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Grundy number and products of graphs

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Thème COM

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*Rapport
de recherche*

Grundy number and products of graphs

Marie Asté^{*}, Frédéric Havet^{*}, Claudia Linhares-Sales[†]

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Abstract: The *Grundy number* of a graph G , denoted by $\Gamma(G)$, is the largest k such that G has a *greedy k -colouring*, that is a colouring with k colours obtained by applying the greedy algorithm according to some ordering of the vertices of G . In this paper, we study the Grundy number of the lexicographic, cartesian and direct products of two graphs in terms of the Grundy numbers of these graphs.

Regarding the lexicographic product, we show that $\Gamma(G) \times \Gamma(H) \leq \Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G) - 1$. In addition, we show that if G is a tree or $\Gamma(G) = \Delta(G) + 1$, then $\Gamma(G[H]) = \Gamma(G) \times \Gamma(H)$. We then deduce that for every fixed $c \leq 1$, given a graph G , it is CoNP-Complete to decide if $\Gamma(G) \leq c \times \chi(G)$ and it is CoNP-Complete to decide if $\Gamma(G) \leq c \times \omega(G)$.

Regarding the cartesian product, we show that there is no upper bound of $\Gamma(G \square H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$. Nevertheless, we prove that for any fixed graph G , there is a function h_G such that, for any graph H , $\Gamma(G \square H) \leq h_G(\Gamma(H))$.

Regarding the direct product, we show that $\Gamma(G \times H) \geq \Gamma(G) + \Gamma(H) - 2$ and construct for any k some graph G_k such that $\Gamma(G_k) = 2k + 1$ and $\Gamma(G_k \times K_2) = 3k + 1$.

Key-words: colouring, greedy algorithm, on-line algorithm, graph product

[†] Universidade Federal do Ceará, Departamento de Computação, Bloco 910, Campus do Pici, Fortaleza, Ceará, CEP 60455-760, Brasil. linhares@lia.ufc.br

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Nombre Grundy et produit de graphes

Résumé : Le *nombre Grundy* d'un graphe G , noté $\Gamma(G)$, est le plus grand entier k pour lequel G admette une k -coloration *gloutonne*, i.e. une coloration avec k couleurs obtenue en appliquant l'algorithme glouton suivant un certain ordre des sommets de G . Dans ce rapport, nous étudions le nombre Grundy des produits lexicographique, cartésien et direct de deux graphes en fonction des nombres Grundy de ces deux graphes.

Pour le produit lexicographique, nous montrons que $\Gamma(G) \times \Gamma(H) \leq \Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G) - 1$. De plus, nous montrons que si G est un arbre ou $\Gamma(G) = \Delta(G) + 1$, alors $\Gamma(G[H]) = \Gamma(G) \times \Gamma(H)$. Nous en déduisons que pour tout $c \geq 1$, étant donné un graphe G , il est CoNP-Complet de décider si $\Gamma(G) \leq c \times \chi(G)$ et il est CoNP-Complet de décider si $\Gamma(G) \leq c \times \omega(G)$.

A propos du produit cartésien, nous montrons qu'il n'existe aucune borne supérieure pour $\Gamma(G \square H)$ qui soit une fonction de $\Gamma(G)$ et $\Gamma(H)$. Néanmoins, nous prouvons que pour tout graphe G fixé, il existe une fonction h_G telle que, pour tout graphe H , $\Gamma(G \square H) \leq h_G(\Gamma(H))$.

Pour le produit direct, nous montrons que $\Gamma(G \times H) \geq \Gamma(G) + \Gamma(H) - 2$ et nous construisons pour tout k un graphe G_k tel que $\Gamma(G_k) = 2k + 1$ et $\Gamma(G_k \times K_2) = 3k + 1$.

Mots-clés : coloration, algorithme glouton, algorithme on-line, produit de graphes

1 Introduction

Graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise). The definitions and notations used in this paper are standard and may be found in any textbook on graph theory. See [2] for example.

A (proper) k -colouring of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, \dots, k\}$, such that for any edge $uv \in E(G)$, $c(u) \neq c(v)$. A k -colouring may also be seen as a partition of the vertex set of G into k disjoint *stable sets* (i.e. sets of pairwise non-adjacent vertices) $S_i = \{v \mid c(v) = i\}$ for $1 \leq i \leq k$. For convenience (and with a slight abuse of terminology), by k -colouring we mean either the mapping c or the partition $(S_1, \dots, S_k)$. The elements of $\{1, \dots, k\}$ are called *colours*. A graph is k -colourable if it has a k -colouring. The *chromatic number* $\chi(G)$ is the least k such that G is k -colourable. Several on-line algorithms producing colourings have been designed. The most basic and most widespread one is the greedy algorithm. A *greedy colouring* relative to a vertex ordering $\sigma = v_1 < v_2 < \dots < v_n$ of $V(G)$ is obtained by colouring the vertices in the order $v_1, \dots, v_n$, assigning to v_i the smallest positive integer not already used on its lowered-indexed neighbours. Denoting by S_i the stable set of vertices coloured i , a greedy colouring has the following property:

For every $j < i$, every vertex in S_i has a neighbour in S_j . (★)

Otherwise the vertex in S_i would have been coloured j . Conversely, a colouring satisfying Property (★) is a greedy colouring relative to any vertex ordering in which the vertices of S_i precede those of S_j when $i < j$. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k -colouring.

It is well known that

$$\omega(G) \leq \chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$$

where $\omega(G)$ denotes the clique number of G and $\Delta(G)$ the maximum degree of G .

The inequality $\chi(G) \leq \Gamma(G)$ may be tight, but it can also be very loose. Zaker [14] showed that for any fixed $k \geq 0$, given a graph G it is CoNP-Complete to decide whether $\Gamma(G) \leq \chi(G) + k$. He also showed that, given a graph G which is the complement of bipartite graph, it is CoNP-Complete to decide if $\Gamma(G) = \chi(G)$. This implies that it is CoNP-Complete to decide if $\Gamma(G) = \omega(G)$. Indeed, if G is the complement of a bipartite graph, then it is perfect, so $\chi(G) = \omega(G)$.

The Grundy number of various classes of graphs has been studied (see the introduction of [1]). In this paper, we study the Grundy number of different usual products of two graphs G and H . The *lexicographic product* $G[H]$, the *direct product* $G \times H$, and the *cartesian product* $G \square H$, of G by H are the graphs with vertex set $V(G) \times V(H)$ and the following edge set:

$$\begin{aligned} E(G[H]) &= \{(a, x)(b, y) \mid ab \in E(G), \text{ or } a = b \text{ and } xy \in E(H)\}; \\ E(G \times H) &= \{(a, x)(b, y) \mid ab \in E(G), \text{ and } xy \in E(H)\}; \\ E(G \square H) &= \{(a, x)(b, y) \mid a = b \text{ and } xy \in E(H) \text{ or } ab \in E(G) \text{ and } x = y\}. \end{aligned}$$

It follows from the definition that $G \times H$ (resp. $G \square H$) and $H \times G$ (resp. $H \square G$) are isomorphic. But $G[H]$ and $H[G]$ are generally not isomorphic. Moreover $G[H]$ may be seen as the graph obtained by blowing up each vertex of G into a copy of H .

Regarding the lexicographic product, we prove in Section 3 that for any graphs G and H ,

$$\Gamma(G) \times \Gamma(H) \leq \Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G) - 1.$$

In addition, we show that if G is a tree or $\Gamma(G) = \Delta(G) + 1$, then $\Gamma(G[H]) = \Gamma(G) \times \Gamma(H)$. Using these results, we prove a stronger complexity result than the one of Zaker [14] mentioned above: for every fixed $c \geq 1$, it is CoNP-Complete to decide if $\Gamma(G) \leq c \times \chi(G)$ for a given graph G . Analogously, we show that it is CoNP-Complete to decide if $\Gamma(G) \leq c \times \omega(G)$.

In Section 4, we investigate the Grundy number of the cartesian product of two graphs. We show that $\Gamma(G \square H) \geq \max\{\Gamma(G), \Gamma(H)\}$ and increase this lower bound in some particular cases. We prove that there is no upper bound of $\Gamma(G \square H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$. More precisely, we show that for the complete bipartite $K_{p,p}$, $\Gamma(K_{p,p}) = 2$ but $\Gamma(K_{p,p} \square K_{p,p}) \geq p + 1$. Nevertheless, we show that for any fixed graph G , there is a function h_G such that, for any graph H , $\Gamma(G \square H) \leq h_G(\Gamma(H))$; in fact, we show that $h_G(k) \leq \Delta(G) \cdot 2^{k-1} + k$. We then give a better upper bound for $h_G(2)$ for some graphs G .

