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# Classroom Examples of Robustness Problems in Geometric Computations<sup>\*</sup> (Extended Abstract)

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**Abstract.** The algorithms of computational geometry are designed for a machine model with exact real arithmetic. Substituting floating point arithmetic for the assumed real arithmetic may cause implementations to fail. Although this is well known, there is no comprehensive documentation of what can go wrong and why. In this extended abstract, we study a simple incremental algorithm for planar convex hulls and give examples which make the algorithm fail in all possible ways. We also show how to construct failure-examples semi-systematically and discuss the geometry of the floating point implementation of the orientation predicate. We hope that our work will be useful for teaching computational geometry. The full paper is available at [www.mpi-sb.mpg.de/~mehlhorn/ftp/ClassRoomExamples.ps](http://www.mpi-sb.mpg.de/~mehlhorn/ftp/ClassRoomExamples.ps). It contains further examples, more theory, and color pictures. We strongly recommend to read the full paper instead of this extended abstract.

## 1 Introduction

The algorithms of computational geometry are designed for a machine model with exact real arithmetic. Substituting floating point arithmetic for the assumed real arithmetic may cause implementations to fail. Although this is well known, it is not common knowledge. There is no paper which systematically discusses what can go wrong and provides simple examples for the different ways in which floating point implementations can fail. Due to this lack of examples, instructors of computational geometry have little material for demonstrating the inadequacy of floating point arithmetic for geometric computations, students of computational geometry and implementers of geometric algorithms still underestimate the seriousness of the problem, and researchers in our and neighboring disciplines, e.g., numerical analysis, still believe, that simple approaches are able to overcome the problem.

In this extended abstract, we study a simple incremental algorithm for planar convex hulls and show how it can fail and why it fails when executed with floating point arithmetic. The convex hull  $CH(S)$  of a set  $S$  of points in the plane is the smallest convex polygon containing  $S$ . A point  $p \in S$  is called *extreme* in  $S$  if  $CH(S) \neq CH(S \setminus p)$ .

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The extreme points of  $S$  form the vertices of the convex hull polygon. Convex hulls can be constructed incrementally. One starts with three non-collinear points in  $S$  and then considers the remaining points in arbitrary order. When a point is considered and lies inside the current hull, the point is simply discarded. When the point lies outside, the tangents to the current hull are constructed and the hull is updated appropriately. We give a more detailed description of the algorithm in Section 4 and the complete C++ program in the full paper.

Figures 2 and 5 show point sets (we give the numerical coordinates of the points in Section 4) and the respective convex hulls computed by the floating point implementation of our algorithm. In each case the input points are indicated by small circles, the computed convex hull polygon is shown in light grey, and the alleged extreme points are shown as filled circles. The examples show that the implementation may make gross mistakes. It may leave out points which are clearly extreme, it may compute polygons which are clearly non-convex, and it may even run forever (example not shown here).

The first contribution of this paper is to provide a set of instances that make the floating point implementation fail in disastrous ways. The computed results do not resemble the correct results in any reasonable sense.

Our second contribution is to explain why these disasters happen. The correctness of geometric algorithms depends on geometric properties, e.g., a point lies outside a convex polygon if and only if it can see one of the edges. We give examples, for which a floating point implementation violates these properties: a point outside a convex polygon which sees no edge (in a floating point implementation of “sees”) and a point not outside which sees some edges (in a floating point implementation of “sees”). We give examples for all possible violations of the correctness properties of our algorithm.

Our third contribution is to show how difficult examples can be constructed systematically or at least semi-systematically. This should allow others to do similar studies.

We believe that the paper and its companion web page will be useful in teaching computational geometry, and that even experts will find it surprising and instructing in how many ways and how badly even simple algorithms can be made to fail. The companion web page (<http://www.mpi-sb.mpg.de/~kettner/proj/NonRobust/>) contains the source code of all programs, the input files for all examples, and installation procedures. It allows the reader to perform our and further experiments.

This paper is structured as follows. In Section 2 we discuss the ground rules for our experiments, in Section 3 we study the effect of floating point arithmetic on one of the most basic predicates of planar geometry, the orientation predicate, in Section 4 we discuss an incremental algorithm for planar convex hulls. Finally, Section 5 offers a short conclusion. In the full paper, we also discuss the gift wrapping algorithm for planar convex hulls and an incremental algorithm for 3d Delaunay triangulations, and also give additional theory.

*Related Work:* The literature contains a small number of documented failures due to numerical imprecision, e.g., Forrest’s seminal paper on implementing the point-in-polygon test [For85], Shewchuk’s example for divide-and-conquer Delaunay triangulation [She97], Ramshaw’s braided lines [MN99, Section 9.6.2], Schirra’s example for convex hulls [MN99, Section 9.6.1], and the sweep line algorithm for line segment intersection and boolean operations on polygons [MN99, Sections 10.7.4 and 10.8.3].

