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Clair Poignard. Asymptotics for steady state voltage potentials in a bidimensional highly contrasted medium with thin layer. *Mathematical Methods in the Applied Sciences*, 2008, 31 (4), pp.443-479. 10.1002/mma.923 . inria-00334770

HAL Id: inria-00334770

<https://inria.hal.science/inria-00334770>

Submitted on 27 Oct 2008

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ASYMPTOTICS FOR STEADY STATE VOLTAGE POTENTIALS IN A BIDIMENSIONAL HIGHLY CONTRASTED MEDIUM WITH THIN LAYER

CLAIR POIGNARD

ABSTRACT. We study the behavior of steady state voltage potentials in two kinds of bidimensional media composed of material of complex permittivity equal to 1 (respectively α) surrounded by a thin membrane of thickness h and of complex permittivity α (respectively 1). We provide in both cases a rigorous derivation of the asymptotic expansion of steady state voltage potentials at any order as h tends to zero, when Neumann boundary condition is imposed on the exterior boundary of the thin layer. Our complex parameter α is bounded but may be very small compared to 1, hence our results describe the asymptotics of steady state voltage potentials in all heterogeneous and highly heterogeneous media with thin layer. The asymptotic terms of the potential in the membrane are given explicitly in local coordinates in terms of the boundary data and of the curvature of the domain, while these of the inner potential are the solutions to the so-called dielectric formulation with appropriate boundary conditions. The error estimates are given explicitly in terms of h and α with appropriate Sobolev norm of the boundary data. We show that the two situations described above lead to completely different asymptotic behaviors of the potentials.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. Motivations. The daily exposure to an electromagnetic environment raises the question of the effects of electromagnetic fields on human health. The accurate assessment of the currents induced by these fields and more generally of their effects in the living tissues are major issues, both for their relevance in medical research and for their implications on the definition of industrial standards. Compared to materials usually studied in classical electromagnetic systems, the human body is a highly heterogeneous medium made of a large number of materials with specific properties. These materials, which are very highly heterogeneous, have non-usual electromagnetic constants. This changes the way that Maxwell's equations are dealt with [19] and introduces numerical difficulties.

At the microscopic scale, electromagnetic modelizations of the biological cell have been developed the last twenty years. The knowledge of the distribution of electromagnetic fields in biological cells has become extremely important in bioelectromagnetic investigations. In particular, the calculation of the transmembranar potential (TMP), which is the difference of the electric potentials through the membrane is of great interest in the modelization of the influency of electromagnetic fields on living cells [19]. Actually, a sufficiently large amplitude of the TMP leads to an increase of the cell membrane permeability [21], [23]. This phenomenon, called electropermeabilization, holds great potential for application in many fields of medicine. It has been already used in oncology and holds promises in gene

1991 *Mathematics Subject Classification.* 34E05, 34E10, 35J05.

Key words and phrases. Asymptotics, Approximated Boundary Condition, Voltage Perturbations, Laplace Equation, Thin Layer, Highly Contrasted Medium.

The author thanks Michelle Schatzman for many fertile discussions and good advices.

Published in Math. Meth. Appl. Sci., DOI: 10.1002/mma.923. electronic version.

therapy and also in carcinogenesis [24], justifying that precise assessments of the TMP are crucial.

Experimental measurements of TMP on living cells are limited, due to the thinness of the membrane (few nanometers). Moreover, the presence of electrodes near the membrane may perturb significantly the TMP, providing non-accurate experimental data. Therefore Fear, Stuchly [12] and Foster and Schwan [14] have developed a simple electromagnetic model of the biological cell so as to perform numerical calculations of the TMP. In their model, the cell is a highly heterogeneous medium of relatively regular shape (there is no corner) composed by a homogeneous conducting cytoplasm of radius of few micrometers surrounded by a thin very insulating membrane with constant thickness of few nanometers (see Fig 1).

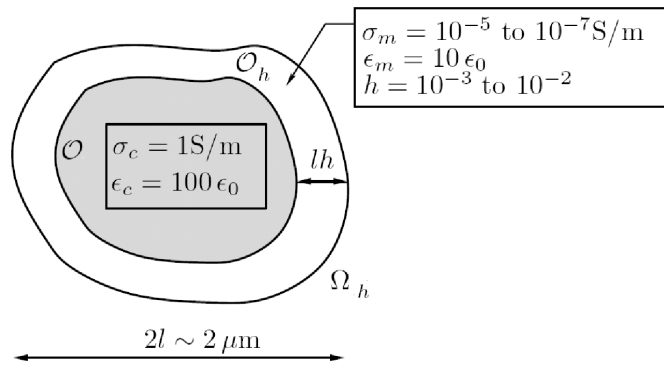


FIGURE 1. The electric model of the biological cell (see Fear *et al.* [12], [14]).

Recall the respective values of the void permeability and permittivity:

$$\mu_0 = 4\pi 10^{-7} \text{ S.I.}, \quad \epsilon_0 = \frac{1}{36\pi} 10^{-9} \text{ S.I.}$$

This model is the authoritative work in bioelectromagnetic research area. Several researchers have numerically computed the steady state potentials in this cell model with simple geometry using finite elements methods. Sebastian, Muñoz *et al.* have shown in [18] and [22] the influence of the geometry on the electric field distribution by computing steady state voltage potentials in few simple shapes of cells (spherical, cubic and ellipsoidal cells). However the thinness of the membrane and the high contrast between cytoplasm and membrane conductivities lead to numerical difficulties, in terms of the meshing of the membrane and computer times of calculations. Moreover the accuracy of the numerical results are not rigorously justified.

To avoid these numerical difficulties and to perform computations in realistic shapes of cell, Pucihar *et al.* [21] propose to replace the membrane by a condition on the boundary of the cytoplasm, using electromagnetic and optic considerations but they do not give any error estimate to prove the accuracy of their method.

For all these considerations, it seems of great interest to develop a rigorous asymptotic analysis to replace the thin insulating membrane surrounding a conducting inner domain. We also choose to present approximated boundary conditions equivalent to a very conducting thin layer, to obtain asymptotic results in all highly heterogeneous domains.

We study in this paper the behavior of the steady state voltage potentials in highly contrasted media composed by an inner domain surrounded by a thin layer.

Two materials are considered. The first medium consists of a conducting inner domain (say that its complex permittivity is equal to 1) surrounded by a thin membrane; we denote by α the membrane complex permittivity. The parameter α is bounded but it may tend to zero. This is the reason why we say that the thin layer is an insulating membrane. This material corresponds to the cell modelization of Fear and Stuchly. The second material consists of an insulating inner domain of permittivity α surrounded by a conducting thin membrane (say that its complex permittivity is equal to 1). In this case, we suppose that α tends to zero. These two media describe all the possible media with thin layer.

We perform an asymptotic expansion of the potentials in terms of the membrane thickness. The approached inner potential is then the finite sum of the solutions to elementary problems in the inner domain with appropriate conditions on its boundary, which approximate the effect of the thin layer. Our method leads to the construction of so-called “approximated boundary conditions” at any order [11]. We estimate precisely the error performed by this method in terms of an appropriate power of the relative thinness and with a precise Sobolev norm of the boundary data. This method is well-known for non highly contrasted media. It is formally described in some particular cases in [1] and [16]. We also refer to Krähenbühl and Muller [17] for electromagnetic considerations. Usually, when it is estimated (see for example [11]), the norm of the error involves an imprecise norm of the boundary data (a \mathcal{C}^∞ norm while a weaker norm is enough) and mainly, the constant of the estimate depends strongly on the dielectric parameters of the domain. It is not obvious (it is even false in general!) that such results hold for highly contrasted domains with thin layer.

The aim of this paper is to derive full rigorous asymptotic expansions of steady state voltage potentials with respect to the thinness h for upper bounded α .

1.2. Problems studied. Let us write mathematically our problem. Let Ω_h be a smooth bounded bidimensional domain (see Fig. 2), composed of a smooth domain \mathcal{O} surrounded by a thin membrane \mathcal{O}_h with a small constant thickness h :

$$\Omega_h = \mathcal{O} \cup \mathcal{O}_h.$$

Let α be a non null complex parameter with positive real part; α is bounded but it may be very small. Without loss of generality, we suppose that $|\alpha| \leq 1$. Denote by q_h and γ_h the following piecewise constant functions

$$\forall x \in \Omega_h, \quad q_h(x) = \begin{cases} 1, & \text{if } x \in \mathcal{O}, \\ \alpha, & \text{if } x \in \mathcal{O}_h, \end{cases}$$

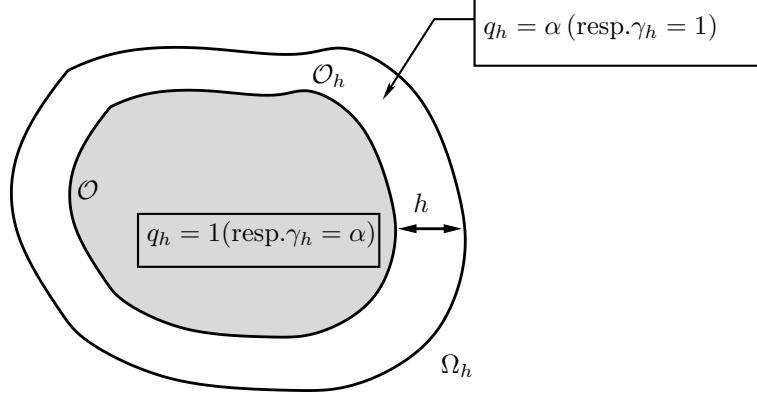
$$\forall x \in \Omega_h, \quad \gamma_h(x) = \begin{cases} \alpha, & \text{if } x \in \mathcal{O}, \\ 1, & \text{if } x \in \mathcal{O}_h. \end{cases}$$

We would like to understand the behavior for h tending to zero and uniformly with respect to $|\alpha| \leq 1$ of V_h and u_h the respective solutions to the following problems (1) and (2) with Neumann boundary condition; V_h satisfies

$$(1a) \quad \nabla \cdot (q_h \nabla V_h) = 0 \text{ in } \Omega_h,$$

$$(1b) \quad \frac{\partial V_h}{\partial n} = \phi \text{ on } \partial\Omega_h,$$

$$(1c) \quad \int_{\partial\mathcal{O}} V_h \, d\sigma = 0;$$

FIGURE 2. Parameters of Ω_h .

and u_h satisfies

$$(2a) \quad \nabla \cdot (\gamma_h \nabla u_h) = 0 \text{ in } \Omega_h,$$

$$(2b) \quad \frac{\partial u_h}{\partial n} = \phi \text{ on } \partial\Omega_h,$$

$$(2c) \quad \int_{\partial\mathcal{O}} u_h \, d\sigma = 0.$$

Since we impose a Neumann boundary conditions on $\partial\Omega_h$ the boundary data ϕ must satisfy the compatibility condition:

$$\int_{\partial\Omega_h} \phi \, d\sigma = 0.$$

The above functions V_h and u_h are well-defined and belong to $H^1(\Omega_h)$ as soon as ϕ belongs to $H^{-1/2}(\partial\Omega_h)$.

Several authors have worked on similar problems (see for instance Beretta *et al.* [5] and [6]). They compared the exact solution to the so-called background solution defined by replacing the material of the membrane by the inner material. The difference between these two solutions has then been given through an integral involving the polarization tensor defined for instance in [2], [3], [5], [6], [7], plus some remainder terms. The remainder terms are estimated in terms of the measure of the inhomogeneity. In this paper, we do not use this approach, for several reasons.

The Beretta *et al.* estimate of the remainder terms depends linearly on α and $1/\alpha$: their results are no more valid in a highly contrasted domain (*i.e.* for α very large or very small). Secondly, α is complex-valued, hence differential operators involved in our case are not self-adjoint, so that the Γ -convergence techniques of Beretta *et al.* do not apply. Thirdly, the potential in the membrane is not given explicitly in [5], [6] or [7], while we are definitely interested in this potential, in order to obtain the transmembranar potential (see Fear and Stuchly [12]). Finally, the asymptotics of Beretta *et al.* are valid on the boundary of the domain, while we are interested in the potentials in the inner domain.

The heuristics of this work consist in performing a change of coordinates in the membrane \mathcal{O}_h , so as to parameterize it by local coordinates (η, θ) , which vary in a domain independent on h ; in particular, if we denote by L the length of $\partial\mathcal{O}$ (in the following, without any restriction, we suppose that L is equal to 2π), the variables (η, θ) should vary in $[0, 1] \times \mathbb{R}/L\mathbb{Z}$. This change of coordinates leads to an expression

of the Laplacian in the membrane, which depends on h . Once the transmission conditions of the new problem are derived, we perform a formal asymptotic expansion of the solution to Problem (1) (respectively to Problem (2)) in terms of h . Then we validate our expansions. In this paper we work with bidimensional domain and we are confident that the same analysis could be performed in higher dimensions.

This paper is structured as follows. In Section 2, we make precise our geometric conventions. We perform a change of variables in the membrane, and with the help of some differential geometry results, we write Problem (1) and Problem (2) in the language of differential forms. We refer the reader to Flanders [13] or Dubrovin *et al.* [9] (or [8] for the french version) for courses on differential geometry. We derive transmission and boundary conditions in the intrinsic language of differential forms, and we express these relations in local coordinates.

In Section 4 we study Problem (1). In paragraph 4.1 we derive formally all the terms of the asymptotic expansion of the solution to our problem in terms of h . Paragraph 4.2 is devoted to a proof of the estimate of the error.

Problem (2) is considered in Section 5. We supposed that α tends to zero: a boundary layer phenomenon appears. To obtain our error estimates, we link the parameters h and α . We introduce a complex parameter β such that

$$\operatorname{Re}(\beta) > 0, \text{ or } (\operatorname{Re}(\beta) = 0, \text{ and } \Im(\beta) \neq 0),$$

and

$$|\beta| = o\left(\frac{1}{h}\right), \quad \text{and} \quad \frac{1}{|\beta|} = o\left(\frac{1}{h}\right).$$

We distinguish two different cases, depending on the convergence of $|\alpha|$ to zero: $\alpha = \beta h^q$, for $q \in \mathbb{N}^*$ and $\alpha = o(h^N)$ for all $N \in \mathbb{N}$.

For $q = 1$ we obtain mixed boundary conditions for the asymptotic terms of the inner potential, and as soon as $q \geq 1$, appropriate Dirichlet boundary conditions are obtained. We end this section by error estimates.

