} + \widetilde{N}_{\phi(\tau_{i-1})}|.\end{aligned}$$

On the event A^n,

$$\begin{aligned}\sup_{t \in [0,T]} |\widetilde{N}_t - \widetilde{N}_{\phi^n(t)}| &\leq \sup_{i=0,\ldots,\ell-1} \sup_{t \in [\tau_i, \tau_{i+1})} |\lambda(t - \tau_i) + (1 - \kappa)\lfloor(\phi^n(t) - \phi^n(\tau_i))/h\rfloor| \\ &\quad + \sum_{i=0,\ldots,\ell-1} |\lambda(\tau_{i+1} - \tau_i) + (1 - \kappa)\lfloor(\phi^n(\tau_{i+1}) - \phi^n(\tau_i))/h\rfloor| \\ &\quad + \ell|\kappa - 1|.\end{aligned}$$

As $\mathbb{E}(\ell) = T/\lambda$, one gets that

$$\mathbb{E}\left(\sup_{t \in [0,T]} |\widetilde{N}_t - \widetilde{N}_{\phi^n(t)}|; A^n\right) \xrightarrow[n \to \infty]{} 0.$$

On $\overline{A}^n$, then $\phi^n(t) = t$ and then

$$\widetilde{N}_{ih} - \widetilde{N}^n_{ih} = -\lambda ih - h(\kappa - 1)i + \sum_{j=0}^{i-1} (N_{(j+1)h} - N_j - 1)\mathbf{1}_{N_{(j+1)h} - N_j \geq 2}.$$

Thus, for any $C > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} |\widetilde{N}_t - \widetilde{N}_t^n| > C\right) \leq \mathbb{P}[\overline{A}^n] + \frac{1}{C}\mathbb{E}\left(\sup_{t\in[0,T]} |\widetilde{N}_t - \widetilde{N}_t^n|; A^n\right).$$

On the other hand, from the very definition of A^n and since $N_{(i+1)h} - N_{ih}$ has the distribution of a Poisson random variable with intensity λh,

$$\mathbb{P}(\overline{A}^n) \leq \sum_{i=0}^{n-1} \mathbb{P}(|N_{(i+1)h} - N_{ih}| \geq 2) \leq n(1 - e^{-\lambda h} - \lambda h e^{-\lambda h}) \leq \frac{\lambda^2 T^2}{n}.$$

Point (II) follows from the Donsker theorem, and (III) holds since W is continuous so that the 2-dimensional path $(W^n, \widetilde{N}^n)$ converges in the J_1-topology to $(W, \widetilde{N})$. □

Remark 3. With a bit more work, it may be proved that $\widetilde{N}$ converges indeed to N in $\mathrm{L}^1(\mathbb{P})$.

3.3 Convergence of martingales

Following the ideas given in [5, Theorem 3.1], consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the following processes defined on it:

- two independent random walks with independent increments, $(W_t^n)_{0\leq t\leq 1}$, $(\tilde{N}_t^n)_{0\leq t\leq 1}$, $(H^n = (W^n, \tilde{N}^n))$;
- $(W_t)_{0\leq t\leq 1}$ a Brownian motion, and $(\tilde{N}_t)_{0\leq t\leq 1}$ a compensated Poisson process, $(H = (W, \tilde{N}))$;

We denote by $(\mathcal{F}_t^n)$ (resp. $(\mathcal{F}_t)$) the right continuous filtration generated by H^n (resp. H).

Let us consider finally a sequence X^n of $\mathcal{F}_T^n$ measurable integrable random variables, and X an $\mathcal{F}_T$ measurable integrable random variables, together with the càdlàg martingales

$$M_t^n = \mathbb{E}\left(X^n \middle| \mathcal{F}_t^n\right), \qquad M_t = \mathbb{E}\left(X \middle| \mathcal{F}_t\right). \tag{3.26}$$

We denote by $[M^n, M^n]$ (resp. $[M, M]$) the quadratic variation of M^n (resp. M) and by $[M^n, W^n], [M^n, \tilde{N}^n]$ (resp. $[M, W], [M, \tilde{N}]$) the cross variation of M^n and W^n (resp. $\tilde{N}^n$).

Proposition 2. *Let us consider the following assumptions:*

(H1) for each n, W^n and $\tilde{N}^n$ are two independent square integrable $\mathcal{F}^n$-martingales with independent increments;

(H2) $W^n \to W$ in probability for the topology of uniform convergence of càdlàg processes indexed by $t \in [0,T]$;

(H3) $\tilde{N}^n \to \tilde{N}$ in probability for the J_1-Skorokhod topology;

(H4) $\mathbb{E}[X^2] + \sup_n \mathbb{E}[(X^n)^2] < \infty$, $\mathbb{E}[|X^n - X|] \to 0$;

Then, if conditions (H1)—(H4) are satisfied,

$$(H^n, M^n, [M^n, M^n], [M^n, W^n], [M^n, \tilde{N}^n]) \to (H, M, [M, M], [M, W], [M, \tilde{N}])$$

in probability for the J_1-Skorokhod topology.