## 2 Ground Rules for our Experiments

Our codes are written in C++ and the results are reproducible on any platform compliant with IEEE floating-point standard 754 for double precision (see [Gol90]), and also with other programming languages. All programs and input data can be found on the companion web page. Numerical computations are based on IEEE arithmetic. In particular, numbers are machine floating point numbers (called *doubles* for short). Such numbers have the form  $\pm m2^e$  where  $m = 1.m_1m_2\dots m_{52}$  ( $m_i = \{0, 1\}$ ) is the mantissa in binary and  $e$  is the exponent satisfying  $-1024 < e < 1024$ . The results of arithmetic operations are rounded to the nearest double (with ties broken using some fixed rule).

Our numerical example data will be written in decimals (for human consumption). Such decimal values, when read into the machine, are internally represented by the nearest double. We have made sure that our data can be represented exactly by doubles and their conversion to binary and back to decimal is the identity operation.

## 3 The Orientation Predicate

Three points  $p$ ,  $q$ , and  $r$  in the plane either lie on a common line or form a left or right turn. The triple  $(p, q, r)$  forms a left (right) turn, if  $r$  lies left (right) of the line through  $p$  and  $q$  and oriented in the direction from  $p$  to  $q$ . Analytically, the orientation of the triple  $(p, q, r)$  is tantamount to the sign of a determinant:

$$\text{orientation}(p, q, r) = \text{sign}\left(\det \begin{bmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{bmatrix}\right), \quad (1)$$

where  $p = (p_x, p_y)$ , etc. We have  $\text{orientation}(p, q, r) = +1$  (resp.,  $-1, 0$ ) iff the polyline  $(p, q, r)$  represents a left turn (resp., right turn, collinearity). Interchanging two points in the triple changes the sign of the orientation. We denote the  $x$ -coordinate of a point  $p$  also by  $x(p)$ . We implement the orientation predicate in the straightforward way:

$$\text{orientation}(p, q, r) = \text{sign}((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x)). \quad (2)$$

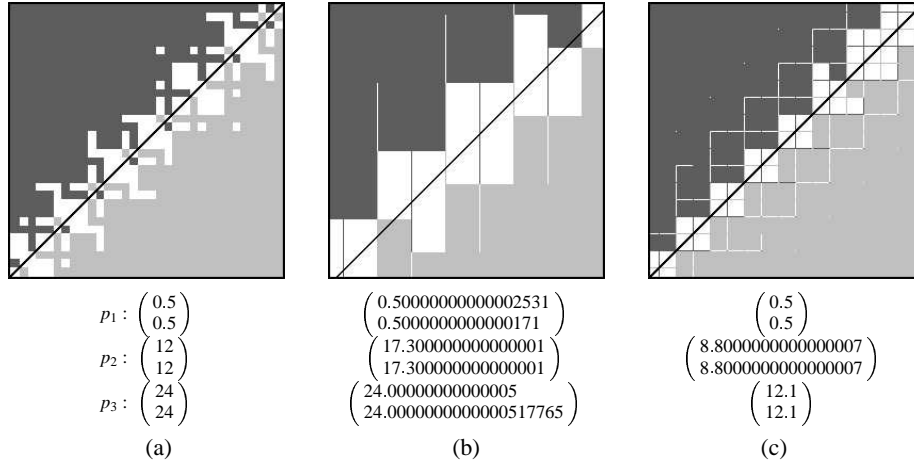
When the orientation predicate is implemented in this obvious way and evaluated with floating point arithmetic, we call it *float\_orient* $(p, q, r)$  to distinguish it from the ideal predicate. Since floating point arithmetic incurs round-off error, there are potentially three ways in which the result of *float\_orient* could differ from the correct orientation:

- *rounding to zero*: we mis-classify a  $+$  or  $-$  as a  $0$ ;
- *perturbed zero*: we mis-classify  $0$  as  $+$  or  $-$ ;
- *sign inversion*: we mis-classify a  $+$  as  $-$  or vice-versa.