In conclusion we present few numerical simulations using a finite element method, which have been presented in the Conference NUMELEC2006 [20]. Using the scientific software GetDP [10], we illustrate the asymptotics at the orders 0 and 1 for a non-highly heterogenous cell. In collaboration with P.Dular from Liège University and R.Perrussel from Ampère Laboratory of Lyon, we are working on the implementation in GetDP of higher order asymptotics in all possible highly heterogeneous domains. Appendix gives some useful differential geometry formulae.

Remark 1. *The use of the formalism of differential forms $\delta(q_h \text{d})$ could seem futile for the study of the operator $\nabla \cdot (q_h \nabla)$. In particular the expression of Laplace operator in local coordinates is well known. However we wanted to present this point of view to show how simple it is to write a Laplacian in curved coordinates once the metric is known.*

Moreover once this formalism is understood for the functions (or 0-forms), it is easy to study $\delta(q_h \text{d})$ applied to 1-forms. This leads directly to the study of the operator $\operatorname{rot}(q_h \operatorname{rot})$, whose expression in local coordinates is less usual.

1.3. Main results. We choose to present our two main theorems in this introduction so that the reader interested in our results without their proves might find them easily.

For sake of simplicity, we suppose that $\partial\mathcal{O}$ is smooth, however this assumption may be weakened, see Remark 3. We denote by Φ the \mathcal{C}^∞ -diffeomorphism, which maps a neighborhood of cylinder $\mathcal{C} = [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ unto a neighborhood of the thin layer. The diffeomorphism $\Phi_0 = \Phi(0, \cdot)$ maps the torus unto the boundary $\partial\mathcal{O}$ of the inner domain while $\Phi_1 = \Phi(1, \cdot)$ is the \mathcal{C}^∞ -diffeomorphism from the torus

unto $\partial\Omega_h$. We denote by κ the curvature of $\partial\mathcal{O}$ written in local coordinates, and let h_0 be such that

$$h_0 < \frac{1}{\sup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} |\kappa(\theta)|}.$$

1.3.1. *Asymptotic for an insulating thin layer.* The first theorem gives the asymptotic expansion of the solution V_h of (1), for h tending to zero, for bounded α .

Theorem 1. *Let h belong to $(0, h_0)$. The complex parameter α satisfies*

$$(3) \quad |\alpha| \leq 1,$$

$$(4) \quad \Re(\alpha) > 0 \text{ or } \left\{ \Re(\alpha) = 0 \text{ and } \Im(\alpha) \neq 0 \right\}.$$

Let $N \in \mathbb{N}$ and ϕ belong to $H^{N+3/2}(\partial\Omega_h)$. Denote by f and \mathfrak{f} the following functions:

$$\begin{aligned} \forall \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad f(\theta) &= \phi \circ \Phi_1(\theta), \\ \forall x \in \partial\mathcal{O}, \quad \mathfrak{f}(x) &= \phi \circ \Phi_1 \circ \Phi_0^{-1}(x). \end{aligned}$$

Define the sequence of potentials $(V_k^c, V_k^m)_{k=0}^N$ as follows. We impose

$$\forall (\eta, \theta) \in \mathbf{C}, \quad \partial_\eta V_0^m = 0,$$

and we use the convention

$$\begin{cases} V_l^c = 0, & \text{if } l \leq -1, \\ V_l^m = 0, & \text{if } l \leq -1. \end{cases}$$

For $0 \leq k \leq N$ we define for all $0 \leq s \leq 1$ the function $\partial_\eta V_k^m(s, \cdot)$ on $\mathbb{R}/2\pi\mathbb{Z}$:

$$\begin{aligned} \partial_\eta V_{k+1}^m(s, \cdot) &= \delta_{1, k+1} f + \int_s^1 \left\{ \kappa \left\{ 3\eta \partial_\eta^2 V_k^m + \partial_\eta V_k^m \right\} \right. \\ &\quad + 3\eta^2 \kappa^2 \partial_\eta^2 V_{k-1}^m + 2\eta \kappa^2 \partial_\eta V_{k-1}^m + \partial_\theta^2 V_{k-1}^m \\ &\quad \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{k-2}^m + \eta^2 \kappa^3 \partial_\eta V_{k-2}^m + \eta \kappa \partial_\theta^2 V_{k-2}^m - \eta \kappa' \partial_\theta V_{k-2}^m \right\} d\eta, \end{aligned}$$

and the functions V_k^c and V_k^m are then defined by

$$\begin{aligned} \Delta V_k^c &= 0, \\ \partial_n V_k^c|_{\partial\mathcal{O}} &= \alpha \partial_\eta V_{k+1}^m \circ \Phi_0^{-1}, \\ \int_{\partial\mathcal{O}} V_k^c d\sigma &= 0, \\ \forall s \in (0, 1), \quad V_k^m(s, \cdot) &= \int_0^s \partial_\eta V_k^m(\eta, \cdot) d\eta + V_k^c \circ \Phi_0. \end{aligned}$$

Let R_N^c and R_N^m be the functions defined by:

$$\begin{cases} R_N^c = V_h - \sum_{k=0}^N V_k^c h^k, & \text{in } \mathcal{O}, \\ R_N^m = V_h \circ \Phi - \sum_{k=0}^N V_k^m h^k, & \text{in } \mathbf{C}. \end{cases}$$

Then, there exists a constant $C_{\mathcal{O}, N} > 0$ depending only on the domain \mathcal{O} and on N such that

$$(5a) \quad \|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} |\alpha| h^{N+1/2},$$

$$(5b) \quad \|R_N^m\|_{H^1_{\mathfrak{g}}(\mathbf{C})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}.$$

Moreover, if ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, then we have

$$(6a) \quad \|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+5/2}(\partial\mathcal{O})} |\alpha| h^{N+1},$$

$$(6b) \quad \|R_N^m\|_{H^1_{\mathfrak{g}}(\mathcal{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+5/2}(\partial\mathcal{O})} h^{N+1/2}.$$

In this theorem, we approach the potential in the inner domain at the order N by solving N elementary problems with appropriate boundary condition. From these results, we may build an approximated boundary condition on $\partial\mathcal{O}$ at any order, in order to solve only one problem. However, this kind of conditions lead to numerical unstabilities, this is the reason why we think that the method to obtain the potential step by step is more useful.

Since it is classical to write approximated boundary conditions we make precise these conditions at the orders 0 and 1. Denote by \mathfrak{K} the curvature of $\partial\mathcal{O}$ in Euclidean coordinates and by V_{app}^0 and V_{app}^1 the approximated potentials with approximated boundary condition at the order 0 and 1 respectively. We have:

$$(7) \quad \Delta V_{app}^0 = 0, \text{ in } \mathcal{O},$$

$$(8) \quad \partial_n V_{app}^0 = \alpha f,$$

and

$$(9) \quad \Delta V_{app}^1 = 0, \text{ in } \mathcal{O},$$

$$(10) \quad \partial_n V_{app}^1 - \alpha h \partial_t^2 V_{app}^1 = \alpha(1 + h\mathfrak{K})f,$$

where ∂_t denotes the tangential derivative on $\partial\mathcal{O}$. The boundary condition (10) imposed to $\partial_n V_{app}^1$ is well-known for non highly contrasted media. It might be found in [17]. With our theorem, we prove that it remains valid for a very insulating membrane, and we give precise norm estimates. Moreover we give complete asymptotic expansion of the potential in both domains (the inner domain and the thin layer).

We perform numerical simulations in a circle of radius 1 surrounded by a thin layer of thickness h . Fig 3 illustrates the asymptotic estimates at the orders 0 and 1 of Theorem 1 for an insulating thin layer. However, Fig 4 shows that as soon as the thin layer becomes very conducting, for example as soon as $\alpha = i/h$, these asymptotics are no more valid: we have to use the asymptotics of Theorem 2.

1.3.2. Asymptotics for an insulating inner domain. Let β be a complex parameter satisfying:

$$Re(\beta) > 0, \text{ or } (Re(\beta) = 0, \text{ and } \Im(\beta) \neq 0).$$

The modulus of β may tend to infinity, or to zero but it must satisfy:

$$|\beta| = o\left(\frac{1}{h}\right), \quad \text{and} \quad \frac{1}{|\beta|} = o\left(\frac{1}{h}\right).$$

Theorem 2. Let h belong to $(0, h_0)$. Let $q \in \mathbb{N}^*$ and $N \in \mathbb{N}$. We suppose that α satisfies:

$$(11) \quad \alpha = \beta h^q.$$

Let ϕ belong to $H^{N+3/2+q}(\partial\Omega_h)$ and denote by f and \mathfrak{f} the following functions:

$$\begin{aligned} \forall \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad f(\theta) &= \phi \circ \Phi_1(\theta), \\ \forall x \in \partial\mathcal{O}, \quad \mathfrak{f}(x) &= \phi \circ \Phi_1 \circ \Phi_0^{-1}(x). \end{aligned}$$

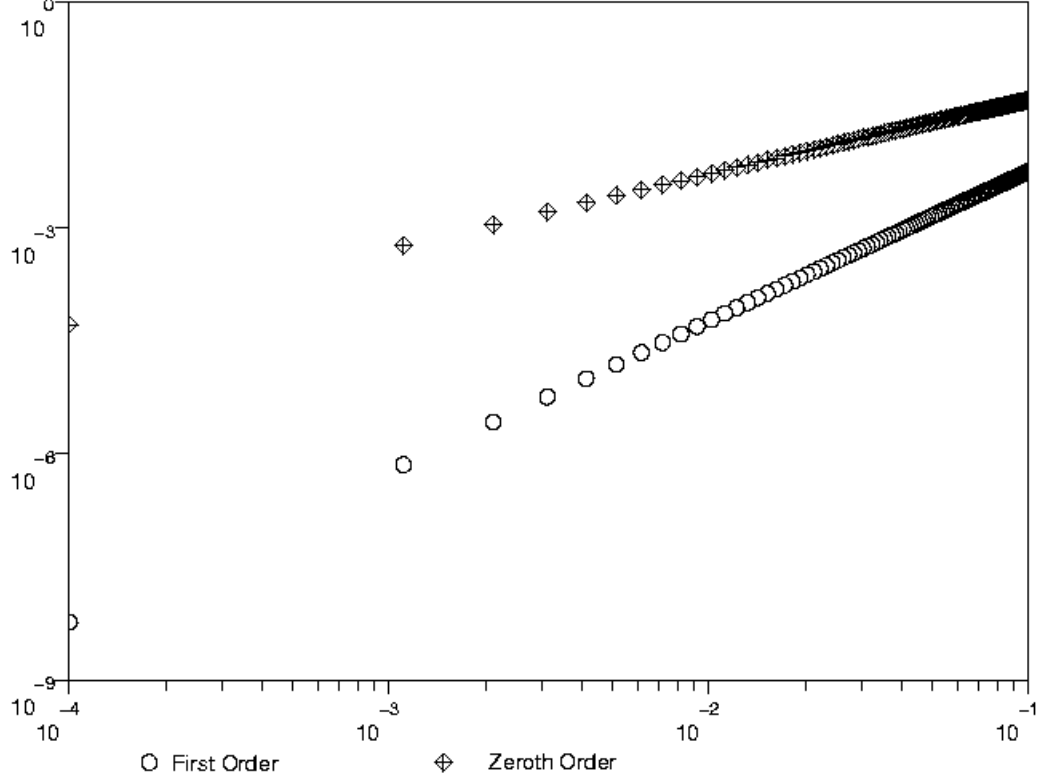


FIGURE 3. H^1 norm of the error at the orders 0 and 1 for $\alpha = i$.

Define the function $(u_k^{c,q}, u_k^{m,q})_{k=-1}^N$ by induction as follows, with the convention

$$\begin{cases} u_l^{c,q} = 0, & \text{if } l \leq -2, \\ u_l^{m,q} = 0, & \text{if } l \leq -2. \end{cases}$$

- If $q = 1$

$$\begin{cases} \Delta u_{-1}^{c,1} = 0, & \text{in } \mathcal{O}, \\ -\partial_t^2 u_{-1}^{c,1}|_{\partial\mathcal{O}} + \beta \partial_n u_{-1}^{c,1}|_{\partial\mathcal{O}} = f, \\ \int_{\partial\mathcal{O}} u_{-1}^{c,1} d\partial\mathcal{O} = 0. \end{cases}$$

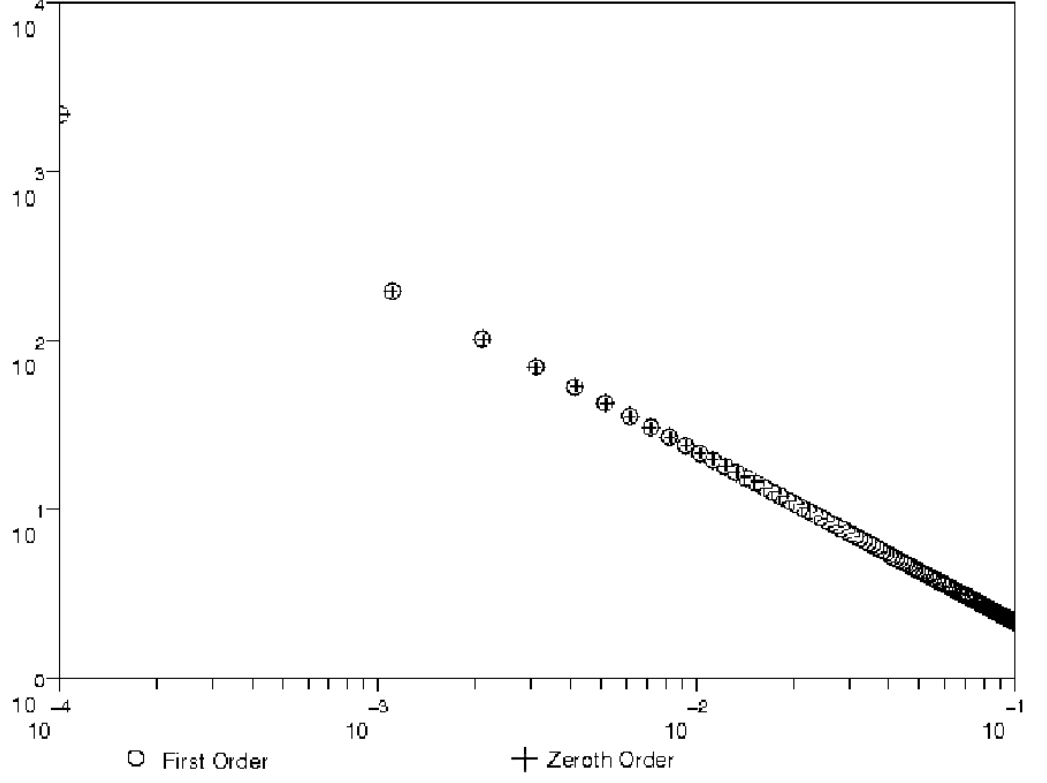
$$\forall (\eta, \theta) \in \mathbf{C}, \quad u_{-1}^{m,1} = u_{-1}^{c,1}|_{\partial\mathcal{O}} \circ \Phi_0.$$

