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Control-Flow Analysis of Function Calls and Returns by Abstract Interpretation

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Abstract: We derive a control-flow analysis that approximates the interprocedural control-flow of both function calls and returns in the presence of first-class functions and tail-call optimization. In addition to an abstract environment, our analysis computes for each expression an abstract control stack, effectively approximating where function calls return across optimized tail calls. The analysis is systematically calculated by abstract interpretation of the stack-based C_aEK abstract machine of Flanagan et al. using a series of Galois connections. Abstract interpretation provides a unifying setting in which we *1)* prove the analysis equivalent to the composition of a continuation-passing style (CPS) transformation followed by an abstract interpretation of a stack-less CPS machine, and *2)* extract an equivalent constraint-based analysis formulation, thereby providing a rational reconstruction of a constraint-based control-flow analysis from abstract interpretation principles.

Key-words: control-flow analysis, abstract interpretation, tail-call optimization, continuation-passing style, direct style, constraint-based analysis

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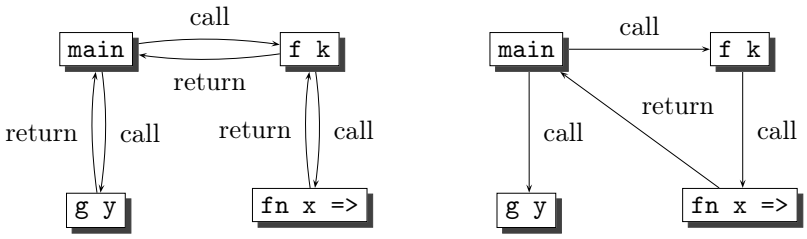
Résumé : –

Mots-clés : –

```
let g z = z in
 let f k = if b then k 1 else k 2 in
  let y = f (fn x => x) in
     g y
```

(a) Example program



(b) Call-return call graph (c) Optimized call graph

Figure 1: The corresponding call graphs

1 Introduction

The control flow of a functional program is expressed in terms of function calls and returns. As a result, iteration in functional programs is expressed using recursive functions. In order for this approach to be feasible, language implementations perform *tail-call optimization* of function calls [Clinger, 1998], by not pushing a stack frame on the control stack at call-sites in *tail position*. Consequently functions do not necessarily return control to their caller. Control-flow analysis (CFA) has long been a staple of program optimization and verification. Surprisingly, research on control-flow analysis has focused on calls: A textbook CFA *"will determine where the flow of control may be transferred to in the case [...] of a function application."* [Nielson et al., 1999]. Our systematic approximation of a known operational semantics leads to a CFA that *"will determine where the flow of control may be transferred to in the case of a function return."* The resulting analysis thereby approximates both call and return information for a higher-order, direct-style language. Interestingly it does so by approximating the control-stack.

Consider the example program in Fig. 1(a). The program contains three functions: two named function `g` and `f` and an anonymous function `fn x => x`. A standard direct-style CFA can determine that the applications of `k` in each branch of the conditional will call the anonymous function `fn x => x` at run time. Building a call-graph based on this output gives rise to Fig. 1(b), where we have named the main expression of the program `main`. In addition to the above resolved call, our analysis will determine that the anonymous function returns to the let-binding of `y` in `main` upon completion, rather than to its caller. The analysis hence gives rise to the call graph in Fig. 1(c).

On a methodological level, we derive the analysis systematically by Cousot-Cousot-style *abstract interpretation*. The analysis approximates the reachable states of an existing abstract machine from the literature: the C_aEK machine of Flanagan et al. [1993]. We obtain the analysis as the result of composing the collecting semantics induced by the abstract machine with a series of Galois connections that each specifies one aspect of the abstraction in the analysis.

We show how the abstract interpretation formulation lends itself to a lock-step equivalence proof between our analysis and a previously derived CPS-based CFA. More precisely, we define a relation between the abstract domains of the analyses that is a simulation between the two, reducing the proof to a fixpoint induction over the abstract interpretations.

To sum up, the main contributions of this article are:

- An abstract interpretation-derivation of a CFA for a higher-order functional language from a well-known operational semantics,
- a resulting CFA with *reachability* which computes both call *and return* control-flow,
- a proof of equivalence of the analysis of programs in direct style and the CPS analysis of their CPS counterparts,
- an equivalent constraint-based analysis extracted from the above.

1.1 Related work

We separate the discussion of related analyses in two: direct-style analyses and analyses based on CPS.

Direct-style CFA has a long research history. Jones [1981] initially developed methods for approximating the control flow of lambda terms. Since then Sestoft [1989] conceived the related *closure analysis*. Palsberg [1995] simplified the analysis and formulated an equivalent constraint-based analysis. At the same time Heintze [1994] developed a related set-based analysis formulated in terms of set constraints. For a detailed account of related work, we refer to a recent survey of the area [Midtgaard, 2007]. It is worth emphasizing that all of the above analyses focus on calls, in that they approximate the source lambdas being called at each call-site. As such they do not directly determine return flow for programs in direct style.

CPS-based CFA was pioneered by Shivers [1988] who formulated control-flow analysis for Scheme. Since then a number of analyses have been formulated for CPS [Ashley and Dybvig, 1998, Might and Shivers, 2006]. In CPS all calls are tail calls, and even returns are encoded as calls to the current continuation. By determining "call flow" and hence the receiver functions of such continuation calls, a CPS-based CFA thereby determines return flow without additional effort.

The impact of CPS transformation on static analyses originates in binding-time analysis, for which the transformation is known to have a positive effect [Consel and Danvy, 1991, Damian and Danvy, 2003]. As to the impact of CPS transformation on CFA we separate the previous work on the subject in two:

1. results relating an analysis *specialized* to the source language to an analysis *specialized* to the target language (CPS), and
2. results relating the analysis of a program to the *same analysis* of the CPS transformed program.

Sabry and Felleisen [1994] designed and compared specialized analyses and hence falls into the first category as does the present paper. Damian and Danvy [2003] related the analysis of a program and its CPS counterpart for a standard flow-logic CFA (as well as for two binding-time analyses), and Palsberg and Wand [2003] related the analysis of a program and its CPS counterpart for a standard conditional constraint CFA. Hence the latter two fall into the second category.

We paraphrase the relevant theorems of Sabry and Felleisen [1994], of Damian and Danvy [2003], of Palsberg and Wand [2003], and of the present paper in order to underline the difference between the contributions (C refers to non-trivial, 0-CFA-like analyses defined in the cited papers, p ranges over direct-style programs, *cps* denotes CPS transformation, and $\sim$ denotes analysis equivalence). Our formulations should not be read as a formal system, but only as a means for elucidating the difference between the contributions.