### 3.1 The Geometry of Float-Orientation

What is the geometry of *float\_orient*, i.e., which triples of points are classified as left-turns, right-turns, or straight? The following type of experiment answers the question: We choose three points  $p_1$ ,  $p_2$ , and  $p_3$  and then compute

$$\text{float\_orient}((x(p_1) + x \cdot u, y(p_1) + y \cdot u), p_2, p_3)$$



**Fig. 1.** The weird geometry of the float-orientation predicate: The figure shows the results of  $float\_orient((x(p_1) + x \cdot u, y(p_1) + y \cdot u, p_2, p_3)$  for  $0 \leq x, y \leq 255$ , where  $u$  is the increment between adjacent floating point numbers in the considered range. The result is color coded: White (light grey, dark grey, resp.) pixels represent collinear (negative, positive, resp.) orientation. The line through  $p_2$  and  $p_3$  is also shown.

for  $0 \leq x, y \leq 255$ , where  $u$  is the increment between adjacent floating point numbers in the considered range; for example,  $u = 2^{-53}$  if  $x(p_1) = 1/2$  and  $u = 4 \cdot 2^{-53}$  if  $x(p_1) = 2 = 4 \cdot 1/2$ . We visualize the resulting  $256 \times 256$  array of signs as a  $256 \times 256$  grid of pixels: A white (light grey, dark grey) pixel represents collinear (negative, positive, respectively) orientation. In the figures in this section we also indicate an approximation of the line through  $p_2$  and  $p_3$ .

Figure 1(a) shows the result of our first experiment: We use the line defined by the points  $p_2 = (12, 12)$  and  $p_3 = (24, 24)$  and query it near  $p_1 = (0.5, 0.5)$ . We urge the reader to pause for a moment and to sketch what he/she expects to see. The authors expected to see a white band around the diagonal with nearly straight boundaries. Even for points with such simple coordinates the geometry of  $float\_orient$  is quite weird: the set of white points (= the points classified as on the line) does not resemble a straight line and the sets of light grey or dark grey points do not resemble half-spaces. We even have points that change the side of the line, i.e., are lying left (right) of the line and being classified as right (left) of the line.

In Figures 1(b) and (c) we have given our base points  $p_i$  more complex coordinates by adding some digits behind the binary point. This enhances the cancellation effects in the evaluation of  $float\_orient$  and leads to even more striking pictures. In (b), the light grey region looks like a step function at first sight. Note however, it is not monotone, has white rays extending into it, and light grey lines extruding from it. The white region (= on-region) forms blocks along the line. Strangely enough, these blocks are separated by dark and light grey lines. Finally, many points change sides. In Figure (c), we have white blocks of varying sizes along the diagonal, thin white and partly light

grey lines extending into the dark grey region (similarly for the light grey region), light grey points (the left upper corners of the white structures extending into the dark grey region) deep inside the blue region, and isolated white points almost 100 units away from the diagonal.

All diagrams in Figure 1 exhibit block structure. We now explain why. Assume we keep  $y$  fixed and vary only  $x$ . We evaluate  $\text{float\_orient}((x(p_1) + x \cdot u, y(p_1) + y \cdot u), p_2, p_3)$  for  $0 \leq x \leq 255$ , where  $u$  is the increment between adjacent floating point numbers in the considered range. Also  $\text{orientation}(p, q, r) = \text{sign}((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x))$ . We incur round-off error in the additions/subtractions and also in the multiplications. Consider first one of the differences, say  $q_x - p_x$ . In (a), we have  $q_x \approx 12$  and  $p_x \approx 0.5$ . Since 12 has four binary digits, we lose the last four bits of  $x$  in the subtraction, in other words, the result of the subtraction  $q_x - p_x$  is constant for  $2^4$  consecutive values of  $x$ . Because of rounding to nearest, the intervals of constant value are  $[8, 23]$ ,  $[24, 39]$ ,  $[40, 55]$  ... Similarly, the floating point result of  $r_x - p_x$  is constant for  $2^5$  consecutive values of  $x$ . Because of rounding to nearest, the intervals of constant value are  $[16, 47]$ ,  $[48, 69]$ , ... Overlaying the two progressions gives intervals  $[16, 23]$ ,  $[24, 39]$ ,  $[40, 47]$ ,  $[48, 55]$ , ... and this explains the structure we see in the rows of (a). We see short blocks of length 8, 16, 24, ... in (a). In (b) and (c), the situation is somewhat more complicated. It is again true that we have intervals for  $x$ , where the results of the subtractions are constant. However, since  $q$  and  $r$  have more complex coordinates, the relative shifts of these intervals are different and hence we see narrow and broad features.

## 4 The Convex Hull Problem

We discuss a simple incremental convex hull algorithm. We describe the algorithm, state the underlying geometric assumptions, give instances which violate the assumptions when used with floating point arithmetic, and finally show which disastrous effects these violations may have on the result of the computation.

