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Abstract: This work is devoted to the well–posedness of a particle position–velocity system which is nonlinear in the sense of McKean. As the dynamics of the velocity depends on the conditional expectation w.r.t. its position, we have to deal with a singular interaction kernel between particles. This study is motivated by a new class of SDEs–PDEs systems, the so called Lagrangian stochastic models which are commonly used in the simulation of turbulent flows.

After a short presentation of these systems, we prove existence and uniqueness results for a simplified model, and a propagation of chaos result for the corresponding interacting particle system.

Key-words: Lagrangian stochastic model, propagation of chaos, conditional Mckean nonlinearity

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Sur les modèles lagrangiens stochastiques McKean conditionnels

Résumé : Dans ce travail, nous nous étudions le caractère bien-posé d'un système de particules en (position, vitesse) nonliéaire au sense de McKean. Dans ce type de système, l'équation de la vitesse fait apparaître une espérance conditionnelle par rapport à sa position courante. Ainsi, la nonlinearité au sense de McKean est singulière.

Cette étude est motivée par une classe de modèles, appelés modèles lagrangiens stochastiques, utilisé pour la simulation d'écoulements turbulents en mécanique des fluides.

Apès une courte présentation de ces modèles, nous montrons un résultat d'existence et d'unicité pour un modèle lagrangien simple, ainsi qu'un résultat de propagation du chaos pour le système de particules associé.

Mots-clés : modèle stochastique lagrangien, propagation du chaos, nonlinéarité conditionnelle au sens de McKean

1 Introduction

In this paper, we prove the well–posedness, and construct an interacting particle approximation for the simplified Lagrangian stochastic model describing the time evolution of the position and velocity of a fluid–particle: for finite horizon time $T > 0$, we consider a d-dimensional standard Brownian motion $(W_t;\ t \in [0,T])$, and a $\mathbb{R}^{2d}$–valued r.v. $(X_0,\mathcal{U}_0)$ independent of W, and $((X_t,\mathcal{U}_t);\ t \in [0,T])$ satisfying

$$\begin{cases} X_t = X_0 + \int_0^t \mathcal{U}_s\, ds, \\ \mathcal{U}_t = \mathcal{U}_0 + \int_0^t B\left[X_s,\mathcal{U}_s;\rho_s\right]\, ds + \int_0^t \sigma(s,X_s,\mathcal{U}_s)\, dW_s, \\ \rho_t \text{ is the density distribution of } (X_t,\mathcal{U}_t) \text{ for all } t \in (0,T]. \end{cases} \tag{1.1}$$

Here, B is the mapping from $\mathbb{R}^d \times \mathbb{R}^d \times L^1(\mathbb{R}^{2d})$ to $\mathbb{R}^d$ defined by

$$B\left[x,u;\gamma\right] = \begin{cases} \dfrac{\int_{\mathbb{R}^d} b(v,u)\gamma(x,v)\, dv}{\int_{\mathbb{R}^d} \gamma(x,v)\, dv} \text{ if } \int_{\mathbb{R}^d} \gamma(x,v)\, dv \neq 0, \\ 0 \text{ elsewhere,} \end{cases} \tag{1.2}$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded interaction kernel. Formally, the drift component of (1.1) involves the function

$$(x,u) \mapsto \mathbb{E}\left[b\left(\mathcal{U}_t,u\right) \Big/ X_t = x\right]. \tag{1.3}$$

Such nonlinearity is typical of Lagrangian stochastic models. A Lagrangian stochastic model is a system of SDEs describing characteristics of a generic fluid–particle in a turbulent flow. These characteristics include the position X_t and the velocity $\mathcal{U}_t$ of a particle.

The present work is strongly motivated by the Lagrangian stochastic approach for the simulation of turbulent flows and the related Probability Density Function (PDF) methods for turbulent flows. The Lagrangian stochastic models for turbulent flows have a dramatic complexity (see description, Section 2). To our knowledge, the mathematical study of these models is an open problem. Since the model aims to be an alternative approach to the Navier–Stokes equations for turbulent flows, it is not surprising that its study presents difficulties. Several recent works separately face some of the difficulties. For example, Bossy & Fontbona [2] study the Poisson PDE (2.2) and its relation with the incompressibility of the mean field velocity; Bossy and Jabir [3] study (1.1) with boundary condition. In this paper, we initiate the theoretical justification of these Lagrangian models. We have started their theoretical study by showing that the simplified model (1.1) is well–posed. Part of our analysis consists in proving a propagation of chaos result for the solution of a particle system approaching the solution of (1.1) which is a first step to prove the convergence of stochastic particle methods. Although simple, the model (1.1) inherits two important features of the Lagrangian stochastic model. First, the infinitesimal generator of the process is non–elliptic due to the Langevin dynamic. Second, the drift coefficient of the velocity involve a conditional expectation w.r.t. the particle position. Consequently, Equation (1.3) is a degenerate and singular nonlinear equation in the sense of McKean.

We prove that (1.1) has a unique solution constructed as the limit of solutions of smoothed systems (see Theorem 3.2). The existence of the solutions of the smoothed systems is obtained by means of the propagation of chaos of interacting particle systems (see Section 3). The uniqueness results are proven in Section 4. In the next section, we briefly present the Lagrangian stochastic models in fluid dynamics, and list some references on the models and their numerical issues.

2 A brief description of Lagrangian stochastic models for turbulent flows

We start this section with a short reminder on the notion of statistical solutions to the Navier–Stokes equation for turbulent flows. For the sake of simplicity, we limit our presentation to monophasic flows. These statistical solutions are random fields, the velocity and the pressure, which are decomposed into their averaged and fluctuating parts. The so called Reynolds decomposition of the Eulerian velocity U is

$$U(t,x,\omega) = \langle U\rangle(t,x) + \mathbf{u}(t,x,\omega),$$

where $\langle U\rangle$ is the (ensemble) averaged part, and $\mathbf{u}$ is the fluctuating part. The Reynolds average $\langle\ \rangle$ is a linear operator applied to the random fields, which is assumed to commute with spatial and times derivatives. Formally applying the Reynolds average to the Navier–Stokes equation, one obtains the so called Reynolds Averaged Navier–Stokes (RANS) equations:

$$\begin{cases} \nabla_x \cdot \langle U \rangle = 0, \\ \partial_t \langle U^{(i)} \rangle + \langle U \rangle \cdot \nabla_x \langle U^{(i)} \rangle = -\frac{1}{\varrho} \nabla_x \langle \mathcal{P}^{(i)} \rangle + \nu \triangle_x \langle U^{(i)} \rangle - \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle, \\ \langle U \rangle (0, x) = \langle U_0 \rangle (x). \end{cases} \tag{2.1}$$

The averaged pressure $\langle \mathcal{P} \rangle$ solves the Poisson equation

$$-\frac{1}{\varrho} \triangle_x \langle \mathcal{P} \rangle = \partial^2_{x_i, x_j} \langle U^{(i)} \rangle \langle U^{(j)} \rangle + \partial^2_{x_i x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle. \tag{2.2}$$

The Reynolds stress tensor stands for the covariance of velocity components:

$$\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle U^{(i)} U^{(j)} \rangle - \langle U^{(i)} \rangle \langle U^{(j)} \rangle.$$

These terms are not closed in Equation (2.1). This problem has led to the introduction of closure models based on Kolmogorov's theory for turbulent flows and experimental observations. For example, the $k - \mathscr{E}$ closures consist in a set of closed equations for the turbulent kinetic k and the dissipation rate $\mathscr{E}$ defined as

$$\begin{aligned} k(t,x) &= \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle (t,x), \\ \mathscr{E}(t,x) &= \nu \langle \partial_{x_j} \mathbf{u}^{(i)} \partial_{x_j} \mathbf{u}^{(i)} \rangle (t,x), \end{aligned}$$

(see, e.g., Mohammadi & Pironneau [9], Pope [13]).

Lagrangian stochastic models have been successfully proposed to provide an alternative approach to the numerical resolution of RANS equations combined with closure models to simulate complex flows for which PDE solvers are inefficient.

In a series of papers initiated in 1985, Stephen B. Pope has proposed Lagrangian stochastic models to describe the position and the instantaneous velocity $(X_t, \mathcal{U}_t)$ of a fluid–particle. Depending on the flow, other Lagrangian characteristics of the turbulence are added to the model. For a fluid with constant mass density ϱ, Lagrangian and Eulerian quantities are related as follows: for all suitable measurable function $g : \mathbb{R}^d \to \mathbb{R}^d$,

$$\langle g(U)\rangle(t,x) = \mathbb{E}\left[g(\mathcal{U}_t)/X_t = x\right].$$

Assuming that $(X,\mathcal{U})$ is a diffusion process, the coefficients of its generator are designed such that the Lagrangian laws are consistent with closed RANS equations and other relevant Physic laws in turbulence theory (see Pope [12], [13] for details). This methodology is known as *PDF method for turbulent flows* in the literature. The simplest model proposed by Pope is the so called simplified Langevin model (see [13]):

$$\begin{cases} X_t = X_0 + \int_0^t \mathcal{U}_s\,ds, \\ \mathcal{U}_t = \mathcal{U}_0 - \dfrac{1}{\varrho}\int_0^t \nabla_x\langle\mathcal{P}\rangle(s,X_s)\,ds + \nu\int_0^t \triangle_x\langle U\rangle(s,X_s)\,ds \\ \quad +C_1\int_0^t \dfrac{\mathscr{E}(s,X_s)}{k(s,X_s)}\left(\langle U\rangle(s,X_s) - \mathcal{U}_s\right)\,ds + \int_0^t \sqrt{C_2\mathscr{E}(s,X_s)}\,dW_s. \end{cases}$$

Here, C_1 and C_2 are positive constants, and $(W_t;\ t\geq 0)$ is a standard $\mathbb{R}^3$–valued Brownian motion. The Poisson equation (2.2) provides the averaged pressure $\langle\mathcal{P}\rangle(t,x)$, and k and $\mathcal{E}$ are assumed to be known.