Sabry and Felleisen [1994]:

$$\text{exists analyses } C_1,\, C_2 : \text{exists } p,\, C_1(p) \nsim C_2(cps(p))$$

Damian and Danvy [2003], Palsberg and Wand [2003]:

$$\text{exists analysis } C : \text{for all } p,\, C(p) \sim C(cps(p))$$

Present paper, Theorem 6.1:

$$\text{exists analyses } C_1,\, C_2 : \text{for all } p,\, C_1(p) \sim C_2(cps(p))$$

Our work relates to all of the above contributions. The disciplined derivation of specialized CPS and direct-style analyses results in comparable analyses, contrary to Sabry and Felleisen [1994]. Furthermore our equivalence proof extends the results of Damian and Danvy [2003] and Palsberg and Wand [2003] in that we relate both call flow, *return flow*, and *reachability*, contrary to their relating only the call flow of standard CFAs. In addition, the systematic abstract interpretation-based approach suggests a strategy for obtaining similar equivalence results for other CFAs derived in this fashion.

Formulating CFA in the traditional abstract interpretation framework was stated as an open problem by Nielson and Nielson [1997]. It has been a recurring theme in the work of the present authors. In an earlier paper Spoto and Jensen [2003] investigated class analysis of object-oriented programs as a Galois connection-based abstraction of a trace semantics. In a recent article [Midtgaard and Jensen, 2008], the authors systematically derived a CPS-based CFA from the collecting semantics of a stack-less machine. While investigating how to derive a corresponding direct-style analysis we discovered the mismatch between the computed return information.

As tail calls are identified syntactically, the additional information could also have been obtained by a subsequent analysis after a traditional direct-style CFA. However we view the need for such a subsequent analysis as a strong indication of a mismatch between the two analysis formulations. Debray and Proebsting [1997] have investigated such a *"return analysis"* for a first-order language with tail-call optimization. The present paper builds a semantics-based CFA that determines such information, and for a higher-order language.

The systematic design of constraint-based analyses is a goal shared with the *flow logic* framework of Nielson and Nielson [2002]. In flow logic an

$$
\begin{array}{lll}
P \ni \mathtt{p} ::= \mathtt{s} & & \text{(programs)} \\
T \ni \mathtt{t} ::= c \mid \mathtt{x} \mid \mathtt{fn\ x\ =>\ s} & & \text{(trivial expressions)} \\
C \ni \mathtt{s} ::= \mathtt{t} & & \text{(serious expressions)} \\
\qquad\quad \mid \mathtt{let\ x{=}t\ in\ s} & & \\
\qquad\quad \mid \mathtt{t_0\ t_1} & & \\
\qquad\quad \mid \mathtt{let\ x{=}t_0\ t_1\ in\ s} & &
\end{array}
$$

Figure 2: ANF grammar

analysis specification can be systematically transformed into a constraint-based analysis. The present paper instead extracts a constraint-based analysis from an analysis developed in the original abstract interpretation framework.

The idea of CFA by control-stack approximation, applies equally well to imperative or object-oriented programs, but it is beyond the scope of this paper to argue this point. The rest of this article is organized as follows. In Section 2 we present the syntax and semantics of the language. In Section 3 we briefly recall basic principles of abstract interpretation. In Section 4 we formulate the collecting semantics of the analysis, which we systematically approximate into an analysis in Section 4 and Section 5. Section 6 relates the derived analysis to an earlier derived CPS-based analysis. In Section 7 we extract an equivalent constraint-based formulation. Section 8 explores applications of the analysis before we conclude.

2 Language and semantics

Our source language is a simple call-by-value core language known as *administrative normal form* (ANF). The grammar of ANF terms is given in Fig. 2. Following Reynolds [1998], the grammar distinguishes *serious* expressions, i.e., terms whose evaluation may diverge, from *trivial* expressions, i.e., terms without risk of divergence. Trivial expressions include constants, variables, and functions, and serious expressions include returns, let-bindings, tail calls, and non-tail calls. Programs are serious expressions.

The analysis is calculated from a simple operational semantics in the form of an abstract machine. We use the environment-based C_aEK abstract machine of Flanagan et al. [1993] given in Fig. 3 in which functional values are represented using *closures* [Landin, 1964], i.e., pairs of a lambda-expression and an environment. The environment-component captures the (values of the) free variables of the lambda. Machine states are triples consisting of a serious expression, an environment and a control stack. The control stack is composed of elements ("stack frames") of the form $[\mathtt{x},\ \mathtt{s},\ e]$ where $\mathtt{x}$ is the variable receiving the return value w of the current function call, and $\mathtt{s}$ is a serious expression whose evaluation in the environment $e[\mathtt{x} \mapsto w]$ represents the rest of the computation in that stack frame. The empty stack is represented by `stop`. The machine has a helper function μ for evaluation of trivial expressions. The machine is initialized

$$\mu : T \times Env \rightharpoonup Val$$

$$\begin{array}{rcl} Val \ni w & ::= & c \mid [\texttt{fn x => s},\ e] \\ Env \ni e & ::= & \bullet \mid e[\texttt{x} \mapsto w] \\ K \ni k & ::= & \texttt{stop} \mid [\texttt{x},\ \texttt{s},\ e] :: k \end{array}$$

(a) Values, environments, and stacks

$$\begin{array}{rcl} \mu(c, e) & = & c \\ \mu(\texttt{x}, e) & = & e(\texttt{x}) \\ \mu(\texttt{fn x => s}, e) & = & [\texttt{fn x => s},\ e] \end{array}$$

(b) Helper function

$$\begin{array}{rcl} \langle \texttt{t},\ e,\ [\texttt{x},\ \texttt{s}',\ e'] :: k' \rangle & \longrightarrow & \langle \texttt{s}',\ e'[\texttt{x} \mapsto \mu(\texttt{t}, e)],\ k' \rangle \\ \langle \texttt{let x=t in s},\ e,\ k \rangle & \longrightarrow & \langle \texttt{s},\ e[\texttt{x} \mapsto \mu(\texttt{t}, e)],\ k \rangle \\ \langle \texttt{t}_0\ \texttt{t}_1,\ e,\ k \rangle & \longrightarrow & \langle \texttt{s}',\ e'[\texttt{x} \mapsto w],\ k \rangle \\ & & \text{if } [\texttt{fn x => s}',\ e'] = \mu(\texttt{t}_0, e) \text{ and } w = \mu(\texttt{t}_1, e) \\ \langle \texttt{let x=t}_0\ \texttt{t}_1\ \texttt{in s},\ e,\ k \rangle & \longrightarrow & \langle \texttt{s}',\ e'[\texttt{y} \mapsto w],\ [\texttt{x},\ \texttt{s},\ e] :: k \rangle \\ & & \text{if } [\texttt{fn y => s}',\ e'] = \mu(\texttt{t}_0, e) \text{ and } w = \mu(\texttt{t}_1, e) \end{array}$$