The incremental algorithm maintains the *current convex hull*  $CH$  of the points seen so far. Initially,  $CH$  is formed by choosing three non-collinear points in  $S$ . It then considers the remaining points one by one. When considering a point  $r$ , it first determines whether  $r$  is outside the current convex hull polygon. If not,  $r$  is discarded. Otherwise, the hull is updated by forming the tangents from  $r$  to  $CH$  and updating  $CH$  appropriately. The incremental paradigm is used in Andrew's variant [And79] of Graham's scan [Gra72] and also in the randomized incremental algorithm [CS89].

The algorithm maintains the current hull as a circular list  $L = (v_0, v_1, \dots, v_{k-1})$  of its extreme points in counter-clockwise order. The line segments  $(v_i, v_{i+1})$ ,  $0 \leq i \leq k-1$  (indices are modulo  $k$ ) are the *edges* of the current hull. If  $\text{orientation}(v_i, v_{i+1}, r) < 0$ , we say  $r$  *sees* the edge  $(v_i, v_{i+1})$ , and say the edge  $(v_i, v_{i+1})$  is *visible* from  $r$ . If  $\text{orientation}(v_i, v_{i+1}, r) \leq 0$ , we say that the edge  $(v_i, v_{i+1})$  is *weakly visible* from  $r$ . After initialization,  $k \geq 3$ . The following properties are key to the operation of the algorithm.

**Property A:** A point  $r$  is outside  $CH$  iff  $r$  can see an edge of  $CH$ .

**Property B:** If  $r$  is outside  $CH$ , the edges weakly visible from  $r$  form a non-empty consecutive subchain; so do the edges that are not weakly visible from  $r$ .

If  $(v_i, v_{i+1}), \dots, (v_{j-1}, v_j)$  is the subsequence of weakly visible edges, the updated hull is obtained by replacing the subsequence  $(v_{i+1}, \dots, v_{j-1})$  by  $r$ . The subsequence  $(v_i, \dots, v_j)$  is taken in the circular sense. E.g., if  $i > j$  then the subsequence is  $(v_i, \dots, v_{k-1}, v_0, \dots, v_j)$ . From these properties, we derive the following algorithm:

```

Initialize  $L$  to the counter-clockwise triangle  $(a, b, c)$ . Remove  $a, b, c$  from  $S$ .
for all  $r \in S$  do
  if there is an edge  $e$  visible from  $r$  then
    Compute the sequence  $(v_i, \dots, v_j)$  of edges that are weakly visible from  $r$ ;
    Replace the subsequence  $(v_{i+1}, \dots, v_{j-1})$  by  $r$ ;
  end if
end for

```

To turn the sketch into an algorithm, we provide more information about the substeps.

1. How does one determine whether there is an edge visible from  $r$ ? We iterate over the edges in  $L$ , checking each edge using the orientation predicate. If no visible edge is found, we discard  $r$ . Otherwise, we take any one of the visible edges as the starting edge for the next item.
2. How does one identify the subsequence  $(v_i, \dots, v_j)$ ? Starting from a visible edge  $e$ , we move counter-clockwise along the boundary until a non-weakly-visible edge is encountered. Similarly, move clockwise from  $e$  until a non-weakly-visible edge is encountered.
3. How to update the list  $L$ ? We can delete the vertices in  $(v_{i+1}, \dots, v_{j-1})$  after all visible edges are found, as suggested in the above sketch (“the off-line strategy”) or we can delete them concurrently with the search for weakly visible edges (“the on-line strategy”).

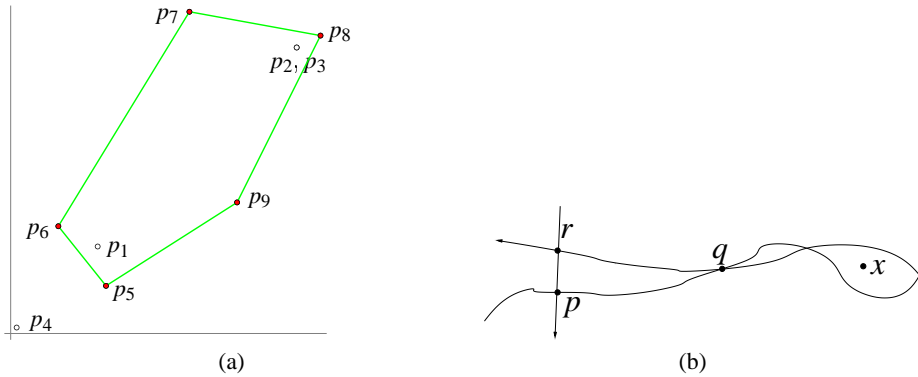
There are four logical ways to negate Properties A and B:

- **Failure (A<sub>1</sub>):** A point outside the current hull sees no edge of the current hull.
- **Failure (A<sub>2</sub>):** A point inside the current hull sees an edge of the current hull.
- **Failure (B<sub>1</sub>):** A point outside the current hull sees all edges of the convex hull.
- **Failure (B<sub>2</sub>):** A point outside the current hull sees a non-contiguous set of edges.