Proof. The proof of this proposition follows directly from [5] Theorem 3.1, and [7] for the convergence of filtrations. We point out two remarks:

- First, since $W^n \xrightarrow{\|\cdot\|_\infty} W$ and $\tilde{N}^n \xrightarrow{J_1} \tilde{N}$ as $n \to \infty$ in probability, then $H^n \xrightarrow{J_1} H$, as $n \to \infty$. So, we prove the convergence in probability for the J_1-Skorokhod topology instead the uniform convergence.
- Second, since $H^n \xrightarrow{J_1} H$, as $n \to \infty$, then we can get the convergence in probability for sequence of martingale M^n to M in the J_1-Skorokhod topology.

□

Corollary 1. *Let W, W^n, $\tilde{N}$ and $\tilde{N}^n$, $n \in \mathbb{N}$, be the standard Brownian motion, the symmetric random walk defined in* (2.3)*, a compensated Poisson Process and the random walk defined in* (2.4)*. Let us consider, on the same space, X and X^n satisfying the hypothesis* (H4) *of proposition 2. Then there exists three sequences of $(Z_t^n)_{0\le t\le 1}$, $(v_t^n)_{0\le t\le 1}$ and $(U_t^n)_{0\le t\le 1}$ $\mathcal{F}_\cdot^n$-predictable processes, and two independent $(Z_t)_{0\le t\le 1}$ and $(U_t)_{0\le t\le 1}$ $\mathcal{F}_\cdot$-predictable processes such that:*

$$\forall t \in [0,1], \begin{cases} M_t^n = \mathbb{E}[X^n] + \int_0^t Z_{s-}^n dW_s^n + \int_0^t U_{s-}^n d\tilde{N}_s^n + \int_0^t v_{s-}^n d\tilde{M}_s^n \\ M_t = \mathbb{E}[X] + \int_0^t Z_{s-} dW_s + \int_0^t U_{s-} d\tilde{N}_s \end{cases}$$

with

$$\int_0^1 (Z_t^n - Z_t)^2 dt \to 0, \quad \lambda \int_0^1 (U_t^n - U_t)^2 dt \to 0$$

in probability and $\int_0^\cdot v_{s-}^n d\tilde{M}_s^n \to 0$ *in probability for the* J_1*-Skorokhod topology.*

Proof. The first part is related to the predictable representation of $\mathcal{F}^n$-martingales in terms of stochastic integrals with respect to three independent random walks, W^n, $\tilde{N}^n$ and $\tilde{M}^n$. The increments of $\tilde{M}^n$ may take up to four different values, which means that we need three orthogonal martingales to represent it (see [8]). It is then easily obtained that $\tilde{M}^n$ is a martingale which is orthogonal to both $\tilde{N}^n$ and W^n. This is why we introduce it. The predictable representation of M with respect to W and $\tilde{N}$ is classical.

For the second part we have from Proposition 2, $[M^n, W^n] \to [M, W]$ in probability for the J_1-Skorokhod topology. Then we have:

$$\sup_{0 \le t \le T} \left| \int_0^{\psi^n(t)} Z_{s-}^n dc_n(s) - \int_0^t Z_s ds \right| \to 0, \tag{3.27}$$

in probability, where $\psi^n(t) \uparrow t$. From Jensen inequality we get that:

$$\sup_{0 \le t \le T} \left| \int_0^t Z_s^n ds - \int_0^t Z_s ds \right| \to 0 \tag{3.28}$$

in probability.

On the other hand since $[M^n, \tilde{N}^n] \to [M, \tilde{N}]$ in probability for the J_1-Skorokhod topology. This implies that

$$\sup_{0 \le t \le T} \left| \int_0^{\psi^n(t)} \eta_{c_n(s)}^2 U_{s-}^n dc_n(s) - \lambda \int_0^t U_s ds \right| \to 0. \tag{3.29}$$

We can apply the same arguments used for (3.28), Burkholder-Davis-Gundy inequality to control the distance between η_k^2 and $\kappa(1-\kappa)$, and the fact that $\kappa \to 1$ as $n \to \infty$ the to get:

$$\sup_{0 \le t \le T} \lambda \left| \int_0^t U_s^n ds - \int_0^t U_s ds \right| \to 0. \tag{3.30}$$

in probability.