Finally, in Section 5, we study the Grundy number of the direct product of two graphs. We show that $\Gamma(G \times H) \geq \Gamma(G) + \Gamma(H) - 2$ and construct for any k some graph G_k such that $\Gamma(G_k) = 2k + 1$ and $\Gamma(G_k \times K_2) = 3k + 1$.

2 Preliminaries

In this section, we present some definitions and preliminary results.

A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Note that since H is a graph we have $E(H) \subset E(G) \cap [V(H)]^2$. If H contains all the edges of G between vertices of $V(H)$, that is $E(H) = E(G) \cap [V(H)]^2$, then H is the subgraph *induced* by $V(H)$. If S is a set of vertices, we denote by $G \langle S \rangle$ the graph induced by S and by $G - S$ the graph induced by $V(G) \setminus S$. For simplicity, we write $G - v$ rather than $G - \{v\}$. For a subset F of $E(G)$, we write $G \setminus F = (V(G), E(G) \setminus F)$. As above $G \setminus \{e\}$ is abbreviated to $G \setminus e$.

If H is a subgraph of G then $\chi(H) \leq \chi(G)$. This assertion cannot be transposed to the Grundy number. For example, the path P_4 of order 4 is a subgraph of the cycle C_4 of order 4 but one can easily check that $\Gamma(P_4) = 3$ and $\Gamma(C_4) = 2$. However such an assertion holds if we add the extra condition of being an induced subgraph.

Proposition 1 *If H is an induced subgraph of G then $\Gamma(H) \leq \Gamma(G)$.*

Proof. Let σ be an ordering for which the corresponding greedy colouring of H uses $\Gamma(H)$ colours. Then a colouring with respect to any ordering of $V(G)$ beginning with σ will use at least $\Gamma(H)$ to colour H , hence at least $\Gamma(H)$ to colour G . $\square$

Lemma 2 *Let G be a graph and u and v two vertices G . The the following hold:*

- (i) *For any edge e , $\Gamma(G) - 1 \leq \Gamma(G \setminus e) \leq \Gamma(G) + 1$.*
- (ii) *If $N(u) \subset N(v)$ then in every greedy colouring c of G , $c(u) \leq c(v)$. In particular, if $N(u) = N(v)$ then $c(u) = c(v)$.*
- (iii) *If $N(u) = N(v)$ then $\Gamma(G) = \Gamma(G - u)$.*

Proof. (i) Set $e = xy$ and $p = \Gamma(G)$. Let $(S_1, \dots, S_p)$ be a greedy p -colouring of G . It satisfies Property $(\star)$. Let i be the integer such that $x \in S_i$ and let $T_j = S_j$ for $1 \leq j < i$ and $T_j = S_{j+1}$ for $i \leq j \leq p - 1$. It is a simple matter to check that $(T_1, \dots, T_{p-1})$ satisfies Property $(\star)$. Hence $\Gamma(G - S_i) \geq p - 1$. As $G - S_i$ is an induced subgraph of $G \setminus e$, by Proposition 1, $\Gamma(G \setminus e) \geq p - 1$.

Set $q = \Gamma(G)$. Let $(S'_1, \dots, S'_q)$ be a greedy q -colouring of $G \setminus e$. It satisfies Property $(\star)$. Now let i be the integer such that $x \in S'_i$. Let $T'_j = S'_j$ for $1 \leq j < i$ and $T'_j = S'_{j+1}$ for $i \leq j \leq q - 1$. It is a simple matter to check that $(T'_1, \dots, T'_{q-1})$ satisfies Property $(\star)$. Hence $\Gamma(G - S_i) \geq q - 1$. As $G - S_i$ is an induced subgraph of G , by Proposition 1, $\Gamma(G) \geq q - 1$.

(ii) Let $c = (S_1, \dots, S_p)$. Suppose $u \in S_j$ and $v \in S_i$. Since $v \in S_i$, then v has no neighbour in S_i . So u has no neighbour in S_j because $N(u) \subset N(v)$. Thus $j \leq i$ because c satisfies Property $(\star)$.

(iii) Let $S_1, \dots, S_p$ be the stable sets of a greedy colouring. By (ii), u and v are in the same stable set S_i . Now $S_1, \dots, S_{i-1}, S_i \setminus \{u\}, S_{i+1}, \dots, S_p$ are the stable sets of a greedy colouring of $G - u$. Indeed as $N_G(u) = N_G(v)$ it is a simple matter to check that they satisfy Property $(\star)$. $\square$

A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$ where the x_i are all distinct. The vertices x_0 and x_k are the *endvertices* of P . A (u, v) -path is a path with endvertices u and v . A graph is *connected* if for any two vertices u and v there is a (u, v) -path.

Proposition 3 *Let G be a connected graph. Then $\Gamma(G) = 2$ if and only if G is complete bipartite.*

Proof. It is easy to see that if G is complete bipartite then $\Gamma(G) = 2$: indeed applying several times Lemma 2 (iii), we obtain that $\Gamma(G) = \Gamma(K_2) = 2$.

Conversely, if $\Gamma(G) = 2$, then G has to be bipartite because $\Gamma(G) \geq \chi(G)$. Suppose now that G is not complete bipartite. Then there exist two vertices u and v in different parts of the partition which are not adjacent. Let P be a shortest (u, v) -path. Then P has odd length, so length at least 3 and because it is a shortest path it is an induced path. Hence G contains an induced P_4 . So by Proposition 1, $\Gamma(G) \geq 3$. $\square$

This proposition implies that one can decide in polynomial time if the Grundy number of a graph is 2. More generally, Zaker [14] showed that for any fixed k , it is decidable in polynomial time if a given graph has Grundy number at most k . To show this, he proved that there is a finite number of graphs called *k-atoms* such that if $\Gamma(G) \geq k$ then G contains a *k-atom* as induced subgraph. The *k-atoms* may easily be found using Proposition 5 below.

Definition 4 Let G be a graph and W a subset of $V(G)$. A set S is *W-dominating* if $S \subset V(G) \setminus W$ and every vertex of W has a neighbour in S .

The following proposition follows immediately from the Property $(\star)$ of greedy colouring.

Proposition 5 *Let G be a graph and W a subset of $V(G)$. If S is a W -dominating stable set then $\Gamma(G(W \cup S)) \geq \Gamma(G(W)) + 1$.*

Note that if S is a W -dominating set then $\Gamma(G(W \cup S))$ cannot be bounded by a function of $\Gamma(G(W))$. For example, a tree may be partitioned into two stable sets S and T . Moreover, because the tree is connected S is T -dominating (and vice-versa). But the Grundy number of a stable set is 1 whereas the Grundy number of a tree may be arbitrarily large. Consider for example the *binomial tree of index k* T_k which may be defined recursively as follows:

- T_1 is the graph with one vertex and no edge;
- T_k is constructed from T_{k-1} by joining each vertex to a new leaf.

The binomial tree T_k has chromatic number 2 and Grundy number k . It is the unique *k-atom* which is a tree. Hence, as shown in [8], the Grundy number of a tree is the largest index of a binomial tree it contains.

The *union* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint (i.e. $V(G_1) \cap V(G_2) = \emptyset$), we refer to their union as a *disjoint union* and denote it $G_1 + G_2$. The *join* of two disjoint graphs G_1 and G_2 is the graph $G_1 \oplus G_2$ obtained from $G_1 + G_2$ by joining all the vertices of G_1 to all the vertices of G_2 .

Proposition 6 *If $G = G_1 + G_2$ then $\Gamma(G) = \max(\Gamma(G_1), \Gamma(G_2))$.
If $G = G_1 \oplus G_2$ then $\Gamma(G) = \Gamma(G_1) + \Gamma(G_2)$.*

This proposition and an immediate induction yield a result of Gyárfás and Lehel [6] stating that for every cograph (graph without induced P_4) $\Gamma(G) = \chi(G)$ because every cograph of order at least two is either the disjoint union or the join of two cographs.