A less elementary model was proposed by Dreeben & Pope [6] where

$$k(t,x) = \mathbb{E}\left((\mathcal{U}_t^{(i)})^2/X_t = x\right) - \left(\mathbb{E}\left(\mathcal{U}_t^{(i)}/X_t = x\right)\right)^2,$$

and $\mathscr{E}$ is defined as $\mathscr{E}(t,x) = \langle\omega\rangle(t,x)k(t,x)$ where $\langle\omega\rangle(t,x) = \mathbb{E}\left(\omega_t/X_t = x\right)$ and $(\omega_t;\ t\geq 0)$ is the solution of the following SDE:

$$\begin{aligned} \omega_t = \omega_0 + C_3\int_0^t \langle\omega\rangle(s,X_s)\left(\langle\omega\rangle(s,X_s) - \omega_s\right)\,ds \\ - \int_0^t \omega_s S_\omega(s,X_s)\langle\omega\rangle(s,X_s)\,ds + \int_0^t \sqrt{C_4\omega_s\left(\langle\omega\rangle(s,X_s)\right)^2}\,d\widetilde{W}_s. \end{aligned}$$

Here, C_3, C_4 are positive constants, $\widetilde{W}$ is a one–dimensional standard Brownian motion independent of W, and

$$S_\omega(t,x) = C_{\omega 2} + C_{\omega 1}\frac{\left(\langle U^{(i)}U^{(j)}\rangle(t,x) - \langle U^{(i)}\rangle(t,x)\langle U^{(j)}\rangle(t,x)\right)\partial_{x_j}\langle U^{(i)}\rangle(t,x)}{\langle\omega\rangle(t,x)k(t,x)}.$$

A description of the numerical issues can be found in Pope & Xu [14]. A recent application to meteorology is developed by one of the authors: see, e.g., Bossy *et al.* [1].

Notation. Let $0 < T < +\infty$ be fixed.

- For all $t \in (0,T]$, we set $Q_t = (0,t) \times \mathbb{R}^{2d}$.
- For all $q \geq 1$, $\mathcal{C}([0,T]\,;\mathbb{R}^q)$ denotes the space of $\mathbb{R}^q$–valued continuous functions equipped with the compact uniform metric $\|\ \|_\infty$.
- Given a metric space E, $\mathcal{C}^k_b(E)$ denotes the set of real–valued bounded functions defined on E with continuous derivatives up to order k; $\mathcal{C}^k_c(E)$ denotes the set of real–valued functions with continuous derivatives up to order k and with compact support.
- Given a metric space E, $\mathcal{M}(E)$ denotes the set of probability measures defined on E, equipped with the weak convergence topology.
- In all the paper, C is a constant which does not depend on the various parameters ϵ, $N, \ldots$, but does vary from line to line.

3 Main result

In the study of (1.1), difficulties come from the dependency of the drift coefficient on the conditional expectation (1.3). Related situations have been studied by Sznitman [17], Oelschlager [10] and Dermoune [4]: Sznitman [17] has considered the one–dimensional nonlinear SDE

$$d\zeta_t = p_t(\zeta_t)\,dt + dW_t,$$

where p_t is the Lebesgue density of ζ_t. Oelschlager [8] has considered the family of models

$$d\zeta_t = F\left(\zeta_t, p_t(\zeta_t)\right)\,dt + dW_t,$$

where $F : \mathbb{R}^d \times \mathbb{R}$ is a bounded Lipschitz function, and

$$d\zeta_t = \nabla p_t(\zeta_t)\,dt + dW_t.$$

Dermoune [4] has studied the system

$$d\zeta_t = \mathbb{E}\left(v(\zeta_0)/\zeta_t\right)\, dt + dW_t,$$

where $v : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Our situation substantially differs from the above: our drift coefficient depends on conditional density rather than the density and the infinitesimal generator of $(X_t, \mathcal{U})$ is not strongly elliptic.

In the sequel, we suppose the following hypotheses:

(H)

- b is a bounded continuous function and the law μ_0 of $(X_0, \mathcal{U}_0)$ is such that
$$\int_{\mathbb{R}^{2d}} \left(|x| + |u|^2\right) \mu_0(dx\, du) < +\infty.$$
- The velocity diffusion coefficient σ is bounded and strongly elliptic: for $a(t,x,u) := \sigma(t,x,u)\sigma^*(t,x,u)$, there exists $\lambda > 0$ such that, for all $t \in (0,T],\ x,\ u,\ v \in \mathbb{R}^d$,
$$\frac{|v|^2}{\lambda} \leq \sum_{i,j=1}^{d} a^{(i,j)}(t,x,u) v_i v_j \leq \lambda\, |v|^2 . \tag{3.1}$$
- For all $1 \leq i,j \leq d$, $a^{(i,j)}$ is Hölder continuous in the following sense: there exist $\alpha \in (0,1]$ and K depending only on T and d such that, for all $(s,x,u), (t,y,v) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,
$$|a^{(i,j)}(t,x,u) - a^{(i,j)}(s,y,v)| \leq C(|t-s|^{\frac{\alpha}{2}} + |x-y-v(t-s)|^\alpha + |u-v|^\alpha). \tag{3.2}$$

Remark 3.1. *The hypothesis (3.2) on a is classical in the literature on ultraparabolic PDEs, see e.g. Theorem A.1 and Subsection A.2. In particular, it is satisfied when $\sigma(t,x,u)$ is a function of (t,u) only.*

For fixed $N \geq 1$ and $\epsilon > 0$, we consider the interacting particle system $\{(X_t^{i,\epsilon,N}, \mathcal{U}_t^{i,\epsilon,N};\ t \in [0,T]);\ 1 \leq i \leq N\}$ as defined by

$$\begin{cases} X_t^{i,\epsilon,N} & = X_0^i + \int_0^t \mathcal{U}_s^{i,\epsilon,N}\, ds, \\ \mathcal{U}_t^{i,\epsilon,N} & = \mathcal{U}_0^i + \int_0^t \dfrac{\sum_{j=1}^N b(\mathcal{U}_s^{j,\epsilon,N}, \mathcal{U}_s^{i,\epsilon,N}) \phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}{\sum_{j=1}^N \left(\phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N}) + \epsilon\right)}\, ds \\ & \quad + \int_0^t \sigma(s, X_s^{i,\epsilon,N}, \mathcal{U}_s^{i,\epsilon,N})\, dW_s^i,\ i = 1, \ldots, N. \end{cases} \tag{3.3}$$

Here, $\{(X_0^i,\mathcal{U}_0^i),\ (W_t^i;\ t\in[0,T]);\ i\geq 1\}$ are independent copies of $((X_0,\mathcal{U}_0),\ (W_t;\ t\in[0,T]))$, and $\{\phi_\epsilon;\ \epsilon>0\}$ denotes a family of mollifiers of the type $\phi_\epsilon(x)=\dfrac{1}{\epsilon^d}\phi(\dfrac{x}{\epsilon})$, where $\phi\in\mathcal{C}_c^1(\mathbb{R}^d)$ is such that $\phi\geq 0$ and $\int_{\mathbb{R}^d}\phi(z)\,dz=1$. As the drift coefficient of the particle system (3.3) is uniformly bounded, the well–posedness of (3.3) follows from Proposition 4.4 (see Section 4.1) and Girsanov's theorem.

In Section 5, we prove that the particles propagate chaos. In particular, as N tends to infinity, $(X^{1,\epsilon,N},\mathcal{U}^{1,\epsilon,N})$ converges weakly to the solution of

$$\begin{cases} X_t^\epsilon = X_0+\int_0^t \mathcal{U}_s^\epsilon\,ds,\\ \mathcal{U}_t^\epsilon=\mathcal{U}_0+\int_0^t B_\epsilon\left[X_s^\epsilon,\mathcal{U}_s^\epsilon;\rho_s^\epsilon\right]\,ds+\int_0^t\sigma(s,X_s^\epsilon,\mathcal{U}_s^\epsilon)\,dW_s,\\ \rho_t^\epsilon \text{ is the density of } (X_t^\epsilon,\mathcal{U}_t^\epsilon) \text{ for all } t\in(0,T], \end{cases} \tag{3.4}$$

where the kernel $B_\epsilon\left[x,u;\gamma\right]$ is defined as follows: for all nonnegative $\gamma\in L^1(\mathbb{R}^{2d})$, $(x,u)\in\mathbb{R}^{2d}$,

$$B_\epsilon\left[x,u;\gamma\right]=\frac{\int_{\mathbb{R}^d} b(v,u)\phi_\epsilon\star\gamma(x,v)\,dv}{\int_{\mathbb{R}^d}\phi_\epsilon\star\gamma(x,v)\,dv+\epsilon},$$

where

$$\phi_\epsilon\star\gamma(x,u)=\int_{\mathbb{R}^d}\phi_\epsilon(x-y)\gamma(y,u)\,dy.$$

Our main result is as follows.

Theorem 3.2. *Assume (H).*

(i) For all $\epsilon>0$, the sequence $\{(X^{1,\epsilon,N},\mathcal{U}^{1,\epsilon,N});\ N\geq 1\}$ converges weakly to a weak solution $(X^\epsilon,\mathcal{U}^\epsilon)$ of (3.4). This solution is unique and, if $\mathbb{P}^\epsilon$ denotes the law of $(X^\epsilon,\mathcal{U}^\epsilon)$, the interacting particle system is $\mathbb{P}^\epsilon$–chaotic; that is, for every integer $k\geq 2$ and every finite family $\{\psi_l;\ l=1,\ldots,k\}$ of $\mathcal{C}_b(\mathcal{C}([0,T];\mathbb{R}^{2d}))$,

$$\langle\mathbb{P}^{\epsilon,N},\psi_1\otimes...\psi_k\otimes\ldots\rangle\longrightarrow\prod_{l=1}^k\langle\mathbb{P}^\epsilon,\psi_l\rangle,\ \text{when } N\longrightarrow+\infty.$$

(*ii*) *When ϵ tends to* 0*, $(X^\epsilon,\mathcal{U}^\epsilon)$ converges weakly to the unique solution $(X,\mathcal{U})$ of (1.1).*

Weak solutions of (3.4) and (1.1) are defined by appropriate martingale problems in the next section (see Definition 4.1 and Definition 4.2).

The rest of the paper is organized as follows: in Section 4, we prove weak uniqueness results for Equations (3.4) and (1.1). In Section 5, we get existence: we show that, for all $\epsilon > 0$, the law of the particle system (3.3) converges weakly and we identify the limit as the weak solution of (3.4); we then get existence of a weak solution of (1.1) by letting ϵ decreases to 0.

4 Uniqueness results

We introduce the notions of weak solutions to (1.1) and (3.4). Let $((x_t,u_t);\ t\in(0,T])$ be the canonical processes in the sample space $\mathcal{C}([0,T];\mathbb{R}^{2d})$. The martingale problem related to the smoothed particle system (3.4) is stated as follows.

Definition 4.1. *A probability measure $\mathbf{P}^\epsilon$ on the canonical space $\mathcal{C}([0,T];\mathbb{R}^{2d})$ is a weak solution of (3.4),or equivalently, a solution to the martingale problem (MP_ϵ) if:*

(*i*) $\mathbf{P}^\epsilon\circ(x_0,u_0)^{-1}=\mu_0$.