The last part of the Corollary 1 is given by the joint convergence of $(M^n, \mathbb{E}(X^n), \int_0^\cdot Z_{s-}^n dW_s^n, \int_0^\cdot U_{s-}^n d\tilde{N}_s^n)$ to $(M, \mathbb{E}(X), \int_0^\cdot Z_{s-} dW_s, \int_0^\cdot U_{s-} d\tilde{N}_s)$. $\square$

3.4 Convergence of the solution of the BSDE

Proposition 3. *Under the* **(A1), (A2)**, *and hypothesis* **(H)**. *Let us consider the scaled random walks W^n and $\tilde{N}^n$. We have that,*

$$\sup_{0\le t\le 1}\left|Y^{n,p}_{\phi^n(t)}-Y^p_t\right|+\int_0^1|Z^{n,p}_{s-}-Z^p_{s-}|^2\,ds+\lambda\int_0^1|U^{n,p}_{s-}-U^p_{s-}|^2\,ds\xrightarrow[n\to\infty]{}0 \tag{3.31}$$

in probability, where ϕ^n is a random one-to-one continuous mapping from $[0,1]$ to $[0,1]$ that converges uniformly to $t\mapsto t$ almost surely.

Proof. This will be done by induction on p. We rewrite (3.16) as

$$\begin{aligned}Y^{n,p+1}_t=\xi^n&+\int_t^1 f\left(s-,Y^{n,p}_{s-},Z^{n,p}_{s-},U^{n,p}_{s-}\right)dc_n(s)\\&-\int_t^1 Z^{n,p+1}_{s-}dW^n_s-\int_t^1 U^{n,p+1}_{s-}d\tilde{N}^n_s-\int_t^1 v^n_s d\tilde{M}^n_s,\end{aligned} \tag{3.32}$$

where $Y^{\cdot,n}$ denotes the càglàd process associated to $Y^{\cdot,n}$. The assumption is that $(Y^{n,p},Z^{n,p},U^{n,p})$ converges to (Y^p,Z^p,U^p) in the sense of (3.31) so that our aim is to prove that $(Y^{n,p+1},Z^{n,p+1},U^{n,p+1})$ converges to $(Y^{p+1},Z^{p+1},U^{p+1})$ in the same sense.

Taking conditional expectations w.r.t. $\mathcal{F}^n_k$ in (3.16) and using the fact that $Y^{n,p+1}_{t_k}$ is $\mathcal{F}^n_k$ measurable, we find that for $t_k\le c_n(t)<t_{k+1}$:

$$Y^{n,p+1}_t=\mathbb{E}\left[\xi^n+\int_t^1 f\left(s-,Y^{n,p}_{s-},Z^{n,p}_{s-},U^{n,p}_{s-}\right)dc_n(s)\,|\mathcal{F}^n_k\right].$$

So that

$$\begin{aligned}M^{n,p+1}_t&:=Y^{n,p+1}_t+\int_0^t f\left(s-,Y^{n,p}_{s-},Z^{n,p}_{s-},U^{n,p}_{s-}\right)dc_n(s)\\&=\mathbb{E}\left[\xi^n+\int_0^1 f\left(s-,Y^{n,p}_{s-},Z^{n,p}_{s-},U^{n,p}_{s-}\right)dc_n(s)\Big|\mathcal{F}^n_k\right]\\&=\mathbb{E}\left[M^{n,p+1}_1\Big|\mathcal{F}^n_k\right]\quad\text{is a }\mathcal{F}^n_\cdot\text{ martingale.}\end{aligned}$$

Moreover, we have the representation

$$\begin{aligned} M_t^{n,p+1} &= \mathbb{E}\left[\int_0^1 Z_{s-}^{n,p+1} dW_s^n + \int_0^1 U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^1 v_{s-}^n d\tilde{M}_s^n \big| \mathcal{F}_k^n\right] \\ &= \int_0^{t_k} Z_{s-}^{n,p+1} dW_s^n + \int_0^{t_k} U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^{t_k} v_{s-}^n d\tilde{M}_s^n \\ &= \int_0^t Z_{s-}^{n,p+1} dW_s^n + \int_0^t U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^t v_{s-}^n d\tilde{M}_s^n. \end{aligned}$$

The last decomposition corresponds to the martingale representation theorem given in Corollary 1.

In order to apply Corollary 1 to the sequence of martingales $\left\{\left(M_t^{n,p+1}\right)_{0\le t\le 1}; n\in\mathbb{N}\right\}$, we have to prove the $L^1(\mathbb{P})$ convergence of $M_1^{n,p+1}$ (the terminal value).

