



# On mean discounted numbers of passage times in small balls of Ito processes observed at discrete times

Frédéric Bernardin, Mireille Bossy, Miguel Martinez, Denis Talay

## ► To cite this version:

Frédéric Bernardin, Mireille Bossy, Miguel Martinez, Denis Talay. On mean discounted numbers of passage times in small balls of Ito processes observed at discrete times. [Research Report] 2008, pp.19. inria-00347610v1

**HAL Id: inria-00347610**

**<https://inria.hal.science/inria-00347610v1>**

Submitted on 16 Dec 2008 (v1), last revised 9 May 2009 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*On mean discounted numbers of passage times in  
small balls of Itô processes observed at discrete times*

Frédéric Bernardin — Mireille Bossy — Miguel Martinez — Denis Talay

N° ????

December 2008

Thème NUM

 *rapport  
de recherche*





# On mean discounted numbers of passage times in small balls of Itô processes observed at discrete times

Frédéric Bernardin<sup>\*</sup>, Mireille Bossy<sup>†</sup>, Miguel Martinez<sup>‡</sup>, Denis  
Talay<sup>§</sup>

Thème NUM — Systèmes numériques  
Projets Tosca

Rapport de recherche n° 1000 — December 2008 — 19 pages

**Abstract:** The aim of this note is to prove estimates on mean values of the number of times that Itô processes observed at discrete times visit small balls in  $\mathbb{R}^d$ . Our technique, in the infinite horizon case, is inspired by Krylov's arguments in [2, Chap.2]. In the finite horizon case, motivated by an application in stochastic numerics, we discount the number of visits by a locally exploding coefficient, and our proof involves accurate properties of last passage times at 0 of one dimensional semimartingales.

**Key-words:** last passage times, Itô processes observed at discrete times

<sup>\*</sup> CETE de Lyon, Laboratoire Régional des Ponts et Chaussées, Clermont-Ferrand, France; frederic.bernardin@developpement-durable.fr

<sup>†</sup> INRIA Sophia Antipolis, EPI TOSCA; Mireille.Bossy@sophia.inria.fr

<sup>‡</sup> Université Paris-Est - Marne-la-Vallée, France; Miguel.Martinez@univ-mlv.fr

<sup>§</sup> INRIA Sophia Antipolis, EPI TOSCA; Denis.Talay@sophia.inria.fr

# Sur les espérances de temps de passage pondérés dans des petites boules pour des processus d'Itô observés en temps discret

**Résumé :** L'objectif de cette note est de démontrer des estimations sur le nombre moyen de visites dans des petites boules de  $\mathbb{R}^d$  d'un processus d'Itô, observé à des instants discrets. En horizon de temps infini, notre technique est inspiré des arguments de Krylov dans [2, Chap.2]. En horizon de temps fini, motivé par une application en analyse numérique stochastique, nous pondérons le nombre de visites par un coefficient localement explosif; notre preuve utilise des propriétés fines sur le dernier temps de passage en zéro d'une semimartingale unidimensionnelle.

**Mots-clés :** dernier temps de passage, processus d'Itô observé à temps discret

# 1 Introduction

The present work is motivated by two convergence rate analyses concerning discretization schemes for diffusion processes whose generators have non smooth coefficients: Bernardin et al. [1] study discretization schemes for stochastic differential equations with multivalued drift coefficients; Martinez and Talay [3] study discretization schemes for diffusion processes whose infinitesimal generators are of divergence form with discontinuous coefficients. In both case, to have reasonable accuracies, a scheme needs to mimic the local behaviour of the exact solution in small neighborhoods of the discontinuities of the coefficients, in the sense that the expectations of the total times spent by the scheme (considered as a piecewise constant process) and the exact solution in these neighborhoods need to be close to each other. The objective of this paper is to provide two estimates for the expectations of such total times spent by fairly general Itô processes observed at discrete times; these estimates concern in particular the discretization schemes studied in the above references.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space satisfying the usual conditions. Let  $(W_t)_{t \geq 0}$  be a  $m$ -dimensional standard Brownian motion on this space. Consider two progressively measurable processes  $(b_t)_{t \geq 0}$  and  $(\sigma_t)_{t \geq 0}$  taking values respectively in  $\mathbb{R}^d$  and in the space  $\mathcal{M}_{d,m}(\mathbb{R})$  of real  $d \times m$  dimensional matrices. For  $X_0$  in  $\mathbb{R}^d$ , consider the Itô process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s. \quad (1)$$

We assume the following hypotheses:

**Hypotheses 1.1.** *There exists a positive number  $K \geq 1$  such that,  $\mathbb{P}$ -a.s.,*

$$\forall t \geq 0, \|b_t\| \leq K, \quad (2)$$

and

$$\forall 0 \leq s \leq t, \frac{1}{K^2} \int_s^t \psi(u) du \leq \int_s^t \psi(u) \|\sigma_u \sigma_u^*\| du \leq K^2 \int_s^t \psi(u) du \quad (3)$$

for all positive locally integrable map  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ .

Notice that (3) is satisfied when  $(\sigma_t)$  is a bounded continuous process; our slightly more general framework is motivated by the reduction of our study, without loss of generality, to one dimensional Itô processes: see section 2.

The aim of this note is to prove the following two results.

Our first result concerns discounted expectations of the total number of times that an Itô process observed at discrete times visits small neighborhoods of an arbitrary point in  $\mathbb{R}^d$ .