(*ii*) *For all $t\in(0,T]$, the time marginal $\mathbf{P}^\epsilon\circ(x_t,u_t)^{-1}$ has a density ρ^ϵ_t w.r.t. Lebesgue measure on $\mathbb{R}^{2d}$.*

(*iii*) *For all $f\in\mathcal{C}^2_b(\mathbb{R}^{2d})$, the process*

$$f(x_t,u_t)-f(x_0,u_0)-\int_0^t\mathcal{A}^\epsilon_{\rho^\epsilon_s}f(s,x_s,u_s)\,ds$$

is a $\mathbf{P}^\epsilon$–martingale where, for all $\gamma\in L^1(\mathbb{R}^{2d})$, $\mathcal{A}^\epsilon_\gamma$ is the differential operator

$$\begin{aligned}\mathcal{A}^\epsilon_\gamma f(t,x,u):=\ & u\cdot\nabla_x f(x,u)+B_\epsilon\left[x,u;\gamma\right]\cdot\nabla_u f(x,u)\\ &+\frac{1}{2}\textstyle\sum_{i,j=1}^d a^{(i,j)}(t,x,u)\partial^2_{u_i,u_j}f(x,u).\end{aligned}$$

The martingale problem related to (1.1) is stated as follows.

Definition 4.2. *A probability measure* $\mathbf{P}$ *on the canonical space* $\mathcal{C}([0,T]\,;\mathbb{R}^{2d})$ *is a weak solution of (1.1), or equivalently, a solution to the martingale problem (MP) if*

(i) $\mathbf{P}\circ(x_0,u_0)^{-1}=\mu_0$.

(ii) For all $t\in(0,T]$, *the time marginal* $\mathbf{P}\circ(x_t,u_t)^{-1}$ *has a positive density* ρ_t *w.r.t. Lebesgue measure on* $\mathbb{R}^{2d}$.

(iii) For all $f\in\mathcal{C}^2_b(\mathbb{R}^{2d})$, *the process*

$$f(x_t,u_t)-f(x_0,u_0)-\int_0^t\mathcal{A}_{\rho_s}f(s,x_s,u_s)\,ds$$

is a $\mathbf{P}$*–martingale, where, for each positive* $\gamma\in L^1(\mathbb{R}^{2d})$, $\mathcal{A}_\gamma$ *is the differential operator*

$$\begin{aligned}\mathcal{A}_\gamma f(t,x,u):=\quad & u\cdot\nabla_x f(x,u)+\left(B\left[x,u;\gamma\right]\cdot\nabla_u f(x,u)\right)\\ & +\frac{1}{2}\textstyle\sum_{i,j=1}^d a^{(i,j)}(t,x,u)\partial^2_{u_i,u_j}f(x,u).\end{aligned}$$

We prove the following uniqueness result.

Proposition 4.3. *There is at most one weak solution to Equation (1.1) and one weak solution to Equation (3.4).*

For a weak solution $\mathbf{P}$ of (1.1), and a weak solution $\mathbf{P}^\epsilon$ of (3.4), we consider the densities ρ_t and ρ^ϵ_t as in Definitions (4.2) and (4.1). We prove that ρ_t and ρ^ϵ_t are the unique solutions of nonlinear mild equations (see Lemma 4.5) which implies Proposition 4.3. A preliminary step consists in studying the linear case $(b=0)$.

4.1 Study of a Langevin system

For $(y,v)\in\mathbb{R}^{2d}$, consider the pair of processes $(Y^{s,y,v}_t,V^{s,v}_t;\ t\geq s\geq 0)$ solution of the Langevin equation

$$\begin{cases}Y^{s,y,v}_t & =y+\int_s^t V^{s,y,v}_\theta\,d\theta,\\ V^{s,y,v}_t & =v+\int_s^t\sigma(\theta,Y^{s,y,v}_\theta,V^{s,y,v}_\theta)\,dW_\theta.\end{cases}\tag{4.1}$$

The following result is a slight extension of a theorem due to Francesco and Pascucci [5]. We postpone the statement of this theorem and the proof of Proposition 4.4 in Subsection A.2.

Proposition 4.4. *Assume (H)–(3.1) and (H)–(3.2). There exists a unique weak solution to (4.1). In addition, this solution admits a density* $\Gamma(s, y, v; t, x, u)$ *w.r.t. Lebesgue measure such that:*

(i) For all $(t, x, u) \in (0, T] \times \mathbb{R}^{2d}$, $1 \le i,\ j \le d$, *the derivatives* $\partial_{v_i}\Gamma(s, y, v; t, x, u)$ *exist and are continuous for all* $(s, y, v) \in \mathbb{R} \times \mathbb{R}^{2d}$ *such that* $(s, y, v) \neq (t, x, u)$.

(ii) Let $f : \mathbb{R}^{2d} \to \mathbb{R}$ *be a bounded continuous function. Then the function* $G_{t,f}$ *defined by*

$$G_{t,f}(s, y, v) = \int_{\mathbb{R}^{2d}} \Gamma(s, y, v; t, x, u) f(x, u)\, dx\, du,$$

is the unique classical solution of the Cauchy problem

$$\begin{cases} \partial_s G_{t,f} + \mathcal{L} G_{t,f} = 0 \ \ in\ [0, t) \times \mathbb{R}^{2d}, \\ \lim_{s \to t^-} G_{t,f}(s, y, v) = f(y, v) \ \ in\ \ (\in \mathbb{R}^{2d}, \end{cases} \tag{4.2}$$

where

$$\mathcal{L}\psi(s, y, v) := v \cdot \nabla_x \psi(s, y, v) + \frac{1}{2} \sum_{i,j=1}^{d} a^{(i,j)}(s, y, v) \partial^2_{v_i, v_j} \psi(s, y, v). \tag{4.3}$$

(iii) There exists a constant $C > 0$ *depending only on* T *and* λ *such that*

$$\sup_{(y,v) \in \mathbb{R}^{2d}} \int_{\mathbb{R}^d} |\nabla_v \Gamma(s, y, v; t, x, u)|\ dx\, du \le \frac{C}{\sqrt{t - s}},\ \forall\ 0 \le s < t \le T. \tag{4.4}$$

Let us now identify mild equations satisfied by $(\rho_t;\ t \in (0, T])$ and $(\rho^\epsilon_t;\ t \in (0, T])$.

4.2 Mild equations for the densities of $(X,\mathcal{U})$ and $(X^\epsilon,\mathcal{U}^\epsilon)$

Consider a weak solution $(X,\mathcal{U})$ of (1.1). For all $f \in \mathcal{C}_b(\mathbb{R}^{2d})$, since $G_{t,f}$ is a classical solution of (4.2), Itô's formula leads to

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[f(X_{t\wedge\tau_M},\mathcal{U}_{t\wedge\tau_M})\right] &= \mathbb{E}_{\mathbb{P}}\left[G_{t,f}(0,X_0,\mathcal{U}_0)\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}\left[\int_0^{t\wedge\tau_M}\left(\nabla_v G_{t,f}(s,X_s,\mathcal{U}_s)\cdot B\left[X_s,\mathcal{U}_s;\rho_s\right]\right)ds\right],\end{aligned} \tag{4.5}$$

where $\{\tau_M;\ M\geq 1\}$ is the sequence of stopping times

$$\tau_M = \inf\left\{t>0\ ;\ \ |X_t|+|\mathcal{U}_t|\geq M\right\}.$$

The boundedness of b and σ implies that $\lim_{M\to+\infty}\tau_M = +\infty$, $\mathbb{P}$–a.s. By the dominated convergence theorem, the left–hand side of (4.5) converges to $\mathbb{E}_{\mathbb{P}}\left[f(X_t,\mathcal{U}_t)\right]$ as M tends to infinity. For the right–hand side, Proposition 4.4 shows that, for $s\neq t$

$$\nabla_v G_{t,f}(s,y,v) = \int_{\mathbb{R}^{2d}}\nabla_v\Gamma(s,y,v;t,x,u)f(x,u)\,dx\,du,$$

and, $\mathbb{P}$–a.s.,

$$\sup_{M\geq 1}\int_0^{t\wedge\tau_M}\left|\nabla_v G_{t,f}(s,X_s,\mathcal{U}_s)\cdot B\left[X_s,\mathcal{U}_s;\rho_s\right]\right|ds \leq \|b\|_\infty\|f\|_\infty\int_0^t\frac{C}{\sqrt{t-s}}\,ds.$$

Letting M tends to infinity, we get

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f(x,u)\rho_t(x,u)\,dx\,du &= \int_{\mathbb{R}^{2d}}G_{t,f}(0,y,v)\mu_0(dy\,dv) \\ &\quad + \int_{Q_t}\left(\nabla_v G_{t,f}(s,y,v)\cdot\rho_s(y,v)B\left[y,v;\rho_s\right]\right)\,ds\,dy\,dv.\end{aligned} \tag{4.6}$$

We denote by $(S^*_{t,s};\ 0\leq s<t\leq T)$ the adjoint of the transition operator of $(Y_t^{s,y,v},V_t^{s,y,v})$, that is

$$S^*_{t,s}(f)(x,u) = \int_{\mathbb{R}^{2d}}\Gamma(s,y,v;t,x,u)f(y,v)\,dy\,dv.$$

In view of Proposition 4.4, for all $0 \leq s < t \leq T$, $S^*_{t,s}$ is a linear operator from $\mathcal{M}(\mathbb{R}^{2d})$ to $L^1(\mathbb{R}^{2d})$. In particular, $S^*_{t,0}(\mu_0) \in L^1(\mathbb{R}^{2d})$ and the first term in the right–hand side in (4.6) can be rewritten as

$$\int_{\mathbb{R}^{2d}} S^*_{t,0}(\mu_0)(x,u) f(x,u)\, dx\, du.$$

In addition, for all $0 \leq s < t \leq T$, we define the operator $S'_{t,s} : L^1(\mathbb{R}^{2d};\mathbb{R}^d) \to L^1(\mathbb{R}^{2d};\mathbb{R})$ by

$$S'_{t,s}(h(\cdot))(x,u) = \int_{\mathbb{R}^{2d}} \left(\nabla_v \Gamma(s,y,v;t,x,u) \cdot h(y,v)\right)\, dy\, dv.$$