Using the fact that $Y^{n,p}$, $Z^{n,p}$ and $U^{n,p}$ are piecewise constant, we have that

$$\begin{aligned} &\left|M_1^{n,p+1} - \xi - \int_0^1 f\left(s, Y_s^p, Z_s^p, U_s^p\right) ds\right| \\ \le\ & \left|\xi^n - \xi\right| + \int_0^1 \left|f\left(s, Y_s^p, Z_s^p, U_s^p\right) - f\left(s, Y_s^{n,p}, Z_s^{n,p}, U_s^{n,p}\right)\right| ds. \end{aligned}$$

Now using the fact that f is Lipschitz with constant K,

$$\begin{aligned} &\left|M_1^{n,p+1} - \xi - \int_0^1 f\left(s, Y_s^p, Z_s^p, U_s^p\right) ds\right| \\ \le\ & (1+K)\sup_{0\le s\le 1}\left|Y_s^{n,p} - Y_s^p\right| + K\int_0^1 \left|Z_s^p - Z_s^{n,p}\right| ds + K\int_0^1 \left|U_s^p - U_s^{n,p}\right| ds, \end{aligned}$$

and by the induction hypothesis, this last quantity tends to zero in probability and then in $L^1(\mathbb{P})$ (using the $L^2(\mathbb{P})$-boundedness).

Applying Corollary 1, we see that $M^{n,p+1}$ converges to

$$\begin{aligned} M_t^{p+1} &:= \mathbb{E}\left(\xi + \int_0^1 f\left(s, Y_s^p, Z_s^p, U_s^p\right) ds \big| \mathcal{F}_t\right) \\ &= Y_t^{p+1} + \int_0^t f\left(s, Y_s^p, Z_s^p, U_s^p\right) ds, \end{aligned}$$

in the sense that

$$\sup_{0\le t\le 1}\left|M_{\phi^n(t)}^{n,p+1} - M_t^{p+1}\right| + \int_0^1 \left|Z_s^{n,p+1} - Z_s^{p+1}\right| ds + \lambda\int_0^1 \left|U_s^{n,p+1} - U_s^{p+1}\right| ds \to 0$$

in probability, where ϕ^n is a random one-to-one continuous mapping from $[0,1]$ to $[0,1]$ that converges uniformly to $t \mapsto t$ almost surely. Since we want to prove

$$\sup_{0\le t\le 1}\left|Y^{n,p+1}_{\phi^n(t)} - Y^{p+1}_t\right| + \int_0^1 \left|Z^{n,p+1}_s - Z^{p+1}_s\right| ds + \lambda \int_0^1 \left|U^{n,p+1}_s - U^{p+1}_s\right| ds \to 0$$

in probability, the theorem will be demonstrated if we prove that

$$\sup_{0\le t\le 1}\left|\int_0^t f\left(s-, Y^{n,p}_{s-}, Z^{n,p}_{s-}\right) dc_n(s) - \int_0^t f\left(s, Y^{n,p}_s, Z^{n,p}_s, Z^{n,p}_s\right) ds\right| \to 0 \quad (3.33)$$

in probability. We check (3.33). We have that

$$\begin{aligned}\sup_{0\le t\le 1}&\left|\int_0^t f\left(s-, Y^{n,p}_{s-}, Z^{n,p}_{s-}, U^{n,p}_{s-}\right) dc_n(s) - \int_0^t f\left(s, Y^{n,p}_s, Z^{n,p}_s, Z^{n,p}_s\right) ds\right| \\ &\le \frac{1}{n}\left(1 + \sup_{0\le t\le 1}\left|Y^{n,p}_{c_n(t)}\right| + \sup_{0\le t\le 1}\left|Z^{n,p}_{c_n(t)}\right| + \sup_{0\le t\le 1}\left|U^{n,p}_{c_n(t)}\right|\right) \\ &+ K\int_0^1 |Y^{n,p}_s - Y^p_s| ds + K\int_0^1 |Z^{n,p}_s - Z^p_s| ds + K\int_0^1 |U^{n,p}_s - U^p_s| ds. \quad (3.34)\end{aligned}$$

The terms in (3.34) (4.21) tend to 0 with as n tends to ∞ in probability because of **(H)** and the recurrence assumption.