**Theorem 1.2** (Infinite horizon). *Let  $(X_t)$  be as in (1). Suppose that the hypotheses 1.1 are satisfied. Let  $h > 0$ . There exists a constant  $C$ , depending only on  $K$ , such that, for all  $\xi \in \mathbb{R}^d$  and  $0 < \varepsilon < 1/2$ , there exists  $h_0 > 0$  (depending on  $\varepsilon$ ) satisfying*

$$\forall h \leq h_0, \quad h \sum_{p=0}^{\infty} e^{-ph} \mathbb{P}(\|X_{ph} - \xi\| \leq \delta_h) \leq C\delta_h, \quad (4)$$

where  $\delta_h := h^{1/2-\varepsilon}$ .

Our second result concerns expectations of the number of times that an Itô process, observed at discrete times between 0 and a finite horizon  $T$ , visits small neighborhoods of an arbitrary point  $\xi$  in  $\mathbb{R}^d$ . In this setting, the expectations are discounted with a multiplying function  $f(t)$  that may tend to infinity when  $t$  tends to  $T$ , which induces technical difficulties which might seem artificial to the reader. However this framework is necessary to analyze the convergence rate of simulation methods for Markov processes whose generators are of divergence form with discontinuous coefficients: see Martinez and Talay [3]. More precisely, we consider functions  $f$  satisfying the following hypotheses:

### Hypotheses 1.3.

1.  $f$  is positive and increasing,
2.  $f$  belongs to  $C^1([0, T]; \mathbb{R}^+)$ ,
3.  $f^\alpha$  is integrable on  $[0, T)$  for all  $1 \leq \alpha < 2$ ,

4. there exists  $1 < \beta < 1 + \eta$ , where  $\eta := \frac{1}{4K^4}$ , such that

$$\int_0^T f^{2\beta-1}(v) f'(v) \frac{(T-v)^{1+\eta}}{v^\eta} dv < +\infty. \quad (5)$$

Remember that we have supposed  $K \geq 1$ . An example of a suitable function  $f$  is the function  $t \rightarrow \frac{1}{\sqrt{T-t}}$  (for which one can choose  $\beta = 1 + \frac{1}{8K^4}$ ).

**Theorem 1.4** (Finite horizon). *Let  $(X_t)$  be as in (1). Suppose that the hypotheses 1.1 are satisfied. Let  $f$  be a function on  $[0, T)$  satisfying the hypotheses 1.3. There exists a constant  $C$ , depending only on  $\beta$ ,  $K$  and  $T$ , such that, for all  $\xi \in \mathbb{R}^d$  and  $0 < \varepsilon < 1/2$ , there exists  $h_0 > 0$  (depending on  $\varepsilon$ ) satisfying*

$$\forall h \leq h_0, \quad h \sum_{p=0}^{N_h} f(ph) \mathbb{P}(\|X_{ph} - \xi\| \leq \delta_h) \leq C\delta_h, \quad (6)$$

where  $N_h := \lfloor T/h \rfloor - 1$  and where  $\delta_h := h^{1/2-\varepsilon}$ .

**Remark 1.5.** *Since all the norms in  $\mathbb{R}^d$  are equivalent, without loss of generality we may choose  $\|Y\| = \sup_{1 \leq i \leq d} |Y^i|$ ,  $Y^i$  denoting the  $i^{\text{th}}$  component of  $Y$ .*

**Remark 1.6.** *Changing  $X_0$  into  $X_0 - \xi$  allows us to limit ourselves to the case  $\xi = 0$  without loss of generality, which we actually do in the sequel.*

## 2 Reduction to one dimensional $(X_t)$ 's

Notice that it suffices to prove that, for all  $1 \leq i \leq d$  and  $h \leq h_0$ ,

$$h \sum_{p=0}^{\infty} e^{-ph} \mathbb{P}(|X_{ph}^i| \leq \delta_h) \leq C\delta_h,$$

or, similarly,

$$h \sum_{p=0}^{N_h} f(ph) \mathbb{P}(|X_{ph}^i| \leq \delta_h) \leq C\delta_h.$$



Set

$$M_t^i := \sum_{j=1}^m \left( \int_0^t \sigma_s^{ij} dW_s^j \right).$$

The martingale representation theorem implies that there exist a process  $(f_t^i)_{t \geq 0}$  and a standard  $\mathcal{F}_t$ -Brownian motion  $(B_t)_{t \geq 0}$  on an extension of the original probability space, such that

$$M_t^i = \int_0^t f_s^i dB_s,$$

and

$$\forall t > 0, \int_0^t (f_\theta^i)^2 d\theta = \sum_{j=1}^m \int_0^t (\sigma_\theta^{ij})^2 d\theta.$$

Therefore it suffices to prove Theorems 1.2 and 1.4 for all one dimensional Itô process

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s \tag{7}$$

such that,  $\exists K \geq 1$ ,  $\mathbb{P}$ -a.s.,

$$\forall t \geq 0, |b_t| \leq K, \tag{8}$$

and

$$\forall 0 \leq s \leq t, \frac{1}{K^2} \int_s^t \psi(u) du \leq \int_s^t \psi(u) \sigma_u^2 du \leq K^2 \int_s^t \psi(u) du \tag{9}$$

for all positive locally integrable map  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ .

### 3 An estimate for a localizing stopping time

In this section we prove a key proposition which is a variant of Theorem 4 in [2, Chap.2]: here we need to consider time indices and stopping times which take values in a mesh with step-size  $h$ . We carefully modify Krylov's arguments.