Lemma 7 *Let G be a graph and x a vertex of G . If there is a greedy colouring c such that x is coloured p then for any $1 \leq i \leq p$, there is a greedy colouring such that x is coloured i .*

Proof. For $1 \leq i \leq p-1$, let S_i be the stable set of vertices coloured i by c . Then for any $1 \leq i \leq p$, $(S_1, \dots, S_{i-1}, \{x\})$ is a greedy i -colouring of $G[\{x\} \cup \bigcup_{j=1}^{i-1} S_j]$ in which x is coloured i . This partial greedy colouring of G may be extended into a greedy colouring of G in which x is coloured i . $\square$

Lemma 8 *Let G be a graph with at least one edge. There are two adjacent vertices x and y such that there is two greedy colourings c_x and c_y such that $c_x(x) = c_y(y) = \Gamma(G)$.*

Proof. Set $p = \Gamma(G)$ and let c_x be a greedy p -colouring of G with stable sets $S_1, \dots, S_p$. Let x a vertex of S_p and y a neighbour of x in S_{p-1} . Then $S_1, S_2, \dots, S_{p-2}, S_p, \{y\}$ is a partial greedy colouring c_y of G with $c_y(x) = p-1$ and $c_y(y) = p$. This colouring may trivially be extended to G . $\square$

3 Lexicographic product

Obviously, $\chi(G[H]) \leq \chi(G) \times \chi(H)$ and Stahl [13] showed $\chi(G[H]) \geq \chi(G) + 2\chi(H) - 2$. In this section, we establish some bounds on $\Gamma(G[H])$ in terms of $\Gamma(G)$ and $\Gamma(H)$.

Definition 9 In the lexicographic product $G[H]$, for every vertex $x \in G$, we call *copy of H at x* the graph $H(x)$ isomorphic to H which is induced by the vertices of $\{x\} \times V(H)$.

Proposition 10 *Let G and H be two graphs. In a greedy colouring of $G[H]$, at most $\Gamma(H)$ colours appear on each $H(x)$, $x \in V(G)$.*

Proof. Consider a greedy colouring of $G[H]$ and let $n_1, n_2, \dots, n_p$ be the p colours appearing on a particular copy $H(x)$ of H . For any $1 \leq i \leq p$, let S_i be the stable set of vertices of $H(x)$ coloured n_i . Let u be a vertex of S_i . For any $1 \leq j < i$, by the Property $(\star)$, in $G[H]$, u has a neighbour v coloured n_j . The vertex v must be in $H(x)$ because the neighbours of x not in $H(x)$ are also neighbours of the vertex z of $H(x)$ coloured n_j . Hence $v \in S_j$. It follows that the colouring $(S_1, \dots, S_p)$ satisfies the Property $(\star)$. Hence $\Gamma(H) = \Gamma(H(x)) \geq p$. $\square$

Geller and Stahl [5] showed that if $\chi(H) = k$ then $\chi(G[H]) = \chi(G[K_k])$ for any graph G . We now prove a similar result for the Grundy number.

Theorem 11 *Let H be a graph such that $\Gamma(H) = k$. Then for any graph G , $\Gamma(G[H]) = \Gamma(G[K_k])$.*

Proof. Set $V(G) = \{v_1, \dots, v_n\}$.

Let c be a greedy colouring of $G[H]$. For every $1 \leq i \leq n$, let $A_i = c(H(V_i)) = \{\alpha_i^1, \dots, \alpha_i^{|A_i|}\}$ be the set of colours appearing on $H(v_i)$. Let F be the graph obtained from $G[H]$ by replacing each $H(v_i)$ by a complete graph on $|A_i|$ vertices, $w_i^1, \dots, w_i^{|A_i|}$ and c' be the colouring of F defined by $c'(w_i^j) = \alpha_i^j$ for any $1 \leq i \leq n$ and $1 \leq j \leq |A_i|$. By construction F is an induced subgraph of $G[K_k]$ because for each $1 \leq i \leq n$, $|A_i| \leq k$ by Proposition 10. Moreover, it is a simple matter to check that c' is a greedy colouring of F . Hence $\Gamma(G[K_k]) \geq \Gamma(F) \geq \Gamma(G[H])$.

Now let $(S_1, \dots, S_k)$ be a greedy k -colouring of H and c be a greedy $\Gamma(G[K_k])$ -colouring of $G[K_k]$. For every $1 \leq i \leq n$, let $B_i = c(H(V_i)) = \{\beta_i^1, \dots, \beta_i^k\}$ be the set of colours appearing on $K_k(v_i)$ with $\beta_i^1 < \dots < \beta_i^k$. Let c' be the colouring of $G[H]$ which, for every $1 \leq i \leq n$ and every $1 \leq j \leq k$, assigns the colour β_i^j to the vertices of $\{v_i\} \times S_j$. Clearly, c' is a greedy $\Gamma(G[K_k])$ -colouring of $G[H]$. So $\Gamma(G[H]) \geq \Gamma(G[K_k])$. $\square$

3.1 Lower bounds

Proposition 12 *Let G and H be two graphs. Then $\Gamma(G[H]) \geq \Gamma(G) \times \Gamma(H)$.*

Proof of Proposition 12. Let c_G (resp. c_H) be a greedy colouring of G (resp. H) with $\Gamma(H)$ (resp. $\Gamma(G)$) colours. Then the colouring $c = (c_G, c_H)$ with the pairs of colours ordered according to the lexicographic product is a greedy colouring of $G[H]$. $\square$

Proposition 12 is tight as there are pairs of graphs (G, H) for which $\Gamma(G[H]) = \Gamma(G) \times \Gamma(H)$. In particular, we shall prove that if G is a tree or satisfies $\Gamma(G) = \Delta(G) + 1$ this is the case.

Theorem 13 *Let G be and H be two graphs. If $\Gamma(G) = \Delta(G) + 1$ then $\Gamma(G[H]) = \Gamma(G) \times \Gamma(H)$.*

Proof. By Proposition 12, $\Gamma(G[H]) \geq \Gamma(G) \times \Gamma(H)$.

Let us now show that $\Gamma(G[H]) \leq \Gamma(G) \times \Gamma(H)$. Consider a greedy colouring of $G[H]$. Let u be a vertex of $G[H]$ coloured with the largest colour c_{max} and $H(x)$ the copy of H containing u . Since the maximum degree of G is $\Gamma(G) - 1$, by Proposition 10, at most $(\Gamma(G) - 1)\Gamma(H)$ colours appear on the vertices of $\bigcup_{y \in N_G(x)} H(y)$ and at most $\Gamma(H) - 1$ colours distinct from c_{max} appear in $H(x)$. By definition of lexicographic product, the neighbourhood of u in $G[H]$ is included in $H(x) \cup \bigcup_{y \in N_G(x)} H(y)$. Moreover by the Property $(\star)$, every colour but c_{max} must appear on the neighbourhood of u . Hence $c_{max} \leq \Gamma(G) \times \Gamma(H)$. $\square$

Theorem 14 *Let T be a tree and H be an arbitrary graph. Then $\Gamma(T[H]) = \Gamma(T) \times \Gamma(H)$.*

Proof. Let k be the integer such that $k\Gamma(H) \geq \Gamma(T[H]) \geq (k-1)\Gamma(H) + 1$. We will prove that $\Gamma(T) \geq k$ by showing that T contains a binomial tree of index k as an induced subgraph. This implies that $\Gamma(T[H]) \leq \Gamma(T) \times \Gamma(H)$. So by Proposition 12, $\Gamma(T[H]) = \Gamma(T) \times \Gamma(H)$.

Let f be a greedy colouring of $T[H]$ with $\Gamma(T[H])$ colours. In the following, by colour we should understand colour assigned by f . We shall construct step by step a binomial tree of order k in T .

Step 1: Let v_1 be a vertex of T such that a vertex of $H(v_1)$ is coloured $c_1 = \Gamma(T[H])$. Then the subtree of T with unique vertex v_1 is T_1 . Let $P_1(v_1)$ be the sequence (v_1) .

Step i : (for $2 \leq i \leq k$) We have the binomial subtree T_{i-1} of T . Moreover, to each vertex v of T_{i-1} is associated a sequence $P_{i-1}(v) = (v_{i-1}, v_{i-2}, \dots, v_2, v_1)$ of $i-1$ vertices in T_{i-1} such that

- (a) $P_{i-1}(v)$ contains v and all its neighbours in T_{i-1} , and
- (b) for all $1 \leq j \leq i-1$, $H(v_j)$ contains the greatest colour not appearing on $\bigcup_{l=1}^{j-1} H(v_l)$.