In particular, Proposition 4.4 shows that

$$\int_0^t \|S'_{t,s}(h(s,\cdot))\|_{L^1(\mathbb{R}^{2d})}\, ds \leq \int_0^t \frac{C}{\sqrt{t-s}} \|h(s,\cdot)\|_{L^1(\mathbb{R}^{2d})}\, ds \tag{4.7}$$

for all $h \in L^\infty((0,T);L^1(\mathbb{R}^{2d}))$. Thus the second term in the right–hand side of (4.6) writes

$$\int_{\mathbb{R}^{2d}} f(x,u) \int_0^t S'_{t,s}\left(\rho_s(\cdot) B\left[\cdot\ ;\rho_s\right]\right)(x,u)\, ds\, dx\, du.$$

Therefore the marginal distributions $(\rho_t;\ t \in (0,T])$ of the solution of (1.1) satisfy the mild equation

$$\forall\ t \in (0,T],\ \rho_t = S^*_{t,0}(\mu_0) + \int_0^t S'_{t,s}\left(\rho_s(\cdot) B\left[\cdot\ ;\rho_s\right]\right) ds, \quad \text{in } L^1(\mathbb{R}^{2d}). \tag{4.8}$$

The preceding calculations hold true when ρ and $B[\cdot\ ;\rho_\cdot]$ are replaced by ρ^ϵ and $B_\epsilon[\cdot\ ;\rho^\epsilon_\cdot]$. Therefore the marginal distributions $(\rho^\epsilon_t;\ t \in (0,T])$ satisfy the mild equation

$$\forall\ t \in (0,T],\ \rho^\epsilon_t = S^*_{t,0}(\mu_0) + \int_0^t S'_{t,s}\left(\rho^\epsilon_s(\cdot) B_\epsilon\left[\cdot\ ;\rho^\epsilon_s\right]\right) ds, \quad \text{in } L^1(\mathbb{R}^{2d}). \tag{4.9}$$

4.3 Uniqueness of the solutions to the mild equations (4.8) and (4.9)

Lemma 4.5. *There exists at most one positive solution (ρ_t) to Equation (4.8), and at most one solution (ρ^ϵ_t) to (4.9).*

Proof. Let (ρ^1_t) and (ρ^2_t) be two positive solutions of (4.8). We set

$$\overline{\rho}^i_t(x) := \int_{\mathbb{R}^d} \rho^i_t(x,u)\,du.$$

For a.e. (t,x) in $(0,T]\times\mathbb{R}^d$, $\overline{\rho}^i_t(x) > 0$ for $i = 1,\ 2$ and so, $B[x,u;\rho^i_t]$ in (1.2) is well–defined for a.e. $(t,x,u)\in(0,T]\times\mathbb{R}^{2d}$. In view of (4.8), we have

$$\|\rho^1_t - \rho^2_t\|_{L^1(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}} \left| \int_0^t S'_{t,s}\left(\rho^1_s(\cdot)B\left[\,\cdot\,;\rho^1_s\right] - \rho^2_s(\cdot)B\left[\,\cdot\,;\rho^2_s\right]\right)(x,u)\,ds\right| dx\,du. \tag{4.10}$$

We aim to prove the following estimate which implies the uniqueness result by Gronwall's lemma: there exists $C > 0$ such that, for all $t\in(0,T]$,

$$\|\rho^1_t - \rho^2_t\|_{L^1(\mathbb{R}^{2d})} \leq C\int_0^t \frac{\|\rho^1_s - \rho^2_s\|_{L^1(\mathbb{R}^d)}}{\sqrt{t-s}}\,ds. \tag{4.11}$$

From (4.10), we have

$$\begin{aligned}\|\rho^1_t - \rho^2_t\|_{L^1(\mathbb{R}^{2d})} &\leq \int_{Q_t} \left|S'_{t,s}\left(\left(\rho^1_s(\cdot) - \rho^2_s(\cdot)\right)B\left[\,\cdot\,;\rho^1_s\right]\right)(x,u)\right|\,ds\,dx\,du \\ &\quad + \int_{Q_t} \left|S'_{t,s}\left(\rho^2_s(\cdot)\left(B\left[\,\cdot\,;\rho^1_s\right] - B\left[\,\cdot\,;\rho^2_s\right]\right)\right)(x,u)\right|\,ds\,dx\,du \\ &=: A_1 + A_2.\end{aligned} \tag{4.12}$$

In view of (4.7) and the boundedness of b, we get

$$A_1 \leq \|b\|_\infty \int_0^t \frac{C}{\sqrt{t-s}}\|\rho^1_s - \rho^2_s\|_{L^1(\mathbb{R}^{2d})}\,ds. \tag{4.13}$$

We now consider A_2. As

$$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 - a_2}{b_2} + \frac{a_1(b_2 - b_1)}{b_2 b_1},\ \forall\ a_1,\ a_2 \in \mathbb{R},\ b_1,\ b_2 > 0, \tag{4.14}$$

we observe that

$$\begin{aligned} B\left[y, v; \rho_s^1\right] - B\left[y, v; \rho_s^2\right] = &\frac{1}{\overline{\rho}_s^2(y)} \int_{\mathbb{R}^d} b(v', v)\left(\rho_s^1(y, v') - \rho_s^2(y, v')\right)\, dv' \\ &+ \frac{\int_{\mathbb{R}^d} b(v', v)\rho_s^1(y, v')\, dv'}{\overline{\rho}_s^2(y)\overline{\rho}_s^1(y)} \left(\overline{\rho}_s^2(y) - \overline{\rho}_s^1(y)\right). \end{aligned}$$

Hence, for all $0 \leq s < t$,

$$\begin{aligned} &\|\rho_s^2(\cdot)\left(B\left[\cdot; \rho_s^1\right] - B\left[\cdot; \rho_s^2\right]\right)\|_{L^1(\mathbb{R}^{2d})} \\ &\leq 2\|b\|_\infty \int_{\mathbb{R}^{2d}} \frac{\rho_s^2(y, v)}{\overline{\rho}_s^2(y)} \int_{\mathbb{R}^d} \left|\rho_s^1(y, v') - \rho_s^2(y, v')\right|\, dv'\, dy\, dv \leq 2\|b\|_\infty \|\rho_s^1 - \rho_s^2\|_{L^1(\mathbb{R}^{2d})}. \end{aligned}$$

In view of (4.7) we thus have

$$A_2 \leq \|b\|_\infty \int_0^t \frac{C}{\sqrt{t - s}} \|\rho_s^1 - \rho_s^2\|_{L^1(\mathbb{R}^{2d})}\, ds. \tag{4.15}$$

It remains to gather (4.13) and (4.15) to get (4.11).

We now prove uniqueness for (4.9). First, let us observe that, by using (4.7) in Equation (4.9), one can found $C > 0$ such that, for all solution (ρ_t^ϵ) of (4.9),

$$\|\rho_t^\epsilon\|_{L^1(\mathbb{R}^{2d})} \leq C,\ \forall\ t \in (0, T].$$

Next, consider two nonnegative solutions $(\rho_t^{\epsilon,1})$ and $(\rho_t^{\epsilon,2})$ of (4.9). As in (4.12), we have

$$\begin{aligned} \|\rho_t^{\epsilon,1} - \rho_t^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})} \leq &\int_{Q_t} \left|S'_{t,s}\left(\left(\rho_s^{\epsilon,1}(\cdot) - \rho_s^{\epsilon,2}(\cdot)\right) B_\epsilon\left[\cdot\ ; \rho_s^{\epsilon,1}\right]\right)(x, u)\right|\, ds\, dx\, du \\ &+ \int_{Q_t} \left|S'_{t,s}\left(\rho_s^{\epsilon,2}(\cdot)\left(B_\epsilon\left[\cdot\ ; \rho_s^{\epsilon,1}\right] - B_\epsilon\left[\cdot\ ; \rho_s^{\epsilon,2}\right]\right)\right)(x, u)\right|\, ds\, dx\, du \\ &=: A_1^\epsilon + A_2^\epsilon. \end{aligned}$$

Obviously,

$$A_1^\epsilon \leq \|b\|_\infty \int_0^t \frac{C}{\sqrt{t-s}} \|\rho_s^{\epsilon,1} - \rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\, ds.$$

In order to estimate A_2^ϵ, we use again (4.14) and we observe that: for a.e. $(s, y, v) \in Q_t$,

$$\begin{aligned}
&\rho_s^{\epsilon,2}(y,v)\left(B_\epsilon[y,v;\rho_s^{\epsilon,1}] - B_\epsilon\left[y,v;\rho_s^{\epsilon,2}\right]\right)\\
&= \frac{\rho_s^{\epsilon,2}(y,v)}{\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y) + \epsilon} \int_{\mathbb{R}^d} b(v',v)\left(\phi_\epsilon \star \rho_s^{\epsilon,1}(y,v') - \phi_\epsilon \star \rho_s^{\epsilon,2}(y,v')\right) dv'\\
&\quad + \frac{\rho_s^{\epsilon,2}(y,v)\int_{\mathbb{R}^d} b(v',v)\phi_\epsilon \star \rho_s^{\epsilon,1}(y,v')\,dv'}{(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y) + \epsilon)(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,1}(y) + \epsilon)} \left(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y) - \phi_\epsilon \star \overline{\rho}_s^{\epsilon,1}(y)\right)\\
&\leq \frac{\|b\|_\infty \rho_s^{\epsilon,2}(y,v)}{(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y) + \epsilon)} \int_{\mathbb{R}^{2d}} \phi_\epsilon(y-y')\left|\rho_s^{\epsilon,1}(y',v') - \rho_s^{\epsilon,2}(y',v')\right| dy'\,dv'.
\end{aligned}$$

In view of (4.7), it follows that

$$A_2^\epsilon \leq \|b\|_\infty \frac{\|\phi\|_\infty}{\epsilon^{d+1}} \int_{Q_t} \frac{C}{\sqrt{t-s}} \|\rho_s^{\epsilon,1} - \rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\, ds.$$

Hence

$$\|\rho_t^{\epsilon,1} - \rho_t^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})} \leq C\|b\|_\infty \left(1 + \frac{2\|\phi\|_\infty}{\epsilon^{d+1}}\right) \int_0^t \frac{\|\rho_s^{\epsilon,1} - \rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}}{\sqrt{t-s}}\, ds.$$

We conclude on the uniqueness result for (4.9) by applying Gronwall's lemma. □

5 Existence results

In this section, we establish that Equations (3.4) and (1.1) admit a solution.

Proposition 5.1. *The martingale problem (MP_ϵ) in Definition 4.1 has a unique solution $\mathbb{P}^\epsilon$. Furthermore, when ϵ tends to 0, $\mathbb{P}^\epsilon$ converges to a solution of the martingale problem (MP) introduced in Definition 4.2.*

The proof of the proposition 5.1 proceeds in two steps.