**Proposition 3.1.** *Let  $(X_t)$  be as in (7). Suppose that conditions (8) and (9) hold true. Let  $\theta : \Omega \rightarrow h\bar{\mathbb{N}}$  be a  $(\mathcal{F}_t)$ -stopping time. Then  $\tau := \inf\{t \in h\mathbb{N}, t \geq \theta, |X_t - X_\theta| \geq 1\}$  is a  $(\mathcal{F}_t)$ -stopping time, and*

$$\mathbb{E}(\mathbf{1}_{\tau < +\infty} e^{-\tau} | \mathcal{F}_\theta) \leq \frac{1}{\text{ch}(\mu)} e^{-\theta} \mathbf{1}_{\theta < +\infty} \text{ a.s.}, \quad (10)$$

where

$$\mu > 0 \text{ and } 1 - K\mu - (1/2)K^2\mu^2 = 0.$$

*Proof.* Set  $\pi(z) := \text{ch}(\mu z)$  for all  $z$  in  $\mathbb{R}$  and

$$L_u \pi(z) := b_u \pi'(z) + \frac{1}{2} \sigma_u^2 \pi''(z).$$

Let  $A \in \mathcal{F}_\theta$ . Suppose that we have proven that, for all  $t > 0$ ,

$$e^{-t \wedge \theta} \geq \mathbb{E}(e^{-\tau} \pi(X_{t \wedge \tau} - X_{t \wedge \theta}) | \mathcal{F}_{t \wedge \theta}). \quad (11)$$

Then we would have

$$\begin{aligned} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta < +\infty} e^{-\theta}) &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta \leq t} e^{-\theta \wedge t}) \\ &\geq \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta \leq t} \mathbb{E}(e^{-\tau} \pi(X_{t \wedge \tau} - X_{t \wedge \theta}) | \mathcal{F}_{t \wedge \theta})) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta \leq t} \mathbb{E}(e^{-\tau} \pi(X_{t \wedge \tau} - X_{t \wedge \theta}) | \mathcal{F}_\theta)) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta \leq t} e^{-\tau} \pi(X_{t \wedge \tau} - X_{t \wedge \theta})) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\theta < +\infty} e^{-\tau} \pi(1)) \\ &\geq \mathbb{E}(\mathbf{1}_A \text{ch}(\mu) \mathbf{1}_{\tau < +\infty} e^{-\tau}), \end{aligned}$$

and thus the desired result would have been obtained. We now prove the inequality (11).

Fix  $t \geq 0$  arbitrarily and set

$$B := \{t \leq \theta\} \cap \{t \leq \tau\} = \{\theta \geq t\},$$

and

$$C := \{t > \theta\} \cap \{t > \tau\} = \{t > \theta\} \cap \{\forall s \in [\theta, t] \cap h\mathbb{N}, |X_s - X_\theta| < 1\}.$$

The sets  $B$  and  $C$  are  $\mathcal{F}_t$ -measurable. Consequently the set  $\{t < \tau\}$  is  $\mathcal{F}_t$ -measurable and thus  $\tau$  is a  $(\mathcal{F}_t)$ -stopping time.

To show (11), we start with checking that,  $\mathbb{P}$ -a.s, for all  $0 \leq s \leq t$  and  $z \in \mathbb{R}$ ,

$$\int_s^t \psi(u) (\pi(z) - L_u \pi(z)) du \geq \text{ch}(\mu z) \left( 1 - K\mu - \frac{1}{2}K^2\mu^2 \right) \int_s^t \psi(u) du = 0 \quad (12)$$

for all positive locally bounded map  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . Indeed, for all  $z \in \mathbb{R}$  and  $u \geq 0$ ,

$$\pi(z) - L_u \pi(z) = \text{ch}(\mu z) - \mu b_u \text{sh}(\mu z) - \frac{1}{2}\mu^2 \sigma_u^2 \text{ch}(\mu z).$$

As  $\text{sh}(\mu |z|) \leq \text{ch}(\mu z)$ , we get (12) by using (8) and (9).

Now, we observe that  $e^{\gamma X_t}$  is integrable for all  $\gamma > 0$  since  $(b_t)$  and  $(\sigma_t)$  are bounded processes. For all  $y \in \mathbb{R}$  Itô's formula implies

$$\begin{aligned} e^{-t \wedge \theta} \pi(X_{t \wedge \theta} + y) &= \mathbb{E} \left( e^{-t \wedge \tau} \pi(X_{t \wedge \tau} + y) \middle| \mathcal{F}_{t \wedge \theta} \right) \\ &\quad + \mathbb{E} \left( \int_{t \wedge \theta}^{t \wedge \tau} e^{-s} (\pi(X_s + y) - L_s \pi(X_s + y)) ds \middle| \mathcal{F}_{t \wedge \theta} \right) \text{ a.s.} \end{aligned}$$

Therefore, in view of (12), there exists a set  $\mathcal{N}_y$  with  $\mathbb{P}(\mathcal{N}_y) = 0$  such that

$$e^{-t \wedge \theta} \pi(X_{t \wedge \theta} + y) \geq \mathbb{E} \left( e^{-\tau} \pi(X_{t \wedge \tau} + y) \middle| \mathcal{F}_{t \wedge \theta} \right) \text{ on } \Omega - \mathcal{N}_y. \quad (13)$$

We now use the continuity of  $\pi$  in order to replace  $y$  by  $-X_{t \wedge \theta}$ . For all positive integer  $q$ , let  $k_q(y)$  be the unique integer  $\ell$  such that  $\ell/2^q \leq y < (\ell + 1)/2^q$ . Then

$$\pi(X_{t \wedge \tau} - X_{t \wedge \theta}) = \lim_{q \rightarrow +\infty} \pi \left( X_{t \wedge \tau} - \frac{k_q(X_{t \wedge \theta})}{2^q} \right).$$