Failures (A<sub>1</sub>) and (A<sub>2</sub>) are equivalent to the negation of Property A. Similarly, Failures (B<sub>1</sub>) and (B<sub>2</sub>) are complete for Property B if we take (A<sub>1</sub>) into account. Are all these failures realizable? We now affirm this.

#### 4.1 Single Step Failures

We give instances violating the correctness properties of the algorithm. More precisely, we give sequences  $p_1, p_2, p_3, \dots$  of points such that the first three points form a counter-clockwise triangle (and *float\_orient* correctly discovers this) and such that the insertion of some later point leads to a violation of a correctness property (in the computations with doubles). We also discuss how we arrived at the examples. All our examples involve nearly or truly collinear points; instances without nearly collinear or truly collinear points cause no problems in view of the error analysis of the preceding section



**Fig. 2.** (a) The convex hull illustrating Failure (A<sub>1</sub>). The point in the lower left corner is left out of the hull. (b) schematically indicates the ridiculous situation of a point outside the current hull and seeing no edge of the hull:  $x$  lies to the left of all sides of the triangle  $(p, q, r)$ .

(omitted in this extended abstract). Does this make our examples unrealistic? We believe not. Many point sets contain nearly collinear points or truly collinear points which become nearly collinear by conversion to floating point representation.

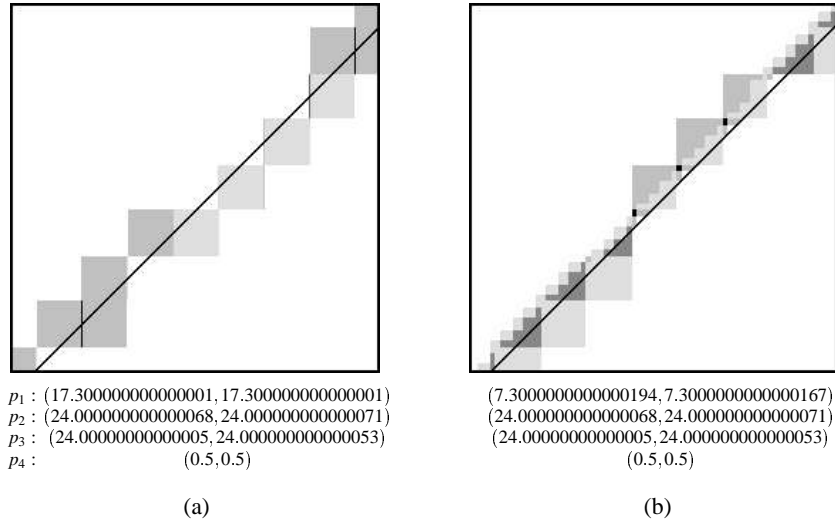
(A<sub>1</sub>) *A point outside the current hull sees no edge of the current hull:* Consider the set of points below. Figure 2(a) shows the computed convex hull, where a point which is clearly extreme was left out of the hull.

$$\begin{array}{ll}
 p_1 = (7.3000000000000194, 7.3000000000000167) & \text{float\_orient}(p_1, p_2, p_3) > 0, \\
 p_2 = (24.000000000000068, 24.000000000000071) & \text{float\_orient}(p_1, p_2, p_4) > 0, \\
 p_3 = (24.000000000000005, 24.000000000000053) & \text{float\_orient}(p_2, p_3, p_4) > 0, \\
 p_4 = (0.50000000000001621, 0.50000000000001243) & \text{float\_orient}(p_3, p_1, p_4) > 0 (!!), \\
 p_5 = (8, 4) \quad p_6 = (4, 9) \quad p_7 = (15, 27) & \\
 p_8 = (26, 25) \quad p_9 = (19, 11) &
 \end{array}$$

*What went wrong?* The culprits are the first four points. They lie almost on the line  $y = x$ , and *float\_orient* gives the results shown above. Only the last evaluation is wrong, indicated by “(!)”. Geometrically, these four evaluations say that  $p_4$  sees no edge of the triangle  $(p_1, p_2, p_3)$ . Figure 2(b) gives a schematic view of this ridiculous situation. The points  $p_5, \dots, p_9$  are then correctly identified as extreme points and are added to the hull. However, the algorithm never recovers from the error made when considering  $p_4$  and the result of the computation differs drastically from the correct hull.