For each of the terms in (3.34) we have that:

$$\frac{1}{n}\sup_{0\le t\le 1}\left|Y^{n,p}_{\phi^n(c_n(t))}\right| \le \frac{1}{n}\sup_{0\le t\le 1}\left|Y^{n,p}_{c_n(t)} - Y^p_{c_n(t)}\right| + \frac{1}{n}\sup_{0\le t\le 1}\left|Y^p_{c_n(t)}\right|. \quad (3.35)$$

The first term in the right hand side of (3.35) tends to 0 with n in probability because of **(H)** and the induction hypothesis. The second term of (3.35) converges to zero in probability because $|Y^p|$ is a L^2 finite random variable and Jensen inequality.

Thanks to Corollary 1, we may use the fact that

$$\sup_{0\le t\le 1}\left|\int_0^t Z^{n,p}_{s-} dc_n(s) - \int_0^t Z^p_s ds\right| \xrightarrow[n\to\infty]{} 0$$

in probability and

$$\sup_{0\le t\le 1}\left|\int_0^t \eta^2_{c_n(s)} U^{n,p}_{s-} dc_n(s) - \lambda\int_0^t U^p_s ds\right| \xrightarrow[n\to\infty]{} 0$$

in probability in order to conclude the proof. □

4 Applications to decoupled system of SDE and BSDEJ and to the numerical computations of the solutions of PIDE

Let X be the solution of the d-dimensional SDE with jumps

$$X_t = x + \int_0^t \sigma(s-, X_{s-})dW_s + \int_0^t b(s-, X_{s-})ds + \int_0^t c(s-, X_{s-})d\widetilde{N}_s, \quad (4.36)$$

where we have assumed that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| + |c(t,x) - c(t,y)| \leq K'|x-y|, \quad (4.37)$$

$$\sup_{t\in[0,T]} (|b(t,0)| + |\sigma(t,0)| + |c(t,0)|) \leq K'' \quad (4.38)$$

for all $t \in [0,T]$ and for alll $x, y \in \mathbb{R}$. Of course, the BSDEJ

$$Y_t = \xi + \int_t^T f(s-, X_{s-}, Y_{s-}, Z_{s-}, U_{s-})ds - \int_s^T Z_{s-}\sigma(s-, X_{s-})dW_s - \int_s^T U_{s-}d\widetilde{N}_s \quad (4.39)$$

is linked to the non-linear PIDE (with $a = \sigma \cdot \sigma^{\mathrm{T}}$) by

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} + \sum_{i,j=1}^d \frac{1}{2} a_{i,j}(t,x) \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x) \frac{\partial u(t,x)}{\partial x_i} \\ &\quad + \int_{\mathbb{R}} \left(u(t, x + c(t,x,z)) - u(t,x) - c_i(t,x,z) \frac{\partial u(t,x)}{\partial x_i} \right) \Pi(dz) \\ &\quad = f(t, x, u(t,x), \nabla u(t,x)\sigma(t,x), u(t, x + c(t,x,\cdot)) - u(t,x)) \quad (4.40) \end{aligned}$$

with the terminal condition $u(T,x) = g(x)$. It is standard in the theory of BSDE that $u(t, X_t) = Y_t$ and thus $u(0,x) = Y_0$. One can compute similarly $u(s,x)$ for any $s \in [0,t)$ by using the solution to (4.36) starting from (s,x) instead of $(0,x)$.

We use now for $\Pi(dz)$ the measure $\Pi(dz) = \sum_{x_i} \pi_i \delta_{x_i}$, where x_i belongs to the interval I_i.

Following Remark 2, we assume for the sake of simplicity that indeed $\Pi(dz) = \lambda\delta_0$ and that the dimension of the Brownian motion W is 1. Hence, we rewrite $c(t,x,z)$ as $c(t,x)$ since only $c(t,x,0)$ is used.

The SDE (4.36) may be discretized the following way for an integer n: we set $\chi_0^n = x$ and for $i = 0, \dots, n$,

$$\begin{aligned}\chi_{i+1}^n = \chi_i^n + hb((i+1)h, \chi_i^n) \\ + \sqrt{h}\sigma(((i+1)h, \chi_i^n))\epsilon_{i+1}^n + c((i+1)h, \chi_i^n)\eta_{i+1}^n \end{aligned} \quad (4.41)$$

where η^n is a Bernoulli approximation of the compensated Poisson process with intensity λ and ϵ^n is a Bernoulli approximation of the Brownian motion W. Of course, the χ_i^n's are easily simulated.

Using our algorithm, it is then possible to find $(y_i^n, z_i^n, u_i^n)_{i=1,\dots,n}$ adapted to $(\mathcal{F}_i^n)_{i=0,\dots,n}$ that solves the discrete BSDE

$$y_i^n = y_{i+1}^n + hf((i+1)h, \chi_i^n, y_i^n, z_i^n, u_{\cdot,i}^n) - z_i^n\epsilon_{i+1}^n - u_i^n\eta_{i+1}^n - v_i^n\epsilon_{i+1}^n\eta_{i+1}^n \quad (4.42)$$

for $i = 0, \dots, n-1$ with the final condition $y_n^n = g(\chi_n^n)$.