Therefore, by Fatou's lemma and (13),

$$\begin{aligned} \mathbb{E} \left( e^{-\tau} \pi(X_{t \wedge \tau} - X_{t \wedge \theta}) \middle| \mathcal{F}_{t \wedge \theta} \right) &\leq \liminf_{q \rightarrow +\infty} \mathbb{E} \left( e^{-\tau} \pi \left( X_{t \wedge \tau} + \frac{k_q(-X_{t \wedge \theta})}{2^q} \right) \middle| \mathcal{F}_{t \wedge \theta} \right) \\ &\leq \liminf_{q \rightarrow +\infty} e^{-t \wedge \theta} \pi \left( X_{t \wedge \theta} + \frac{k_q(-X_{t \wedge \theta})}{2^q} \right) \\ &= e^{-t \wedge \theta}. \end{aligned}$$

□

We now define two sequences of stopping times  $(\theta_p^h)_p$  and  $(\tau_p^h)_p$  taking values in  $h\bar{\mathbb{N}}$  as

$$\begin{cases} \theta_0^h &= \tau_0^h = 0 \\ \theta_{p+1}^h &= \inf\{t \geq \tau_p^h, t \in h\mathbb{N}, |X_t| \leq \delta_h\}, p \geq 0 \\ \tau_p^h &= \inf\{t \geq \theta_p^h, t \in h\mathbb{N}, |X_t - X_{\theta_p^h}| \geq 1\}, p \geq 1. \end{cases} \quad (14)$$

**Lemma 3.2.** *For all  $p \geq 0$ ,*

$$\mathbb{E} \left( \mathbf{1}_{\theta_p^h < +\infty} e^{-\theta_p^h} \right) \leq \left( \frac{1}{\text{ch}(\mu)} \right)^p, \quad (15)$$

where  $\mu$  is defined as in Proposition 3.1.

*Proof.* Apply Lemma 3.1 to the stopping times  $\theta_p^h$  and  $\tau_p^h$ . It comes

$$\mathbb{E} \left( \mathbf{1}_{\tau_p^h < +\infty} e^{-\tau_p^h} \middle| \mathcal{F}_{\theta_p^h} \right) \leq \frac{1}{\text{ch}(\mu)} e^{-\theta_p^h} \mathbf{1}_{\theta_p^h < +\infty} \mathbb{P} - a.s.,$$

from which

$$\begin{aligned} \mathbb{E} \left( \mathbf{1}_{\theta_{p+1}^h < +\infty} e^{-\theta_{p+1}^h} \right) &\leq \mathbb{E} \left( \mathbf{1}_{\tau_p^h < +\infty} e^{-\tau_p^h} \right) = \mathbb{E} \left( \mathbb{E} \left( \mathbf{1}_{\tau_p^h < +\infty} e^{-\tau_p^h} \middle| \mathcal{F}_{\theta_p^h} \right) \right) \\ &\leq \frac{1}{\text{ch}(\mu)} \mathbb{E} \left( \mathbf{1}_{\theta_p^h < +\infty} e^{-\theta_p^h} \right) \leq \dots \leq \left( \frac{1}{\text{ch}(\mu)} \right)^{p+1}. \end{aligned}$$

□

Any subsequence of  $(\theta_p^h)_p$  takes an infinite value in a finite time, or converges to infinity when  $p$  tends to infinity. Therefore

$$\mathbb{E} \left( h \sum_{p=0}^{+\infty} e^{-ph} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{ph}) \right) = \sum_{p=0}^{+\infty} \mathbb{E} \left( \mathbf{1}_{\theta_p^h < +\infty} \int_{\theta_p^h}^{\tau_p^h} e^{-\eta_s} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) ds \right), \quad (16)$$

where

$$\eta_s := h \lfloor s/h \rfloor \text{ for all } s \geq 0. \quad (17)$$

In the next section we carefully estimate the right-hand side of (16).

## 4 Estimate for the localized problem in the infinite horizon setting

In this section we aim to prove the following lemma.

**Lemma 4.1.** *Consider a one dimensional Itô process (7) satisfying (8) and (9). Let  $\theta : \Omega \rightarrow h\mathbb{N}$  be a  $\mathcal{F}_t$ -stopping time such that  $X_\theta \in [-\delta_h, \delta_h]$  almost surely. Let  $\tau$  be the stopping time  $\tau := \inf\{t \geq \theta, t \in h\mathbb{N}, |X_t - X_\theta| \geq 1\}$ . There exists a constant  $C$ , independent of  $\theta$  and  $\tau$ , such that, for all  $h$  small enough,*

$$\mathbb{E} \left( \mathbf{1}_{\theta < +\infty} \int_{\theta}^{\tau} e^{-\eta_s} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) ds \right) \leq C \delta_h \sqrt{\mathbb{E}(\mathbf{1}_{\theta < +\infty} e^{-\theta})}. \quad (18)$$

*Proof.* To start with, we construct a function  $I$  whose second derivative is  $\mathbf{1}_{[-2\delta_h, 2\delta_h]}$  in order to be able to apply Itô's formula. Let  $I : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$I(z) = \begin{cases} 0 & \text{if } z \leq -2\delta_h, \\ \frac{1}{2}z^2 + 2\delta_h z + 2\delta_h^2 & \text{if } -2\delta_h \leq z \leq 2\delta_h, \\ 4\delta_h z & \text{if } 2\delta_h \leq z. \end{cases} \quad (19)$$