We shall construct T_i , that is add a leaf to each vertex of T_{i-1} , and also describe the sequences P_i satisfying the conditions (a) and (b). Let v be a vertex of T_{i-1} . As $P_{i-1}(v)$ contains $i-1$ vertices, at most $(i-1)\Gamma(H)$ colours appear on $\bigcup_{l=1}^{i-1} H(v_l)$ by Proposition 10. Thus, for $i \leq k$, there exists at least one colour that does not appear on $\bigcup_{l=1}^{i-1} H(v_l)$. Let c_i be the largest such colour and $n(v)$ a neighbour of v such that c_i appears on $H(n(v))$. Such a vertex exists because for every vertex x of $P_{i-1}(v)$ (and in particular for v), thanks to the condition (a), $H(x)$ contains a colour larger than c_i . Moreover, as $P_{i-1}(v)$ contains v and all its neighbours; the vertex $n(v)$ is not in T_{i-1} . Finally, since T is a tree all the $n(v)$, $v \in V(T_{i-1})$, are distinct. Hence the subtree of T induced by $V(T_{i-1}) \cup \{n(v) \mid v \in T_{i-1}\}$ is the binomial tree T_i . Let us now define the P_i . For all $v \in V(T_{i-1})$, set $P_i(v) = P_i(n(v)) = (n(v), P_{i-1}(v))$. One can check easily that the P_i fulfil the conditions (a) and (b).

After Step k , one obtains a binomial tree of index k contained in T . So $\Gamma(T) \geq k$. $\square$

3.2 Upper bounds

There are pairs of graphs (G, H) for which $\Gamma(G[H])$ is greater than $\Gamma(G) \times \Gamma(H)$ as shown by the following proposition.

Proposition 15 *Let G_3 be the graph depicted in Figure 1. Then $\Gamma(G_3) = 3$ and $\Gamma(G_3[K_{2p}]) \geq 7p$.*

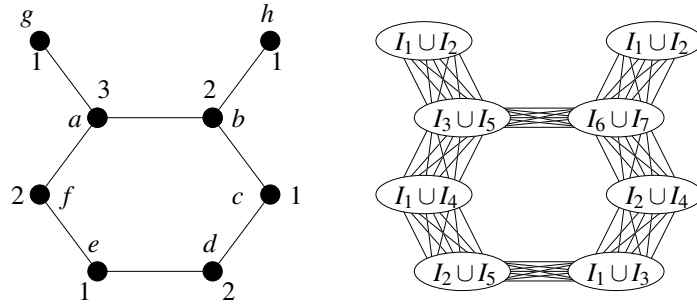


Figure 1: A greedy 3-colouring of G_3 and a greedy $7p$ -colouring of $G_3[K_{2p}]$.

Proof.

Let us first show that $\Gamma(G_3) = 3$.

The greedy 3-colouring of G_3 depicted Figure 15 shows that $\Gamma(G_3) \geq 3$.

Suppose, by way of contradiction, that G_3 admits a greedy 4-colouring. Then one of the two vertices of degree three, namely a and b , is coloured 4. By symmetry, we may assume that it is a . This vertex must have a neighbour coloured 3. This neighbour is necessarily b which is the unique one having degree at least two in $G_3 - a$. The vertices a and b must each have a neighbour coloured 2 which must have degree at least one in $G - \{a, b\}$. Hence f and c are coloured 2. These two vertices must have a neighbour coloured 1. So d and e are coloured 1, which is a contradiction as they are adjacent.

Let us now show that $\Gamma(G_3[K_{2p}]) \geq 7p$. For every vertex $v \in V(G_3)$, let us assign $2p$ colours to the $2p$ vertices of $K_2(v)$ as follows (See Figure 1). $I_3 \cup I_5$ to $K_2(a)$, $I_6 \cup I_7$ to $K_2(b)$, $I_2 \cup I_4$ to $K_2(c)$,

$I_1 \cup I_3$ to $K_2(d)$, $I_2 \cup I_5$ to $K_2(e)$, $I_1 \cup I_4$ to $K_2(f)$, $I_1 \cup I_2$ to $K_2(g)$ and $K_2(h)$. It is a simple matter to check that this is a greedy $7p$ -colouring of $G_3[K_{2p}]$. $\square$

We would like to find upper bounds on $\Gamma(G[H])$ in terms of $\Gamma(G)$ and $\Gamma(H)$. Ideally we would like to determine exactly

$$\begin{aligned}\psi(k, l) &= \max\{\Gamma(G[H]) \mid \Gamma(G) = k \text{ and } \Gamma(H) = l\} \\ &= \max\{\Gamma(G[K_l]) \mid \Gamma(G) = k\}\end{aligned}$$

by Theorem 11. In the remainder of this section we give upper and lower bounds on ψ . Note that $\Gamma(G) = 1$ if and only if G has no edge. Thus if $\Gamma(H) = 1$ then $\Gamma(G[H]) = \Gamma(G)$ using Lemma 2-(iii). Moreover if $\Gamma(G) = 1$ then $\Gamma(G[H]) = \Gamma(H)$ by Proposition 6. In the remainder of the section, we will assume that **all the graphs we consider have Grundy number at least 2**.

Theorem 16 $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G) - 1$

Proof. Let k be the integer such that $2^{k-1}(\Gamma(H) - 1) + k - 1 \geq \Gamma(G[H]) > 2^{k-2}(\Gamma(H) - 1) + k - 2$. We will show that $\Gamma(G) \geq k$, which implies that $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G) - 1$.

Let f be a greedy colouring of $G[H]$ with $\Gamma(G[H])$ colours. We shall construct step by step an induced subgraph of G which has Grundy number at least k .

Step 1: Let v_1 be a vertex such that the largest colour $c_1 = \Gamma(G[H])$ appears on $H(v_1)$. Let $G_1 = G[\{v_1\}]$. Then $\Gamma(G_1) = 1$.

Step 2: Since $\Gamma(G[H]) > 2^{k-2}(\Gamma(H) - 1) + k - 2 \geq \Gamma(H)$, by Proposition 10, there are colours that do not appear on $H(v_1)$. Let c_2 be the largest such colour. For $c_1 > c_2$, there is a vertex $v_2 \in N_G(v_1)$ such that c_2 appears on $H(v_2)$. Let $G_2 = G[\{v_1, v_2\}]$. Since $v_1 v_2$ is an edge, $\Gamma(G_2) = 2$.

Step i : (for $3 \leq i \leq k$): We have a subgraph G_{i-1} of G of at most 2^{i-2} vertices such that $\Gamma(G_{i-1}) \geq i - 1$ and at most $2^{i-2}(\Gamma(H) - 1) + i - 2$ colours appear on $G_{i-1}[H]$. For $\Gamma(G[H]) > 2^{k-2}(\Gamma(H) - 1) + k - 2 \geq 2^{i-2}(\Gamma(H) - 1) + i - 2$, there are colours that do not appear on $G_{i-1}[H]$. Let c_i be the greatest such colour. Since $c_0 > c_1 > \dots > c_i$ and c_i does not appear on $G_{i-1}[H]$, any vertex $v \in V(G_i)$ has a neighbour $n(v)$ in $V(G) \setminus V(G_i)$ such that the colour c_i appears on $H(n(v))$. Let $S_i = \{n(v), v \in V(G_i)\}$ and $G_i = G[V(G_{i-1}) \cup S_i]$. Then $|S_i| \leq |G_{i-1}|$ so $|G_i| \leq 2^{i-1}$. Moreover S_i is a stable set since the colour c_i appears on the copy of H at each vertex of S_i . So by Proposition 5, $\Gamma(G_i) \geq \Gamma(G_{i-1}) + 1 \geq i$. Now at most $2^{i-2}(\Gamma(H) - 1) + i - 2$ colours appear on $G_{i-1}[H]$ and at most $2^{i-2}(\Gamma(H) - 1) + 1$ colours appear on $S_i[H]$ by Proposition 10 and because c_i appears in all the $H(v)$ for $v \in S_i$. So in total at most $2^{i-1}(\Gamma(H) - 1) + i - 1$ colours appear on $G_i[H]$. $\square$

Corollary 17 (a) $\psi(k, l) \leq 2^{k-1}(l - 1) + k - 1$.