We next explain how we arrived at the instance above. Intuition told us that an example (if it exists at all) would be a triangle with two almost parallel sides and with a query point near the wedge defined by the two nearly parallel edges. In view of Figure 1 such a point might be mis-classified with respect to one of the edges and hence would be unable to see any edge of the triangle. So we started with the points used in Figure 1(b), i.e.,  $p_1 \approx (17, 17)$ ,  $p_2 \approx (24, 24) \approx p_3$ , where we moved  $p_2$  slightly to the right so as to guarantee that we obtain a counter-clockwise triangle. We then probed the edges incident to  $p_1$  with points  $p_4$  in and near the wedge formed by these edges. Figure 3(a) visualizes the outcomes of the two relevant orientation tests. Each black pixel (not lying on the line indicating the nearly parallel edges of the triangle) is a candidate for Failure





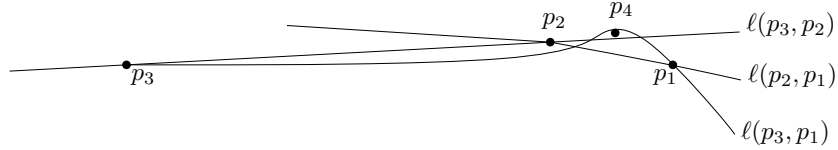
**Fig. 3.** The points  $(p_1, p_2, p_3)$  form a counter-clockwise triangle and we are interested in the classification of points  $(x(p_4) + xu, y(p_4) + yu)$  with respect to the edges  $(p_1, p_2)$  and  $(p_3, p_1)$  incident to  $p_1$ . The extensions of these edges are indistinguishable in the pictures and are drawn as a single black line. The black points not lying on the black diagonal do not “float-see” either one of the edges (Failure  $A_1$ ). Points collinear with one of the edges are middle grey, those collinear with both edges are light grey, those classified as seeing one but not the other edge are white, and those seeing both edges are dark grey. (a) Example starting from points in Figure 1. (b) Example that achieves “robustness” with respect to the first three points.

( $A_1$ ). The example obtained in this way was not completely satisfactory, since some orientation tests on the initial triangle  $(p_1, p_2, p_3)$  were evaluating to zero.

We perturbed the example further, aided by visualizing  $float\_orient(p_1, p_2, p_3)$ , until we found the example shown in (b); by our error analysis, this test incurs the largest error among the six possible ways of performing the orientation test for three points and is hence most likely to return the incorrect result. The final example has the nice property that all possible  $float\_orient$  tests on the first three points are correct. So this example is pretty much independent from any conceivable initialization an algorithm could use to create the first valid triangle. Figure 3(b) shows the outcomes of the two orientations tests for our final example.

( $A_2$ ) *A point inside the current hull sees an edge of the current hull:* Such examples are plenty. We take any counter-clockwise triangle and chose a fourth point inside the triangle but close to one of the edges. By Figure 1 there is the chance of sign reversal. A concrete example follows:

$$\begin{array}{ll}
 p_1 = (27.643564356435643, -21.881188118811881) & float\_orient(p_1, p_2, p_3) > 0 \\
 p_2 = (83.366336633663366, 15.544554455445542) & float\_orient(p_1, p_2, p_4) < 0 (!! ) \\
 p_3 = (4, 4) & float\_orient(p_2, p_3, p_4) > 0 \\
 p_4 = (73.415841584158414, 8.8613861386138595) & float\_orient(p_3, p_1, p_4) > 0
 \end{array}$$



**Fig. 4.** Schematics: The point  $p_4$  sees all edges of the triangle  $(p_1, p_2, p_3)$ .

The convex hull is correctly initialized to  $(p_1, p_2, p_3)$ . The point  $p_4$  is inside the current convex hull, but the algorithm incorrectly believes that  $p_4$  can see the edge  $(p_1, p_2)$  and hence changes the hull to  $(p_1, p_4, p_2, p_3)$ , a slightly non-convex polygon.

*(B<sub>1</sub>) A point outside the current hull sees all edges of the convex hull:* Intuition told us that an example (if it exists) would consist of a triangle with one angle close to  $\pi$  and hence three almost parallel sides. Where should one place the query point? We first placed it in the extension of the three parallel sides and quite a distance away from the triangle. This did not work. The choice which worked is to place the point near one of the sides so that it could see two of the sides and “fbat-see” the third. Figure 4 illustrates this choice. A concrete example follows:

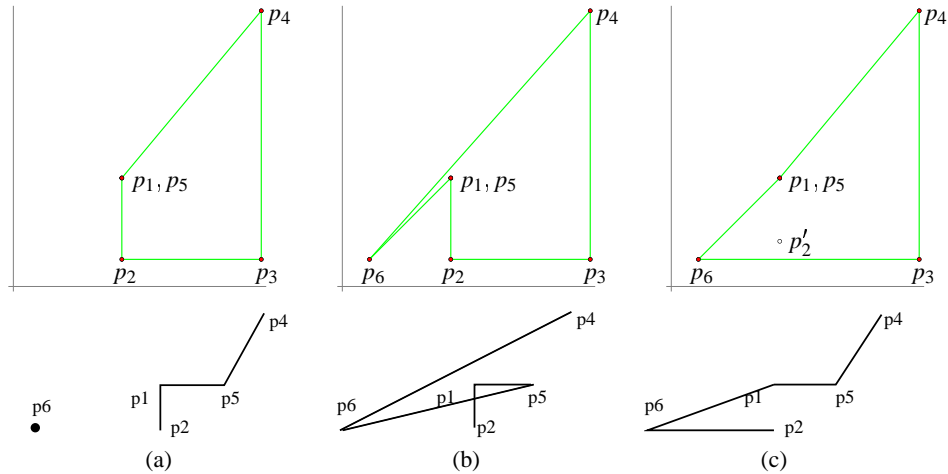
$p_1 = (200, 49.200000000000003)$	$float\_orient(p_1, p_2, p_3) > 0$
$p_2 = (100, 49.600000000000001)$	$float\_orient(p_1, p_2, p_4) < 0$
$p_3 = (-233.33333333333334, 50.933333333333333)$	$float\_orient(p_2, p_3, p_4) < 0$
$p_4 = (166.66666666666669, 49.333333333333336)$	$float\_orient(p_3, p_1, p_4) < 0 (!!)$

The first three points form a counter-clockwise oriented triangle and according to *float\_orient*, the algorithm believes that  $p_4$  can see all edges of the triangle. What will our algorithm do? It depends on the implementation details. If the algorithm first searches for an invisible edge, it will search forever and never terminate. If it deletes points on-line from  $L$  it will crash or compute nonsense depending on the details of the implementation.

*(B<sub>2</sub>) A point outside the current hull sees a non-contiguous set of edges:* Consider the following points:

$p_1 = (0.50000000000001243, 0.50000000000000189)$	$float\_orient(p_1, p_4, p_5) < 0 (!!)$
$p_2 = (0.50000000000001243, 0.50000000000000333)$	$float\_orient(p_4, p_3, p_5) > 0$
$p_3 = (24.000000000000005, 24.000000000000053)$	$float\_orient(p_3, p_2, p_5) < 0$
$p_4 = (24.000000000000068, 24.000000000000071)$	$float\_orient(p_2, p_1, p_5) > 0$
$p_5 = (17.300000000000001, 17.300000000000001)$	

Inserting the first four points results in the convex quadrilateral  $(p_1, p_4, p_3, p_2)$ ; this is correct. The last point  $p_5$  sees only the edge  $(p_3, p_2)$  and none of the other three. However, *float\_orient* makes  $p_5$  see also the edge  $(p_1, p_4)$ . The subsequences of visible and invisible edges are not contiguous. Since the falsely classified edge  $(p_1, p_4)$



**Fig. 5.** (a) The hull constructed after processing points  $p_1$  to  $p_5$ . Points  $p_1$  and  $p_5$  lie close to each other and are indistinguishable in the upper figure. The schematic sketch below shows that we have a concave corner at  $p_5$ . The point  $p_6$  sees the edges  $(p_1, p_2)$  and  $(p_4, p_5)$ , but does **not** see the edge  $(p_5, p_1)$ . One of the former edges will be chosen by the algorithm as the chain of edges visible from  $p_6$ . Depending on the choice, we obtain the hulls shown in (b) or (c). In (b),  $(p_4, p_5)$  is found as the visible edge, and in (c),  $(p_1, p_2)$  is found. We refer the reader to the text for further explanations. The figures show the coordinate axes for orientation.

comes first, our algorithm inserts  $p_5$  at this edge, removes no other vertex, and returns a polygon that has self-intersections and is not simple.