(c) Machine transitions

$$eval(\texttt{p}) = w \ \text{ iff } \ \langle \texttt{p},\ \bullet,\ [\texttt{x}_\texttt{r},\ \texttt{x}_\texttt{r},\ \bullet] :: \texttt{stop} \rangle \longrightarrow^* \langle \texttt{x}_\texttt{r},\ \bullet[\texttt{x}_\texttt{r} \mapsto w],\ \texttt{stop} \rangle$$

(d) Machine evaluation

Figure 3: The C_aEK abstract machine

with the input program, with an empty environment, and with an initial stack, that will bind the result of the program to a special variable $\texttt{x}_\texttt{r}$ before halting. Evaluation follows by repeated application of the machine transitions.

3 Abstract interpretation basics

We assume some familiarity with the basic mathematical facts recalled in Appendix A. Canonical abstract interpretation approximates the *collecting semantics* of a transition system [Cousot, 1981]. A standard example of a collecting semantics is the *reachable states* from a given set of initial states I. Given a transition function T defined as:

$$T(\Sigma) = I \cup \{\sigma \mid \exists \sigma' \in \Sigma : \sigma' \to \sigma\}$$

we can compute the reachable states of T as the least fixed-point $\mathrm{lfp}\, T$ of T. The collecting semantics is ideal, in that it is the most precise analysis. Unfortunately it is in general uncomputable. Abstract interpretation therefore approximates the collecting semantics, by instead computing a fixed-point over an alternative and perhaps simpler domain. For this reason, abstract interpretation is also referred to as a theory of fixed-point approximation.

Abstractions are formally represented as Galois connections which connect complete lattices through a pair of adjoint functions α and γ (see Appendix A). Galois connection-based abstract interpretation suggests that one may derive an analysis systematically by composing the transition function with these adjoints:

$\alpha \circ T \circ \gamma$. In this setting Galois connections allow us to gradually refine the collecting semantics into a computable analysis function by mere calculation. An alternative "recipe" consists in rewriting the composition of the abstraction function and transition function $\alpha \circ T$ into something on the form $T^\sharp \circ \alpha$, from which the analysis function $T^\sharp$ can be read off [Cousot and Cousot, 1992a]. Cousot [1999] has shown how to systematically construct a static analyser for a first-order imperative language using calculational abstract interpretation.

Rather than insisting on simplifying the abstract domains into finite ones, an alternative *widening* technique permits infinite ones, while still ensuring termination. Abstract interpretation with widening [Cousot and Cousot, 1977] can be formulated as computing the limit of the sequence:

$$\begin{aligned} X_0 &= \bot \\ X_{i+1} &= X_i \triangledown T(X_i) \end{aligned}$$

where $\triangledown$ denotes the *widening operator*: an operator not decreasing in its second argument, which must not give rise to an infinite, strictly increasing sequence: $X_0 \sqsubset X_1 \sqsubset \ldots$.

4 Approximating the C_aEK collecting semantics

As our collecting semantics we consider the reachable states of the C_aEK machine, expressed as the least fixed point lfp F of the following transition function.

$$\begin{aligned} F &: \wp(C \times Env \times K) \to \wp(C \times Env \times K) \\ F(S) &= I_{\mathtt{p}} \cup \{s \mid \exists s' \in S : s' \longrightarrow s\} \\ &\quad \text{where } \; I_{\mathtt{p}} = \{\langle \mathtt{p}, \bullet, [\mathtt{x_r}, \mathtt{x_r}, \bullet] :: \mathtt{stop} \rangle\} \end{aligned}$$

First we formulate in Fig. 4(a) an equivalent helper function μ_c extended to work on sets of environments.

Lemma 4.1. $\forall \mathtt{t}, e : \{\mu(\mathtt{t}, e)\} = \mu_c(\mathtt{t}, \{e\})$

The equivalence of the two helper functions follow straight forwardly. This lemma enables us to express an equivalent collecting semantics based on μ_c, which appears in Fig. 4.

Lemma 4.2. $\forall S : F(S) = F^c(S)$

Proof. By above lemma and unfolding the definitions. □

The abstraction of the collecting semantics is staged in several steps. Figure 5 provides an overview. Intuitively, the analysis extracts three pieces of information from the set of reachable states.

1. An approximation of the set of reachable expressions.
2. A relation between expressions and control stacks that represents where the values of expressions are returned to.
3. An abstract environment mapping variables to the expressions that may be bound to that variable. This is standard in CFA and allows to determine which functions are called at a given call site.

$$\mu_c : T \times \wp(Env) \to \wp(Val)$$
$$\mu_c(c, E) = \{c\}$$
$$\mu_c(\mathtt{x}, E) = \{w \mid \exists e \in E : w = e(\mathtt{x})\}$$
$$\mu_c(\mathtt{fn\ x\ =>\ s}, E) = \{[\mathtt{fn\ x\ =>\ s},\ e] \mid \exists e \in E\}$$

(a) Helper function

$$F^c : \wp(C \times Env \times K) \to \wp(C \times Env \times K)$$
$$\begin{aligned} F^c(S) = I_{\mathtt{p}} & \\ & \cup \bigcup_{\substack{\langle \mathtt{t},\, e,\, [\mathtt{x},\, \mathtt{s}',\, e'] :: k' \rangle \in S \\ w \in \mu_c(\mathtt{t}, \{e\})}} \{\langle \mathtt{s}',\ e'[\mathtt{x} \mapsto w],\ k' \rangle\} \\ & \cup \bigcup_{\substack{\langle \mathtt{let\ x=t\ in\ s},\, e,\, k \rangle \in S \\ w \in \mu_c(\mathtt{t}, \{e\})}} \{\langle \mathtt{s},\ e[\mathtt{x} \mapsto w],\ k \rangle\} \\ & \cup \bigcup_{\substack{\langle \mathtt{t_0\ t_1},\, e,\, k \rangle \in S \\ [\mathtt{fn\ x\ =>\ s'},\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}} \{\langle \mathtt{s}',\ e'[\mathtt{x} \mapsto w],\ k \rangle\} \\ & \cup \bigcup_{\substack{\langle \mathtt{let\ x=t_0\ t_1\ in\ s},\, e,\, k \rangle \in S \\ [\mathtt{fn\ y\ =>\ s'},\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}} \{\langle \mathtt{s}',\ e'[\mathtt{y} \mapsto w],\ [\mathtt{x},\ \mathtt{s},\ e] :: k \rangle\} \end{aligned}$$