The map  $I$  is of class  $C^1(\mathbb{R})$  and of class  $C^2(\mathbb{R} \setminus (\{-2\delta_h\} \cup \{2\delta_h\}))$ . In addition, for all  $z \in \mathbb{R}$ ,  $I''(z) = \mathbf{1}_{[-\delta_h, \delta_h]}(z)$  (we set  $I''(-2\delta_h) = I''(2\delta_h) = 1$ ). For  $h$  small enough, there exists a constant  $C$  independent of  $h$  such that

$$\sup_{-1-\delta_h \leq z \leq 1+\delta_h} |I(z)| \leq C\delta_h, \quad \sup_{z \in \mathbb{R}} |I'(z)| \leq C\delta_h. \quad (20)$$

Set

$$q_h := \mathbf{1}_{\theta < +\infty} \int_{\theta}^{\tau} e^{-\eta_s} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) ds, \quad (21)$$

and observe that, for all  $s \geq 0$ ,

$$\begin{aligned} & \{X_{\eta_s} \in [-\delta_h, \delta_h]\} \\ &= (\{X_{\eta_s} \in [-\delta_h, \delta_h]\} \cap \{X_s \in [-2\delta_h, 2\delta_h]\}) \cup (\{X_{\eta_s} \in [-\delta_h, \delta_h]\} \cap \{X_s \notin [-2\delta_h, 2\delta_h]\}), \end{aligned}$$

so that

$$\mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \leq \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) + \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \mathbf{1}_{\mathbb{R} \setminus [-2\delta_h, 2\delta_h]}(X_s). \quad (22)$$

From (21) and (22),

$$q_h \leq A^h + B^h, \quad (23)$$

where

$$A^h = \mathbf{1}_{\theta < +\infty} \int_{\theta}^{\tau} e^{-\eta_s} \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) ds \quad (24)$$

and

$$B^h = \mathbf{1}_{\theta < +\infty} \int_{\theta}^{\tau} e^{-\eta_s} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \mathbf{1}_{\mathbb{R} \setminus [-2\delta_h, 2\delta_h]}(X_s) ds. \quad (25)$$

**An upper bound for  $A^h$ .** In view of (9), one has

$$A^h \leq \mathbf{1}_{\theta < +\infty} K^2 e^{-h} \int_{\theta}^{\tau} e^{-s} \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) \sigma_s^2 ds = \mathbf{1}_{\theta < +\infty} K^2 e^{-h} \int_{\theta}^{\tau} e^{-s} I''(X_s) \sigma_s^2 ds.$$

Notice that  $(\int_0^t e^{-s} I'(X_s) \sigma_s dW_s)$  is a uniformly square integrable martingale. Thus the limit  $\int_0^{\infty} e^{-s} I'(X_s) \sigma_s dW_s$  exists a.s. As  $I''$  is continuous except at points  $-\delta_h$  and  $\delta_h$ , an extended Itô's formula (see, e.g., Protter [4, Thm.71, Chap.IV]) implies

$$A^h \leq \mathbf{1}_{\theta < +\infty} K^2 e^{-h} \left\{ e^{-\tau} I(X_{\tau}) - e^{-\theta} I(X_{\theta}) + \int_{\theta}^{\tau} e^{-s} I(X_s) ds - \int_{\theta}^{\tau} e^{-s} I'(X_s) b_s ds - \int_{\theta}^{\tau} e^{-s} I'(X_s) \sigma_s dW_s \right\}. \quad (26)$$

In view of (20) we have  $|I(X_s)| \leq C\delta_h$  even on the random set  $\{\tau = +\infty\}$ . In addition,

$$\mathbb{E} \left[ \mathbf{1}_{\tau < \infty} \int_0^{\tau} e^{-s} I(X_s) ds \right] \leq \mathbb{E} \left[ \mathbf{1}_{\tau < \infty} \int_0^{\tau} e^{-s} |I(X_{\eta_s})| ds \right] + \mathbb{E} \left[ \mathbf{1}_{\tau < \infty} \int_0^{+\infty} e^{-s} |I(X_s) - I(X_{\eta_s})| ds \right].$$

The first expectation is bounded from above by  $C\delta_h$  since, for all integer  $k$  such that  $\theta \leq kh \leq \tau - h$ , one has  $|X_{ph}| \leq 1 + \delta_h$ . The second one is obviously bounded from above by  $C\sqrt{h}$ . Thus

$$\mathbb{E}(A^h) \leq C\delta_h \mathbb{E}(\mathbf{1}_{\theta < +\infty} e^{-\theta}). \quad (27)$$

**An upper bound for  $B^h$ .** Setting  $M_s := \int_{\eta_s}^s \sigma_u dW_u$ , one has

$$\begin{aligned}
B^h &\leq \mathbf{1}_{\theta < +\infty} \int_{\theta}^{\tau} e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)} (|X_s - X_{\eta_s}|) ds \\
&= \mathbf{1}_{\theta < +\infty} e^{-\theta} \int_0^{\tau-\theta} e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)} (|X_{s+\theta} - X_{\eta_{s+\theta}}|) ds \\
&\leq \mathbf{1}_{\theta < +\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)} \left( \left| \int_{\eta_s+\theta}^{s+\theta} b_u du + M_{s+\theta} \right| \right) ds \\
&\leq \mathbf{1}_{\theta < +\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h - Kh, +\infty)} (|M_{s+\theta}|) ds \\
&\leq \mathbf{1}_{\theta < +\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h/2, +\infty)} (|M_{s+\theta}|) ds,
\end{aligned}$$