Moreover,

$$\partial_\eta u_0^{m,1} = 0, \quad \partial_\eta u_1^{m,1} = (1 - \eta) \partial_\theta^2 u_{-1}^{m,1} + f.$$

For $0 \leq k \leq N$, denote by ϕ_k^1 the following function:

$$\phi_k^1 = \int_0^1 \left(\kappa \left(3\eta \partial_\eta^2 u_{k+1}^{m,1} + \partial_\eta u_{k+1}^{m,1} \right) + \eta \kappa \partial_\theta^2 u_{k-1}^{m,1} - \eta \kappa' \partial_\theta u_{k-1}^{m,1} \right) d\eta.$$

FIGURE 4. H^1 norm of the error at the orders 0 and 1 for $\alpha = i/h$.

and define $u_k^{c,1}$ by

$$\begin{cases} \Delta u_k^{c,1} = 0, & \text{in } \mathcal{O}, \\ -\partial_t^2 u_k^{c,1}|_{\partial\mathcal{O}} + \beta \partial_n u_k^{c,1}|_{\partial\mathcal{O}} = \left(\phi_k^1 - \int_0^1 (\eta-1) \partial_\theta^2 \partial_\eta u_k^{m,1} d\eta \right) \circ \Phi_0^{-1}, \\ \int_{\partial\mathcal{O}} u_k^{c,1} d\partial\mathcal{O} = 0. \end{cases}$$

In the membrane $u_k^{m,1}$ is defined by

$$u_k^{m,1} = \int_0^s \partial_\eta u_k^{m,q} d\eta + u_k^{c,q} \circ \Phi_0,$$

and $\partial_\eta u_{k+i}^{m,1}$ for $i = 1, 2$ is determined by:

$$(12) \quad \begin{aligned} \partial_\eta u_{k+i}^{m,1} = \int_1^s & \left(-\kappa \left(3\eta \partial_\eta^2 u_{k+i-1}^{m,1} + \partial_\eta u_{k+i-1}^{m,1} \right) \right. \\ & \left. - \partial_\theta^2 u_{k+i-2}^{m,1} - \eta \kappa \partial_\theta^2 u_{k+i-3}^{m,1} + \eta \kappa' \partial_\theta u_{k+i-3}^{m,1} \right) d\eta. \end{aligned}$$

- If $q \geq 2$. The function $u_1^{m,q}$ is defined by

$$\begin{aligned} \int_{\mathbb{T}} u_{-1}^{m,q} d\theta &= 0, \\ -\partial_\theta^2 u_{-1}^{m,q} &= f. \end{aligned}$$

The potential $u_{-1}^{c,q}$ is solution to the following problem:

$$\begin{cases} \Delta u_{-1}^{c,q} = 0, & \text{in } \mathcal{O}, \\ u_{-1}^{c,q}|_{\partial\mathcal{O}} = u_{-1}^{m,q} \circ \Phi_0^{-1}. \end{cases}$$

Moreover,

$$\partial_\eta u_0^{m,q} = 0, \quad \partial_\eta u_1^{m,q} = (1-\eta)\partial_\theta^2 u_{-1}^{m,q} + f.$$

For $0 \leq k \leq N$, denote by ϕ_k^q the following function:

$$\phi_k^q = \int_0^1 \left(\kappa (3\eta\partial_\eta^2 u_{k+1}^{m,q} + \partial_\eta u_{k+1}^{m,q}) + \eta\kappa\partial_\theta^2 u_{k-1}^{m,q} - \eta\kappa' \partial_\theta u_{k-1}^{m,q} \right) d\eta.$$

$u_k^{m,q}|_{\eta=1}$ is entirely determined by the equality:

$$-\partial_\theta^2 u_k^{m,q}|_{\eta=1} = \beta\partial_n u_{k+1-q}^{c,q} \circ \Phi_0 + \phi_k^q - \int_0^1 \eta\partial_\theta^2 \partial_\eta u_k^{m,q} d\eta,$$

hence

$$u_k^{m,q}(s, \theta) = \int_1^s \partial_\eta u_k^{m,q} d\eta + u_k^{m,q}|_{\eta=1}.$$

The potential $u_k^{c,q}$ satisfies the following boundary value problem:

$$\begin{cases} \Delta u_k^{c,q} = 0, & \text{in } \mathcal{O}, \\ u_k^{c,q}|_{\partial\mathcal{O}} = u_k^{m,q} \circ \Phi_0^{-1}. \end{cases}$$

The functions $(\partial_\eta u_{k+i}^{m,q})_{i=1,2}$ satisfies equation (12), in which $u^{m,1}$ is replaced by $u^{m,q}$.

Let $r_N^{c,q}$ and $r_N^{m,q}$ be the functions defined by:

$$\begin{cases} r_N^{c,q} = u_h - \sum_{k=-1}^N u_k^{c,q} h^k, & \text{in } \mathcal{O}, \\ r_N^{m,q} = u_h \circ \Phi - \sum_{k=-1}^N u_k^{m,q} h^k, & \text{in } \mathcal{C}. \end{cases}$$

Then, there exists a constant $C_{\mathcal{O},N} > 0$ depending only on the domain \mathcal{O} and on N such that

$$\|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2+q}(\partial\mathcal{O})} \max\left(\sqrt{\frac{h}{|\beta|}}, \sqrt{h}\right) h^{N+1/2},$$

$$\|r_N^{m,q}\|_{H_1^1(\mathcal{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}.$$

If ϕ belongs to $H^{N+5/2+q}(\partial\Omega_h)$, we have

$$\|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+5/2+q}(\mathbb{T})} h^{N+1}.$$

Suppose that $q = 1$ and let us give now the approximated boundary conditions at the order -1 and 0 . The approximated boundary condition at the order -1 is given by:

$$-\partial_t^2 u_{-1,app}^1 + \beta\partial_n u_{-1,app}^1 = \phi \circ \Phi_1 \circ \Phi_0^{-1},$$

while those at the order 0 is:

$$-(1 - h\mathfrak{K}/2)\partial_t^2 u_{0,app}^1 + \frac{h\partial_t \mathfrak{K}}{2}\partial_t u_{0,app}^1 + \beta\partial_n u_{0,app}^1 = \frac{1 + h\mathfrak{K}}{h}\phi \circ \Phi_1 \circ \Phi_0^{-1}.$$

Thus it is very different from the approximated boundary condition (8) imposed to V_{app}^0 in the case of an insulating membrane. This is a feature of the conducting thin layer. Observe on Fig 5 that the numerical computations in a circle confirm our theoretical results.

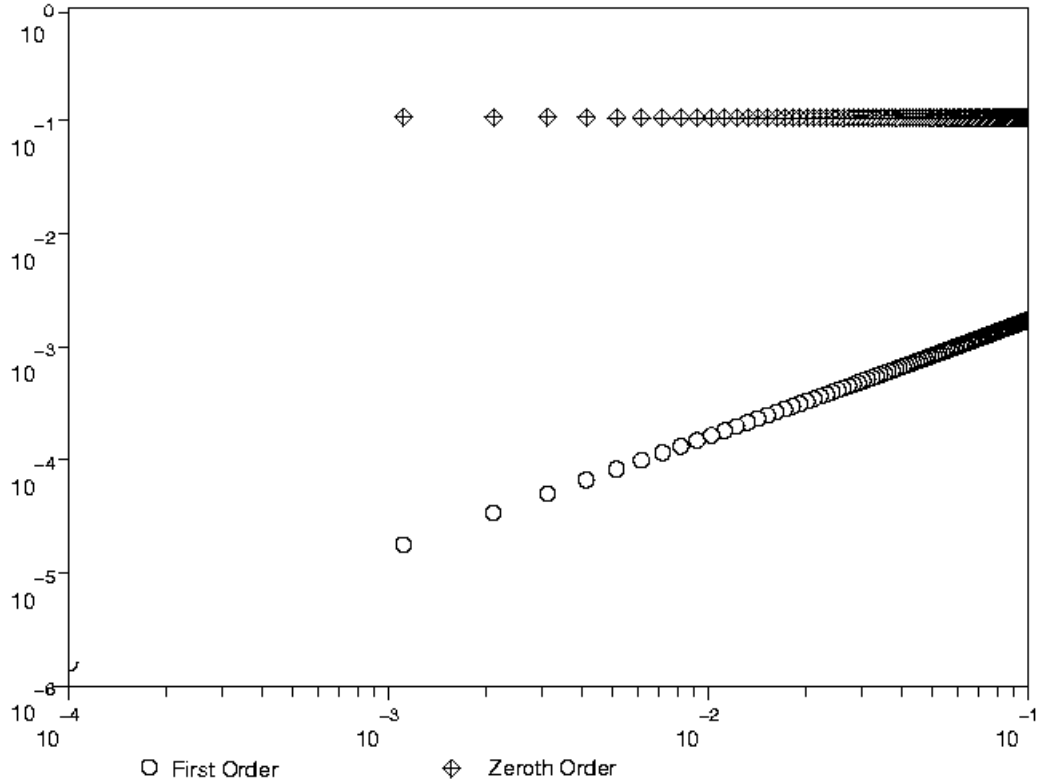


FIGURE 5. H^1 Norm of the error at the order -1 and 0 for an insulating inner domain : $\alpha = ih$.

Remark 3 (Regularity of the domain). *Our asymptotic method is “derivative consuming” in the sense that the definition of the asymptotic coefficient at the order $n \in \mathbb{N}$ involves derivatives of the previous asymptotic coefficients. Hence, for sake of simplicity we suppose that $\partial\mathcal{O}$ is smooth. This assumption may be weakened to $\mathcal{C}^{p,1}$ -regularity for $p \geq 1$, which depends on the order of the approximate boundary condition. For example, in Theorem 1, if $N \geq 1$, a $\mathcal{C}^{N+1,1}$ -regularity for $\partial\mathcal{O}$ is enough to define the first two terms $(V_k^c, V_k^m)_{k=0, \dots, N}$, while in Theorem 2, a $\mathcal{C}^{N+3,1}$ -regularity is enough. However with these regularities, the proofs of the error estimates are more technical and less optimal than in our theorems (the estimates in the inner domain are of order $O(h^{N+1/2})$ instead of $O(h^{N+1})$).*

Thanks to our previous results by comparing the parameters $|\alpha|$ and h of a heterogeneous medium with thin layer, we know *a priori*, which asymptotic formula (Theorem 1 or Theorem 2) has to be computed. We emphasize that our method

might be easily implemented by iterative process as soon as the geometry of the domain is precisely known.

In the following, we show how the asymptotics are built, and then we prove our theorems. Let us now make precise the geometric conventions.

2. GEOMETRY

The boundary of the domain \mathcal{O} is assumed to be smooth. The orientation of the boundary $\partial\mathcal{O}$ is the trigonometric orientation. To simplify, we suppose that the length of $\partial\mathcal{O}$ is equal to 2π . We denote by \mathbb{T} the flat torus:

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Since $\partial\mathcal{O}$ is smooth, we can parameterize it by a function Ψ of class \mathcal{C}^∞ from \mathbb{T} to \mathbb{R}^2 satisfying:

$$\forall \theta \in \mathbb{T}, \quad |\Psi'(\theta)| = 1.$$

Since the boundary $\partial\Omega_h$ of the cell is parallel to the boundary $\partial\mathcal{O}$ of the inner domain the following identities hold:

$$\partial\mathcal{O} = \{\Psi(\theta), \theta \in \mathbb{T}\},$$

and

$$\partial\Omega_h = \{\Psi(\theta) + hn(\theta), \theta \in \mathbb{T}\}.$$

Here $n(\theta)$ is the unitary exterior normal at $\Psi(\theta)$ to $\partial\mathcal{O}$. Therefore the membrane \mathcal{O}_h is parameterized by:

$$\mathcal{O}_h = \{\Phi(\eta, \theta), (\eta, \theta) \in]0, 1[\times \mathbb{T}\},$$

where

$$\Phi(\eta, \theta) = \Psi(\theta) + h\eta n(\theta).$$

Denote by κ the curvature of $\partial\mathcal{O}$. Let h_0 belong to $(0, 1)$ such that:

$$(13) \quad h_0 < \frac{1}{\|\kappa\|_\infty}.$$

Thus for all h in $[0, h_0]$, there exists an open interval I containing $(0, 1)$ such that Φ is a smooth diffeomorphism from $I \times \mathbb{R}/2\pi\mathbb{Z}$ to its image, which is a neighborhood of the membrane. The metric in \mathcal{O}_h is:

$$(14) \quad h^2 d\eta^2 + (1 + h\eta\kappa)^2 d\theta^2.$$

Thus, we use two systems of coordinates, depending on the domains \mathcal{O} and \mathcal{O}_h : in the interior domain \mathcal{O} , we use Euclidean coordinates (x, y) and in the membrane \mathcal{O}_h , we use local (η, θ) coordinates with metric (14).

We translate into the language of differential forms Problem (1) and Problem (2). We refer the reader to Dubrovin, Fomenko and Novikov [9] or Flanders [13] for the definition of the exterior derivative denoted by d , the exterior product denoted by ext , the interior derivative denoted by δ and the interior product denoted by int . In Appendix we give the formulae describing these operators in the case of a general 2D metric. Our aim, while rewriting our problems (1) and (2) is to take into account nicely the change of coordinates in the thin membrane.