(b) Transition function

Figure 4: Collecting semantics

$$
\begin{array}{ccc}
\wp(C \times Env \times K) & \text{collecting semantics} & F^c \\
\alpha_\times \downarrow \uparrow \gamma_\times & & \\
\wp(C) \times \wp(C \times K) \times \wp(Env) & \text{-} & F^\times \\
\rho \downarrow \uparrow 1 & & \\
\rho(\wp(C) \times \wp(C \times K) \times \wp(Env)) & \text{-} & F^\rho \\
\alpha_\otimes \downarrow \uparrow \gamma_\otimes & & \\
\wp(C) \times (C/_\equiv \to \wp(K^\sharp)) \times Env^\sharp & \text{0-CFA} & F^\sharp
\end{array}
$$

Figure 5: Overview of abstraction

Keeping an explicit set of reachable expressions is more precise than leaving it out, once we further approximate the expression-stack pairs. Alternatively the reachable expressions would be approximated by the expressions present in the expression-stack relation. However expressions may be in the expression-stack relation without ever being reached. An example hereof would be a diverging non-tail call.

To formalize this intuition, we first perform a Cartesian abstraction of the machine states, however keeping the relation between expressions and their corresponding control stacks. The second step in the approximation consists in closing the triples by a closure operator, to ensure that (a) any saved environment on the stack or nested within another environment is itself part of the environment set, and (b) that all expression-control stack pairs that appear further down in a control stack are also contained in the expression-stack relation. We explain this in more detail below (Section 4.2). Finally as a third step we approximate stacks by their top element, we merge expressions with the same return point into equivalence classes, and we approximate closure values by their lambda expression.

In the following sections we provide a detailed explanation of each abstraction in turn. In order to illustrate the systematic calculation and still remain of a manageable size, we only provide the calculations for the return case `t`. Since we calculate with Galois connections on complete lattices, the abstraction functions are complete join morphisms (CJMs), and hence distribute over each element of a join, permitting us to do such case division. The remaining cases are proved similarly.

4.1 Projecting machine states

The mapping that extracts the three kinds of information described above is defined formally as follows.

$$\wp(C \times Env \times K) \underset{\alpha_\times}{\overset{\gamma_\times}{\leftrightarrows}} \wp(C) \times \wp(C \times K) \times \wp(Env)$$

$$\alpha_\times(S) = \langle \pi_1 S, \{\langle \mathtt{s},\, k\rangle \mid \exists e : \langle \mathtt{s},\, e,\, k\rangle \in S\}, \pi_2 S\rangle$$

$$\gamma_\times(\langle C,\, F,\, E\rangle) = \{\langle \mathtt{s},\, e,\, k\rangle \mid \mathtt{s} \in C \ \wedge\ \langle \mathtt{s},\, k\rangle \in F \ \wedge\ e \in E\}$$

Lemma 4.3. $\alpha_\times$, $\gamma_\times$ is a Galois connection.

The above Galois connection and the proof hereof closely resembles the independent attributes abstraction, which is a known Galois connection. We use the notation $\cup_\times$ and $\subseteq_\times$ for the componentwise join and componentwise inclusion of triples.

As traditional [Cousot and Cousot, 1979, 1992a, 1994], we will assume that the abstract product domains throughout this article have been *reduced*, i.e., all triples $\langle A,\, B,\, C\rangle$ representing the empty set $(\gamma_a(A) = \emptyset \vee \gamma_b(B) = \emptyset \vee \gamma_c(C) = \emptyset)$ have been eliminated and replaced by a single bottom element $\langle \bot_a,\, \bot_b,\, \bot_c\rangle$.

Based on the partly-relational abstraction we now calculate a new transfer function. Let $\langle C,\, F,\, E\rangle \in \wp(C) \times \wp(C \times K) \times \wp(Env)$ be given.

$$
\begin{aligned}
&\alpha_\times\Big(\bigcup_{\substack{\langle \mathtt{t},\, e,\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle \in \gamma_\times(\langle C,\, F,\, E\rangle)\\ w \in \mu_c(\mathtt{t},\{e\})}} \{\langle \mathtt{s}',\, e'[\mathtt{x} \mapsto w],\, k'\rangle\}\Big) \\
&= \bigcup_{\substack{\langle \mathtt{t},\, e,\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle \in \gamma_\times(\langle C,\, F,\, E\rangle)\\ w \in \mu_c(\mathtt{t},\{e\})}}^{\times} \alpha_\times(\{\langle \mathtt{s}',\, e'[\mathtt{x} \mapsto w],\, k'\rangle\}) && (\alpha_\times \text{ a CJM}) \\
&= \bigcup_{\substack{\langle \mathtt{t},\, e,\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle \in \gamma_\times(\langle C,\, F,\, E\rangle)\\ w \in \mu_c(\mathtt{t},\{e\})}}^{\times} \langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle && (\text{def. } \alpha_\times) \\
&= \bigcup_{\substack{\alpha_\times(\{\langle \mathtt{t},\, e,\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle\}) \subseteq_\times \langle C,\, F,\, E\rangle\\ w \in \mu_c(\mathtt{t},\{e\})}}^{\times} \langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle && (\text{Galois conn.}) \\
&= \bigcup_{\substack{\langle \{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle\\ w \in \mu_c(\mathtt{t},\{e\})}}^{\times} \langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle && (\text{def. } \alpha_\times)
\end{aligned}
$$

The resulting transition function appears in Fig. 6. By construction, the transition function satisfies the following theorem.