In the first step, we construct a weak solution to (3.4) by studying the interacting system (3.3) as the number of particles tends to infinity. As in Sznitman [18], we prove the relative compactness of the sequence of the empirical measures of the particles (see Lemma 5.3). We then show that the support of the limit probability measure is the set of solutions of the martingale problem (MP_ϵ) (see Lemma 5.4). Using the uniqueness result in Proposition 4.3, we then get the propagation of chaos result.

The second step consists in exhibiting a solution to the martingale problem (MP) as the limit of the solution to martingale problem (MP_ϵ) when ϵ tends to 0.

5.1 A propagation of chaos result

Throughout this section we fix $\epsilon > 0$.

Proposition 5.2. *There exists a unique probability measure $\mathbb{P}^\epsilon$ solution to the martingale problem (MP_ϵ). Moreover, the sequence of probability laws $\{\mathbb{P}^{\epsilon,N};\ N \geq 1\}$ of the processes $\{(X^{i,\epsilon,N}, \mathcal{U}^{i,\epsilon,N});\ 1 \leq i \leq N\}$ is $\mathbb{P}^\epsilon$–chaotic.*

Let $\overline{\mu}^{\epsilon,N}$ be the empirical measure valued in $\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d}))$ by

$$\overline{\mu}^{\epsilon,N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{X^{i,\epsilon,N}, \mathcal{U}^{i,\epsilon,N}\}}.$$

Let $\overline{\mathbb{P}}^{\epsilon,N}$ in $\mathcal{M}(\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d})))$ be the probability law of $\overline{\mu}^{\epsilon,N}$.

Lemma 5.3. *The sequence $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ is tight.*

Proof. We proceed as in Sznitman [18]: since the particle systems is exchangeable, the tightness of $\overline{\mathbb{P}}^{\epsilon,N}$ is equivalent to the tightness of the probability laws of $\{(X^{1,\epsilon,N}, \mathcal{U}^{1,\epsilon,N});\ N \geq 1\}$. Let $((x_t^{(i)}, u_t^{(i)});\ t \in (0,T], i = 1,\ldots,N)$ be the canonical processes in the sample space $\mathcal{C}([0,T];\mathbb{R}^{2dN})$. In view of the boundedness of b and σ,

$$\mathbb{E}_{\mathbb{P}^{\epsilon,N}}[|u_t^{(1)} - u_s^{(1)}|^4] \leq C(t-s)^4 \quad \text{and} \quad \mathbb{E}_{\mathbb{P}^{\epsilon,N}}[|x_t^{(1)} - x_s^{(1)}|^2] \leq C(t-s)^2.$$

The result follows from the Kolmogorov criterion. □

We have shown that $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ is relatively compact. We still denote by $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ a weakly convergent subsequence. Let $\overline{\mathbb{P}}^{\epsilon,\infty}$ be the limit of such a subsequence.

Lemma 5.4. $\overline{\mathbb{P}}^{\epsilon,\infty}$ *assigns full measure to the set of the solutions to the martingale problem* (MP_ϵ).

Proof. We denote by m the canonical measure on $\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d}))$. Since $\{(X_0^{i,\epsilon,N}, \mathcal{U}_0^{i,\epsilon,N});\ 1 \leq i \leq N\}$ are μ_0–i.i.d., it is easy to check that

$$\overline{\mathbb{P}}^{\epsilon,N} \circ (m \circ (x_0, u_0)^{-1})^{-1} = \delta_{\mu_0},$$

and the same holds for $\overline{\mathbb{P}}^{\epsilon,\infty}$, by the weak convergence of $\overline{\mathbb{P}}^{\epsilon,N}$ to $\overline{\mathbb{P}}^{\epsilon,\infty}$. The property (i) of (MP_ϵ) is then true for all m, $\mathbb{P}^{\epsilon,\infty}$–a.e.

Now we prove that, $\overline{\mathbb{P}}^{\epsilon,\infty}$–a.e., m satisfies the properties (ii) and (iii) of (MP_ϵ). We define $\alpha : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d})) \mapsto \mathbb{R}^d$ by

$$\alpha(t,x,u,m) := \frac{\mathbb{E}_m\left[b(u_t,u)\phi_\epsilon(x - x_t)\right]}{\mathbb{E}_m\left[\phi_\epsilon(x - x_t)\right] + \epsilon},$$

For any $f \in \mathcal{C}_c^2(\mathbb{R}^{2d})$, for any $0 \leq t_1 \leq \ldots \leq t_n \leq s < t \leq T$, and any finite family of functions $\{\psi_j\ ;\ 1 \leq j \leq n\}$ in $\mathcal{C}_b(\mathbb{R}^{2d})$, we set

$$F^\epsilon(m) := \Bigg| \mathbb{E}_m \Bigg[\prod_{j=1}^n \psi_j(x_{t_j}, u_{t_j}) \Bigg(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \left(u_\theta \cdot \nabla_x f(x_\theta,u_\theta)\right) d\theta \\ - \int_s^t \left(\alpha(\theta,x_\theta,u_\theta;m_\theta) \cdot \nabla_u f(x_\theta,u_\theta)\right) + \frac{1}{2}\sum_{i,j=1}^d a^{(i,j)}(\theta,x_\theta,u_\theta)\partial^2_{u_i,u_j} f(x_\theta,u_\theta)\, d\theta \Bigg) \Bigg] \Bigg|.$$

We aim to prove that $F^\epsilon = 0$, $\overline{\mathbb{P}}^{\epsilon,\infty}$–a.s. Let us first check that, for $\overline{\mathbb{P}}^{\epsilon,\infty}$–a.e. m,

$$\int_{\mathcal{C}([0,T];\mathbb{R}^d)} \max_{t\in[0,T]} |u_t(\omega)|\, dm(\omega) < +\infty,$$

and therefore $F^\epsilon < +\infty$, $\overline{\mathbb{P}}^{\epsilon,\infty}$–a.s. Indeed, by setting $G(m) = \mathbb{E}_m\left[\max_{t\in[0,T]}|u_t|\right]$, we observe that, from the boundedness of b and σ,

$$\sup_{N\geq 1} \mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,N}}[G(m)] = \sup_{N\geq 1} \mathbb{E}_{\mathbb{P}^{\epsilon,N}}\left[G(\frac{1}{N}\sum_{i=1}^N \delta_{\{x^{(i)},u^{(i)}\}})\right] \leq C.$$

The upper bound for F^ϵ follows by weak convergence.

It is clear that α is a weakly continuous function of m, and therefore $\mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,N}}[F^\epsilon(m)]$ tends to $\mathbb{E}_{\mathbb{P}^{\epsilon,\infty}}[F^\epsilon(m)]$. Moreover, in view of (3.3), $\mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,N}}[F^\epsilon(m)] \leq C/N$, and then $F^\epsilon = 0$, $\overline{\mathbb{P}}^{\epsilon,\infty}$–a.s. Consequently,

$$
\begin{aligned}
&f(x_t,u_t) - f(x_0,u_0) - \int_0^t (u_\theta \cdot \nabla_x f(x_\theta,u_\theta))\,d\theta \\
&\quad - \int_s^t (\alpha(\theta,x_\theta,u_\theta,m)\cdot\nabla_u f(x_\theta,u_\theta)) + \frac{1}{2}\sum_{i,j=1}^d a^{(i,j)}(\theta,x_\theta,u_\theta)\partial^2_{u_i,u_j} f(x_\theta,u_\theta)\,d\theta
\end{aligned}
$$

is a m–martingale. As α is uniformly bounded, by Cameron–Martin's formula, we easily check that $m\circ(x_\theta,u_\theta)^{-1}$ has a density ρ^ϵ_θ, so that

$$
f(x_t,u_t) - f(x_0,u_0) - \int_0^t \mathcal{A}_{\rho^\epsilon_\theta} f(\theta,x_\theta,u_\theta)\,d\theta
$$

is a m–martingale. This ends the proof of *(ii)* and *(iii)*. $\square$

Proposition 4.3 ensure that $\overline{\mathbb{P}}^{\epsilon,\infty}$ is reduced to a Dirac mass, and we denote $\mathbb{P}^\epsilon$ the point such that $\overline{\mathbb{P}}^{\epsilon,\infty} = \delta_{\{\mathbb{P}^\epsilon\}}$. This implies that $\mathbb{P}^\epsilon$ is the unique solution to (MP_ϵ). Moreover, this implies the $\mathbb{P}^\epsilon$–chaoticity of the particle system (see Sznitman [18, Prop. 2.2]).

5.2 Convergence of the smoothed system

In this subsection, we prove that the probability measure $\mathbb{P}^\epsilon$, solution of the martingale problem (MP_ϵ), converges to the solution to the martingale problem (MP). We start with studying the probability measure $\widetilde{\mathbb{P}}^\epsilon$ defined on $\mathcal{C}([0,T];\mathbb{R}^{3d})$ by

$$
\widetilde{\mathbb{P}}^\epsilon = \mathbb{P}^\epsilon \circ ((x_t,u_t,u_t-u_0-\int_0^t B_\epsilon[x_s,u_s;\rho^\epsilon_s]\,ds);\ t\in[0,T])^{-1}.
$$

As in the proof of Lemma 5.3, using (3.4), the Kolmogorov criterion implies that the sequence $\{\widetilde{\mathbb{P}}^\epsilon;\ \epsilon>0\}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Let $\widetilde{\mathbb{P}}^\epsilon$ be a converging subsequence and denote its limit by $\widetilde{\mathbb{P}}$. Let us characterize the support of $\widetilde{\mathbb{P}}$. To this aim, we introduce the subset $\mathcal{H}_{\|b\|_\infty}$ of $\mathcal{C}([0,T]\,;\mathbb{R}^{3d})$ defined by

$$\mathcal{H}_{\|b\|_\infty} = \left\{ \begin{array}{l} (x,u,D) \text{ in } \mathcal{C}([0,T];\mathbb{R}^{3d}), \text{ s.t. } x(t) = x(0) + \int_0^t u(s)\,ds, \text{ and} \\ u(t) - u(0) - D(t) = \int_0^t \beta(s)\,ds, \text{ for a measurable function } \beta : [0,T] \to \mathbb{R}^d \\ \text{ s.t. } \sup_{t\in[0,T]} |\beta(t)| \le \|b\|_\infty. \end{array} \right\}$$

We now prove that

Lemma 5.5. *$\widetilde{\mathbb{P}}$ has full measure on $\mathcal{H}_{\|b\|_\infty}$.*