## 4.2 Global Effects

By now, we have seen examples which cover the negation space of the correctness properties on the incremental algorithm and we have seen the effect of an incorrect orientation test for a single update step. We next study global effects. *The goal is to refute the myth that the algorithm will always compute an approximation of the true convex hull.*

*The algorithm computes a convex polygon, but misses some of the extreme points:* We have already seen such an example in Failure (A<sub>1</sub>). We can modify this example so that the ratio of the areas of the true hull and the computed hull becomes arbitrarily large. We do as in Failure (A<sub>1</sub>), but move the fourth point to infinity. The true convex hull has four extreme points. The algorithm misses  $q_4$ .

$$\begin{array}{ll}
 q_1 = (0.100000000000000001, 0.100000000000000001) & \text{float\_orient}(q_1, q_2, q_3) < 0 \\
 q_2 = (0.200000000000000001, 0.200000000000000004) & \text{float\_orient}(q_1, q_2, q_4) = 0 (!! ) \\
 q_3 = (0.79999999999999993, 0.80000000000000004) & \text{float\_orient}(q_2, q_3, q_4) = 0 (!! ) \\
 q_4 = (1.267650600228229 \cdot 10^{30}, 1.2676506002282291 \cdot 10^{30}) & \text{float\_orient}(q_3, q_1, q_4) > 0
 \end{array}$$

*The algorithm crashes or does not terminate:* See Failure (B<sub>1</sub>).

*The algorithm computes a non-convex polygon:* We have already given such an example in Failure (A<sub>2</sub>). However, this failure is not visible to the naked eye. We next give examples where non-convexity is visible to the naked eye. We consider the points:

$$\begin{aligned} p_1 &= (24.00000000000005, 24.00000000000053) & p_2 &= (24.0, 6.0) \\ p_3 &= (54.85, 6.0) & p_4 &= (54.85000000000357, 61.00000000000121) \\ p_5 &= (24.00000000000068, 24.00000000000071) & p_6 &= (6, 6). \end{aligned}$$

After the insertion of  $p_1$  to  $p_4$ , we have the convex hull  $(p_1, p_2, p_3, p_4)$ . This is correct. Point  $p_5$  lies inside the convex hull of the first four points; but  $\text{float\_orient}(p_4, p_1, p_5) < 0$ . Thus  $p_5$  is inserted between  $p_4$  and  $p_1$ . and we obtain  $(p_1, p_2, p_3, p_4, p_5)$ . However, this error is not visible yet to the eye, see Figure 5(a).

The point  $p_6$  sees the edges  $(p_4, p_5)$  and  $(p_1, p_2)$ , but does not see the edge  $(p_5, p_1)$ . All of this is correctly determined by  $\text{float\_orient}$ . Consider now the insertion process for point  $p_6$ . Depending on where we start the search for a visible edge, we will either find the edge  $(p_4, p_5)$  or the edge  $(p_1, p_2)$ . In the former case, we insert  $p_6$  between  $p_4$  and  $p_5$  and obtain the polygon shown in (b). It is visibly non-convex and has a self-intersection. In the latter case, we insert  $p_6$  between  $p_1$  and  $p_2$  and obtain the polygon shown in (c). It is visibly non-convex.

Of course, in a deterministic implementation, we will see only one of the errors, namely (b). This is because in our example, the search for a visible edge starts at edge  $(p_2, p_3)$ . In order to produce (c) with the implementation we replace the point  $p_2$  by the point  $p'_2 = (24.0, 10.0)$ . Then  $p_6$  sees  $(p'_2, p_3)$  and identifies  $(p_1, p'_2, p_3)$  as the chain of visible edges and hence constructs (c).

## 5 Conclusion

We provided instances which cause the floating point implementation of a simple convex hull algorithm to fail in many different ways. We showed how to construct such instances semi-systematically. We hope that our paper and its companion web page will be useful for classroom use and that it will alert students and researchers to the intricacies of implementing geometric algorithms.

What can be done to overcome the robust problem of floating point arithmetic? There are essentially three approaches: (1) make sure that the implementation of the orientation predicate always returns the correct result or (2) change the algorithm so that it can cope with the floating point implementation of the orientation predicate and still computes something meaningful or (3) perturb the input so that the floating point implementation is guaranteed to produce the correct result on the perturbed input. The first approach is the most general and is known under the exact geometric computation (EGC) paradigm and was first used by Jünger, Reinelt and Zepf [JRZ91] and Karasick, Lieber and Nackmann [KLN91]. For detailed discussions, we refer the reader to [Yap04] and Chapter 9 of [MN99]. The EGC approach has been adopted for the software libraries LEDA and CGAL and other successful implementations. The second and third approaches have been successfully applied to a number of geometric

problems, see for example [Mil89,FM91,LM90,DSB92,SIII00,HS98]. But in the second approach the interpretation of “meaningful” is a crucial and difficult problem. The third approach should in principle be applicable to all problems with purely numerical input. There are also suggested approaches, e.g., epsilon-tweaking, which do not work. Epsilon-tweaking simply activates rounding to zero, both for correct and mis-classified orientations. E.g., it is now more likely for points outside the current hull not to see any edges because of enforced collinearity.

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