Theorem 4.1.

$$\forall C, F, E : \alpha_\times(F^c(\gamma_\times(\langle C,\, F,\, E\rangle))) = F^\times(\langle C,\, F,\, E\rangle)$$

4.2 A closure operator on machine states

For the final analysis, we are only interested in an abstraction of the information present in an expression-stack pair. More precisely, we aim at only keeping track of the link between an expression and the top stack frame in effect during its evaluation, throwing away everything below. However, we need to make this information explicit for all expressions appearing on the control stack, *i.e.*, for a pair $\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle$ we also want to retain that $\mathtt{s}'$ will be evaluated with control stack k. Similarly, environments can be stored on the stack or inside other environments and will have to be extracted. We achieve this by defining a suitable *closure operator* on these nested structures.

$$F^{\times} : \wp(C) \times \wp(C \times K) \times \wp(Env) \to \wp(C) \times \wp(C \times K) \times \wp(Env)$$

$$\begin{aligned}
F^{\times}(\langle C,\, F,\, E\rangle) = \langle \{\mathtt{p}\},\, \{\langle \mathtt{p},\, [\mathtt{x_r},\, \mathtt{x_r},\, \bullet] :: \mathtt{stop}\rangle\},\, \{\bullet\}\rangle \\
\cup_{\times} \bigcup_{\substack{\langle \{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e'] :: k'\rangle\},\, \{e\}\rangle \subseteq_{\times} \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}{}_{\times} \langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle \\
\cup_{\times} \bigcup_{\substack{\langle \{\mathtt{let\ x{=}t\ in\ s}\},\, \{\langle \mathtt{let\ x{=}t\ in\ s},\, k\rangle\},\, \{e\}\rangle \subseteq_{\times} \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}{}_{\times} \langle \{\mathtt{s}\},\, \{\langle \mathtt{s},\, k\rangle\},\, \{e[\mathtt{x} \mapsto w]\}\rangle \\
\cup_{\times} \bigcup_{\substack{\langle \{\mathtt{t_0\ t_1}\},\, \{\langle \mathtt{t_0\ t_1},\, k\rangle\},\, \{e\}\rangle \subseteq_{\times} \langle C,\, F,\, E\rangle \\ [\mathtt{fn\ x\ =>\ s}',\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}}{}_{\times} \langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle \\
\cup_{\times} \bigcup_{\substack{\langle \{\mathtt{let\ x{=}t_0\ t_1\ in\ s}\},\, \{\langle \mathtt{let\ x{=}t_0\ t_1\ in\ s},\, k\rangle\},\, \{e\}\rangle \subseteq_{\times} \langle C,\, F,\, E\rangle \\ [\mathtt{fn\ y\ =>\ s}',\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}}{}_{\times} \langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, [\mathtt{x},\, \mathtt{s},\, e] :: k\rangle\},\, \{e'[\mathtt{y} \mapsto w]\}\rangle
\end{aligned}$$

Figure 6: Abstract transition function

Milner and Tofte's constituent relation: For environments, we adapt the definition of a constituent relation due to Milner and Tofte [1991]. We say that each component x_i of a tuple $\langle x_0, \ldots, x_n\rangle$ is a *constituent* of the tuple, written $\langle x_0, \ldots, x_n\rangle \succ x_i$. For a partial function[1] $f = [x_0 \mapsto w_0, \ldots, x_n \mapsto w_n]$, we say that each w_i is a constituent of the function, written $f \succ w_i$. We write $\succ^*$ for the reflexive, transitive closure of the constituent relation.

An order on expression-stack pairs: To deal with the control stack, we define an order on expression-stack pairs. Two pairs are ordered if (a) the stack component of the second is the tail of the first's stack component, and (b) the expression component of the second, resides on the top stack frame of the first pair: $\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle \gtrdot \langle \mathtt{s}',\, k\rangle$. We write $\gtrdot^*$ for the reflexive, transitive closure of the expression-stack pair ordering.

Next, we consider an operator ρ, defined in terms of the constituent relation and the expression-stack pair ordering. The operator ρ ensures that all constituent environments will themselves belong to the set of environments, and that any structurally smaller expression-stack pairs are also contained in the expression-stack relation.

Definition 4.1.

$$\begin{aligned}
\rho(\langle C,\, F,\, E\rangle) = \langle C, \{\langle \mathtt{s},\, k\rangle \mid \exists \langle \mathtt{s}',\, k'\rangle \in F : \langle \mathtt{s}',\, k'\rangle \gtrdot^* \langle \mathtt{s},\, k\rangle\}, \\
\{e \mid \exists \langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e \ \vee\ \exists e' \in E : e' \succ^* e\}\rangle
\end{aligned}$$

We need to relate the expression-stack ordering to the constituent relation. By case analysis one can prove the following lemma.

[1] Milner and Tofte define the constituent relation for finite functions.

Lemma 4.4. $\forall \langle \mathtt{s},\, k\rangle, \langle \mathtt{s}',\, k'\rangle : \langle \mathtt{s},\, k\rangle \gtrdot \langle \mathtt{s}',\, k'\rangle \implies k \succ k'$

By structural induction (on the stack component) the following lemma now follows.

Lemma 4.5. $\forall \langle \mathtt{s},\, k\rangle, \langle \mathtt{s}',\, k'\rangle : \langle \mathtt{s},\, k\rangle \gtrdot^* \langle \mathtt{s}',\, k'\rangle \implies k \succ^* k'$

Using the above lemmas, we can verify that ρ is a closure operator.

Lemma 4.6. ρ is a closure operator

We can now formulate an abstraction on the triples:

$$\wp(C) \times \wp(C \times K) \times \wp(Env) \underset{\rho}{\overset{1}{\leftrightarrows}} \rho(\wp(C) \times \wp(C \times K) \times \wp(Env))$$

We use the notation $\cup_\rho$ for the join operation $\lambda X.\, \rho(\cup_\times X)$ on the closure operator-induced complete lattice. First observe that in our case:

$$\cup_\rho = \lambda X.\, \rho(\bigcup_i{}_\times\, X_i) = \lambda X.\, \bigcup_i{}_\times\, \rho(X_i) = \lambda X.\, \bigcup_i{}_\times\, X_i = \cup_\times$$

Based on the closure operator-based Galois connection, we calculate a new intermediate transfer function F^ρ. Now let $\langle C,\, F,\, E\rangle \in \rho(\wp(C) \times \wp(C \times K) \times \wp(Env))$ be given.

$$\begin{aligned}
&\rho\Big(\bigcup_{\substack{\langle \{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}{}_\times \langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle\Big) \\
&= \bigcup_{\substack{\langle \{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}{}_\rho \rho(\langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) \qquad (\rho \text{ a CJM}) \\
&= \bigcup_{\substack{\langle \{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e']::k'\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}{}_\times \rho(\langle \{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) \quad \text{(by observation)}
\end{aligned}$$

The resulting transfer function appears in Fig. 7. This transfer function differs only minimally from the one in Fig. 6, in that (a) the signature has changed, (b) the set of initial states has been "closed" and now contains the structurally smaller pair $\langle \mathtt{x_r},\, \mathtt{stop}\rangle$, and (c) the four indexed joins now each join "closed" triples in the image of the closure operator.