Proof. According to the Portemanteau theorem, the weak convergence of $\widetilde{\mathbb{P}}^\epsilon$ to $\widetilde{\mathbb{P}}$ yields that, for all closed subset F of $\mathcal{C}([0,T];\mathbb{R}^{3d})$,

$$\limsup_{\epsilon\to 0} \widetilde{\mathbb{P}}^\epsilon(F) \le \widetilde{\mathbb{P}}(F).$$

Since $\widetilde{\mathbb{P}}^\epsilon(\mathcal{H}_{\|b\|_\infty}) = 1$ for all $\epsilon > 0$, it suffices to show that $\mathcal{H}_{\|b\|_\infty}$ is a closed subset of $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Let $\{(x_n,u_n,D_n);\ n \in \mathbb{N}\}$ be a sequence of $\mathcal{H}_{\|b\|_\infty}$ converging to (x,u,D) in $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Set $A_n(t) = u_n(t) - u_n(0) - D_n(t)$ and $A(t) = u(t) - u(0) - D(t)$. By uniform convergence, it holds that

$$\begin{aligned} & x(t) = x(0) + \int_0^t u(s)\,ds,\ \forall\, t \in [0,T], \\ \text{and } & \lim_{n\to+\infty} \max_{t\in[0,T]} |A_n(t) - A(t)| = 0. \end{aligned}$$

To prove that (x,u,D) belongs to $\mathcal{H}_{\|b\|_\infty}$, it remains to show that A is a.e. differentiable with a time derivative uniformly bounded by $\|b\|_\infty$. By the Riesz representation theorem, it is enough to prove that

$$\left| \int_0^T A(t) f'(t)\,dt \right| \le \|b\|_\infty \int_0^T |f(t)|\,dt,\ \forall f \in \mathcal{C}_c^1([0,T];\mathbb{R}^d).$$

As $A_n(t) = \int_0^t \beta_n(s)\,ds$ for some measurable function β_n satisfying $\sup_{t\in[0,T]} |\beta_n(t)| \le \|b\|_\infty$, an integration by parts allows us to write

$$\left| \int_0^T A(t) f'(t)\,dt \right| \le \left| \lim_{n\to+\infty} \int_0^T \beta_n(t) f(t)\,dt \right| \le \|b\|_\infty \int_0^T |f(t)|\,dt,$$

which ends the proof. $\square$

Consider the marginal distribution $\mathbb{P}$ of $\widetilde{\mathbb{P}}$ on $\mathcal{C}([0,T];\mathbb{R}^{2d})$, defined by

$$\mathbb{P} = \widetilde{\mathbb{P}} \circ ((x_t, u_t);\ t \in [0,T])^{-1}.$$

We have the following result:

Proposition 5.6. $\mathbb{P}$ *solves the martingale problem* (*MP*) *in Definition 4.2.*

Proof. The point (i) of (*MP*) is obvious. To prove (ii), consider $((x_t, u_t, D_t);\ t \in [0,T])$ the canonical processes of $\mathcal{C}([0,T];\mathbb{R}^{3d})$. In view of Lemma 5.5, we know that, $\widetilde{\mathbb{P}}$–a.s., for all $t \in [0,T]$,

$$\begin{aligned} x_t &= x_0 + \textstyle\int_0^t u_s\, ds, \\ u_t &= u_0 + \textstyle\int_0^t \beta_s\, ds + D_t, \end{aligned}$$

with $\sup_{t\in[0,T]} |\beta_t| \leq \|b\|_\infty$. Since σ is bounded continuous, the weak convergence of $\widetilde{\mathbb{P}}^\epsilon$ to $\widetilde{\mathbb{P}}$ yields that, for all function f in $\mathcal{C}_b^2(\mathbb{R}^d)$,

$$f(D_t) - f(D_0) - \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t a^{(i,j)}(s, x_s, u_s)\partial^2_{u_i,u_j} f(D_s) ds$$

is a $\widetilde{\mathbb{P}}$–martingale. We deduce that $D_t = \int_0^t \sigma(\theta, x_\theta, u_\theta)\, dw_\theta$ for some d–dimensional Wiener process $(w_t;\ t \in [0,T])$.

In view of (3.1), Girsanov's theorem allows one to define a new probability $\mathbb{Q}$ absolutely continuous to $\widetilde{\mathbb{P}}$ on $\mathcal{C}([0,T];\mathbb{R}^{3d})$ such that $\mathbb{Q} \circ (x_t, u_t)^{-1}$ is the law of the Langevin system

$$(y_t, v_t) = \left(x_0 + \int_0^t v_s\, ds, u_0 + \int_0^t \sigma(s, y_s, v_s)\, dw_s \right).$$

In view of Proposition 4.4, for all $t \in [0,T]$ the law of (y_t, v_t) is absolutely continuous w.r.t. to Lebesgue measure. Thus the measure $\mathbb{P} \circ (x_t, u_t)^{-1}$ has also a density ρ_t which satisfies

$$\gamma_t(x,u) = \rho_t(x,u)\mathbb{E}_{\mathbb{P}}\left(Z_t \Big/ (x_t, u_t) = (x,u) \right),\ \text{for a.e. } (x,u) \in \mathbb{R}^{2d},$$

where Z_t is the restriction to $\mathcal{C}([0,t];\mathbb{R}^{3d})$ of the density $\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}$, and

$$\gamma_t(x,u) := \int_{\mathbb{R}^{2d}} \Gamma(0,y,v;t,x,u)\mu_0(y,v)\,dy\,dv,$$

where $\Gamma(0,y,v;t,x,u)$ is defined in Proposition 4.4. We now recall the following estimate (see Pascucci & Polidoro [11]): there exists $\eta > 0$, depending only on λ and d, such that

$$\Gamma(t,x,u;s,y,v) \geq \Gamma_\eta(t,x,u;s,y,v),$$

where Γ_η is defined in (A.2). Hence, the function $\rho_t(x,u)$ is strictly positive. We thus have proven *(ii)*.

We now prove *(iii)*. Observe that there exists $C > 0$ such that

$$\sup_{\epsilon>0} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\max_{t\in[0,T]} |u_t|\right] \leq C.$$

Therefore it suffices to prove that, for all $f \in \mathcal{C}^2_c(\mathbb{R}^{2d})$ and all process (Ψ_s) of the form

$$\Psi_s = \Psi_s(x_\cdot, u_\cdot) := \prod_{j=1}^{n} \psi_j(x_{t_j}, u_{t_j}),$$

where the ψ_j's are bounded continuous functions, one has

$$\mathbb{E}_{\mathbb{P}}\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{A}_{\rho_\theta} f(x_\theta,u_\theta)d\theta\right)\right] = 0, \tag{5.1}$$

Since $\widetilde{\mathbb{P}}^\epsilon$ converges weakly to $\widetilde{\mathbb{P}}$, we have

$$\begin{aligned}\lim_{\epsilon\to0^+} \mathbb{E}_{\mathbb{P}^\epsilon}&\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{L}f(x_\theta,u_\theta)d\theta\right)\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{L}f(x_\theta,u_\theta)d\theta\right)\right],\end{aligned}$$

for $\mathcal{L}$ defined in (4.3). To obtain (5.1), it thus remains to show

$$\lim_{\epsilon\to 0^+} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]. \quad (5.2)$$

If $\mathbb{P}^\epsilon$ were the law of a strongly elliptic diffusion process and the coefficient B_ϵ would not depend on ρ^ϵ, (5.2) would result from Stroock & Varadhan's results on limits of martingale problems: see Lemma 11.3.2 and Lemma 9.1.15 in [16]. In our situation, we prove that (5.2) holds true by adapting Stroock & Varadhan's techniques and by taking advantage of the mild equation (4.9). By adding and subtracting to (5.2) the terms

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right],$$
$$\text{and} \quad \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right],$$

we obtain that

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\right.\\
&\left.-\mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\right|\\
&\leq \left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right)\, d\theta\right]\right.\\
&\quad \left.-\ \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right)\, d\theta\right]\right|\\
&\quad + \left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] - B_\xi\left[x_\theta, u_\theta; \rho^\epsilon_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right)\, d\theta\right]\right|\\
&\quad + \left|\mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] - B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right)\, d\theta\right]\right|\\
&=: I_{\epsilon,\xi} + J_{\epsilon,\xi} + K_\xi.
\end{aligned}$$

We aim to show that

$$\forall \xi > 0, \ \lim_{\epsilon \to 0^+} I_{\epsilon,\xi} = 0, \tag{5.3}$$

$$\lim_{\xi \to 0^+} K_\xi = 0, \tag{5.4}$$

$$\forall \xi > 0, \ \limsup_{\epsilon \to 0^+} J_{\epsilon,\xi} = 0. \tag{5.5}$$

A crucial step to get these results is to prove the following proposition.

Proposition 5.7. *For all $0 < t \leq T$, ρ_t^ϵ converges to ρ_t in $L^1(\mathbb{R}^{2d})$ when $\epsilon \to 0^+$.*

In view of Lemma A.2 in the Appendix, Proposition 5.7 results from the following lemma:

Lemma 5.8. *For all $t \in (0, T]$, $h, \delta \in \mathbb{R}^d$, it holds that*

$$\lim_{|h|,|\delta| \to 0} \limsup_{\epsilon \to 0^+} \int_{\mathbb{R}^{2d}} |\rho_t^\epsilon(x+h, u+\delta) - \rho_t^\epsilon(x,u)| \, dx \, du = 0.$$

Proof. As $\mathbb{P}^\epsilon$ is the unique solution to the martingale problem (MP_ϵ), its time marginals satisfy the mild equation (4.9). Thus, for all $t \in (0, T]$,

$$\begin{aligned}
&\limsup_{\epsilon \to 0^+} \int_{\mathbb{R}^{2d}} |\rho_t^\epsilon(x+h, u+\delta) - \rho_t^\epsilon(x,u)| \, dx \, du \\
&\leq \int_{\mathbb{R}^{2d}} \left| S_{t,0}^*(\mu_0)(x+h, u+\delta) - S_{t,0}^*(\mu_0)(x,u) \right| \, dx \, du \\
&\quad + \limsup_{\epsilon \to 0^+} \int_{Q_t} \left| S_{t,s}' \left(\rho_s^\epsilon(\cdot) B_\epsilon \left[\cdot \, ; \rho_s^\epsilon \right] \right)(x+h, u+\delta) - S_{t,s}' \left(\rho_s^\epsilon(\cdot) B_\epsilon \left[\cdot \, ; \rho_s^\epsilon \right] \right)(x,u) \right| \, ds \, dx \, du.
\end{aligned}$$