Let \mathbb{V} be the 0-form on Ω_h such that, in the Euclidean coordinates (x, y) , \mathbb{V} is equal to V , and let F be the 0-form, which is equal to ϕ on $\partial\Omega_h$. We denote by \mathbb{N}

the 1-form corresponding to the inward unit normal on the boundary Ω_h (see for instance Gilkey *et al.* [15] p.33):

$$\begin{aligned}\mathbb{N} &= N_x dx + N_y dy, \\ &= N_\eta d\eta.\end{aligned}$$

\mathbb{N}^* is the inward unit normal 1-form. Problem (1) takes now the intrinsic form:

$$(15a) \quad \delta(q_h d\mathbb{V}) = 0, \text{ in } \Omega_h,$$

$$(15b) \quad \text{int}(\mathbb{N}^*)d\mathbb{V} = F, \text{ on } \partial\Omega_h.$$

According to Green's formula (Lemma 1.5.1 of [15]), we obtain the following transmission conditions for \mathbb{V} along $\partial\mathcal{O}$:

$$(15c) \quad \begin{aligned}\text{int}(\mathbb{N}^*)d\mathbb{V}|_{\partial\mathcal{O}} &= \alpha \text{int}(\mathbb{N}^*)d\mathbb{V}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}, \\ \text{ext}(\mathbb{N}^*)\mathbb{V}|_{\partial\mathcal{O}} &= \text{ext}(\mathbb{N}^*)\mathbb{V}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}.\end{aligned}$$

Similarly, denoting by \mathbb{U} the 0-form equal to u in Euclidean coordinates we rewrite Problem (2) as follows:

$$(16a) \quad \delta(\gamma_h d\mathbb{U}) = 0, \text{ in } \Omega_h,$$

$$(16b) \quad \text{int}(\mathbb{N}^*)d\mathbb{U} = F, \text{ on } \partial\Omega_h;$$

the following transmission conditions hold on $\partial\mathcal{O}$:

$$(16c) \quad \begin{aligned}\alpha \text{int}(\mathbb{N}^*)d\mathbb{U}|_{\partial\mathcal{O}} &= \text{int}(\mathbb{N}^*)d\mathbb{U}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}, \\ \text{ext}(\mathbb{N}^*)\mathbb{U}|_{\partial\mathcal{O}} &= \text{ext}(\mathbb{N}^*)\mathbb{U}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}.\end{aligned}$$

3. STATEMENT OF THE PROBLEM

In this section, we write Problem (15) and Problem (16) in local coordinates, with the help of differential forms. It is convenient to write:

$$\forall \theta \in \mathbb{T}, \quad \Phi_0(\theta) = \Phi(0, \theta), \quad \Phi_1(\theta) = \Phi(1, \theta),$$

and to denote by \mathcal{C} the cylinder:

$$\mathcal{C} = [0, 1] \times \mathbb{T}.$$

We denote by \mathfrak{R} , f and \mathfrak{f} the following functions:

$$(17) \quad \forall (x, y) \in \partial\mathcal{O}, \quad \mathfrak{R}(x, y) = \kappa \circ \Phi_0^{-1}(x, y),$$

$$(18) \quad \forall \theta \in \mathbb{T}, \quad f(\theta) = \phi \circ \Phi_1(\theta),$$

$$(19) \quad \forall x \in \partial\mathcal{O}, \quad \mathfrak{f} = f \circ \Phi_0^{-1}(x).$$

Using the expressions of the differential operators d and δ , which are respectively the exterior and the interior derivatives (see Appendix), applied to the metric (14), the Laplacian in the membrane is given in the local coordinates (η, θ) by:

$$(20) \quad \begin{aligned}\forall (\eta, \theta) \in \mathcal{C}, \\ \Delta|_{\Phi(\eta, \theta)} &= \frac{1}{h(1+h\eta\kappa)} \partial_\eta \left(\frac{1+h\eta\kappa}{h} \partial_\eta \right) + \frac{1}{1+h\eta\kappa} \partial_\theta \left(\frac{1}{1+h\eta\kappa} \partial_\theta \right).\end{aligned}$$

Moreover, for a 0-form z defined in \mathcal{O}_h , we have:

$$\begin{aligned}\text{int}(\mathbb{N}^*)dz|_{\partial\mathcal{O}} &= \frac{1}{h} \partial_\eta z|_{\eta=0}, \\ \text{int}(\mathbb{N}^*)dz|_{\partial\Omega_h} &= \frac{1}{h} \partial_\eta z|_{\eta=1}.\end{aligned}$$

Denote by

$$\begin{aligned} V^c &= V, \text{ in } \mathcal{O}, \\ V^m &= V \circ \Phi, \text{ in } \mathcal{C}, \end{aligned}$$

and by

$$\begin{aligned} u^c &= u, \text{ in } \mathcal{O}, \\ u^m &= u \circ \Phi, \text{ in } \mathcal{C}. \end{aligned}$$

We infer that Problem (15) may be rewritten as follows:

$$\begin{aligned} (21a) \quad & \Delta V^c = 0, \text{ in } \mathcal{O}, \\ (21b) \quad & \forall (\eta, \theta) \in \mathcal{C}, \quad \frac{1}{h^2} \partial_\eta \left((1 + h\eta\kappa) \partial_\eta V^m \right) + \partial_\theta \left(\frac{1}{1 + h\eta\kappa} \partial_\theta V^m \right) = 0, \\ (21c) \quad & \partial_n V^c \circ \Phi_0 = \frac{\alpha}{h} \partial_\eta V^m \Big|_{\eta=0}, \\ (21d) \quad & V^c \circ \Phi_0 = V^m \Big|_{\eta=0}, \\ (21e) \quad & \partial_\eta V^m \Big|_{\eta=1} = hf. \\ & \int_{\partial\mathcal{O}} V \, d\sigma = 0. \end{aligned}$$

Similarly the couple (u^c, u^m) satisfies

$$\begin{aligned} (22a) \quad & \Delta u^c = 0, \text{ in } \mathcal{O}, \\ (22b) \quad & \forall (\eta, \theta) \in \mathcal{C}, \quad \frac{1}{h^2} \partial_\eta \left((1 + h\eta\kappa) \partial_\eta u^m \right) + \partial_\theta \left(\frac{1}{1 + h\eta\kappa} \partial_\theta u^m \right) = 0, \\ (22c) \quad & \alpha \partial_n u^c \circ \Phi_0 = \frac{1}{h} \partial_\eta u^m \Big|_{\eta=0}, \\ (22d) \quad & u^c \circ \Phi_0 = u^m \Big|_{\eta=0}, \\ (22e) \quad & \partial_\eta u^m \Big|_{\eta=1} = hf, \\ & \int_{\partial\mathcal{O}} u \, d\sigma = 0. \end{aligned}$$

Remark 4. *In the following, the parameter α is such that:*

$$\Re(\alpha) > 0 \text{ or } \left\{ \Re(\alpha) = 0 \text{ and } \Im(\alpha) \neq 0 \right\}.$$

Since α represents a complex permittivity it may be written (see Balanis [4]) as follows:

$$\alpha = \varepsilon - i\sigma/\omega,$$

with ε , σ , and ω positive. Thus this hypothesis is always satisfied for dielectric materials.

Notation 5. *We provide \mathcal{C} with the metric (14). The L^2 - norm of a 0- form u in \mathcal{C} , denoted by $\|u\|_{\Lambda^0 L_m^2(\mathcal{C})}$, is equal to:*

$$\begin{aligned} \|u\|_{\Lambda^0 L_m^2(\mathcal{C})} &= \left(\int_0^1 \int_0^{2\pi} h(1 + h\eta\kappa) |u(\eta, \theta)|^2 \, d\eta \, d\theta \right)^{1/2}, \\ &= \|u\|_{L^2(\mathcal{O}_h)}, \end{aligned}$$

and the L^2 norm of its exterior derivative du , denoted by $\|du\|_{\Lambda^1 L^2_m}$ is equal to

$$\begin{aligned} \|du\|_{\Lambda^1 L^2_g(\mathcal{C})} &= \left(\int_0^1 \int_0^{2\pi} \frac{1+h\eta\kappa}{h} |\partial_\eta u(\eta, \theta)|^2 + \frac{h}{1+h\eta\kappa} |\partial_\theta u(\eta, \theta)|^2 d\eta d\theta \right)^{1/2}, \\ &= \|\text{grad } u\|_{L^2(\mathcal{O}_h)}. \end{aligned}$$

To simplify our notations, for a 0-form u defined on \mathcal{C} , we define by $\|u\|_{H^1_g(\mathcal{C})}$ the following quantity

$$\|u\|_{H^1_g(\mathcal{C})} = \|u\|_{\Lambda^0 L^2_m(\mathcal{C})} + \|du\|_{\Lambda^1 L^2_m(\mathcal{C})},$$

when the above integrals are well-defined. Observe that for a function $u \in H^1(\mathcal{O}_h)$, we have:

$$\|u\|_{H^1(\mathcal{O}_h)} = \|u \circ \Phi\|_{H^1_g(\mathcal{C})}.$$

Remark 6 (Poincaré inequality in the thin layer). *Let z belong to $H^1_g(\mathcal{C})$, such that*

$$(23) \quad \int_0^{2\pi} z(0, \theta) d\theta = 0.$$

Then, there exists an h -independant constant $C_{\mathcal{O}}$ such that

$$(24) \quad \|z\|_{\Lambda^0 L^2_g(\mathcal{C})} \leq C_{\mathcal{O}} \|dz\|_{\Lambda^1 L^2_g(\mathcal{C})}.$$

We prove (24) using Fourier analysis. According to the definition (13) of h_0 there exists two constants $C_{\mathcal{O}}$ and $c_{\mathcal{O}}$ depending on the domain \mathcal{O} such that the following inequalities hold:

$$(25a) \quad \|z\|_{\Lambda^0 L^2_g(D)}^2 \leq C_{\mathcal{O}} h \int_0^1 \int_0^{2\pi} |z(\eta, \theta)|^2 d\theta d\eta,$$

$$(25b) \quad \|dz\|_{\Lambda^1 L^2_g(D)}^2 \geq c_{\mathcal{O}} \left(\int_0^1 \int_0^{2\pi} \frac{|\partial_\eta z(\eta, \theta)|^2}{h} + h |\partial_\theta z|^2 d\theta d\eta \right).$$

For $k \in \mathbb{Z}$, we denote by \widehat{z}_k the k^{th} -Fourier coefficient (with respect to θ) of z :

$$\widehat{z}_k = \frac{1}{2\pi} \int_0^{2\pi} z(\theta) e^{-ik\theta} d\theta.$$

Since $(\widehat{\partial_\theta z})_k = ik\widehat{z}_k$, we infer:

$$\forall k \neq 0, \quad \int_0^1 |\widehat{z}_k(\eta)|^2 d\eta \leq \int_0^1 \left| (\widehat{\partial_\theta z})_k(\eta) \right|^2 d\eta.$$

According to gauge condition (23), we have:

$$\widehat{z}_0(0) = 0,$$

thus, using the equality

$$\widehat{z}_0(\eta) = \int_0^\eta (\widehat{\partial_\eta z})_0(s) ds,$$

we infer

$$\int_0^1 |\widehat{z}_0(\eta)|^2 d\eta \leq \int_0^1 \left| (\widehat{\partial_\eta z})_0(\eta) \right|^2 d\eta.$$

Therefore,

$$\sum_{k \in \mathbb{Z}} \int_0^1 |\widehat{z}_k(\eta, \theta)|^2 d\eta \leq \sum_{k \in \mathbb{Z}} \left\{ \int_0^1 \left| \left(\widehat{\partial_\theta z} \right)_k(\eta) \right|^2 d\eta + \int_0^1 \left| \left(\widehat{\partial_\eta z} \right)_k(\eta) \right|^2 d\eta \right\}.$$

We end the proof of (24) by using Parseval inequality and inequalities (25).

4. ASYMPTOTIC EXPANSION OF THE STEADY STATE POTENTIAL FOR AN INSULATING MEMBRANE

We derive asymptotic expansions with respect to h of the potentials (V^c, V^m) solution to Problem (21). The membrane is insulating since the modulus of α is supposed to be smaller than 1. However, our results are still valid if $|\alpha|$ is bounded by a constant C_0 greater than 1. We emphasize that the following results are valid for α tending to zero.

4.1. Formal asymptotic expansion. We write the following ansatz:

$$(26a) \quad V^c = V_0^c + hV_1^c + h^2V_2^c + \dots,$$

$$(26b) \quad V^m = V_0^m + hV_1^m + h^2V_2^m + \dots.$$

We multiply (21b) by $h^2(1 + h\eta\kappa)^2$ and we order the powers of h to obtain:

$$(27) \quad \begin{aligned} & \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ & \partial_\eta^2 V^m + h\kappa \{3\eta \partial_\eta^2 V^m + \partial_\eta V^m\} + h^2 \{3\eta^2 \kappa^2 \partial_\eta^2 V^m + 2\eta\kappa^2 \partial_\eta V^m + \partial_\theta^2 V^m\} \\ & + h^3 \{\eta^3 \kappa^3 \partial_\eta^2 V^m + \eta^2 \kappa^3 \partial_\eta V^m + \eta\kappa \partial_\theta^2 V^m - \eta\kappa' \partial_\theta V^m\} = 0 \end{aligned}$$

We are now ready to derive formally the terms of the asymptotic expansions of V^c and V^m by identifying the terms of the same power in h .

Recall that for (m, n) in \mathbb{N}^2 , $\delta_{m,n}$ is Kronecker symbol equal to 1 if $m = n$ and to 0 if $m \neq n$. By identifying the powers of h , we infer that for $l \in \mathbb{N}$, V_l^c and V_l^m satisfy the following equations:

$$(28a) \quad \Delta V_l^c = 0, \text{ in } \mathcal{O},$$

for all $(\eta, \theta) \in \mathcal{C}$,

$$(28b) \quad \begin{aligned} \partial_\eta^2 V_l^m = - \left\{ \right. & \kappa \{3\eta \partial_\eta^2 V_{l-1}^m + \partial_\eta V_{l-1}^m\} \\ & + 3\eta^2 \kappa^2 \partial_\eta^2 V_{l-2}^m + 2\eta\kappa^2 \partial_\eta V_{l-2}^m + \partial_\theta^2 V_{l-2}^m \\ & \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{l-3}^m + \eta^2 \kappa^3 \partial_\eta V_{l-3}^m + \eta\kappa \partial_\theta^2 V_{l-3}^m - \eta\kappa' \partial_\theta V_{l-3}^m \right\}, \end{aligned}$$

with transmission conditions

$$(28c) \quad \partial_n V_l^c \circ \Phi_0 = \alpha \partial_\eta V_{l+1}^m |_{\eta=0},$$

$$(28d) \quad V_l^c \circ \Phi_0 = V_l^m |_{\eta=0},$$

with boundary condition

$$(28e) \quad \partial_\eta V_l^m |_{\eta=1} = \delta_{l,1} f,$$

and with gauge condition

$$(28f) \quad \int_{\partial\mathcal{O}} V_l^c d\sigma = 0.$$

In equations (28), we have implicitly imposed

$$(29) \quad \begin{cases} V_l^c = 0, & \text{if } l \leq -1, \\ V_l^m = 0, & \text{if } l \leq -1. \end{cases}$$

The next lemma ensures that for each non null integer N , the functions V_N^c and V_N^m are entirely determined if the boundary condition ϕ is enough regular.