By construction, the new transition function satisfies the following theorem.

Theorem 4.2.

$$\forall C, F, E : \rho \circ F^\times \circ 1(\langle C,\, F,\, E\rangle) = F^\rho(\langle C,\, F,\, E\rangle)$$

4.3 Abstracting the expression-stack relation

Since stacks can grow unbounded (for non-tail recursive programs), we need to approximate the stack component and hereby the expression-stack relation.

$$F^\rho : \rho(\wp(C) \times \wp(C \times K) \times \wp(Env)) \to \rho(\wp(C) \times \wp(C \times K) \times \wp(Env))$$

$$\begin{aligned}
F^\rho(\langle C,\, F,\, E\rangle) = {} & \langle\{\mathtt{p}\},\, \{\langle \mathtt{p},\, [\mathtt{x_r},\, \mathtt{x_r},\, \bullet] :: \mathtt{stop}\rangle, \langle \mathtt{x_r},\, \mathtt{stop}\rangle\},\, \{\bullet\}\rangle \\
& \cup_\times \bigcup_{\substack{\langle\{\mathtt{t}\},\, \{\langle \mathtt{t},\, [\mathtt{x},\, \mathtt{s}',\, e'] :: k'\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) \\
& \cup_\times \bigcup_{\substack{\langle\{\mathtt{let\ x{=}t\ in\ s}\},\, \{\langle \mathtt{let\ x{=}t\ in\ s},\, k\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ w \in \mu_c(\mathtt{t}, \{e\})}}^{\times} \rho(\langle\{\mathtt{s}\},\, \{\langle \mathtt{s},\, k\rangle\},\, \{e[\mathtt{x} \mapsto w]\}\rangle) \\
& \cup_\times \bigcup_{\substack{\langle\{\mathtt{t_0\ t_1}\},\, \{\langle \mathtt{t_0\ t_1},\, k\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ [\mathtt{fn\ x\ =>\ s}',\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) \\
& \cup_\times \bigcup_{\substack{\langle\{\mathtt{let\ x{=}t_0\ t_1\ in\ s}\},\, \{\langle \mathtt{let\ x{=}t_0\ t_1\ in\ s},\, k\rangle\},\, \{e\}\rangle \subseteq_\times \langle C,\, F,\, E\rangle \\ [\mathtt{fn\ y\ =>\ s}',\, e'] \in \mu_c(\mathtt{t_0}, \{e\}) \\ w \in \mu_c(\mathtt{t_1}, \{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, [\mathtt{x},\, \mathtt{s},\, e] :: k\rangle\},\, \{e'[\mathtt{y} \mapsto w]\}\rangle)
\end{aligned}$$

Figure 7: The second abstract transition function

We first formulate a grammar of abstract stacks and an elementwise operator $@ : C \times K \to C \times K^\sharp$ operating on expression-stack pairs.

$$K^\sharp \ni k^\sharp ::= \mathtt{stop} \mid [\mathtt{x},\, \mathtt{s}] \qquad \text{(abstract stacks)}$$

$$@(\langle \mathtt{s},\, \mathtt{stop}\rangle) = \langle \mathtt{s},\, \mathtt{stop}\rangle$$

$$@(\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle) = \langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}']\rangle$$

Based on the elementwise operator we can now use an elementwise abstraction.

Elementwise abstraction [Cousot and Cousot, 1997]: A given elementwise operator $@ : C \to A$ induces a Galois connection:

$$\langle \wp(C); \subseteq\rangle \underset{\alpha_@}{\overset{\gamma_@}{\leftrightarrows}} \langle \wp(A); \subseteq\rangle$$

$$\alpha_@(P) = \{@(p) \mid p \in P\}$$

$$\gamma_@(Q) = \{p \mid @(p) \in Q\}$$

Notice how some expressions share the same return point (read: same stack): the expression `let x=t in s` and the expression `s` share the same return point, and `let x=t`$_0$ `t`$_1$ `in s` and `s` share the same return point. In order to eliminate such redundancy we define an equivalence relation on serious expressions grouping together expressions sharing the same return point. We define the smallest equivalence relation $\equiv$ satisfying:

$$\mathtt{let\ x{=}t\ in\ s} \equiv \mathtt{s}$$

$$\mathtt{let\ x{=}t_0\ t_1\ in\ s} \equiv \mathtt{s}$$

Based hereon we define a second elementwise operator $@' : C \times K^\sharp \to C/_\equiv \times K^\sharp$ mapping the first component of an expression-stack pair to a representative of its corresponding equivalence class:

$$@'(\langle \mathtt{s},\, k^\sharp\rangle) = \langle [\mathtt{s}]_\equiv,\, k^\sharp\rangle$$

We can choose the outermost expression as a representative for each equivalence class by a linear top-down traversal of the input program.

Pointwise coding of a relation [Cousot and Cousot, 1994]: A relation can be isomorphically encoded as a set-valued function by a Galois connection:

$$\langle \wp(A \times B); \subseteq\rangle \underset{\alpha_\omega}{\overset{\gamma_\omega}{\leftrightarrows}} \langle A \to \wp(B); \dot{\subseteq}\rangle$$
$$\alpha_\omega(r) = \lambda a.\, \{b \mid \langle a,\, b\rangle \in r\}$$
$$\gamma_\omega(f) = \{\langle a,\, b\rangle \mid b \in f(a)\}$$

By composing the three above Galois connections we obtain our abstraction of the expression-stack relation:

$$\wp(C \times K) \underset{\alpha_{st}}{\overset{\gamma_{st}}{\leftrightarrows}} C/_\equiv \to \wp(K^\sharp)$$

where $\alpha_{st} = \alpha_\omega \circ \alpha_{@'} \circ \alpha_{@} = \lambda F.\, \dot{\bigcup}_{\langle \mathtt{s},\, k\rangle \in F} \alpha_\omega(\{@' \circ @(\langle \mathtt{s},\, k\rangle)\})$ and $\gamma_{st} = \gamma_{@} \circ \gamma_{@'} \circ \gamma_\omega$. We can now prove a lemma relating the concrete and abstract expression-stack relations.