Since $S_{t,0}^*(\mu_0)$ belongs to $L^1(\mathbb{R}^{2d})$, Lemma A.3 implies that

$$\lim_{|h|,|\delta| \to 0} \int_{\mathbb{R}^{2d}} \left| S_{t,0}^*(\mu_0)(x+h, u+\delta) - S_{t,0}^*(\mu_0)(x,u) \right| \, dx \, du = 0.$$

In addition,

$$\int_{Q_t} \left| S'_{t,s}\left(\rho^\epsilon_s(\cdot) B_\epsilon\left[\cdot\ ; \rho^\epsilon_s\right]\right)(x+h,u+\delta) - S'_{t,s}\left(\rho^\epsilon_s(\cdot) B_\epsilon\left[\cdot\ ; \rho^\epsilon_s\right]\right)(x,u)\right| \, ds\, dx\, du$$
$$\leq \|b\|_\infty \int_{Q_t} \left(\int_{\mathbb{R}^{2d}} |\nabla_v \Gamma(s,y,v;t,x+h,u+\delta) - \nabla_v \Gamma(s,y,v;t,x,u)| \, dx\, du \right)$$
$$\times\, \rho^\epsilon_s(y,v)\, ds\, dy\, dv.$$

Set

$$L_{h,\delta}(t,s,y,v) := \int_{\mathbb{R}^{2d}} |\nabla_v \Gamma(s,y,v;t,x+h,u+\delta) - \nabla_v \Gamma(s,y,v;t,x,u)| \, dx\, du.$$

As $\widetilde{\mathbb{P}}^\epsilon$ converges weakly to $\widetilde{\mathbb{P}}$, ρ^ϵ_t converges weakly to ρ_t for all $t \in [0,T]$ and

$$\lim_{\epsilon \to 0^+} \int_{Q_t} L_{h,\delta}(t,s,y,v) \rho^\epsilon_s(y,v)\, ds\, dy\, dv = \int_{Q_t} L_{h,\delta}(t,s,y,v) \rho_s(y,v)\, ds\, dy\, dv.$$

In addition, in view of (4.4), one has

$$\sup_{(y,v)\in\mathbb{R}^{2d}} L_{h,\delta}(t,s,y,v) \leq \frac{C}{\sqrt{t-s}}$$

for all $t > s$, from which

$$\int_{\mathbb{R}^{2d}} L_{h,\delta}(t,s,y,v) \rho^\epsilon_s(y,v) dy dv \leq \frac{C}{\sqrt{t-s}}.$$

It then remains to apply Lebesgue's dominated convergence theorem. □

Now we are in a position to prove (5.3), (5.4), and (5.5).

Proof of (5.3). For all fixed $\xi > 0$, $0 \leq s \leq t \leq T$, the bounded functions $\{\mathrm{f}^{\epsilon',\xi};\ \epsilon' > 0\}$ defined on $\mathcal{C}([0,T];\mathbb{R}^{2d})$ by

$$\mathrm{f}^{\epsilon',\xi}(x_\cdot, u_\cdot) := \Psi_s \int_s^t \left(B_\xi[x_\theta, u_\theta; \rho^{\epsilon'}_\theta] \cdot \nabla_u f(x_\theta, u_\theta) \right) d\theta$$

are equicontinuous (this latter property results from the definition of B_ξ and the proposition 5.7). Moreover, setting

$$\mathrm{f}^{\xi}(x.,u.) := \Psi_s \int_s^t \left(B_\xi[x_\theta, u_\theta; \rho_\theta] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta,$$

we have

$$\begin{aligned} I_{\epsilon,\xi} &= |\mathbb{E}_{\mathbb{P}^\epsilon} \mathrm{f}^{\epsilon,\xi} - \mathbb{E}_{\mathbb{P}} \mathrm{f}^{\xi}| \\ &\leq \sup_{\epsilon'>0} |\mathbb{E}_{\mathbb{P}^\epsilon} \mathrm{f}^{\epsilon',\xi} - \mathbb{E}_{\mathbb{P}} \mathrm{f}^{\epsilon',\xi}| + |\mathbb{E}_{\mathbb{P}} \mathrm{f}^{\epsilon,\xi} - \mathbb{E}_{\mathbb{P}} \mathrm{f}^{\xi}|. \end{aligned}$$

In view of Lemma A.4, the first term in the right–hand side tends to 0 with ϵ. The second term tends also to 0 in view of Proposition 5.7. We thus have proven (5.3).

Proof of (5.4). We recall the notation

$$\overline{\rho}_\theta(x) := \int_{\mathbb{R}^{\mathbb{R}^d}} \rho_\theta(x,v)\, dv.$$

Observe that

$$\begin{aligned} |B_\xi\left[x,u,\rho_\theta\right] - B\left[x,u;\rho_\theta\right]| \leq & \left| \frac{\int_{\mathbb{R}^d} b(v,u) \phi_\xi \star \rho_\theta(x,v)\, dv}{\phi_\xi \star \overline{\rho}_\theta(x) + \xi} - \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x,v)\, dv}{\overline{\rho}_\theta(x) + \xi} \right| \\ & + \left| \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x,v)\, dv}{\overline{\rho}_\theta(x) + \xi} - \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x,v)\, dv}{\overline{\rho}_\theta(x)} \right|. \end{aligned}$$

Then, in view of (4.14), we get

$$\begin{aligned} & \left| \frac{\int_{\mathbb{R}^{2d}} b(v,u) \phi_\xi \star \rho_\theta(x,v)\, dv}{\phi_\xi \star \overline{\rho}_\theta(x) + \xi} - \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x,v)\, dv}{\overline{\rho}_\theta(x) + \xi} \right| \\ & \leq \left| \frac{\int_{\mathbb{R}^d} b(v,u) \left(\phi_\xi \star \rho_\theta(x,v) - \rho_\theta(x,v)\right)\, dv}{\overline{\rho}_\theta(x) + \xi} \right| \\ & \quad + \frac{\left|\int_{\mathbb{R}^d} b(v,u) \phi_\xi \star \rho_\theta(x,v)\, dv\right| \left|\int_{\mathbb{R}^d} \left(\phi_\xi \star \rho_\theta(x,v) - \rho_\theta(x,v)\right)\, dv\right|}{\left(\phi_\xi \star \overline{\rho}_\theta(x) + \xi\right)\left(\overline{\rho}_\theta(x) + \xi\right)} \\ & \leq \frac{2\|b\|_\infty}{\overline{\rho}_\theta(x) + \xi} \int_{\mathbb{R}^d} \left|\phi_\xi \star \rho_\theta(x,v) - \rho_\theta(x,v)\right|\, dv. \end{aligned} \tag{5.6}$$

Thus

$$\begin{aligned}
&\lim_{\xi\to 0^+}\int_{Q_T} |B_\xi\left[x,u,\rho_\theta\right] - B\left[x,u;\rho_\theta\right]| \, \rho_\theta(x,u)\, d\theta\, dx\, du \\
&\leq 2\|b\|_\infty \lim_{\xi\to 0^+}\int_0^T \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\, d\theta \\
&\quad + \lim_{\xi\to 0^+}\int_{Q_T}\left|\frac{1}{\overline{\rho}_\theta(x)+\xi} - \frac{1}{\overline{\rho}_\theta(x)}\right| \int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\, dv \rho_\theta(x,u)\, d\theta\, dx\, du.
\end{aligned} \tag{5.7}$$

As

$$\|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} = \int_{\mathbb{R}^d}\phi(y)\int_{\mathbb{R}^{2d}} |\rho_\theta(x-\xi y, v) - \rho_\theta(x,v)| \, dx\, du\, dy,$$

Lemma A.3 implies that

$$\lim_{\xi\to 0^+}\int_0^T \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\, d\theta = 0. \tag{5.8}$$

The second limit in the right–hand side of (5.7) is null in view of the Lebesgue dominated convergence theorem and the positivity of $(\rho_t;\ t\in(0,T])$. That ends the proof of (5.4).

Proof of (5.5). Observe that

$$\begin{aligned}
J_{\epsilon,\xi} &\leq \|\Psi_s\|_\infty \|\nabla_u f\|_\infty \left(\int_{Q_t} |B_\xi[x,u;\rho^\epsilon_\theta] - B[x,u;\rho^\epsilon_\theta]| \, \rho^\epsilon_\theta(x,u)\, d\theta\, dx\, du \right.\\
&\quad \left. + \int_{Q_t} |B[x,u;\rho^\epsilon_\theta] - B_\epsilon[x,u;\rho^\epsilon_\theta]| \, \rho^\epsilon_\theta(x,u)\, d\theta\, dx\, du\right) \\
&=: \|\Psi_s\|_\infty \|\nabla_u f\|_\infty \left(J^1_{\epsilon,\xi} + J^2_\epsilon\right).
\end{aligned}$$

In order to show that J^1_ϵ tends to 0, observe that

$$\begin{aligned}
&\int_{Q_t} \left|B_\xi\left[x,u,\rho^\epsilon_\theta\right] - B\left[x,u;\rho^\epsilon_\theta\right]\right| \rho^\epsilon_\theta(x,u)\,d\theta\,dx\,du \\
&\leq \int_{Q_t} \left| \frac{\int_{\mathbb{R}^d} b(v,u)\phi_\xi \star \rho^\epsilon_\theta(x,v)\,dv}{\phi_\xi \star \overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi} \right| \rho^\epsilon_\theta(x,u)\,d\theta\,dx\,du \\
&\quad + \int_{Q_t} \left| \frac{\int_{\mathbb{R}^{2d}} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^{2d}} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)} \right| \rho^\epsilon_\theta(x,u)\,d\theta\,dx\,du.
\end{aligned} \tag{5.9}$$

We now check that each term in the right–hand side of (5.9) tends to 0. Using (4.14) again, we get

$$\begin{aligned}
&\left| \frac{\int_{\mathbb{R}^{2d}} b(v,u)\phi_\xi \star \rho^\epsilon_\theta(x,v)\,dv}{\phi_\xi \star \overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi} \right| \\
&\leq \frac{2\|b\|_\infty}{\overline{\rho}^\epsilon_\theta(x)+\xi} \int_{\mathbb{R}^d} \left|\phi_\xi \star \rho^\epsilon_\theta(x,v) - \rho^\epsilon_\theta(x,v)\right|\,dv.
\end{aligned}$$

Therefore the first term in the right–hand side of (5.9) is bounded from above by

$$C \int_{\mathbb{R}^d} \left|\phi_\xi \star \rho^\epsilon_\theta(x,v) - \rho^\epsilon_\theta(x,v)\right|\,dv.$$

Notice that

$$\begin{aligned}
&\int_0^T \|\phi_\xi \star \rho^\epsilon_\theta - \rho^\epsilon_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta \\
&\leq \int_0^T \|\phi_\xi \star \rho^\epsilon_\theta - \phi_\xi \star \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta + \int_0^T \|\rho^\epsilon_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\,d\theta + \int_0^T \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta \\
&\leq 2\int_0^T \|\rho^\epsilon_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta + \int_0^T \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta.
\end{aligned}$$

Use (5.8) and apply Proposition 5.7. It results that the first term in the right–hand side of (5.9) tends to 0 with ϵ and ξ.