where we have used (8) and  $h$  is assumed small enough. We now successively use the Cauchy-Schwarz, Jensen and Bernstein inequalities (see, e.g., Revuz and Yor [5, Ex.3.16, Chap.IV] for the Bernstein inequality for martingales):

$$\begin{aligned}
\mathbb{E} B^h &\leq \sqrt{\mathbb{E} (\mathbf{1}_{\theta < +\infty} e^{-\theta})^2} \sqrt{\mathbb{E} \left( \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h/2, +\infty)} (|M_{s+\theta}|) ds \right)^2} \\
&\leq \sqrt{\mathbb{E} (\mathbf{1}_{\theta < +\infty} e^{-\theta})} \sqrt{\mathbb{E} \left( \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h/2, +\infty)} (|M_{s+\theta}|) ds \right)} \\
&\leq \sqrt{\mathbb{E} (\mathbf{1}_{\theta < +\infty} e^{-\theta})} \sqrt{\int_0^{+\infty} e^{-\eta_s} \mathbb{P} \left( \sup_{\eta_s+\theta \leq u \leq s+\theta} |M_{u+\theta}| \geq \frac{\delta_h}{2} \right) ds} \quad (28) \\
&\leq C \sqrt{\mathbb{E} (\mathbf{1}_{\theta < +\infty} e^{-\theta})} \sqrt{\int_0^{+\infty} e^{-\eta_s} \exp \left( -\frac{1}{2K^2} \frac{\delta_h^2}{h} \right) ds} \\
&\leq C \delta_h \sqrt{\mathbb{E} (\mathbf{1}_{\theta < +\infty} e^{-\theta})} h^{\varepsilon-1/2} \exp \left( -\frac{1}{4K^2} h^{-2\varepsilon} \right).
\end{aligned}$$

One can then deduce (18) from (21), (23), (27) and (28).  $\square$

## 5 End of the proof of Theorem 1.2

We apply Lemmas 4.1 and 3.2 and the equality (16) and deduce

$$\begin{aligned}
h\mathbb{E}\left(\sum_{p=0}^{+\infty} e^{-ph} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{ph})\right) &= \sum_{p=0}^{+\infty} \mathbb{E}\left(\mathbf{1}_{\theta_p^h < +\infty} \int_{\theta_p^h}^{\tau_p^h - h} e^{-\eta_s} \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) ds\right) \\
&\leq C\delta_h \sum_{p=0}^{+\infty} \sqrt{\mathbb{E}\left(\mathbf{1}_{\theta_p^h < +\infty} e^{-\theta_p^h}\right)} \\
&\leq C\delta_h \sum_{p=0}^{+\infty} \left(\frac{1}{\text{ch}(\mu)}\right)^{\frac{p}{2}},
\end{aligned}$$

which ends the proof of Theorem 1.2.

## 6 Proof of Theorem 1.4

In the finite horizon setting, we do not need to localize by means of an appropriate sequence of stopping times as in the preceding section. We may start from

$$\mathbb{E}\left(h \sum_{p=0}^{N_h} f(ph) \mathbf{1}_{[-\delta_h, \delta_h]}(X_{ph})\right) = \sum_{p=0}^{N_h} \mathbb{E}\left(\int_{ph}^{(p+1)h} f(\eta_s) \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) ds\right). \quad (29)$$

In all the sequel,  $C_T$  denotes a constant depending on  $K$  and  $T$  only which may change from line to line.

In view of (22) we have

$$\mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \leq \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) + \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \mathbf{1}_{\mathbb{R} \setminus [-2\delta_h, 2\delta_h]}(X_s), \quad (30)$$



and thus,

$$\begin{aligned} \mathbb{E} \left( h \sum_{p=0}^{N_h} f(ph) \mathbf{1}_{[-\delta_h, \delta_h]}(X_{ph}) \right) &\leq \mathbb{E} \left( \int_0^T f(\eta_s) \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) ds \right) + \sum_{p=0}^{N_h} \mathbb{E} (D_h^p) \\ &=: A^h(T) + \sum_{p=0}^{N_h} \mathbb{E} (D_p^h), \end{aligned} \quad (31)$$

where for all  $p \in \{0, \dots, N-1\}$

$$D_p^h := \int_{ph}^{(p+1)h} f(\eta_s) \mathbf{1}_{[-\delta_h, \delta_h]}(X_{\eta_s}) \mathbf{1}_{\mathbb{R} \setminus [-2\delta_h, 2\delta_h]}(X_s) ds. \quad (32)$$

**An upper bound for  $A^h(T)$ .**

In view of (9), as the function  $f$  is increasing, one has

$$A^h(T) \leq K^2 \mathbb{E} \left( \left( \int_0^T f(s) \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) \sigma_s^2 ds \right) \right).$$

Since we may have  $\lim_{s \uparrow T} f(s) = +\infty$ , the proof of Lemma 4.1 does not apply.

Set  $\tau_y := \inf\{s > 0 : X_s - y = 0\}$ . The generalized occupation time formula (see, e.g., Revuz and Yor [5, Ex.1.15, Chap.VI]) implies that

$$\begin{aligned} \mathbb{E} \left( \int_0^T f(s) \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) \sigma_s^2 ds \right) &= \int_{\mathbb{R}} dy \mathbf{1}_{[-2\delta_h, 2\delta_h]}(y) \mathbb{E} \left( \int_0^T f(s) dL_s^y(X) \right) \\ &= \int_{\mathbb{R}} dy \mathbf{1}_{[-2\delta_h, 2\delta_h]}(y) \mathbb{E} \left( \mathbf{1}_{\tau_y \leq T} \int_{\tau_y}^T f(s) dL_s^y(X) \right), \end{aligned} \quad (33)$$

where  $\{L_s^y(X) : s \in [0, T], y \in \mathbb{R}\}$  denotes a bi-continuous càdlàg version of the local time of the process  $(X_t)_{t \geq 0}$ .