Lemma 4.7. Control stack and saved environments
Let $\langle C,\, F,\, E\rangle \in \rho(\wp(C) \times \wp(C \times K) \times \wp(Env))$ be given.

$$\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle \in F \implies e \in E \;\wedge\; \{\langle \mathtt{s}',\, k\rangle\} \subseteq F \;\wedge\; \{[\mathtt{x},\, \mathtt{s}']\} \subseteq \alpha_{st}(F)([\mathtt{s}]_\equiv)$$

Proof. Assume $\{\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle\} \subseteq F$. Now $\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle \succ^* e$ and hence $e \in E$ by the assumption on E. Furthermore $\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle \succcurlyeq \langle \mathtt{s}',\, k\rangle$ hence $\{\langle \mathtt{s}',\, k\rangle\} \subseteq F$ by the assumption on F. For the last part we reason as follows:

$$\begin{aligned}
&\implies \alpha_{st}(\{\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle\}) \,\dot{\subseteq}\, \alpha_{st}(F) && (\alpha_{st} \text{ monotone})\\
&\iff \dot{\bigcup}_{\langle \mathtt{s}'',\, k''\rangle \in \{\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle\}} \alpha_\omega(\{@' \circ @(\langle \mathtt{s}'',\, k''\rangle)\}) \,\dot{\subseteq}\, \alpha_{st}(F) && (\text{def. } \alpha_{st})\\
&\iff \alpha_\omega(\{@' \circ @(\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}',\, e] :: k\rangle)\}) \,\dot{\subseteq}\, \alpha_{st}(F) && (\text{def. } \dot{\cup})\\
&\iff \alpha_\omega(\{@'(\langle \mathtt{s},\, [\mathtt{x},\, \mathtt{s}']\rangle)\}) \,\dot{\subseteq}\, \alpha_{st}(F) && (\text{def. } @)\\
&\iff \alpha_\omega(\{\langle [\mathtt{s}]_\equiv,\, [\mathtt{x},\, \mathtt{s}']\rangle\}) \,\dot{\subseteq}\, \alpha_{st}(F) && (\text{def. } @')\\
&\iff \lambda_.\, \emptyset[[\mathtt{s}]_\equiv \mapsto \{[\mathtt{x},\, \mathtt{s}']\}] \,\dot{\subseteq}\, \alpha_{st}(F) && (\text{def. } \alpha_\omega)\\
&\iff \{[\mathtt{x},\, \mathtt{s}']\} \subseteq \alpha_{st}(F)([\mathtt{s}]_\equiv) && (\text{def. } \dot{\subseteq})
\end{aligned}$$

□

4.4 Abstracting environments

We also abstract values using an elementwise abstraction. Again we formulate a grammar of abstract values and an elementwise operator $@ : Val \to Val^\sharp$

mapping concrete to abstract values.

$$\begin{aligned} \mathit{Val}^\sharp \ni w^\sharp ::= c \ \mid \ [\texttt{fn x => s}] \\ @(c) = c \\ @([\texttt{fn x => s},\ e]) = [\texttt{fn x => s}] \end{aligned}$$

The abstraction of environments, which are partial functions, can be composed by a series of well-known Galois connections.

Pointwise abstraction of a set of functions [Cousot and Cousot, 1994]: A given Galois connection on the co-domain $\langle \wp(C); \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle C^\sharp; \sqsubseteq \rangle$ induces a Galois connection on a set of functions:

$$\begin{aligned} \langle \wp(D \to C); \subseteq \rangle \underset{\alpha_\Pi}{\overset{\gamma_\Pi}{\leftrightarrows}} \langle D \to C^\sharp; \dot{\sqsubseteq} \rangle \\ \alpha_\Pi(F) = \lambda d.\, \alpha(\{f(d) \mid f \in F\}) \\ \gamma_\Pi(A) = \{f \mid \forall d : f(d) \in \gamma(A(d))\} \end{aligned}$$

Subset abstraction [Cousot and Cousot, 1997]: Given a set C and a strict subset $A \subset C$ hereof, the restriction to the subset induces a Galois connection:

$$\begin{aligned} \langle \wp(C); \subseteq \rangle \underset{\alpha_\subset}{\overset{\gamma_\subset}{\leftrightarrows}} \langle \wp(A); \subseteq \rangle \\ \alpha_\subset(X) = X \cap A \\ \gamma_\subset(Y) = Y \cup (C \setminus A) \end{aligned}$$

A standard trick is to think of partial functions $r : D \rightharpoonup C$ as total functions $r_\perp : D \to (C \cup \perp)$ where $\perp \sqsubseteq \perp \sqsubseteq c$, for all $c \in C$. Now consider environments $e \in \mathit{Var} \rightharpoonup \mathit{Val}$ to be total functions $\mathit{Var} \to (\mathit{Val} \cup \perp)$ using this idea. In this context the bottom element $\perp$ will denote variable lookup failure. Now compose a subset abstraction $\wp(\mathit{Val} \cup \perp) \underset{\alpha_\subset}{\overset{\gamma_\subset}{\leftrightarrows}} \wp(\mathit{Val})$ with the value abstraction from the previous section, and feed the result to the pointwise abstraction above. The result is a pointwise abstraction of a set of environments, that does not explicitly model variable lookup failure:

$$\wp(\mathit{Env}) \underset{\alpha_\Pi}{\overset{\gamma_\Pi}{\leftrightarrows}} \mathit{Var} \to \wp(\mathit{Val}^\sharp)$$

By considering only closed programs, we statically ensure against failure of variable-lookup, hence disregarding $\perp$ loses no information.

4.5 Abstracting the helper function

We calculate an abstract helper function, by "pushing α's" under the function definition, and reading off a resulting abstract definition.

Lemma 4.8. Abstract helper function

$$\forall \texttt{t}, E : \alpha_@(\mu_c(\texttt{t}, E)) = \mu^\sharp(\texttt{t}, \alpha_\Pi(E))$$

The resulting helper function reads:

$$\begin{aligned}
\mu^\sharp &: T \times \mathit{Env}^\sharp \to \wp(\mathit{Val}^\sharp) \\
\mu^\sharp(c, E^\sharp) &= \{c\} \\
\mu^\sharp(\mathtt{x}, E^\sharp) &= E^\sharp(\mathtt{x}) \\
\mu^\sharp(\mathtt{fn\ x\ =>\ s}, E^\sharp) &= \{[\mathtt{fn\ x\ =>\ s}]\}
\end{aligned}$$

where we write $\mathit{Env}^\sharp$ as shorthand for $\mathit{Var} \to \wp(\mathit{Val}^\sharp)$. We shall need a lemma relating the two helper function definitions on closed environments.