(b) If $\Gamma(G) = 2$ then $\Gamma(G[H]) = 2k$.

(c) $\psi(2, k) = 2k$.

(d) $\psi(3, 2) = 7$.

Proof. (a) follows directly Theorem 16; Proposition 12 and Theorem 16 imply (b) and (c); Proposition 15 and Theorem 16 yield (d). $\square$

Lemma 18 Let α be a positive integer. If $\psi(k, l) \geq kl + \alpha$ then $\psi(k', l) \geq k'l + \alpha$ for all $k' > k$.

Proof. To prove this result it suffices to prove that if $\psi(k, l) = kl + \alpha$ then $\psi(k + 1, l) \geq (k + 1)l + \alpha$. Then an easy induction will give the result.

Let G be a graph such that $\Gamma(G[K_l]) = kl + \alpha$. Let x be a vertex of G such that there exists a greedy $(kl + \alpha)$ -colouring c such that the colour $kl + \alpha$ appears on $K_l(x)$. Let G_1 and G_2 be two disjoint copies of G . For $i = 1, 2$, we denote by v_i the vertex $v_i \in V(G_i)$ corresponding to $v \in V(G)$. Let G' be the graph obtained from $G_1 + G_2$ by adding an edge between the two vertices x_1 and x_2 .

By Lemma 2 (i) and Proposition 6, $\Gamma(G') \leq \Gamma(G_1 + G_2) + 1 = \Gamma(G) + 1 = k + 1$. Now let c' be the colouring of $G'[K_l]$ defined as follows:

- $c'(v_1) = c(v)$ for $v_1 \in V(G_1[K_l])$;
- $c'(v_2) = c(v)$ for $v_2 \in V(G_2[K_l]) \setminus \{x_2\}$;
- the vertices of $K_l(x_2)$ are assigned distinct colours in $\{kl + \alpha + 1, \dots, (k + 1)l + \alpha\}$.

One can check that c' is a greedy colouring of G' . Indeed as $kl + \alpha$ appears on $K_l(x)$ then all the colours in $\{1, \dots, kl + \alpha\}$ appear in $K_l(x) \cup \bigcup_{y \in N(x)} K_l(y)$. So by definition of c' all the colours in $\{1, \dots, kl + \alpha\}$ appear in $K_l(x_1) \cup \bigcup_{y \in N(x)} K_l(y_2)$. So $\Gamma(G'[K_l]) \geq (k + 1)l + \alpha$. $\square$

Proposition 15-b) and Lemma 18 yield directly the following.

Corollary 19 *Let $k \geq 3$ be an integer. Then $\psi(k, p) \geq (2k + 1)p$.*

3.3 Complexity

According to [14] for any fixed $k \geq 0$, it is CoNP-Complete to decide if $\Gamma(G) \leq \chi(G) + k$ for a given graph G . In other words, we cannot decide (unless $P=NP$) if the Grundy number approximates the chromatic number to within a fixed additive factor. We now show that we cannot decide if the Grundy number approximates the chromatic number to within a fixed multiplicative factor.

Theorem 20 *Let $c \geq 1$ be an integer. The following problem is CoNP-complete:*

- *Instance* : a graph G .
- *Question* : $\Gamma(G) \leq c\chi(G)$?

Proof. Let G be a graph. If c_1 is a colouring of G with t colours and c_2 a greedy colouring of G with more than ct colours, then the pair (c_1, c_2) forms a certificate that $\Gamma(G) > c\chi(G)$. Clearly, it can be checked in polynomial time if a pair (c_1, c_2) is a certificate. So the problem is in CoNP.

Let us now show that this problem is CoNP-complete via a reduction to the problem of deciding if $\Gamma(G) \leq \chi(G)$ for a given graph G , which is known to be CoNP-complete [14]. Let G be a graph. Consider $H = T_{2c}[G]$. Then $\chi(H) = 2\chi(G)$ as $\omega(T_{2c}) = \chi(T_{2c}) = 2$. Moreover $\Gamma(H) = 2c\Gamma(G)$ by Theorem 13 (or Theorem 14). Hence $\Gamma(H) \leq c\chi(H)$ if and only if $\Gamma(G) \leq \chi(G)$. $\square$

A similar proof yields that it is NP-complete to decide if $\Gamma(G) \leq c\omega(G)$ as $\omega(T_{2c}[G]) = 2\omega(G)$ and it is CoNP-complete to decide if $\Gamma(G) \leq \omega(G)$.

Theorem 21 *Let $c \geq 1$ be an integer. The following problem is CoNP-complete:*

- *Instance* : a graph G .
- *Question* : $\Gamma(G) \leq c\omega(G)$?

4 Cartesian product

It is well-known that the chromatic number of the cartesian product of two graphs is the maximum of the chromatic numbers of these graphs: $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. Unfortunately, no such formula holds for the grundy number. In this section, we are looking for bounds on the grundy number of the cartesian product of two graphs in terms of the grundy numbers of these graphs. We first show that such an upper bound does not exist. However, we show that for any graph G there is a function h_G such that for every graph H , $\Gamma(G \square H) \leq h_G(\Gamma(H))$. Regarding lower bounds, we give upper and lower bounds for the function

$$\phi_{\square}(k, l) = \min\{\Gamma(G \square H) \mid \Gamma(G) = k \text{ and } \Gamma(H) = l\}$$

Let G and H be two graphs. For any $v \in V(G)$, the graph H_v of $G \square H$ induced by the vertices of $\{v\} \times V(H)$ is isomorphic to H . Analogously, for any $x \in V(H)$, the subgraph G_x of $G \square H$ induced by the vertices of $V(G) \times \{x\}$ is isomorphic to G .

4.1 Upper bounds

We denote by $K_{p,p}$ the complete bipartite graph with p vertices in each part.

Proposition 22 *Let $p \geq 2$ be an integer. Then $\Gamma(K_{p+1,p+1} \square K_{p+1,p+1}) \geq \Gamma(K_{p,p} \square K_{p,p}) + 1$. So $\Gamma(K_{p,p} \square K_{p,p}) \geq p + 1$.*

Proof. Let $(X \cup \{x\}, Y \cup \{y\})$ be the bipartition of $K_{p+1,p+1}$ with $x \notin X$ and $y \notin Y$. Then $K_{p+1,p+1} - \{x, y\}$ is a $K_{p,p}$, so $K_{p+1,p+1} - \{x, y\} \square K_{p+1,p+1} - \{x, y\}$ is an induced $K_{p,p} \square K_{p,p}$ in $K_{p+1,p+1} \square K_{p+1,p+1}$. Now the set $(\{x\} \times Y \setminus \{y\}) \cup (\{y\} \times X \setminus \{x\}) \cup (X \setminus \{x\} \times \{x\}) \cup (Y \setminus \{y\} \times \{y\})$ is a $(X \cup Y) \times (X \cup Y)$ -dominating stable set. So by Proposition 5, $\Gamma(K_{p+1,p+1} \square K_{p+1,p+1}) \geq \Gamma(K_{p,p} \square K_{p,p}) + 1$.

As $\Gamma(K_{2,2}) = 2$, an easy induction yields $\Gamma(K_{p,p} \square K_{p,p}) \geq p + 1$, for $p \geq 2$, □

Remark 23 Note that one can also prove for $p \geq 3$ that $\Gamma(K_{p,p} \square K_{p,p}) \geq p + 3$ as $\Gamma(K_{2,2} \square K_{2,2}) = 5$.

As $\Gamma(K_{p,p}) = 2$ by Proposition 3, there is no bound of $\Gamma(G \square H)$ in terms of $\Gamma(G)$, and $\Gamma(H)$.

But one may ask the following natural question.

Problem 24 For any fixed graph G , does there exist a function h_G such that for any graph H , $\Gamma(G \square H) \leq h_G(\Gamma(H))$?

We now show that the h_G exists and $h_G(k) \leq \Delta(G) \cdot 2^{k-1} + k$.

Proposition 25 *Let G be a graph then for any positive integer k , $h_G(k) \leq \Delta(G) \cdot 2^{k-1} + k$. In other words, for any graph H , $\Gamma(G \square H) \leq \Delta(G) \cdot 2^{\Gamma(H)-1} + \Gamma(H)$.*

Proof. Let c be a greedy p -colouring of $G \square H$. Let (v, x_1) be a vertex coloured $p = c_1$. For every vertex x of H , set $C(x) := \{c(w, x) \mid w \in N_G(v)\}$. By extension, for every $S \subset V(H)$, we set $C(S) = \bigcup_{x \in S} C(x)$.