Notation 7. For $s \in \mathbb{R}$, we denote by $\mathcal{C}^\infty([0, 1]; H^s(\mathbb{T}))$ the space of functions u defined for $(\eta, \theta) \in [0, 1] \times \mathbb{T}$, such that for almost all $\theta \in \mathbb{T}$, $u(\cdot, \theta)$ belongs to $\mathcal{C}^\infty([0, 1])$, and such that for all $\eta \in [0, 1]$, $u(\eta, \cdot)$ belongs to $H^s(\mathbb{T})$.

Lemma 8. We suppose that $\partial\mathcal{O}$ is smooth.

For $N \in \mathbb{N}$ and $p \geq 0$ we suppose that ϕ belongs to $H^{N+p-1/2}(\partial\Omega_h)$ and let $|\alpha| \leq 1$.

Then the functions V_0^m, \dots, V_N^m and V_0^c, \dots, V_N^c are uniquely determined and they belong to the respective functional spaces:

$$(30a) \quad \forall k = 0, \dots, N, \quad V_k^m \in \mathcal{C}^\infty([0, 1]; H^{N+p-k+1/2}(\mathbb{T})),$$

$$(30b) \quad V_k^c \in H^{N+p-k+1}(\mathcal{O}).$$

Moreover, there exists a constant $C_{N, \mathcal{O}, p}$ such that:

$$(31a) \quad \forall k = 0, \dots, N, \quad \sup_{\eta \in [0, 1]} \|V_k^m(\eta, \cdot)\|_{H^{N+p-k+1/2}(\mathbb{T})} \leq C_{N, \mathcal{O}, p} \|f\|_{H^{N+p-1/2}(\partial\mathcal{O})},$$

$$(31b) \quad \|V_k^c\|_{H^{N+p-k+1}(\mathcal{O})} \leq |\alpha| C_{N, \mathcal{O}, p} \|f\|_{H^{N+p-1/2}(\partial\mathcal{O})}.$$

Remark 9. To simplify, we suppose that $|\alpha| \leq 1$, but the same result may be obtained if there exists $C_0 > 1$ such that $|\alpha| \leq C_0$. In this case, the constant $C_{N, \mathcal{O}, p}$ would also depends on C_0 .

Proof. Since $\partial\mathcal{O}$ is smooth and since ϕ belongs to $H^{N+p-1/2}(\partial\Omega_h)$, for $N \geq 0$ and $p \geq 0$, then the functions f and \mathbf{f} defined by (18) and by (19) belong respectively to $H^{N+p-1/2}(\mathbb{T})$ and to $H^{N+p-1/2}(\partial\mathcal{O})$. We prove this lemma by recursive process.

- $N = 0$. Let $p \geq 0$ and let ϕ belong to $H^{p-1/2}(\partial\Omega_h)$.

Thus f and \mathbf{f} belong respectively to $H^{p-1/2}(\mathbb{T})$ and $H^{p-1/2}(\partial\mathcal{O})$. Using (28b) and (28e), we infer:

$$(32) \quad \begin{cases} \partial_\eta^2 V_0^m = 0, \\ \partial_\eta V_0^m|_{\eta=1} = 0, \end{cases}$$

hence, $\partial_\eta V_0^m = 0$. According to (28b) and to (28e), we straight infer

$$\partial_\eta V_1^m = f.$$

Therefore by (28a) and (28c) the function V_0^c satisfies the following Laplace problem:

$$(33a) \quad \Delta V_0^c = 0,$$

$$(33b) \quad \partial_n V_0^c|_{\partial\mathcal{O}} = \alpha f,$$

with gauge condition

$$(33c) \quad \int_{\partial\mathcal{O}} V_0^c d\sigma = 0.$$

According to (28d), we infer

$$(34) \quad V_0^m = V_0^c \circ \Phi_0,$$

hence V_0^c and V_0^m are entirely determined and they belong to the following spaces:

$$\begin{aligned} V_0^m &\in \mathcal{C}^\infty([0, 1]; H^{p+1/2}(\mathbb{T})), \\ V_0^c &\in H^{p+1}(\mathcal{O}). \end{aligned}$$

Observe also that there exists a constant $C_{\mathcal{O},p}$ such that

$$\begin{aligned} \sup_{\eta \in [0,1]} \|V_0^m(\eta, \cdot)\|_{H^{p+1/2}(\mathbb{T})} &\leq C_{\mathcal{O},p} \|f\|_{H^{p-1/2}(\partial\mathcal{O})}, \\ \|V_0^c\|_{H^{p+1}(\mathcal{O})} &\leq |\alpha| C_{\mathcal{O},p} \|f\|_{H^{p-1/2}(\partial\mathcal{O})}. \end{aligned}$$

• Induction.

Let $N \geq 0$. Suppose that for all $p \geq 0$, for all $\phi \in H^{N+p-1/2}(\partial\Omega_h)$ and for $M = 0, \dots, N$ the functions V_M^c and V_M^m are known. Suppose that they belong respectively to $H^{N+p-M+1}(\mathcal{O})$ and to $V_M^m \in \mathcal{C}^\infty([0, 1]; H^{N+p-M+1/2}(\mathbb{T}))$ and that estimates (31) hold.

Let ϕ belong to $H^{N+p+1/2}(\partial\Omega_h)$. Therefore, for $M = 0, \dots, N$, the functions V_M^c and V_M^m are known, they belong respectively to $H^{N+p-M+2}(\mathcal{O})$ and to $V_M^m \in \mathcal{C}^\infty([0, 1]; H^{N+p-M+3/2}(\mathbb{T}))$ and the following estimates hold:

$$\begin{aligned} \forall M = 0, \dots, N, \\ \sup_{\eta \in [0,1]} \|V_M^m(\eta, \cdot)\|_{H^{N+p-M+3/2}(\mathbb{T})} &\leq C_{N,\mathcal{O},p} \|f\|_{H^{N+p+1/2}(\partial\mathcal{O})}, \\ \|V_M^c\|_{H^{N+p-M+2}(\mathcal{O})} &\leq |\alpha| C_{N,\mathcal{O},p} \|f\|_{H^{N+p+1/2}(\partial\mathcal{O})}. \end{aligned}$$

We are going to build V_{N+1}^c and V_{N+1}^m . From (28b) and (28e), we infer, for all $(\eta, \theta) \in \mathcal{C}$,

$$\begin{aligned} \partial_\eta^2 V_{N+1}^m &= - \left\{ \kappa \{ 3\eta \partial_\eta^2 V_N^m + \partial_\eta V_N^m \} \right. \\ &\quad + 3\eta^2 \kappa^2 \partial_\eta^2 V_{N-1}^m + 2\eta \kappa^2 \partial_\eta V_{N-1}^m + \partial_\theta^2 V_{N-1}^m \\ &\quad \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{N-2}^m + \eta^2 \kappa^3 \partial_\eta V_{N-2}^m + \eta \kappa \partial_\theta^2 V_{N-2}^m - \eta \kappa' \partial_\theta V_{N-2}^m \right\}, \\ \partial_\eta V_{N+1}^m|_{\eta=1} &= 0. \end{aligned}$$

Recall that we use convention (29). Since we have supposed that V_M^m is known for $M \leq N$ and belongs to $\mathcal{C}^\infty([0, 1]; H^{N+1+p-M-1/2}(\mathbb{T}))$, we infer that:

$$(35) \quad \begin{aligned} \forall (s, \theta) \in \mathcal{C}, \\ \partial_\eta V_{N+1}^m(s, \cdot) &= \int_s^1 \left\{ \kappa \{ 3\eta \partial_\eta^2 V_N^m + \partial_\eta V_N^m \} \right. \\ &\quad + 3\eta^2 \kappa^2 \partial_\eta^2 V_{N-1}^m + 2\eta \kappa^2 \partial_\eta V_{N-1}^m + \partial_\theta^2 V_{N-1}^m \\ &\quad \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{N-2}^m + \eta^2 \kappa^3 \partial_\eta V_{N-2}^m + \eta \kappa \partial_\theta^2 V_{N-2}^m - \eta \kappa' \partial_\theta V_{N-2}^m \right\} d\eta, \end{aligned}$$

is entirely determined and belongs to $\mathcal{C}^\infty([0, 1]; H^{p+1/2}(\mathbb{T}))$. Moreover, since $\partial_\eta V_{N+1}^m$ is known, we infer exactly by the same way that $\partial_\eta V_{N+2}^m$ is also determined. Actually, it is equal to

$$\begin{aligned} \forall (s, \theta) \in \mathcal{C}, \\ \partial_\eta V_{N+2}^m(s, \cdot) = \int_s^1 \left\{ \kappa \left\{ 3\eta \partial_\eta^2 V_{N+1}^m + \partial_\eta V_{N+1}^m \right\} \right. \\ \left. + 3\eta^2 \kappa^2 \partial_\eta^2 V_N^m + 2\eta \kappa^2 \partial_\eta V_N^m + \partial_\theta^2 V_N^m \right. \\ \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{N-1}^m + \eta^2 \kappa^3 \partial_\eta V_{N-1}^m + \eta \kappa \partial_\theta^2 V_{N-1}^m - \eta \kappa' \partial_\theta V_{N-1}^m \right\} d\eta, \end{aligned}$$

and it belongs to $\mathcal{C}^\infty([0, 1]; H^{p+1/2}(\mathbb{T}))$. According to (28c), the function V_{N+1}^c is then uniquely determined by

$$(36a) \quad \Delta V_{N+1}^c = 0,$$

$$(36b) \quad \partial_n V_{N+1}^c|_{\partial\mathcal{O}} = \alpha \partial_\eta V_{N+2}^m \circ \Phi_0^{-1},$$

with gauge condition

$$(36c) \quad \int_{\partial\mathcal{O}} V_{N+1}^c d\sigma = 0.$$

Moreover, it belongs to $H^{p+1}(\mathcal{O})$. Transmission condition (28d) implies the following expression of V_{N+1}^m :

$$\forall s \in (0, 1), \quad V_{N+1}^m(s, \cdot) = \int_0^s \partial_\eta V_{N+1}^m(\eta, \cdot) d\eta + V_{N+1}^c \circ \Phi_0,$$

where $\partial_\eta V_{N+1}^m$ is given by (35) and belongs to $\mathcal{C}^\infty([0, 1]; H^{p+1/2}(\mathbb{T}))$. We infer also that there exists $C_{N+1, \mathcal{O}, p} > 0$ such that

$$\begin{aligned} \sup_{\eta \in [0, 1]} \|V_{N+1}^m(\eta, \cdot)\|_{H^{p+1/2}(\mathbb{T})} &\leq C_{N+1, \mathcal{O}, p} \|f\|_{H^{N+p+1/2}(\partial\mathcal{O})}, \\ \|V_{N+1}^c\|_{H^{p+1}(\mathcal{O})} &\leq |\alpha| C_{N+1, \mathcal{O}, p} \|f\|_{H^{N+p+1/2}(\partial\mathcal{O})}, \end{aligned}$$

hence the lemma. \square

Observe that the functions (V_k^c, V_k^m) are these given in Theorem 1.

4.2. Error Estimates of Theorem 1. Let us prove now the estimates of Theorem 1. Let $N \in \mathbb{N}$ and ϕ belong to $H^{N+3/2}(\partial\Omega_h)$. The function f is defined by (19). Let R_N^c and R_N^m be the functions defined by:

$$\begin{cases} R_N^c = V_h - \sum_{k=0}^N V_k^c h^k, & \text{in } \mathcal{O}, \\ R_N^m = V_h \circ \Phi - \sum_{k=0}^N V_k^m h^k, & \text{in } \mathcal{C}. \end{cases}$$

We have to prove that there exists a constant $C_{\mathcal{O}, N} > 0$ depending only on the domain \mathcal{O} and on N such that

$$(37a) \quad \|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} |\alpha| h^{N+1/2},$$

$$(37b) \quad \|R_N^m\|_{H_{\frac{1}{2}}^1(\mathcal{C})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}.$$

Moreover, if ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, then we have

$$(38a) \quad \|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+5/2}(\partial\mathcal{O})} |\alpha| h^{N+1},$$

$$(38b) \quad \|R_N^m\|_{H_{\frac{1}{2}}^1(\mathcal{C})} \leq C_{\mathcal{O}, N} \|f\|_{H^{N+5/2}(\partial\mathcal{O})} h^{N+1/2}.$$

Proof of Theorem 1. Since ϕ belongs to $H^{N+3/2}(\partial\Omega_h)$, according to the previous lemma, the couples of functions (R_N^c, R_N^m) and (R_{N+1}^c, R_{N+1}^m) are well defined and belong to $H^1(\mathcal{O}) \times H_{\mathfrak{g}}^1(\mathbf{C})$. The Sobolev space $H_{\mathfrak{g}}^1(\mathbf{C})$ is defined in Notation 5.