To show that the second term in this right hand–side also tends to 0, we start with setting

$$A_{\xi,\epsilon}(\theta,x,u) := \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi}\rho^\epsilon_\theta(x,u) - \frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\,dv}{\overline{\rho}_\theta(x)+\xi}\rho_\theta(x,u),$$

and inserting

$$\frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta^\epsilon(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)+\xi}\rho_\theta(x,u).$$

We then apply (4.14) to

$$\frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta^\epsilon(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)+\xi}$$

and get

$$\begin{aligned}|A_{\xi,\epsilon}(\theta,x,u)| \leq C\frac{\int_{\mathbb{R}^d}\rho_\theta^\epsilon(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)+\xi}|\rho_\theta^\epsilon(x,u)-\rho_\theta(x,u)| + C\frac{\rho(x,u)}{\overline{\rho}_\theta^\epsilon(x)+\xi}\int_{\mathbb{R}^d}|\rho_\theta^\epsilon(x,v)-\rho_\theta(x,v)|dv \\ + C\frac{\rho(x,u)}{\overline{\rho}_\theta^\epsilon(x)+\xi}\frac{\int_{\mathbb{R}^d}\rho_\theta^\epsilon(x,v)dv}{\overline{\rho}_\theta^\epsilon(x)+\xi}|\overline{\rho}_\theta^\epsilon(x,u)-\overline{\rho}_\theta^\epsilon(x,u)|.\end{aligned}$$

We deduce that

$$\int_{Q_t}|A_{\xi,\epsilon}(\theta,x,u)|d\theta dx du \leq \int_0^T \|\rho_\theta^\epsilon - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}d\theta$$

which tends to 0 in view of Proposition 5.7.

To end the proof that the second term in the right hand–side of (5.9) tends to 0, it now remains to observe that, owing to the positivity of $(\rho_t;\ t\in[0,T])$, we have

$$\begin{aligned}&\lim_{\xi\to 0^+}\limsup_{\epsilon\to 0^+}\int_{Q_t}\left|\frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta^\epsilon(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta^\epsilon(x,v)\,dv}{\overline{\rho}_\theta^\epsilon(x)}\right|\rho_\theta^\epsilon(x,u)\,d\theta\,dx\,du \\ &\leq \lim_{\xi\to 0^+}\int_{Q_t}\left|\frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\,dv}{\overline{\rho}_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\,dv}{\overline{\rho}_\theta(x)}\right|\rho_\theta(x,u)d\theta dx du = 0,\end{aligned}$$

by using Lebesgue's dominated convergence theorem. We thus have proven

$$\lim_{\xi\to 0^+}\limsup_{\epsilon\to 0^+} J^1_{\epsilon,\xi} = 0. \tag{5.10}$$

Similarly, we have

$$\lim_{\epsilon\to 0^+}\int_{Q_t}\left|B\left[x,u,\rho^\epsilon_\theta\right]-B_\epsilon\left[x,u;\rho^\epsilon_\theta\right]\right|\rho^\epsilon_\theta(x,u)\,d\theta\,dx\,du$$

$$\le 2\|b\|_\infty \lim_{\epsilon\to 0^+}\int_0^T \|\phi_\epsilon\star\rho^\epsilon_\theta-\rho^\epsilon_\theta\|_{L^1(\mathbb{R}^{2d})}\,d\theta$$

$$+\lim_{\epsilon\to 0^+}\int_{Q_t}\left|\frac{1}{\overline{\rho}^\epsilon_\theta(x)+\epsilon}-\frac{1}{\overline{\rho}^\epsilon_\theta(x)}\right|\left(\int_{\mathbb{R}^d}|b(v,u)|\,\rho^\epsilon_\theta(x,v)\,dv\right)\rho^\epsilon_\theta(x,u)\,d\theta\,dx\,du=0,$$

so that $\lim_{\epsilon\to 0^+} J^2_\epsilon = 0$. In view of (5.10), the proof is completed. □

6 Conclusion and perspectives

In this paper we have studied a Lagrangian stochastic model and shown its well–posedness. We have proved that the unique weak solution is an hypoelliptic diffusion process whose dynamics depends on the conditional distribution of the velocity component knowing the position component. To our knowledge, this is the first theoretical result on the Lagrangian stochastic models modelling turbulent fluid particles. Bossy and Jabir [3] consider models with specular boundary conditions. See also Jabir [7]. A lot remains to be done to study the complex models developed by e.g. Pope [13].

We also emphasize another possible extension of our result. We conjecture the following PDE analysis result: the estimate (4.4) holds true under classical Hölder conditions rather than (3.2), possibly by using Maxwellian approximations rather than using the parametrix method.

A Appendix

A.1 Pascussi and Francesco´s estimates on fundamental solutions of ultraparabolic PDEs

Before stating the estimate on fundamental solutions of ultraparabolic PDEs which are used in this paper, we need to introduce some new notation.

In [5] Pascussi and Francesco consider ultraparabolic PDEs of the type

$$\partial_s\psi - \sum_{i,j} a^{(i,j)}(s,y,v)\partial^2_{v_iv_j}\psi - v\cdot B\nabla_{(x,v)}\psi = 0,$$

where $a(t,x,u) = \sigma(t,x,u)\sigma^*(t,x,u)$ and B is a $2d\times 2d$ matrix with constant entries. The statement of their results for general matrices B require to introduce a pseudo–metric depending on B and some notational effort. We thus limit ourselves to our context where

$$B = \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{\mathbb{R}} & 0 \end{pmatrix}.$$

In this context, Pascussi and Francesco´s assumption on the coefficient a writes as follows: a is a bounded function and there exist $\alpha\in(0,1]$ and $C>0$ such that, for all (s,x,u) , $(t,y,v)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, the inequality (3.2) holds true.

For all $\eta>0$ where λ as in (3.1), let $\Gamma_\eta(s,y,v;t,x,u)$ be the fundamental solution of

$$\partial_s\psi + v\cdot\nabla_y\psi + \frac{\eta}{2}\triangle_v\psi = 0, \tag{A.1}$$

that is,

$$\begin{aligned}&\Gamma_\eta(s,y,v;t,x,u)\\ &=\frac{\sqrt{3^d}}{(\sqrt{\pi}\eta(t-s))^{2d}}\\ &\quad\times\exp\left\{-\frac{6\left|x-y-(t-s)v\right|^2}{\eta(t-s)^3}+\frac{6(x-y-(t-s)v)\cdot(u-v)}{\eta(t-s)^2}-\frac{2\left|u-v\right|^2}{\eta(t-s)}\right\}.\end{aligned} \tag{A.2}$$

We are in a position to state the following theorem:

Theorem A.1 (Francesco & Pascucci [5]). *Suppose (3.1) and (3.2). There exists a fundamental solution $\Gamma(s,y,v;t,x,u)$ to the operator*

$$\partial_s - \sum_{i,j} a^{(i,j)}(s,y,v)\partial_{v_iv_j} - v\cdot\nabla_x$$

which satisfies all the properties listed in Proposition 4.4. In particular, there exists $\eta>\lambda$ and a constant $C>0$ such that

$$\nabla_v\Gamma(s,y,v;t,x,u) \le \frac{C}{\sqrt{s-t}}\Gamma_\eta(s,y,v;t,x,u).$$

A.2 Proof of Proposition 4.4

To get the existence of a weak solution, we adapt the proof of Theorem 6.3.2 in Stroock & Varadhan [16]: consider a sequence $\{\sigma^n;\ n \in \mathbb{N}\}$ of $\mathbb{R}^d \times \mathbb{R}^d$–valued Lipschitz functions on $[0,T] \times \mathbb{R}^d$ such that $\lim_{n\to+\infty} \sigma^n = \sigma$ uniformly. For all $n \in \mathbb{N}$, one has existence of a strong solution $(Y_t^{n,s,y,v},\ V_t^{n,s,v};\ s \le t \le T)$ to Equation (4.1) when one substitutes σ to σ^n. Then it is easy to check that $(Y_t^{n,s,y,v},\ V_t^{n,s,v};\ s \le t \le T)$ converges in distribution to a weak solution of (4.1).

The uniqueness of the weak solution and the properties (i) to (iii) result from Theorem A.1.

A.3 Technical lemmas

For the reader's convenience we state three technical results which played a key role in our proofs.

The first lemma can be found in Stroock & Varadhan [16, Lemma 11.4.1].

Lemma A.2. *Let $\{f_n;\ n \ge 1\}$ be a sequence of non–negative measurable functions such that $\int_{\mathbb{R}^q} f_n(z)\,dz = 1$ and, for all $h \in \mathbb{R}^q$,*

$$\lim_{|h|\to 0} \sup_{n\ge 1} \int_{\mathbb{R}^q} |f_n(z+h) - f_n(z)|\ dz = 0.$$

Suppose that there exists a density function f such that, for all function $\psi \in \mathcal{C}_c(\mathbb{R}^q)$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^q} f_n(z)\psi(z)\,dz = \int_{\mathbb{R}^q} f(z)\psi(z)\,dz.$$

Then $\{f_n\}$ converges to f in $L^1(\mathbb{R}^q)$.

The next lemma can be found in Rana [15].

Lemma A.3. *Let $1 \le p < +\infty$. For all $f \in L^p(\mathbb{R}^q)$ and $h \in \mathbb{R}^q$ we have*

$$\lim_{|h|\to 0} \int_{\mathbb{R}^q} |f(z+h) - f(z)|^p\ dz = 0. \tag{A.3}$$

The last lemma can be found in Stroock & Varadhan [16, Cor.1.1.5].

Lemma A.4. *Let S be a Polish space and let $\{F_{\epsilon'},\ \epsilon' > 0\}$ be a uniformly bounded set of functions which are equicontinuous at each point of S. For all $\{\mu_\epsilon;\ \epsilon > 0\}$ and μ in $\mathcal{M}(S)$ such that $\lim_{\epsilon\to 0}\mu_\epsilon = \mu$ one has*

$$\lim_{\epsilon\to 0}\sup_{\epsilon'>0}\left|\int_S F_{\epsilon'}\,d\mu_\epsilon - \int_S F_{\epsilon'}\,d\mu\right| = 0.$$

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