Let $T_1 = \{x_1\}$. We have $\Gamma(H \langle T_1 \rangle) = 1$. Now, iteratively, as long as $\{1, \dots, p\} \setminus C(T_i) \cup \{c_1, \dots, c_i\}$ is not empty, let us construct T_{i+1} as follows. Let c_{i+1} be the largest integer of $\{1, \dots, p\} \setminus C(T_i) \cup \{c_1, \dots, c_i\}$. Then for every $x \in T_i$, the vertex (v, x) has a neighbour coloured c_{i+1} which by definition of $C(x)$ is in H_v . Hence there exists a stable set S_{i+1} of size at most $|T_i|$ in H such that $c(v, y) = c_{i+1}$

for every $y \in S_{i+1}$ and every vertex $x \in T_i$ has a neighbour in S_{i+1} . Setting $T_{i+1} = T_i \cup S_{i+1}$, we have $|T_{i+1}| \leq 2|T_i| \leq 2^i$ and by Proposition 5, $\Gamma(H\langle T_{i+1} \rangle) \geq i+1$.

Let i_0 be the integer when the process terminates, i.e. when $\{1, \dots, p\} = C(T_{i_0}) \cup \{c_1, \dots, c_{i_0}\}$. We have $\Gamma(H) \geq \Gamma(H\langle T_{i_0} \rangle) \geq i_0$, $|T_{i_0}| \leq 2^{i_0-1}$ and $|C(T_{i_0})| \leq \Delta(G) \times |T_{i_0}|$. So $p \leq \Delta(G) \cdot 2^{i_0-1} + i_0 \leq \Delta(G) \cdot 2^{\Gamma(H)-1} + \Gamma(H)$. $\square$

We think that the upper bound $\Delta(G) \cdot 2^{k-1} + k$ is far to be tight. For some graphs one can get slightly better upper bounds. Let us show an example when $k = 2$. For a vertex v of graph G , we denote by $d_G^1(v)$ or simply $d^1(v)$ the maximum degree of a neighbour of v , i.e. $d^1(v) = \max\{d(u) \mid u \in N(v)\}$. According to the proof of Theorem 25, $p \leq \max\{d_G(v) + d_G^1(v) + 2 \mid v \in V(G)\}$. We now show a slightly better upper bound.

Proposition 26 *Let G be a graph. Then $h_G(2) \leq \max\{\min\{2d(v) + 2, 2d^1(v) + 3\} \mid v \in V(G)\}$.*

Proof. Let H be a complete bipartite graph and c be a greedy colouring of $G \square H$ with p colours. Let $x = (v, v')$ be a vertex coloured with p and let (X, Y) be the bipartition of H_v with $x \in X$.

Since x has $d_G(v)$ neighbours not in H_v , it has $p - 1 - d_G(v)$ neighbours in Y with distinct colours in $\{1, \dots, p - 1\}$. Let q be the largest integer in $\{1, \dots, p - 1\}$ that is assigned to a vertex in Y and let y be a vertex coloured q . Then x has $p - 2 - d_G(v)$ neighbours in Y with distinct colours in $\{1, \dots, q - 1\}$. Now since y has at most $d_G(v)$ neighbours not in H_v , it has $q - 1 - d_G(v)$ neighbours in X with distinct colours in $\{1, \dots, q - 1\}$. As H_v is complete bipartite, the colours that appear on X do not appear on Y . Thus $p - 2 - d_G(v) + q - 1 - d_G(v) \leq q - 1$, so $p \leq 2d_G(v) + 2$.

We claim that there is a vertex $y = (u, u')$ with $u \in N_G(v)$ such that is assigned a colour $p' \geq p - 2$ and is adjacent to a vertex in H_v coloured p or $p - 1$. Indeed x has a neighbour that is coloured $p - 1$. If this neighbour is not in H_v it is the desired y . If not this neighbour z is in Y . Now both x and z have a neighbour coloured $p - 2$. But these two neighbours are not both in H_v otherwise they would be adjacent. Hence one of them is not in H_v and is the desired y .

Now applying the same reasoning as above and taking into account that y has a neighbour outside H_u with a larger colour than its, we obtain that $p - 2 \leq 2d_G(u) + 1$. So $p \leq 2d^1(v) + 3$. $\square$

If the graph G has two adjacent vertices of maximum degree then Proposition 26 yields the same upper bound $2\Delta(G) + 2$ as Theorem 25. But for graphs in which vertices of high degree form a stable set, this bound is far better. Consider for example a star $K_{1,p}$. By Proposition 26, for any $p \geq 2$, $h_{K_{1,p}}(2) \leq 5$. Moreover $K_{1,p}$ contains $K_{1,2}$ as an induced subgraph, so $K_{1,p} \square K_{3,3}$ contains $K_{1,2} \square K_{3,3}$ as an induced subgraph. But this graph has grundy number 5, as shown by the greedy 5-colouring in Figure 2. So $h_{K_{1,p}}(2) = 5$.

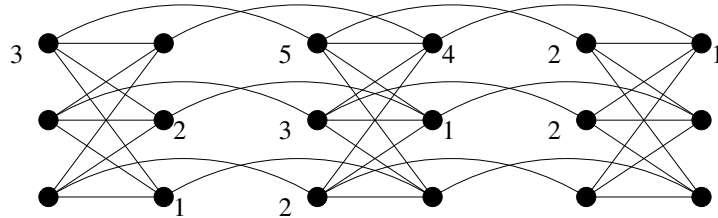


Figure 2: Partial greedy 5-colouring of $K_{1,2} \square K_{3,3}$

With similar arguments, one can improve a little bit the upper bound for h_G for some graphs. However, the upper bound is still exponential in k while we think h_G is linear.

Conjecture 27 For any graph G , there is a constant C_G such that $h_G(k) \leq C_G \times k$ for any k .

A very first step towards this conjecture would be to prove it for K_2 . Balogh et al. [1] showed that $h_{K_2}(G) \geq 2k$ because $\Gamma(K_2 \square K_k[S_2]) = 2\Gamma(K_k[S_2]) = 2k$ with S_2 the edgeless graph on two vertices. They also conjectured that $h_{K_2}(G) \geq 2k$.

Denoting by S_k be the edgeless graph on k vertices, we now generalise both their conjecture and their tightness examples.

Conjecture 28 Let k and n be two positive integers. Then $h_{K_n}(k) = n \times k$.

More generally, we conjecture the following:

Conjecture 29 For any graphs G and H , $\Gamma(G \square H) \leq (\Delta(H) + 1)\Gamma(G)$.

If true these two conjectures would be tight as shown by the following proposition.

Proposition 30 Let k and n be two positive integers. Then $\Gamma(K_n \square K_k[S_n]) = n \times k$.

Proof. $\Delta(K_n \square K_k[S_n]) = n \times k - 1$ so $\Gamma(K_n \square K_k[S_n]) \leq n \times k$.

We now prove by induction on k that $\Gamma(K_n \square K_k[S_n]) \geq n \times k$. The result holds trivially when $k = 1$. Suppose now that $k > 1$. Let us denote the vertices of $K_k[S_n]$ by v_j^i , $1 \leq i \leq k$, $1 \leq j \leq n$ so that for any i the set $\{v_j^i \mid 1 \leq j \leq n\}$ is stable and the vertices of K_n by $x_1, \dots, x_n$. Let $T_1 = \{(x_j, v_j^1) \mid 1 \leq j \leq n\}$. Then T_1 is a $V(G) \setminus T_1$ -dominating stable set. Indeed let (x_j, v_j^1) be a vertex in $V(G) \setminus T_1$. Then it is adjacent to (x_j, v_j^1) if $i \neq 1$ and to (x_i, v_i^1) if $i = 1$. More generally, for $1 \leq i \leq n$, the set $T_i = \{(x_j, v_{i+j-1}^1) \mid 1 \leq j \leq n\}$ is a $V(G) \setminus T_i$ -dominating stable set. Note that $(T_1, \dots, T_n)$ is a partition of $\{(x_j, v_i^1) \mid 1 \leq j \leq n, 1 \leq i \leq n\}$ and that $K_n \square K_k[S_n] - (\bigcup_{i=1}^n T_i)$ is isomorphic to $K_n \square K_{k-1}[S_n]$. Hence applying Proposition 5, to all the T_i one after another, we obtain $\Gamma(K_n \square K_k[S_n]) \geq n + \Gamma(K_n \square K_{k-1}[S_n])$. Now the induction hypothesis yields $\Gamma(K_n \square K_k[S_n]) \geq n \times k$. $\square$

Theorem 31 For any graph G , $h_G(k) \geq \Gamma(G) + 2k - 2$.