Denote by g_N the following function defined on \mathbf{C} :

$$\begin{aligned}
(39) \quad g_N = & \kappa (3\eta \partial_\eta^2 V_N^m + \partial_\eta V_N^m) + 3\eta^2 \kappa^2 \partial_\eta^2 V_{N-1}^m + 2\eta \kappa^2 \partial_\eta V_{N-1}^m + \partial_\theta^2 V_{N-1}^m \\
& + \eta^3 \kappa^3 \partial_\eta^2 V_{N-2}^m + \eta^2 \kappa^3 \partial_\eta V_{N-2}^m + \eta \kappa \partial_\theta^2 V_{N-2}^m - \eta \kappa' \partial_\theta V_{N-2}^m \\
& + h \left(3\eta^2 \kappa^2 \partial_\eta^2 V_N^m + 2\eta \kappa^2 \partial_\eta V_N^m + \partial_\theta^2 V_N^m \right. \\
& \left. + \eta^3 \kappa^3 \partial_\eta^2 V_{N-1}^m + \eta^2 \kappa^3 \partial_\eta V_{N-1}^m + \eta \kappa \partial_\theta^2 V_{N-1}^m - \eta \kappa' \partial_\theta V_{N-1}^m \right) \\
& + h^2 \left(\eta^3 \kappa^3 \partial_\eta^2 V_N^m + \eta^2 \kappa^3 \partial_\eta V_N^m + \eta \kappa \partial_\theta^2 V_N^m - \eta \kappa' \partial_\theta V_N^m \right)
\end{aligned}$$

According to the previous lemma and since ϕ belongs to $H^{N+1/2}(\partial\Omega_h)$, the above function g_N belongs to $\mathcal{C}^\infty([0, 1]; H^{-1/2}(\mathbb{T}))$ and the function $\partial_\eta V_N^m$ belongs to $\mathcal{C}^\infty([0, 1]; H^{3/2}(\mathbb{T}))$. Moreover, there exists a constant $C_{N, \mathcal{O}}$ such that

$$(40) \quad \begin{cases} \sup_{\eta \in [0, 1]} \|g_N(\eta, \cdot)\|_{H^{-1/2}(\mathbb{T})} \leq C_{N, \mathcal{O}} \|f\|_{H^{N+1/2}(\mathbb{T})}, \\ \sup_{\eta \in [0, 1]} \|\partial_\eta V_N^m(\eta, \cdot)\|_{H^{3/2}(\mathbb{T})} \leq C_{N, \mathcal{O}} \|f\|_{H^{N+1/2}(\mathbb{T})}. \end{cases}$$

The functions R_N^c and R_N^m satisfy the following problem:

$$\begin{aligned}
\Delta R_N^c &= 0, \text{ in } \mathcal{O}, \\
\partial_\eta \left(\frac{1 + h\eta\kappa}{h} \partial_\eta R_N^m \right) + \partial_\theta \left(\frac{h}{1 + h\eta\kappa} \partial_\theta R_N^m \right) &= \frac{-h^N}{(1 + h\eta\kappa)} g_N,
\end{aligned}$$

with transmission conditions:

$$\begin{aligned}
\partial_n R_N^c \circ \Phi_0 &= \frac{\alpha}{h} \left(\partial_\eta R_N^m|_{\eta=0} + h^{N+1} \partial_\eta V_N^m|_{\eta=0} \right), \\
R_N^c \circ \Phi_0 &= R_N^m|_{\eta=0},
\end{aligned}$$

with boundary condition

$$\partial_\eta R_N^m|_{\eta=1} = 0,$$

and with gauge condition

$$\int_{\partial\mathcal{O}} R_N^c \, d\sigma = 0.$$

By multiplying the above equality by $\overline{R_N^m}$ and by integration by parts, we infer that:

$$\begin{aligned}
(41) \quad \|dR_N^c\|_{\Lambda^1 L^2(\mathcal{O})}^2 + \alpha \|dR_N^m\|_{\Lambda^1 L_{\mathfrak{g}}^2(\mathbf{C})}^2 &= -\alpha h^N \int_{\mathbf{C}} g_N(\eta, \theta) \overline{R_N^m}(\eta, \theta) \, d\eta \, d\theta \\
&+ \alpha h^{N+1} \int_{\mathbb{T}} \partial_\eta V_N^m|_{\eta=0} \overline{R_N^m}|_{\eta=0} \, d\theta \\
&- \alpha h^{N+1} \int_{\mathbb{T}} \kappa \partial_\eta V_N^m|_{\eta=1} \overline{R_N^m}|_{\eta=1} \, d\theta.
\end{aligned}$$

By hypothesis (4), and using by Cauchy-Schwarz inequality and estimates (40), we infer that there exists a constant $C_{\mathcal{O}, N} > 0$ such that

$$\Re(\alpha) \|dR_N^m\|_{\Lambda^1 L_{\mathfrak{g}}^2(\mathbf{C})}^2 \leq |\alpha| C_{\mathcal{O}, N} h^{N-1/2} \|f\|_{H^{N+1/2}(\mathbb{T})} \|R_N^m\|_{H_{\mathfrak{g}}^1(\mathbf{C})},$$

and

$$|\Im(\alpha)| \|dR_N^m\|_{\Lambda^1 L_{\mathfrak{g}}^2(\mathcal{C})}^2 \leq |\alpha| C_{\mathcal{O},N} h^{N-1/2} \|f\|_{H^{N+1/2}(\mathbb{T})} \|R_N^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})},$$

hence

$$\|dR_N^m\|_{\Lambda^1 L_{\mathfrak{g}}^2(\mathcal{C})}^2 \leq C_{\mathcal{O},N} h^{N-1/2} \|f\|_{H^{N+1/2}(\mathbb{T})} \|R_N^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})}.$$

Since $\int_{\mathbb{T}} R_N^m|_{\eta=0} d\theta = 0$, by Poincaré inequality (24), there exists a strictly positive constant $C_{\mathcal{O}}$, which does not depend on h such that

$$\|R_N^m\|_{\Lambda^0 L_{\mathfrak{g}}^2(\mathcal{C})} \leq C_{\mathcal{O}} \|dR_N^m\|_{\Lambda^1 L_{\mathfrak{g}}^2(\mathcal{C})},$$

hence

$$\|R_N^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+1/2}(\mathbb{T})} h^{N-1/2},$$

and therefore we deduce directly from the above estimate and from (41),

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+1/2}(\mathbb{T})} |\alpha| h^{N-1/2}.$$

The above estimate holds for $\phi \in H^{N+1/2}(\partial\Omega_h)$. Since ϕ belongs to $H^{N+3/2}(\partial\Omega_h)$, we obtain the same result by replacing N by $N+1$:

$$(42) \quad \begin{cases} \|R_{N+1}^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})} \leq C_{\mathcal{O},N+1} \|f\|_{H^{N+3/2}(\mathbb{T})} h^{N+1/2}, \\ \|R_{N+1}^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N+1} \|f\|_{H^{N+3/2}(\mathbb{T})} |\alpha| h^{N+1/2}. \end{cases}$$

According to the previous lemma, the functions V_{N+1}^c and V_{N+1}^m are well-defined and there exists a constant $C_{N,\mathcal{O}}$ such that:

$$\begin{aligned} \|V_{N+1}^c\|_{H^1(\mathcal{O})} &\leq |\alpha| C_{N,\mathcal{O}} \|f\|_{H^{N+3/2}(\partial\mathcal{O})}, \\ \|V_{N+1}^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})} &\leq \frac{C_{N,\mathcal{O}}}{\sqrt{h}} \|f\|_{H^{N+3/2}(\partial\mathcal{O})}. \end{aligned}$$

Writing

$$R_N^c = R_{N+1}^c + V_{N+1}^c h^{N+1},$$

and

$$R_N^m = R_{N+1}^m + V_{N+1}^m h^{N+1},$$

we infer that

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\mathbb{T})} |\alpha| h^{N+1/2},$$

and

$$\|R_N^m\|_{H_{\mathfrak{g}}^1(\mathcal{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\mathbb{T})} h^{N+1/2}.$$

If ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, we write

$$R_N^c = R_{N+2}^c + V_{N+1}^c h^{N+1} + V_{N+2}^c h^{N+2},$$

and

$$R_N^m = R_{N+1}^m + V_{N+1}^m h^{N+1} + V_{N+2}^m h^{N+2},$$

to obtain estimates (38), hence Theorem 1. \square

Remark 10 (Counter-example :the perfectly insulating inner domain). *Consider Problem (2):*

$$\begin{aligned} \operatorname{div}(\gamma_h \operatorname{grad} u) &= 0 \text{ in } \Omega_h, \\ \frac{\partial u}{\partial n} &= \phi \text{ on } \partial\Omega_h, \\ \int_{\partial\mathcal{O}} u \, d\sigma &= 0. \end{aligned}$$

If the inner domain is perfectly insulating (i.e. if γ_h vanishes in \mathcal{O}), the steady state potential in the membrane satisfies:

$$\frac{1}{h^2} \partial_\eta ((1 + h\eta\kappa) \partial_\eta u^m) + \partial_\theta \left(\frac{1}{1 + h\eta\kappa} \partial_\theta u^m \right) = 0, \text{ in } \mathbf{C},$$

with the following boundary conditions:

$$\partial_\eta u^m|_{\eta=0} = 0, \quad \partial_\eta u^m|_{\eta=1} = hf,$$

and with gauge condition

$$\int_0^{2\pi} u^m|_{\eta=0} d\theta = 0.$$

By identifying the terms of the same power of h we would obtain: $u_0^m = 0$, and u_1^m would satisfy:

$$\begin{aligned} \partial_\eta^2 u_1^m &= 0, \text{ in } \mathbf{C}, \\ \partial_\eta u_1^m|_{\eta=0} &= 0, \quad \partial_\eta u_1^m|_{\eta=1} = f, \\ \int_0^{2\pi} u_1^m|_{\eta=0} d\theta &= 0, \end{aligned}$$

which is a non-sense as soon as $f \neq 0$. Our ansatz (26) fails. Actually, the asymptotic expansion of u^m begins at the order -1 : a boundary layer phenomenon appears. This is described in the next section.

5. ASYMPTOTIC EXPANSION OF THE STEADY STATE POTENTIAL FOR AN INSULATING INNER DOMAIN

Consider now the case of an insulating inner domain surrounded by a thin conducting layer. Similarly to the previous section, we derive the asymptotics of the potentials with respect to h . As we see later on, the asymptotics begin at the order -1 instead of 0 in the previous section.

Let solution u_h to Problem (2):

$$\begin{aligned} \nabla \cdot (\gamma_h \nabla u_h) &= 0 \text{ in } \Omega_h, \\ \frac{\partial u_h}{\partial n} &= \phi \text{ on } \partial\Omega_h, \\ \int_{\partial\mathcal{O}} u_h d\sigma &= 0, \end{aligned}$$

where γ_h equals

$$\forall x \in \Omega_h, \quad \gamma_h(x) = \begin{cases} \alpha, & \text{if } x \in \mathcal{O}, \\ 1, & \text{if } x \in \mathcal{O}_h. \end{cases}$$

We suppose

$$(43a) \quad |\alpha| \text{ tends to zero,}$$

$$(43b) \quad \Re(\alpha) > 0 \text{ or } \left\{ \Re(\alpha) = 0 \text{ and } \Im(\alpha) \neq 0 \right\}.$$

Thus the inner domain is insulating. Let β be a complex parameter satisfying:

$$\Re(\beta) > 0, \text{ or } (\Re(\beta) = 0, \text{ and } \Im(\beta) \neq 0).$$

The modulus of β may tend to infinity, or to zero but it must satisfy:

$$|\beta| = o\left(\frac{1}{h}\right), \quad \text{and} \quad \frac{1}{|\beta|} = o\left(\frac{1}{h}\right).$$

We suppose that u may be written as follows:

$$u_h = \frac{1}{h}u_{-1} + u_0 + hu_1 + \dots.$$

We denote by u^c and $u^m \circ \Phi^{-1}$ the respective restrictions of u_h to \mathcal{O} and to \mathcal{O}_h . One of the two following cases holds.

Hypothesis 11 ($\alpha = \beta h^q$). *There exists $q \geq 1$ such that:*

$$(44) \quad \alpha = \beta h^q.$$

Hypothesis 12 ($\alpha = o(h^N)$, $\forall N \in \mathbb{N}$). *The complex parameter α satisfies (43) and for all $N \in \mathbb{N}$,*

$$(45) \quad \forall N \in \mathbb{N}, \quad |\alpha| = o(h^N).$$

First we suppose that Hypothesis 11 holds: we will discuss on Hypothesis 12 later on. We denote by $(u^{c,q}, u^{m,q})$ the solution to Problem 2 under the Hypothesis 11. According to (22), by ordering and identifying the terms of the same power of h , for $k \in \mathbb{N} \cup -1$, for $q \in \mathbb{N}^*$, $u_k^{c,q}$ and $u_k^{m,q}$ satisfy:

$$(46a) \quad \Delta u_l^{c,q} = 0, \text{ in } \mathcal{O},$$

for all $(\eta, \theta) \in \mathcal{C}$,

$$(46b) \quad \begin{aligned} \partial_\eta^2 u_l^{m,q} = & - \left\{ \kappa \{3\eta \partial_\eta^2 u_{l-1}^{m,q} + \partial_\eta u_{l-1}^{m,q}\} \right. \\ & + 3\eta^2 \kappa^2 \partial_\eta^2 u_{l-2}^{m,q} + 2\eta \kappa^2 \partial_\eta u_{l-2}^{m,q} + \partial_\theta^2 u_{l-2}^{m,q} \\ & \left. + \eta^3 \kappa^3 \partial_\eta^2 u_{l-3}^{m,q} + \eta^2 \kappa^3 \partial_\eta u_{l-3}^{m,q} + \eta \kappa \partial_\theta^2 u_{l-3}^{m,q} - \eta \kappa' \partial_\theta u_{l-3}^{m,q} \right\}, \end{aligned}$$

$$(46c) \quad u_l^{c,q} \circ \Phi_0 = u_l^{m,q}|_{\eta=0},$$

$$(46d) \quad \partial_\eta u_l^{m,q}|_{\eta=1} = \delta_{l,1} f,$$

$$(46e) \quad \int_{\partial \mathcal{O}} u_l^{c,q} d\sigma = 0.$$

Transmission condition (22c) coupled with Hypothesis 11 implies:

$$(46f) \quad \beta \partial_n u_{l-1-q}^{c,q} \circ \Phi_0 = \partial_\eta u_l^{m,q}|_{\eta=0},$$

In equations (46), we have implicitly imposed

$$(47) \quad \begin{cases} u_l^{c,q} = 0, & \text{if } l \leq -2, \\ u_l^{m,q} = 0, & \text{if } l \leq -2. \end{cases}$$

Let us now derive the formal asymptotics of u when Hypothesis 11 holds.

5.1. Formal asymptotics.

- $N = -1$.

The functions $u_{-1}^{m,q}$ satisfies

$$\begin{cases} \partial_\eta^2 u_{-1}^{m,q} = 0, \text{ in } \mathcal{C}, \\ \partial_\eta u_{-1}^{m,q}|_{\eta=0} = 0, \quad \partial_\eta u_{-1}^{m,q}|_{\eta=1} = 0, \end{cases}$$

hence $u_{-1}^{m,q}$ depends only on the variable θ . Observe that we have, for almost all $\theta \in \mathbb{T}$ the following equality:

$$u_{-1}^{m,q}(\theta) = u_{-1}^{c,q} \circ \Phi_0(\theta).$$

- $N = 0$.