We are looking for a function $v^n(i, z)$ such that $y_k^n = v^n(i, \chi_i^n)$ and v^n solves a discrete PDE.

For a function v on $\{0, \dots, n\} \times \mathbb{R}$, we define using (2.13) the discrete operators

$$\begin{aligned} D_0^n v(k,x) &= \mathbb{E}[v(k,x) \,|\, \mathcal{F}_k^n] \\ &= \frac{1-\kappa}{2}\Big(v(k, x + hb(kh,x) + \sqrt{h}\sigma(kh,x) + \kappa c(kh,x)) \\ &\quad + v(k, x + hb(kh,x) - \sqrt{h}\sigma(kh,x) + \kappa c(kh,x))\Big) \\ &\quad + \frac{\kappa}{2}\Big(v(k, x + hb(kh,x) + \sqrt{h}\sigma(kh,x) + (\kappa-1)c(kh,x)) \\ &\qquad + v(k, x + hb(kh,x) - \sqrt{h}\sigma(kh,x) + (\kappa-1)c(kh,x))\Big) \end{aligned}$$

and

$$D_1^n v(k,x) = \mathbb{E}[v(k,x)\epsilon_{k+1}^n \,|\, \mathcal{F}_k^n] \text{ and } D_2^n v(k,x) = \mathbb{E}[v(k,x)\eta_{k+1}^n \,|\, \mathcal{F}_k^n],$$

for which formulae similar to the one for $D_0^n v(k,x)$ can be given.

¿From these results, one can deduce the representation of the solution of a discrete PDE with the help of the χ_i^n. This representation is similar to the representation of the solution of the BSDEJ in term of $Y_t = u(t, X_t)$, where u is the solution to the PIDE (4.40).

Proposition 4. *Let v^n be the solution to the discrete PDE*

$$\begin{aligned} v_i^n(i,x) &= D_0^n v^n(i+1,x) \\ &\quad + hf((i+1)h, x, v_i^n(i,x), h^{-1/2} D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,x)) \\ &\qquad\qquad \text{for } i = 0,\dots,n-1,\ x \in \mathbb{R}, \end{aligned} \tag{4.43}$$

with the terminal condition $v_i^n(T,x) = g(x)$. If $hK < 1$, then this solution exists and is unique. In addition, the solution (y^n, z^n, u^n) to the discrete BSDE (2.10) with the final condition $\xi = g(\chi_n^n)$ which we construct using our algorithm satisfies $y_i^n = v^n(i,\chi_i^n)$, $z_i^n = h^{-1/2} D_1^n v^n(i+1,\chi_i^n)$ and $u_i^n = D_2^n v^n(i+1,\chi_i^n)$,

Proof. As $hK < 1$ the existence and uniqueness of $v^n(i,\cdot)$ follows from the existence of the solution $\rho(x)$ to

$$\begin{aligned} \rho(x) = D_0^n v^n(i+1,x) + hf((i+1)h, x, \rho(x), h^{-1/2} D_1^n v^n(i+1,x), \\ (1-\kappa)^{-1} D_2^n v^n(i+1,x)) \end{aligned}$$

for any $x \in \mathbb{R}$, once $v^n(i+1,\cdot)$ is known. Thus, one can proceed recursively with $i = n-1$ down to 0.

Let (y^n, z^n, u^n) be given by our algorithm. We assume that $v^n(i+1,\chi_{i+1}^n) = y_{i+1}^n$, which is true for $i+1 = n$. Using (2.7), (2.8), and the definitions of D_1^n and D_2^n,

$$\begin{aligned} z_i^n &= h^{-1/2}\mathbb{E}[v^n(i+1,\chi_{i+1}^n)\epsilon_{i+1}^n|\mathcal{F}_i^n] = h^{-1/2} D_1^n v^n(i+1,\chi_i^n), \\ u_i^n &= \mathbb{E}[v^n(i+1,\chi_{i+1}^n)\eta_{i+1}^n|\mathcal{F}_i^n] = D_2^n v^n(i+1,\chi_i^n). \end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_i^n$ in (4.43), since $D_0^n v^n(i+1,\chi_i^n) = \mathbb{E}[v^n(i+1,\chi_{i+1}^n)|\mathcal{F}_i^n]$, we get

$$\begin{aligned} v^n(i,\chi_i^n) &= \mathbb{E}[y_{i+1}^n|\mathcal{F}_i^n] \\ &+ hf((i+1)h, \chi_i^n, v^n(i,\chi_i^n), h^{-1/2} D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,\chi_i^n)), \end{aligned} \tag{4.44}$$

while taking the conditional expectation with respect to $\mathcal{F}_i^n$ in (2.10), we get

$$\begin{aligned} y_i^n &= \mathbb{E}[y_{i+1}^n|\mathcal{F}_i^n] \\ &\quad + hf((i+1)h, \chi_i^n, y_i^n, h^{-1/2} D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,\chi_i^n)), \end{aligned} \tag{4.45}$$