Proof. Set $p = \Gamma(G)$ and $n = p + 2k - 2$. We will prove that $\Gamma(K_k[S_n] \square G) \geq \Gamma(G) + 2k - 2$. Let us denote the vertices of $K_k[S_n]$ by v_j^i , $1 \leq i \leq k$, $1 \leq j \leq n$ so that for any i the set $\{v_j^i \mid 1 \leq j \leq n\}$. According to Lemma 8, there are two vertices x and y that receive colour $\Gamma(G)$ by some greedy colouring. Observe that in $K_k[S_n] \square G$ the $G_{v_j^1}$, $1 \leq j \leq n$ are disjoint copies of G . Hence, by Lemma 7, there is a greedy colouring c of $\bigcup_{i=1}^{2p} G_{v_j^1}$ such that for $1 \leq j \leq p$, $c((v_j^1, x)) = c((v_{p+j}^1, y)) = j$. Now setting for $1 \leq l \leq k - 1$, $c((v_{2p+2l-1}^{l+1}, x)) = c((v_{2p+2l}^{l+1}, y)) = p + 2l - 1$ and $c((v_{2p+2l}^{l+1}, x)) = c((v_{2p+2l-1}^{l+1}, y)) = p + 2l$, we obtain a partial greedy n -colouring of $K_k[S_n] \square G$. So $\Gamma(K_k[S_n] \square G) \geq n = \Gamma(G) + 2k - 2$. $\square$

4.2 Lower bounds

As G and H are induced subgraphs of $G \square H$ then $\Gamma(G \square H) \geq \max\{\Gamma(G), \Gamma(H)\}$.

Lemma 32 Let G and H be two graphs. If $\chi(H) \leq \Delta(G)$ then $\Gamma(G \square H) \geq \Gamma(H) + 1$.

Proof. W.l.o.g. we may assume that G and H have no isolated vertices. Let v be a vertex of G of degree $\Delta(G)$ and let $u_1, \dots, u_{\Delta(G)}$ its neighbours. Let $S_1, \dots, S_{\chi(H)}$ be the stable set of a colouring of H with $\chi(H)$ colours. The set $\bigcup_{i=1}^{\chi(H)} \{u_i\} \times S_i$ is a $V(H_v)$ -dominating stable set. So by Proposition 5, $\Gamma(G \square H) \geq \Gamma(H) + 1$. $\square$

Corollary 33 *Let G and H be two connected graphs such that $\Gamma(G) = \Gamma(H) = k$. Then $\Gamma(G \square H) \geq k + 1$ unless $G = H = K_1$ or $G = H = K_2$.*

Proof. If $\chi(H) \leq \Delta(G)$ or $\chi(G) \leq \Delta(H)$, we have the result by Lemma 32. So we may assume that $\chi(H) = \chi(G) = \Delta(G) + 1 = \Delta(H) + 1$. Hence by Brooks Theorem [3], G and H are complete graphs or odd cycles. If $G = H = K_k$, the result follows from Proposition 35 below. If G and H are odd cycles, then one of $P_3 \square K_2$ and $C_3 \square K_2$ is an induced subgraph of $G \square H$. These graphs have Grundy number 4; greedy 4-colourings are given Figure 3. So $\Gamma(G \square H) \geq 4$. $\square$

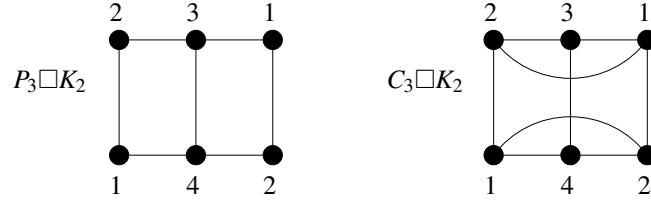


Figure 3: Greedy 4-colouring of $P_3 \square K_2$ and $C_3 \square K_2$

Lemma 32 yields a direct easy proof of a result of Hoffman and Johnson [9] stating that the k -dimensional hypercube Q_k has Grundy number $k + 1$ for $k \geq 3$ and $\Gamma(Q_1) = \Gamma(Q_2) = 2$.

Recall that $Q_1 = K_2$ and for $k \geq 2$ then $Q_k = Q_{k-1} \square K_2$.

Proposition 34 (Hoffman and Johnson [9]) *For $k \geq 3$, $\Gamma(Q_k) = k + 1$.*

Proof. As $\Delta(Q_k) = k$, we have $\Gamma(Q_k) \leq k + 1$.

Let us now prove the by induction that $\Gamma(Q_k) \geq k + 1$. If $k = 3$, a greedy 4-colouring is given in Figure 4. If $k > 3$, then $\chi(K_2) \leq \Delta(Q_{k-1}) = k$. Hence by Lemma 32, $\Gamma(Q_k) \geq \Gamma(Q_{k-1}) + 1 \geq k + 1$. $\square$

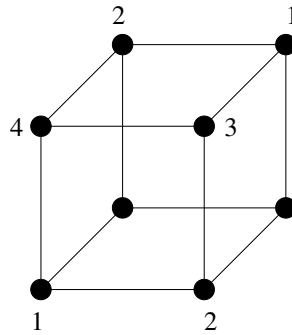


Figure 4: Partial greedy 4-colouring of the 3-dimensional hypercube Q_3

Proposition 35 For any $p \geq 2$ then $\Gamma(K_p \square K_p) = 2p - 2$.

Corollary 33 implies that $\varphi_{\square}(k, k) \geq k + 1$. To have better lower bound in $\varphi_{\square}(k, k)$, one may study the function $g(k) = \min\{\Gamma(G \square G) \mid \Gamma(G) = k\}$. Clearly, $g(2) = 2$ and by Corollary 33 and Proposition 35, if $k \geq 3$, we have

$$k \leq \varphi_{\square}(k, k) \leq g(k) \leq 2k - 2.$$

Moreover every graph with Grundy number 3 has either a K_3 or a P_4 as induced subgraph. But $\Gamma(K_3 \square K_3) = 4$ by Proposition 35 and $\Gamma(P_4 \square P_4) = 5$ (As $\Delta(P_4 \square P_4) \leq 4$ then $\Gamma(P_4 \square P_4) \leq 5$ and it is easy to find a greedy 5-colouring of $P_4 \square P_4$). Hence $g(3) = 4$.

5 Direct product

A well known conjecture on graph colouring regards the chromatic number of the direct product of graphs.

Conjecture 36 (Hedetniemi [7]) $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

Poljak [12] proved that the function f defined by $f(n) = \min\{\chi(G \times H) \mid \chi(G) = \chi(H) = n\}$ is either bounded by 9 or tends to infinity when n tends to infinity.

In this section, our aim is to find upper bounds of the Grundy number of the direct product of two graphs in terms of the Grundy number of these graphs. Ideally, we would like to determine the functions

$$\begin{aligned} \varphi_{\times}(k, l) &= \min\{\Gamma(G \times H) \mid \Gamma(G) = k \text{ and } \Gamma(H) = l\} \\ \Phi_{\times}(k, l) &= \max\{\Gamma(G \times H) \mid \Gamma(G) = k \text{ and } \Gamma(H) = l\} \end{aligned}$$

Let us first observe that if $G = G_1 + G_2$ then $G \times H = (G_1 \times H) + (G_2 \times H)$. Hence it is sufficient to consider connected graphs. Furthermore, the direct product of a graph with K_1 is a graph without any edge of order $|G|$. So $\varphi_{\times}(k, 1) = \Phi_{\times}(k, 1) = 1$. In the remaining of this section, **all the graphs are assumed to be connected of order at least 2**. In particular, their Grundy number is at least two.

5.0.1 Lower bounds

As every graph with Grundy number k contains a k -atom as an induced subgraph then $\varphi_{\times}(k, l) = \min\{\Gamma(G \times H) \mid G \text{ is a } k\text{-atom and } H \text{ is an } l\text{-atom}\}$. Furthermore if $k \geq k'$ and $l \geq l'$ then $\varphi_{\times}(k, l) \geq \varphi_{\times}(k', l')$.