Our aim now is to bound from above  $\mathbb{E} \left( \mathbf{1}_{\tau_y \leq T} \int_{\tau_y}^T f(s) dL_s^y(X) \right)$  uniformly in  $y$  in  $[-2\delta_h, 2\delta_h]$ . It clearly suffices to show that there exists a constant  $C_T$  depending only  $\beta$ ,  $K$  and  $T$ , such that

$$\mathbb{E} \left( \int_0^T f(s) dL_s^0(Y) \right) \leq C_T, \quad (34)$$

where the semimartingale  $Y$  is defined as  $Y := X - y$ .

Because  $f$  is not square integrable over  $[0, T)$ , we arbitrarily choose  $\gamma > 0$ . Now  $f$  is square integrable on  $[0, T - \gamma]$  and the function  $t \mapsto \int_0^t f(s) dL_s^0(Y)$  is the local time of the local semimartingale

$$\{f(g_t)Y_t, t \in [0, T - \gamma]\},$$

where  $g_t := \sup\{s < t : Y_s = 0\}$  (see, e.g., Revuz and Yor [5, Sec.4, Chap.IV]). The Stieltjes measure induced by the function  $s \mapsto f(g_s)$  has support  $\mathcal{Z} := \{s \geq 0 : Y_s = 0\}$ , and therefore

$$\int_0^{T-\gamma} |Y_\theta| df(g_\theta) = 0,$$

so that Itô-Tanaka formula implies

$$\mathbb{E} \left( \int_0^{T-\gamma} f(s) dL_s^0(Y) \right) = \mathbb{E} (|f(g_{T-\gamma})Y_{T-\gamma}| - f(g_0)|Y_0| - \mathbb{E} \left( \int_0^{T-\gamma} f(g_s) \text{sgn}((Y_s)) dY_s \right)). \quad (35)$$

Apply Theorem 4.2 in [5]. It comes:

$$\mathbb{E} (|f(g_{T-\gamma})Y_{T-\gamma}|) = \mathbb{E} \left( |f(0)Y_0 + \int_0^{T-\gamma} f(g_s) dY_s| \right),$$

from which

$$\begin{aligned} \mathbb{E} (|f(g_{T-\gamma})Y_{T-\gamma}|) &\leq |f(0)Y_0| + \mathbb{E} \left( \left| \int_0^{T-\gamma} f(g_s) \sigma_s dW_s \right| \right) + \mathbb{E} \left( \left| \int_0^{T-\gamma} f(s) b_s ds \right| \right) \\ &\leq C_T + K^2 \left( \int_0^T \mathbb{E} (f^2(g_s)) ds \right)^{\frac{1}{2}}. \end{aligned} \quad (36)$$

Coming back to (35) we deduce

$$\mathbb{E} \left( \int_0^{T-\gamma} f(s) dL_s^0(Y) \right) \leq C_T + K^2 \left( \int_0^T \mathbb{E} (f^2(g_s)) ds \right)^{\frac{1}{2}}.$$

Thus, as  $\beta > 1$  by assumption, (34) will be proven if we can show that

$$\int_0^T \tilde{\mathbb{E}} [f^{2\beta}(g_s)] ds < +\infty, \quad (37)$$

where  $\tilde{\mathbb{P}}$  is the probability measure defined by a Girsanov transformation removing the drift of  $(Y_t, 0 \leq t \leq T)$ .

Set  $\Lambda(t) := \langle Y \rangle_t$ . Observe that

$$\sup\{\theta \leq s; Y_\theta = 0\} = \Lambda^{-1}(\sup\{\theta \leq \Lambda(s); Y_{\Lambda^{-1}(\theta)} = 0\}).$$

In addition, the Dubins-Schwarz theorem implies that there exists a  $\tilde{\mathbb{P}}$ -Brownian Motion  $(\tilde{B}_t)_{t \geq 0}$  such that  $Y_s = \tilde{B}_{\Lambda(s)}$ . Therefore,

$$\begin{aligned} \tilde{\mathbb{P}}[g_s \geq u] &= \tilde{\mathbb{P}} \left[ \Lambda_{\sup\{t < \Lambda(s) : X_{\Lambda^{-1}(t)} = 0\}}^{-1} \geq u \right] \\ &= \tilde{\mathbb{P}} \left[ \sup\{t < \Lambda(s) : \tilde{B}_t = 0\} \geq \Lambda(u) \right] \\ &= \tilde{\mathbb{P}} \left[ \inf\{t > \Lambda(u) : \tilde{B}_t = 0\} \leq \Lambda(s) \right] \\ &\leq \tilde{\mathbb{P}} \left[ \inf\{t > 0 : Y_u + \tilde{B}_{u+t} - \tilde{B}_u = 0\} \leq K^2(s - u) \right], \end{aligned}$$

from which

$$\begin{aligned} \tilde{\mathbb{P}}[g_s \geq u] &\leq \tilde{\mathbb{E}} \left[ \int_0^{K^2(s-u)} \frac{1}{\sqrt{2\pi\theta^3}} |Y_u| \exp\left(-\frac{Y_u^2}{2\theta}\right) d\theta \right] \\ &\leq C \tilde{\mathbb{E}} \left[ \exp\left(-\frac{Y_u^2}{4K^2(s-u)}\right) \right]. \end{aligned}$$