As $hf(\cdot,\cdot,\cdot,\cdot,\cdot)$ is Kh-Lipschitz in its third argument with $Kh < 1$, we obtain that y_i^n and $v^n(i,\chi_i^n)$ are equal. □

4.1 Convergence of the discrete SDE

We may also prove the convergence of the solution of the discrete SDE

$$X_t^n = x + \int_0^t \sigma(s-, X_{s-}^n) dW_s^n + \int_0^t b(s-, X_{s-}^n) dc^n(s) + \int_0^t c(s-, X_{s-}^n) d\widetilde{N}_s^n \tag{4.46}$$

to the solution X of (4.36). The solution X^n is equal to the solution χ^n of (4.41). The proof is similar to the one for BSDE, but is much more simpler.

Proposition 5. *The solution X^n to* (4.46) *converges in probability in the J_1-Skorokhod topology to the solution X to* (4.36).

Proof. We use the Picard scheme on the discrete SDE (4.41). For this, we set

$$\begin{aligned}\chi_{i+1}^{n,p+1} = \chi_i^{n,p} + hb((i+1)h, \chi_i^{n,p}) \\ + \sqrt{h}\sigma(((i+1)h, \chi_i^{n,p}))\epsilon_{i+1}^n + c((i+1)h, \chi_i^{n,p})\eta_i^n\end{aligned} \tag{4.47}$$

with $\chi_i^{n,0} = x$ for $i = 0, \dots, n$ and $\chi_0^{n,p} = x$ for $i = 0, \dots, n$.

Let us note that for any $n, p \geq 0$, χ_i^n, ϵ_i^n and η_i^n are independent for any $n = 1, \dots, n$. In addition, the means of ϵ_i^n and η_i^n are equal to 0, and $\mathbb{E}[(\eta_i^n)^2] = \kappa(1-\kappa)$. Thus, for $i = 1, \dots, n-1$, $\Delta^p \chi_i^n = \chi_i^{n,p} - \chi_i^{n,p-1}$,

$$\begin{aligned}\mathbb{E}((\Delta^{p+1}\chi_{i+1}^n)^2) \leq 3h^2\mathbb{E}\left(\left(\sum_{j\leq i}(b((j+1)h, \chi_j^{n,p}) - b((j+1)h, \chi_j^{n,p-1})\right)^2\right) \\ + 3h\mathbb{E}\left(\sum_{j\leq i}(\sigma((j+1)h, \chi_j^{n,p}) - \sigma((j+1)h, \chi_j^{n,p-1})^2\right) \\ + 3\kappa(1-\kappa)\mathbb{E}\left(\sum_{j\leq i}(c((j+1)h, \chi_j^{n,p}) - c((j+1)h, \chi_j^{n,p-1})^2\right).\end{aligned}$$

Since σ, c and b are uniformly Lipschitz with a constant K' with respect to

the x-variable,

$$\mathbb{E}((\Delta^{p+1}\chi^n_{i+1})^2) \leq 3(K')^2h^2\mathbb{E}\left(\left(\sum_{j\leq i}|\Delta^p\chi^n_j|\right)^2\right) + 6K^2(h+\kappa(1-\kappa))\mathbb{E}\left(\sum_{j\leq i}|\Delta^p\chi^n_j|^2\right).$$

With the convexity inequality and since $h = T/n$, we get

$$\mathbb{E}\big((\Delta^{p+1}\chi^n_{i+1})^2\big) \quad \leq \quad 3K^2\left(\frac{T^2}{n} + 2\left(\frac{T}{n} + \kappa(1-\kappa)\right)\right)\mathbb{E}\left(\sum_{j\leq i}|\Delta^p\chi^n_j|^2\right).$$

We set $\|\chi^n\| := \sum_{i=0}^{n-1}\mathbb{E}((\chi^n_i)^2)$ for a family $\chi^n = (\chi^n_0, \ldots, \chi^n_n)$. It defines a norm on such families for which $\chi^n_0 = 0$. Since $\Delta^p\chi^n_0 = 0$,

$$\|\Delta^{p+1}\chi^n\| \leq 3(K')^2(T^2 + 2T + 2n\kappa(1-\kappa))\|\Delta^p\chi^n\|.$$

As $n(1-\kappa)\kappa$ converges to λT, we obtain that there exist $k < 1$ and n_0 such that $\|\Delta^{p+1}\chi^n\| \leq k\|\Delta^p\chi^n\|$ for any $n \geq n_0$ when T is small enough.