The function $u_0^{m,q}$ satisfies:

$$\begin{cases} \partial_\eta^2 u_0^{m,q} = 0, \text{ in } \mathbf{C}, \\ \partial_\eta u_0^{m,q}|_{\eta=0} = 0, \quad \partial_\eta u_0^{m,q}|_{\eta=1} = 0, \end{cases}$$

hence, $\partial_\eta u_0^{m,q}$ vanishes identically in \mathbf{C} .

- $N = 1$.

The functions $u_1^{m,q}$ satisfy:

$$\begin{cases} \partial_\eta^2 u_1^{m,q} = -\partial_\theta^2 u_{-1}^{m,q}, \text{ in } \mathbf{C}, \\ \partial_\eta u_1^{m,q}|_{\eta=0} = \beta \partial_n u_{-1-q}^{c,q} \circ \Phi_0, \quad \partial_\eta u_1^{m,q}|_{\eta=1} = f. \end{cases}$$

Therefore for $q = 1$ we obtain the following equality:

$$-\partial_\theta^2 u_{-1}^{m,1} + \beta \partial_n u_{-1}^{c,1} \circ \Phi_0 = f,$$

hence the following boundary condition imposed to $u_{-1}^{c,1}$ on $\partial\mathcal{O}$:

$$-\partial_\theta^2 u_{-1}^{c,1}|_{\partial\mathcal{O}} + \beta \partial_n u_{-1}^{c,1}|_{\partial\mathcal{O}} = f.$$

Therefore, the function $u_{-1}^{c,1}$ is solution to the following problem:

$$(48) \quad \begin{cases} \Delta u_{-1}^{c,1} = 0, \text{ in } \mathcal{O}, \\ -\partial_\theta^2 u_{-1}^{c,1}|_{\partial\mathcal{O}} + \beta \partial_n u_{-1}^{c,1}|_{\partial\mathcal{O}} = f, \\ \int_{\partial\mathcal{O}} u_{-1}^{c,1} d\partial\mathcal{O} = 0. \end{cases}$$

$$(49) \quad \forall(\eta, \theta) \in \mathbf{C}, \quad u_{-1}^{m,1} = u_{-1}^{c,1}|_{\partial\mathcal{O}} \circ \Phi_0.$$

Since $\Re(\beta) > 0$, a straight application of Lax-Milgram theorem ensures that $u_{-1}^{c,1}$ is uniquely determined and belongs to $H^1(\mathcal{O})$ as soon as the boundary data belongs to $H^{-3/2}(\partial\mathcal{O})$.

If $q \geq 2$, the function $u_1^{m,q}$ satisfies:

$$(50) \quad -\partial_\theta^2 u_{-1}^{m,q} = f.$$

Since $\int_{\mathbb{T}} u_{-1}^{m,q} d\theta = 0$, equality (50) defines uniquely $u_{-1}^{m,q}$. We infer that $u_{-1}^{c,q}$ is solution to the following problem:

$$(51) \quad \begin{cases} \Delta u_{-1}^{c,q} = 0, \text{ in } \mathcal{O}, \\ -u_{-1}^{c,q}|_{\partial\mathcal{O}} = u_{-1}^{m,q} \circ \Phi_0^{-1}. \end{cases}$$

Hence we have determined $u_{-1}^{m,q}$ and $u_{-1}^{c,q}$ for $q \in \mathbb{N}^*$. Observe that $u_{-1}^{c,1}$ is solution to Laplace equation with mixed boundary condition, and for $q \geq 2$ the potential $u_{-1}^{c,q}$ is the solution to Laplace equation with Dirichlet boundary condition, while for an insulating membrane, we obtained Neumann conditions for the approximated steady state potentials.

Let us now determined $u_N^{m,q}$ and $u_N^{c,q}$ for $q \in \mathbb{N}^*$ by recurrence.

- Induction.

Suppose that for $N \geq 0$, the functions $u_{N-1}^{m,q}$, $u_{N-1}^{c,q}$, $\partial_\eta u_N^{m,q}$ and $\partial_\eta u_{N+1}^{m,q}$ are built.

The function $u_{N+2}^{m,q}$ satisfies:

$$\begin{cases} \partial_\eta^2 u_{N+2}^{m,q} = -\kappa (3\eta \partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}) - \partial_\theta^2 u_N^{m,q} - \eta \kappa \partial_\theta^2 u_{N-1}^{m,q} + \eta \kappa' \partial_\theta u_{N-1}^{m,q}, \text{ in } \mathbf{C}, \\ \partial_\eta u_{N+2}^{m,q}|_{\eta=0} = \beta \partial_n u_{N+1-q}^{c,q} \circ \Phi_0, \quad \partial_\eta u_{N+2}^{m,q}|_{\eta=1} = 0. \end{cases}$$

Denote by ϕ_N^q the following function:

$$\phi_N^q = \int_0^1 \left(\kappa (3\eta \partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}) + \eta \kappa \partial_\theta^2 u_{N-1}^{m,q} - \eta \kappa' \partial_\theta u_{-1}^{m,q} \right) d\eta.$$

Since $\partial_\eta^2 u_{N+1}^{m,q}$ and $\partial_\eta u_{N+1}^{m,q}$ are supposed to be known, the function ϕ_N^q is entirely determined. Observe that if $q = 1$, $\partial_\eta u_{N+2}^{m,1}|_{\eta=0}$ is unknown since $\partial_n u_N^{c,1}$ is not yet determined, while as soon as $q \geq 2$, $\partial_\eta u_{N+2}^{m,q}|_{\eta=0}$ is known.

Using transmission condition (46c), we infer the following equality satisfied by $u_N^{m,1}$ in $\eta = 0$:

$$-\partial_\theta^2 u_N^{m,1}|_{\eta=0} + \beta \partial_n u_N^{c,1} \circ \Phi_0 = \phi_N^1 - \int_0^1 (\eta - 1) \partial_\theta^2 \partial_\eta u_N^{m,1} d\eta,$$

hence the boundary condition imposed to $u_N^{c,1}$ on $\partial\mathcal{O}$:

$$\beta \partial_n u_N^{c,1}|_{\partial\mathcal{O}} - \partial_\theta^2 u_N^{c,1}|_{\partial\mathcal{O}} = \left(\phi_N^1 - \int_0^1 (\eta - 1) \partial_\theta^2 \partial_\eta u_N^{m,1} d\eta \right) \circ \Phi_0^{-1}.$$

Thus the function $u_N^{c,1}$ is solution to the following problem:

$$(52) \quad \begin{cases} \Delta u_N^{c,1} = 0, \text{ in } \mathcal{O}, \\ -\partial_\theta^2 u_N^{c,1}|_{\partial\mathcal{O}} + \beta \partial_n u_N^{c,1}|_{\partial\mathcal{O}} = \left(\phi_N^1 - \int_0^1 (\eta - 1) \partial_\theta^2 \partial_\eta u_N^{m,1} d\eta \right) \circ \Phi_0^{-1}, \\ \int_{\partial\mathcal{O}} u_N^{c,1} d\partial\mathcal{O} = 0. \end{cases}$$

In the membrane $u_N^{m,1}$ is defined by

$$(53) \quad u_N^{m,1} = \int_0^s \partial_\eta u_N^{m,q} d\eta + u_N^{c,q} \circ \Phi_0.$$

If $q \geq 2$, $u_N^{m,q}|_{\eta=1}$ is entirely determined by the equality:

$$-\partial_\theta^2 u_N^{m,q}|_{\eta=1} = \beta \partial_n u_{N+1-q}^{c,q} \circ \Phi_0 + \phi_N^q - \int_0^1 (\eta - 1) \partial_\theta^2 \partial_\eta u_N^{m,q} d\eta,$$

hence

$$u_N^{m,q}(s, \theta) = \int_1^s \partial_\eta u_N^{m,q} d\eta + u_N^{m,q}|_{\eta=1}.$$

The potential $u_N^{c,q}$ satisfies the following boundary value problem:

$$(54) \quad \begin{cases} \Delta u_N^{c,q} = 0, \text{ in } \mathcal{O}, \\ u_N^{c,q}|_{\partial\mathcal{O}} = u_N^{m,q} \circ \Phi_0^{-1}. \end{cases}$$

Observe that for $q \geq 1$, $\partial_\eta u_{N+2}^{m,q}$ is then entirely determined by:

$$\begin{aligned} \partial_\eta u_{N+2}^{m,q} = \int_1^s \left(-\kappa (3\eta \partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}) \right. \\ \left. - \partial_\theta^2 u_N^{m,q} - \eta \kappa \partial_\theta^2 u_{N-1}^{m,q} + \eta \kappa' \partial_\theta u_{N-1}^{m,q} \right) d\eta. \end{aligned}$$

Therefore, we have proved that for all $N \geq -1$, for $q \in \mathbb{N}^*$, the functions $u_N^{c,q}$ and $u_N^{m,q}$ are uniquely determined.

Remark 13 (Regularity). *Observe that these functions are the potentials given in Theorem 2. We leave the reader verify by induction that the following regularities hold. Let $q \in \mathbb{N}^*$, $N \geq -1$ and $p \geq 1$. Let ϕ belong to $H^{N+p-3/2}(\partial\Omega_h)$.*

$$\begin{aligned} (55a) \quad & u_{-1}^{c,q} \in H^{1+N+p}(\mathcal{O}), \\ & u_{-1}^{m,q} \in \mathcal{C}^\infty\left([0, 1]; H^{1/2+N+p}(\mathbb{T})\right), \\ & \forall k = 0, \dots, N, \\ (55b) \quad & u_k^{c,q} \in H^{1+N+p-2[k/2]}(\mathcal{O}), \\ (55c) \quad & u_k^{m,q} \in \mathcal{C}^\infty\left([0, 1]; H^{1/2+N+p-2[(k+1)/2]}(\mathbb{T})\right). \end{aligned}$$

Moreover, there exists a constant $C_{N,\mathcal{O},p}$ independant on h and β such that:

$$\begin{aligned} (56a) \quad & \sup_{\eta \in [0,1]} \|u_{-1}^{m,q}(\eta, \cdot)\|_{H^{1/2+N+p}(\mathbb{T})} \leq C_{N,\mathcal{O},p} \|f\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \\ (56b) \quad & \|u_{-1}^c\|_{H^{1+N+p}(\mathcal{O})} \leq C_{N,\mathcal{O},p} \|f\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \\ & \forall k = 0, \dots, N, \\ (56c) \quad & \sup_{\eta \in [0,1]} \|u_k^{m,q}(\eta, \cdot)\|_{H^{1/2+N+p-2[(k+1)/2]}(\mathbb{T})} \leq C_{N,\mathcal{O},p} \|f\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \\ (56d) \quad & \|u_k^{c,q}\|_{H^{1+N+p-[k/2]}(\mathcal{O})} \leq C_{N,\mathcal{O},p} \|f\|_{H^{N+p-3/2}(\partial\mathcal{O})}. \end{aligned}$$

5.2. Error estimates of Theorem 2. Let us now prove Theorem 2. Let $q \in \mathbb{N}^*$ and $N \in \mathbb{N}$. The complex parameter α satisfies (43) with Hypothesis 11. Let ϕ belong to $H^{N+3/2+q}(\partial\Omega_h)$. Let $r_N^{c,q}$ and $r_N^{m,q}$ be the functions defined by:

$$\begin{cases} r_N^{c,q} = u - \sum_{k=-1}^N u_k^{c,q} h^k, & \text{in } \mathcal{O}, \\ r_N^{m,q} = u \circ \Phi - \sum_{k=-1}^N u_k^{m,q} h^k, & \text{in } \mathcal{C}. \end{cases}$$

We have to prove that there exists a constant $C_{\mathcal{O},N} > 0$ depending only on the domain \mathcal{O} and on N such that

$$(57a) \quad \|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2+q}(\partial\mathcal{O})} \max\left(\sqrt{\frac{h}{|\beta|}}, \sqrt{h}\right) h^{N+1/2},$$

$$(57b) \quad \|r_N^{m,q}\|_{H_{\mathfrak{g}}^1(\mathcal{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}.$$

If ϕ belongs to $H^{N+5/2+q}(\partial\Omega_h)$, we have

$$\|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+5/2+q}(\mathbb{T})} h^{N+1}.$$

Proof. The proof of Theorem 2 is similar to the proof of Theorem 1. Since ϕ belongs to $H^{N-1/2}(\partial\Omega_h)$, according to the previous lemma, the couples of functions $(r_N^{c,q}, r_N^{m,q})$ and $(r_{N+1}^{c,q}, r_{N+1}^{m,q})$ are well defined and belong to $H^1(\mathcal{O}) \times H_{\mathfrak{g}}^1(\mathcal{C})$.