Theorem 37 Let G and H be two graphs with at least one edge. Then $\Gamma(G \times H) \geq \Gamma(G) + \Gamma(H) - 2$. Hence if $k \geq 2$ and $l \geq 2$ then $\varphi_{\times}(k, l) \geq k + l - 2$.

Proof. Let $k = \Gamma(G)$ and $l = \Gamma(H)$. We prove the result by induction on $k + l$, the result holding trivially if $k = l = 2$.

Suppose now that $k + l > 4$. Without loss of generality, we may assume that $k \geq l$. Let $S_1, \dots, S_k$ be the stable sets of a greedy p -colouring of G . Set $G' = G - S_1$. Then S_1 is a $(V(G'))$ -dominating stable set and $\Gamma(G') = k - 1$. Now, in $G \times H$, the set $S_1 \times V(H)$ is $V(G' \times H)$ -dominating. Hence, by Proposition 5, $\Gamma(G \times H) \geq \Gamma(G' \times H)$. Now, since $\Gamma(G') + \Gamma(H) = k + l - 1$, by induction hypothesis, $\Gamma(G' \times H) \geq k + l - 3$. So $\Gamma(G \times H) \geq k + l - 2$. $\square$

This lower bound for $\varphi_{k,l}$ is attained when $l = 2$ or $k = l = 3$.

Corollary 38 (i) For any integer $k \geq 2$, $\Phi_{\times}(k, 2) = k$.

(ii) $\Phi_{\times}(3, 3) = 4$.

Proof. (i) The maximum degree of $K_k \times K_2$ is $k - 1$, so $\Gamma(K_k \times K_2) \leq k$. So $\Phi_{\times}(k, 2) \leq k$. But Theorem 37 yields $\Phi_{\times}(k, 2) \geq k$.

(ii) One can easily check that $\Gamma(P_4 \times P_4) = \Gamma(P_4 \times C_3) = \Gamma(C_3 \times C_3) = 4$. $\square$

There are pairs of graphs (G, H) for which $\Gamma(G \times H) > \Gamma(G) + \Gamma(H) - 2$. Consider for example the *jellyfish* J depicted in Figure 5. It is simple matter to check that $\Gamma(J) = 3$ and $\Gamma(J \times K_2) = 4$. A greedy 4-colouring of $J \times K_2$ is given in Figure 5.

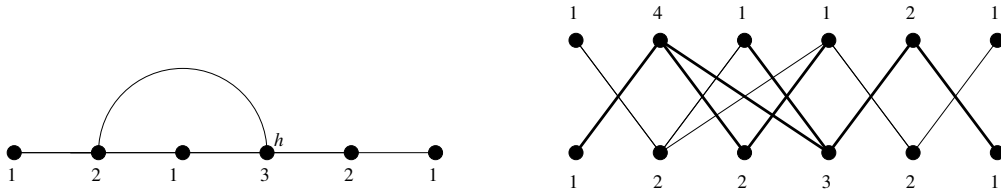


Figure 5: The jellyfish J and a greedy 4-colouring of $J \times K_2$

An interesting question would be to know what are the pairs of graphs (G, H) for which $\Gamma(G \times H) = \Gamma(G) + \Gamma(H) - 2$.

Problem 39 Given a pair of graphs (G, H) , is it polynomial to decide if $\Gamma(G \times H) = \Gamma(G) + \Gamma(H) - 2$?

5.1 Upper bounds

Lemma 40 Let G and H be two graphs, and u and v be two vertices of G . If $N_G(u) = N_G(v)$ then $\Gamma(G \times H) = \Gamma((G - u) \times H)$.

Proposition 3 and Lemma 40 directly imply:

Corollary 41 Let H be a graph such that $\Gamma(H) = 2$. Then for every G , $\Gamma(G \times H) = \Gamma(G \times K_2)$. In particular, $\Phi(k, 2) = \max\{\Gamma(G \times K_2) \mid \Gamma(G) = k\}$.

If G is bipartite then $G \times K_2 = G + G$. So, by Proposition 6, $\Gamma(G \times K_2) = \Gamma(G)$. Then Proposition 3 yields $\Phi_{\times}(2, 2) = 2$. There are non-bipartite graphs G for which $\Gamma(G \times K_2) = \Gamma(G)$. For example, $K_3 \times K_2$ is the 6-cycle so $\Gamma(K_3 \times K_2) = 3 = \Gamma(K_3)$. There are also graphs G for which $\Gamma(G \times K_2) \neq \Gamma(G)$: the jellyfish for example.

Definition 42 The *head* of the jellyfish J is the vertex h on Figure 5. Let G be a graph. Then the *jellyfished* of G is the graph $J(G)$ obtained from G by creating for each vertex $v \in V(G)$ a jellyfish $J(v)$ whose head is identified with v . See Figure 6.

Proposition 43 Let G be a graph. Then $\Gamma(J(G)) = \Gamma(G) + 2$.

Proof. Let c be a greedy colouring of $J(G)$ with $\Gamma(J(G))$ colours. Let u be a vertex such that $c(u) \geq 4$.

We claim that u is in $V(G)$. Suppose not. Let v be the vertex of G such that $u \in J(v)$. Then u must be the vertex of $J(v)$ of degree 3 adjacent to v . Since u has a neighbour of each colour smaller

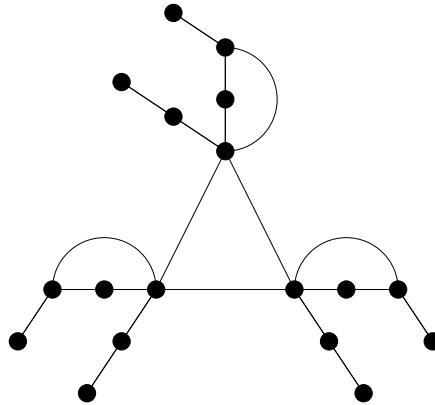


Figure 6: The jellyfished of K_3

than 4 the vertex v must be assigned 3. But then the two others neighbours of u are coloured 1, a contradiction.

Now as $\Gamma(J - h) = 2$ the neighbours of u coloured 3 are in G . Therefore the colouring c' defined on $S = \{v \in V(G) \mid c(v) \geq 3\}$ by $c'(v) = c(v) - 2$ is a greedy colouring of $G[S]$ with $\Gamma(J(G)) - 2$ colours. So $\Gamma(J(G)) \leq \Gamma(G) + 2$. $\square$

Lemma 44 *If $G \times K_2$ contains an induced binomial tree of index k then there is a graph H such that $H \times K_2$ contains an induced binomial tree of index $k + 3$ and $\Gamma(H) = \Gamma(G) + 2$.*

Proof. Let T be an induced T_k of $G \times K_2$. Let $S = \{v \in V(G) \mid \{(v, 1), (v, 2)\} \subset V(T_k)\}$. Let G' be the graph obtained from G by blowing up each vertex v of S with a stable set of size two $\{v_1, v_2\}$. By Proposition 2 (iii), $\Gamma(G') = \Gamma(G)$.

Set $H = J(G')$. By Proposition 43, $\Gamma(H) = \Gamma(G') + 2 = \Gamma(G) + 2$.

Let us now show that $H \times K_2$ contains a T_{k+3} . By construction, the subgraph of $G' \times K_2$ induced by $(V(T_k) \setminus (S \times K_2)) \cup \bigcup_{v \in S} \{(v_1, 1), (v_2, 2)\}$ is a T_k . Note that for every vertex $v \in V(G')$ at most one of $\{(v, 1), (v, 2)\}$ is in $V(T')$. Now every vertex of T' is the root of a T_3 in its associated $J \times K_2$. All these T_3 together with T' form an induced T_{k+3} of $H \times K_2$. $\square$

Corollary 45 $\Phi_{\times}(2k + 1, 2) \geq 3k + 1$ and $\Phi_{\times}(2k, 2) \geq 3k - 1$.

Note that Corollary 41 may not be generalised to graphs H such that $\Gamma(H) = 3$. Indeed $4 = \Gamma(G_3 \times K_2) \neq \Gamma(K_3 \times K_2) = 3$.

Problem 46 Let G and H be two graphs such that $\Gamma(H) = k$. Is it true that $\Gamma(G \times H) \geq \Gamma(G \times K_k)$?

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