This prove that $\chi^{n,p}$ converges to the solution of (4.41) as $p \to \infty$ at a speed that is uniform in n. This also proves that

$$\forall p \geq 1,\ \sup_{n\geq n_0}\mathbb{E}\left(\sup_{i=1,\ldots,n}(\chi^{n,p}_i - x)^2\right) \leq \sup_{n\geq n_0}\mathbb{E}\left(\sum_{i=0}^{n}(\chi^{n,p}_i - x)^2\right) \leq C \quad (4.48)$$

$$\sup_{n\geq n_0}\mathbb{E}\left(\sup_{i=1,\ldots,n}(\chi^{n,p}_i - \chi^n_i)^2\right) \leq \sup_{n\geq n_0}\mathbb{E}\left(\sum_{i=0}^{n}(\chi^{n,p}_i - \chi^n_i)^2\right) \xrightarrow[p\to\infty]{} 0 \quad (4.49)$$

for some constant C that does not depend on n and p The bound (4.48) also holds when $\chi^{n,p}$ is replaced with the solution χ^n to (4.41).

Let us set $X^{n,p}_t = \chi^{n,p}_{h\lfloor t/h\rfloor}$, $X^n_t = \chi^n_{h\lfloor t/h\rfloor}$ and $X^{\infty,p}$ be the p-step in the Picard scheme of the solution $X^{\infty,\infty}$ to (4.36), with $X^{\infty,0}_t = x$. These processes $X^{\infty,p}$ are known to converge to $X^{\infty,\infty}$ in J_1 in probability.

The idea of the proof is the same as the one of Corollary 1. We prove the following by induction: if $(X^{n,p}, W^n, \widetilde{N}^n)$ converges in probability to $(X^{p,\infty}, W, \widetilde{N})$ in the J_1-Skorokhod topology, then $(X^{n,p+1}, W^n, \widetilde{N}^n)$ converges

to $(X^{p+1,\infty}, W, \widetilde{N})$ in the J_1-Skorokhod topology. If this induction is true (it is of course true that $X^{n,0} = x = X^{\infty,0}$), then if follows easily from (4.49) that X^n converges in probability in the J_1-Skorokhod topology to $X^{\infty,\infty}$.

For this, let us remark first that $B(x) = x \mapsto \left(t \mapsto \int_0^t b(s-, x_{s-})ds\right)$ is continuous from the space of càdlàg functions with the J_1-topology to the space of continuous function with the uniform norm. From the continuity of σ and c follows the convergence in probability of the 6-uple

$$(X^{n,p}, B(X^{n,p}), \sigma(\cdot, X^{n,p}), c(\cdot, X^{n,p}), W^n, \widetilde{N}^n)$$

to $(X^{\infty,p}, B(X^{\infty,p}), \sigma(\cdot, X^{\infty,p}), c(\cdot, X^{\infty,p}), W, \widetilde{N})$ in the J_1-Skorokhod topology.

In addition, $[W^n, W^n]_T = T$ and $\mathbb{E}\left[[N^n, N^n]_T\right] = Tn\kappa(1-\kappa) \xrightarrow[n\to\infty]{} \lambda T$. Thus, this is sufficient to assert from the previous convergences and Proposition 2 that

$$\left(X^{n,p}, B(X^{n,p}), W^n, \widetilde{N}^n, \int_0^\cdot \sigma(s-, X^{n,p}_{s-})dW^n_s, \int_0^\cdot c(s-, X^{n,p}_{s-})d\widetilde{N}^n_s\right)$$

converge in probability to

$$\left(X^{\infty,p}, B(X^{\infty,p}), W, \widetilde{N}, \int_0^\cdot \sigma(s-, X^{\infty,p}_{s-})dW_s, \int_0^\cdot c(s-, X^{\infty,p}_{s-})d\widetilde{N}_s\right)$$

in the J_1-Skorokhod topology. This proves the induction step, and then X^n converges in probability in $X^{\infty,\infty}$ in the J_1-Skorokhod topology.

The convergence is proved on a time interval $[0, T]$ where T is small enough, but it is easy to get it, using $X^{n,p}_T$ instead of x as the initial condition, to prove the result on $[0, 2T]$, and so on. □

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