Denote by \tilde{g}_N the following function defined on \mathcal{C} :

$$\begin{aligned}
\tilde{g}_N = & \kappa \left(3\eta \partial_\eta^2 u_N^{m,q} + \partial_\eta u_N^{m,q} \right) + 3\eta^2 \kappa^2 \partial_\eta^2 u_{N-1}^{m,q} + 2\eta \kappa^2 \partial_\eta u_{N-1}^{m,q} + \partial_\theta^2 u_{N-1}^{m,q} \\
& + \eta^3 \kappa^3 \partial_\eta^2 u_{N-2}^{m,q} + \eta^2 \kappa^3 \partial_\eta u_{N-2}^{m,q} + \eta \kappa \partial_\theta^2 u_{N-2}^{m,q} - \eta \kappa' \partial_\theta u_{N-2}^{m,q} \\
& + h \left(3\eta^2 \kappa^2 \partial_\eta^2 u_N^{m,q} + 2\eta \kappa^2 \partial_\eta u_N^{m,q} + \partial_\theta^2 u_N^{m,q} \right. \\
(58) \quad & \left. + \eta^3 \kappa^3 \partial_\eta^2 u_{N-1}^{m,q} + \eta^2 \kappa^3 \partial_\eta u_{N-1}^{m,q} + \eta \kappa \partial_\theta^2 u_{N-1}^{m,q} - \eta \kappa' \partial_\theta u_{N-1}^{m,q} \right) \\
& + h^2 \left(\eta^3 \kappa^3 \partial_\eta^2 u_N^{m,q} + \eta^2 \kappa^3 \partial_\eta u_N^{m,q} + \eta \kappa \partial_\theta^2 u_N^{m,q} - \eta \kappa' \partial_\theta u_N^{m,q} \right)
\end{aligned}$$

According to the previous lemma and since ϕ belongs to $H^{N-1/2}(\partial\Omega_h)$, the above function \tilde{g}_N belongs to $\mathcal{C}^\infty([0, 1]; H^{-1/2}(\mathbb{T}))$ and the function $\partial_\eta V_N^{m,q}$ belongs to $\mathcal{C}^\infty([0, 1]; H^{3/2}(\mathbb{T}))$. Moreover, there exists a constant $C_{N,\mathcal{O}}$ such that

$$(59) \quad \begin{cases} \sup_{\eta \in [0,1]} \|\tilde{g}_N(\eta, \cdot)\|_{H^{-1/2}(\mathbb{T})} \leq C_{N,\mathcal{O}} \|f\|_{H^{N-1/2}(\mathbb{T})}, \\ \sup_{\eta \in [0,1]} \|\partial_\eta u_N^{m,q}(\eta, \cdot)\|_{H^{3/2}(\mathbb{T})} \leq C_{N,\mathcal{O}} \|f\|_{H^{N-1/2}(\mathbb{T})}. \end{cases}$$

The functions $r_N^{c,q}$ and $r_N^{m,q}$ satisfy the following problem:

$$\begin{aligned}
\Delta r_N^{c,q} &= 0, \text{ in } \mathcal{O}, \\
\partial_\eta \left(\frac{1+h\eta\kappa}{h} \partial_\eta r_N^{m,q} \right) + \partial_\theta \left(\frac{h}{1+h\eta\kappa} \partial_\theta r_N^{m,q} \right) &= \frac{-h^N}{(1+h\eta\kappa)} \tilde{g}_N,
\end{aligned}$$

with transmission conditions:

$$\begin{aligned}
\beta h^{1+q} \partial_n r_N^{c,q} \circ \Phi_0 &= \frac{1}{h} \left(\partial_\eta r_N^{m,q} \Big|_{\eta=0} + \beta h^{N+1+q} (\partial_n u_{N-1}^{c,q} \circ \Phi_0 + h \partial_\eta u_N^{m,q} \circ \Phi_0) \right), \\
r_N^{c,q} \circ \Phi_0 &= r_N^{m,q} \Big|_{\eta=0},
\end{aligned}$$

with boundary condition

$$\partial_\eta r_N^{m,q} \Big|_{\eta=1} = 0,$$

and with gauge condition

$$\int_{\partial\mathcal{O}} r_N^{c,q} \, d\sigma = 0.$$

By multiplying the above equality by $\overline{r_N}$ and by integration by parts, we infer that:

$$\begin{aligned}
(60) \quad \beta h^{1+q} \|dr_N^{c,q}\|_{\Lambda^1 L^2(\mathcal{O})}^2 + \|dr_N^{m,q}\|_{\Lambda^1 L^2_g(\mathcal{C})}^2 &= -h^N \int_{\mathcal{C}} \tilde{g}_N(\eta, \theta) \overline{r_N^{m,q}}(\eta, \theta) \, d\eta \, d\theta \\
&+ \beta h^{N+1+q} \int_{\mathbb{T}} (\partial_\eta u_{N-1}^{c,q} \circ \Phi_0 + h \partial_\eta u_N^{c,q} \circ \Phi_0) \overline{r_N^{m,q}} \Big|_{\eta=0} \, d\theta.
\end{aligned}$$

The end of the proof is similar to Theorem 1. Using the positivity of $\Re(\beta)$ and Poincaré-Wirtinger inequality (24), we straight infer estimate (57b) of $r_N^{m,q}$. To obtain the estimates of $r_N^{c,q}$, we write:

$$r_N^{c,q} = r_{N+q}^{c,q} + \sum_{k=1}^q u_{N+k}^{c,q} h^{N+k}.$$

□

5.3. **The case** $\alpha = o(h^N)$, $\forall N \in \mathbb{N}$. Now, we suppose that Hypothesis 12 holds. In this case, we prove that u^c and u^m may be approximated by U^c and U^m , which are solution to:

$$(61a) \quad \Delta U^m = 0, \text{ in } \mathcal{O}_h,$$

$$(61b) \quad \partial_\eta U^m|_{\partial\mathcal{O}} = 0, \quad \partial_\eta U^m|_{\partial\Omega_h} = \phi,$$

$$(61c) \quad \int_{\partial\mathcal{O}} U^m d\sigma = 0.$$

and

$$(62a) \quad \Delta U^c = 0, \text{ in } \mathcal{O},$$

$$(62b) \quad U^c|_{\partial\mathcal{O}} = U^m|_{\partial\mathcal{O}}.$$

Actually, we have the following lemma:

Lemma 14. *Let ϕ belong to $H^{-1/2}(\partial\Omega_h)$. Let (u^c, u^m) be the solution to Problem (2), and U^m and U^c be defined respectively by (61) and (62). Then, we have:*

$$(63) \quad \|u^m - U^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}} |\alpha| |\phi|_{H^{-1/2}(\partial\Omega_h)},$$

$$(64) \quad \|u^c - U^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}} \sqrt{|\alpha|} |\phi|_{H^{-1/2}(\partial\Omega_h)}.$$

Proof. Denote by w^c and w^m the following functions:

$$w^c = u^c - U^c, \quad w^m = u^m - U^m,$$

and let ϕ belong to $H^{-1/2}(\partial\Omega_h)$. We have:

$$(65a) \quad \Delta w^c = 0, \text{ in } \mathcal{O},$$

$$(65b) \quad \Delta w^m = 0, \text{ in } \mathcal{O}_h,$$

$$(65c) \quad \alpha \partial_n w^c|_{\partial\mathcal{O}} = \partial_n w^m|_{\partial\mathcal{O}} - \alpha \partial_n U^c|_{\partial\mathcal{O}},$$

$$(65d) \quad w^c|_{\partial\mathcal{O}} = w^m|_{\partial\mathcal{O}},$$

$$(65e) \quad \partial_\eta w^m|_{\partial\Omega_h} = 0,$$

$$(65f) \quad \int_{\partial\mathcal{O}} w^m d\sigma = 0.$$

Thus we infer:

$$(66) \quad \alpha \int_{\mathcal{O}} |\nabla w^c|^2 d\text{vol}_{\mathcal{O}} + \int_{\mathcal{O}_h} |\nabla w^m|^2 d\text{vol}_{\mathcal{O}_h} = \alpha \int_{\partial\mathcal{O}} \partial_n U^c|_{\partial\mathcal{O}} \overline{w^m} d\sigma.$$

It is well-known that :

$$\|U^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}} |\phi|_{H^{-1/2}(\partial\Omega_h)},$$

and

$$\|U^c\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}} |U^m|_{\partial\mathcal{O}}|_{H^{1/2}(\partial\mathcal{O})}.$$

Since α satisfies (43) we infer,

$$\|w^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}} |\alpha| |\phi|_{H^{-1/2}(\partial\Omega_h)},$$

and thereby

$$\|w^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}} \sqrt{|\alpha|} |\phi|_{H^{-1/2}(\partial\Omega_h)}.$$

□

It remains to derive asymptotics of U^m and then these of U^c . They are similar to asymptotics of $u^{m,q}$ for $q \geq 2$: we just have to replace β by zero. We think the reader may easily derive these asymptotics from our previous results.

CONCLUSION

In this paper, we have studied the steady state potentials in a highly contrasted domain with thin layer when Neumann boundary condition is imposed on the exterior boundary. We derived rigorous asymptotics with respect to the thickness of the potentials in each domain and we gave error estimate in terms of appropriate Sobolev norm of the boundary data, electromagnetic parameters of our domain and a constant depending only on the geometry of the domain. It has to be mentioned that for an insulating inner domain (or equivalently a conducting membrane), the asymptotic expansions start at the order -1 and mixed or Dirichlet boundary conditions has to be imposed on the asymptotic terms of the inner domain.

To illustrate these asymptotics, numerical simulations using FEM are forthcoming work with Patrick Dular from Université de Liège and Ronan Perrussel from Ampère laboratory of Lyon. The main difficulty in illustrating the convergences of our asymptotics consists of the geometrical approximation of the domain: high-order geometric elements seem to be necessary.

Few results have been shown at the conference NUMELEC [20] for a non-highly heterogenous cell. We perform calculus with GetDP [10] in elongated cell. In Fig. 6 we present the steady state potentials when the thin layer is slightly insulating.

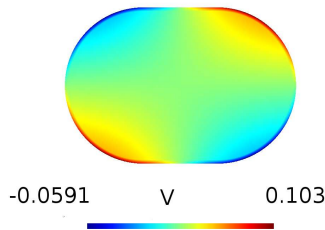


FIGURE 6. Steady state potentials in an elongated cell with insulating membrane.

Asymptotics at the order 0 and 1 of Theorem 1 give the approximated potential in \mathcal{O} without meshing the membrane. In Fig. 7 we illustrate the respective convergence orders of the errors in a semi-logarithm graphic.

Observe for instance that if $h = 5 \cdot 10^{-3}$ the error made by our method is around 10% at the order 0 and 1% at the order 1.

APPENDIX

Let \star denote the Hodge star operator, which maps 0- forms to 2- forms, 1- forms to 1- forms and 2- forms to 0- forms (see Flanders [13]). We give explicit formulae for the operators d , δ , ext and int . These formulae are straightforward consequences of the definition of the operators \star , d and $\delta = \star^{-1}d\star$. We refer the reader to Dubrovin, Fomenko and Novikov [9].

We consider the metric given by the following matrix G

$$(67) \quad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

We denote by $|G|$ the determinant of G . The inverse of G is denoted by G^{-1}

$$G^{-1} = (g^{ij})_{ij},$$

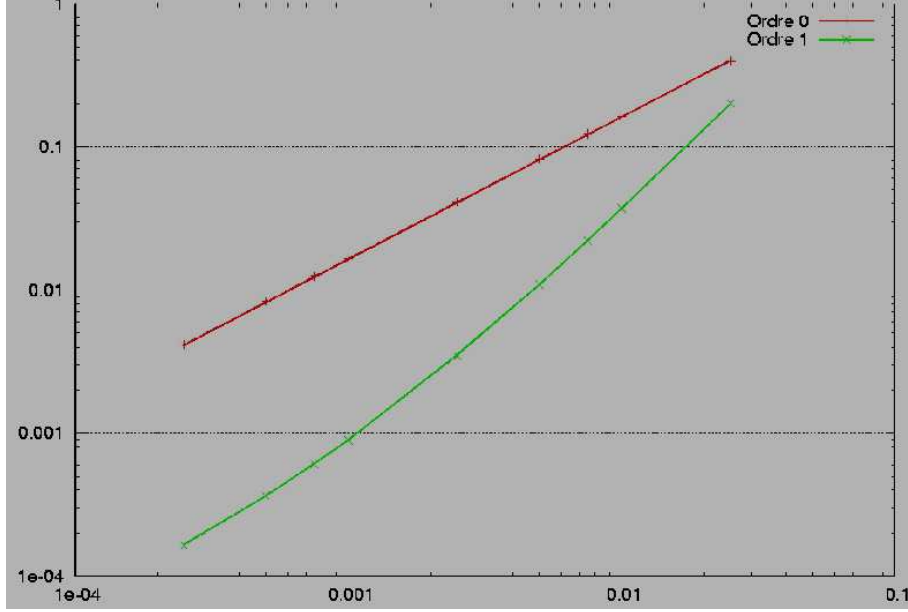


FIGURE 7. Error made by the asymptotics at the order 0 and 1 of Theorem 1.

and we suppose that the signature of G is equal to 1. Thereby, the operator \star^2 is equal to Id on the space of 0-forms and 2-forms and it is equal to $-\text{Id}$ on 1-forms.

5.4. Star operator in \mathbb{R}^2 .

5.4.1. *On 0-forms and on 2-forms.* Let T be a 0-form and let S be the 2-form $\nu dy^1 dy^2$. Then $\star T$ is the 2-form $\mu dy^1 dy^2$ and $\star S$ is the 0-form f . The following identities hold:

$$\begin{aligned}\mu &= \sqrt{|G|}T, \\ f &= \frac{1}{\sqrt{|G|}}\nu.\end{aligned}$$

5.4.2. *On 1-forms.* Let T be the 1-form $T_1 dy^1 + T_2 dy^2$. Then $\star T$ is the 1-form $\mu_1 dy^1 + \mu_2 dy^2$, and we have the following formulae:

$$\begin{aligned}\mu_1 &= -\sqrt{|G|} (g^{12}T_1 + g^{22}T_2), \\ \mu_2 &= \sqrt{|G|} (g^{11}T_1 + g^{12}T_2).\end{aligned}$$

5.5. **The action of d acting on 0-forms in \mathbb{R}^2 .** Let μ be a 0 form, then $d\mu$ has the following expression:

$$d\mu = \frac{\partial\mu}{\partial y^1} dy^1 + \frac{\partial\mu}{\partial y^2} dy^2.$$

5.6. **The action of δ acting on 1-forms on \mathbb{R}^2 .** Let μ be the 1-form $\mu_1 dy^1 + \mu_2 dy^2$, and define $\delta\mu = \alpha$. The 0-form α is equal to:

$$\begin{aligned}\alpha &= -\frac{1}{\sqrt{|G|}} \left\{ \frac{\partial}{\partial y_1} \left(\sqrt{|G|} (g^{11}\mu_1 + g^{12}\mu_2) \right) \right. \\ &\quad \left. + \frac{\partial}{\partial y_2} \left(\sqrt{|G|} (g^{12}\mu_1 + g^{22}\mu_2) \right) \right\}.\end{aligned}$$

5.7. The exterior product of a 1-form with a 0-form. Let N be the 1-form $N_1 dy^1 + N_2 dy^2$ and f be a 0-form. The exterior product of $\text{ext}(N)f$ is:

$$\text{ext}(N)f = fN_1 dy^1 + fN_2 dy^2.$$

5.8. The interior product of a 1-form with a 1-form. Let N and μ be the 1-forms $N_1 dy^1 + N_2 dy^2$, and $\mu_1 dy^1 + \mu_2 dy^2$. Then 0-form $\text{int}(N)\mu$ has the following expression:

$$\text{int}(N)\mu = N_1 (\mu_1 g^{11} + \mu_2 g^{12}) + N_2 (\mu_1 g^{12} + \mu_2 g^{22}).$$

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