For  $0 \leq v < s$  set

$$m_v := - \int_0^v \frac{Y_\theta}{2K^2(s-\theta)} dY_\theta.$$

Itô's formula implies that

$$\begin{aligned} -\frac{Y_v^2}{4K^2(s-v)} &\leq -\frac{1}{2K^2} \int_0^v \frac{Y_\theta}{s-\theta} dY_\theta - \frac{1}{8K^4} \int_0^v \frac{Y_\theta^2}{(s-\theta)^2} d\langle Y \rangle_\theta \\ &\quad - \frac{1}{4K^4} \int_0^v \frac{d\theta}{s-\theta}, \end{aligned}$$

from which

$$\tilde{\mathbb{E}} \left[ \exp \left( -\frac{Y_v^2}{4K^2(s-v)} \right) \right] \leq \exp \left( -\frac{1}{4K^4} \int_0^v \frac{d\theta}{s-\theta} \right).$$

As  $f$  is increasing, we deduce

$$\begin{aligned} \tilde{\mathbb{E}} [f^{2\beta}(g_s)] &= \int_{f^{2\beta}(0)}^{f^{2\beta}(s)} \tilde{\mathbb{P}} [f^{2\beta}(g_s) \geq u] du + f^{2\beta}(0) \\ &= \int_{f^{2\beta}(0)}^{f^{2\beta}(s)} \tilde{\mathbb{P}} [g_s \geq (f^{2\beta})^{-1}(u)] du + f^{2\beta}(0) \\ &= \int_0^s \tilde{\mathbb{P}} [g_s \geq v] 2\beta f^{2\beta-1}(v) f'(v) dv + f^{2\beta}(0) \\ &\leq C_T \int_0^s \left( \frac{s-v}{s} \right)^{\frac{1}{4K^4}} f^{2\beta-1}(v) f'(v) dv + f^{2\beta}(0). \end{aligned}$$

Therefore, using the Fubini-Tonelli theorem for positive functions and the hypothesis (5), we get

$$\begin{aligned} \int_0^T \tilde{\mathbb{E}} (f^{2\beta}(g_s)) ds &\leq C_T \int_0^T dv f^{2\beta-1}(v) f'(v) \int_v^T \left( \frac{s-v}{s} \right)^{\frac{1}{4K^4}} ds + C_T T f^{2\beta}(0) \\ &\leq C_T \int_0^T \frac{dv}{v^{\frac{1}{4K^4}}} f^{2\beta-1}(v) f'(v) (T-v)^{1+\frac{1}{4K^4}} + C_T T f^{2\beta}(0) \\ &< +\infty. \end{aligned} \tag{38}$$

In view of (33), we deduce that

$$A^h(T) \leq C_T \delta_h. \tag{39}$$

**An upper bound for  $D_p^h$ .** We proceed as in the proof of Lemma 4.1. The main modification is the following one: As

$$D_p^h \leq \int_0^h f(ph) \mathbf{1}_{[\delta_h/2, +\infty)} (|M_{s+ph}|) ds,$$

we deduce

$$\begin{aligned}
\mathbb{E}D_h^p &\leq \mathbb{E} \left( \int_0^h f(ph) \mathbf{1}_{\sup_{0 \leq u \leq s} |M_{u+ph}| \geq \delta_h/2} ds \right) \\
&\leq f(ph) \mathbb{E} \left( \int_0^h \mathbf{1}_{\sup_{0 \leq u \leq s} |M_{u+ph}| \geq \delta_h/2} ds \right) \\
&\leq f(ph) \left( \int_0^h 2\mathbb{P} \left( \sup_{0 \leq u \leq s} M_{u+ph} \geq \delta_h/2 \right) ds \right) \\
&\leq C f(ph) \exp \left( -\frac{1}{K^2} \frac{\delta_h^2}{h} \right) \\
&\leq C h f(ph) \delta_h.
\end{aligned} \tag{40}$$

From (31), (39), and (40), we deduce that

$$\begin{aligned}
h\mathbb{E} \left( \sum_{p=0}^{N_h} f(ph) \mathbf{1}_{[-\delta_h, \delta_h]}(X_{ph}) \right) &\leq C_T \delta_h + C \delta_h \sum_{p=0}^{N_h} h f(ph) \\
&\leq C_T \delta_h + C \delta_h \int_0^T f(s) ds \\
&\leq C_T \delta_h.
\end{aligned}$$

This ends the proof of Theorem 1.4.

## References

- [1] F. Bernardin, M. Bossy and D. Talay, *Viscosity solutions of parabolic differential inclusions and weak convergence rate of discretizations of multi-valued stochastic differential equations*. In preparation.
- [2] N.V. Krylov, *Controlled diffusion processes*, Springer-Verlag, New-Yor, Heidelberg, Berlin, 1980.
- [3] M. Martinez and D. Talay, *Time discretization of one dimensional Markov processes with generators under divergence form with discontinuous coefficients*. Submitted for publication, 2008.

- [4] P. Protter, *Stochastic Integration and Differential Equations*, Stoch. Modelling and Appl. Prob., vol. 21, Springer, 2005.
- [5] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, vol. 293, Springer-Verlag, Berlin, 1999.



---

Unité de recherche INRIA Sophia Antipolis  
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes  
4, rue Jacques Monod - 91893 ORSAY Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique  
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)

Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

---

Éditeur  
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399