Lemma 4.9. Helper function on closed environments (1)
Let $\langle C,\, F,\, E\rangle \in \rho(\wp(C) \times \wp(C \times K) \times \wp(\mathit{Env}))$ be given.

$$\{[\mathtt{fn\ x\ =>\ s},\ e]\} \subseteq \mu_c(\mathtt{t}, E) \implies e \in E \ \wedge\ \{[\mathtt{fn\ x\ =>\ s}]\} \subseteq \mu^\sharp(\mathtt{t}, \alpha_\Pi(E))$$

The above lemma is easily extended to capture nested environments in all values returned by the helper function:

Lemma 4.10. Helper function on closed environments (2)
Let $\langle C,\, F,\, E\rangle \in \rho(\wp(C) \times \wp(C \times K) \times \wp(\mathit{Env}))$ be given.

$$\{w\} \subseteq \mu_c(\mathtt{t}, E) \ \wedge\ w \succ^* e'' \implies e'' \in E$$

4.6 Abstracting the machine states

We abstract the triplet of sets into abstract triples by a componentwise abstraction.

Componentwise abstraction [Cousot and Cousot, 1994]: Assuming a series of Galois connections: $\wp(C_i) \underset{\alpha_i}{\overset{\gamma_i}{\leftrightarrows}} A_i$ for $i \in \{1, \ldots, n\}$, their componentwise composition induces a Galois connection on tuples:

$$\begin{gathered}
\langle \wp(C_1) \times \ldots \times \wp(C_n); \subseteq_\times\rangle \underset{\alpha_\otimes}{\overset{\gamma_\otimes}{\leftrightarrows}} \langle A_1 \times \ldots \times A_n; \subseteq_\otimes\rangle \\
\alpha_\otimes(\langle X_1,\, \ldots,\, X_n\rangle) = \langle \alpha_1(X_1),\, \ldots,\, \alpha_n(X_n)\rangle \\
\gamma_\otimes(\langle x_1,\, \ldots,\, x_n\rangle) = \langle \gamma_1(x_1),\, \ldots,\, \gamma_n(x_n)\rangle
\end{gathered}$$

We write $\cup_\otimes$ and $\subseteq_\otimes$ for componentwise join and inclusion, respectively.

For the set of expressions $\wp(C)$ we use the identity abstraction consisting of two identity functions. For the expression-stack relation $\wp(C \times K)$ we use the expression-stack abstraction α_{st} developed in Section 4.3. For the set of environments $\wp(\mathit{Env})$ we use the environment abstraction α_Π developed in Section 4.4.

5 Calculating the analysis

Using the alternative "recipe" we can calculate the analysis by "pushing α's" under the intermediate transition function:

$$\alpha_\otimes(F^\rho(\langle C,\, F,\, E\rangle)) \subseteq_\otimes F^\sharp(\langle C,\, \alpha_{st}(F),\, \alpha_\Pi(E)\rangle)$$

from which the final definition of $F^\sharp$ can be read off. For space-saving purposes the calculation is divided into a number of observations, on which the derivation relies. Let $\langle C,\, F,\, E\rangle \in \rho(\wp(C) \times \wp(C \times K) \times \wp(Env))$ be given. First observe that:

$$
\begin{aligned}
&\{e \mid \exists\langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e \\
&\qquad\qquad \vee\ \exists e' \in (\bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}} \{e'[\mathtt{x} \mapsto w]\}) : e' \succ^* e\} \\
&= \{e \mid \exists\langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e \\
&\qquad\qquad \vee\ \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] \succ^* e\} && \text{(def. } \cup) \\
&= \{e \mid \exists\langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e\} \\
&\qquad\qquad \cup \{e \mid \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] \succ^* e\} && \text{(def. } \vee) \\
&\subseteq E \cup \{e \mid \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] \succ^* e\} && \text{(assumption on } E) \\
&= E \cup \{e \mid \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] = e \\
&\qquad\qquad \vee\ e' \succ^* e\ \vee\ w \succ^* e\} && \text{(case analysis)} \\
&= E \cup \{e \mid \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] = e \\
&\qquad\qquad \vee\ e' \succ^* e\} && \text{(by Lemma 4.10)} \\
&= E \cup \{e \mid \exists e' \in E, w \in \mu_c(\mathtt{t}, E) : e'[\mathtt{x} \mapsto w] = e\} && \text{(assumption on } E) \\
&= E \cup \{e'[\mathtt{x} \mapsto w] \mid e' \in E, w \in \mu_c(\mathtt{t}, E)\} && \text{(def } =)
\end{aligned}
$$

Secondly, observe that:

$$
\begin{aligned}
&\bigcup_{\substack{\{\langle \mathtt{s}',\, k'\rangle\}\subseteq F \\ \{e'\}\subseteq E \quad \{e\}\subseteq E \\ w\in\mu_c(\mathtt{t},\{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) \\
&= \bigcup_{\substack{\{\langle \mathtt{s}',\, k'\rangle\}\subseteq F \\ \{e'\}\subseteq E \quad w\in\mu_c(\mathtt{t},E)}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\text{def. } \mu_c) \\
&= \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}}^{\times} \rho(\bigcup_{\{\langle \mathtt{s}',\, k'\rangle\}\subseteq F}^{\times} \langle\{\mathtt{s}'\},\, \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\rho \text{ a CJM}) \\
&= \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}}^{\times} \rho(\langle\{\mathtt{s}'\},\, \bigcup_{\{\langle \mathtt{s}',\, k'\rangle\}\subseteq F} \{\langle \mathtt{s}',\, k'\rangle\},\, \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\text{def. } \cup_\times) \\
&\subseteq_\times \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}}^{\times} \rho(\langle\{\mathtt{s}'\},\, F,\, \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\text{def. } \cup) \\
&= \rho(\bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}}^{\times} \langle\{\mathtt{s}'\},\, F,\, \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\rho \text{ a CJM}) \\
&= \rho(\langle\{\mathtt{s}'\},\, F,\, \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}} \{e'[\mathtt{x} \mapsto w]\}\rangle) && (\text{def. } \cup_\times)
\end{aligned}
$$

$$
\begin{aligned}
&\langle\{\mathtt{s}'\}, \{\langle \mathtt{s},\, k\rangle \mid \exists\langle \mathtt{s}',\, k'\rangle \in F : \langle \mathtt{s}',\, k'\rangle \succcurlyeq^* \langle \mathtt{s},\, k\rangle\}, \\
&\quad \{e \mid \exists\langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e \\
= &\qquad\qquad \vee\ \exists e' \in \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}} \{e'[\mathtt{x}\mapsto w]\} : e' \succ^* e\}\rangle && \text{(def. } \rho) \\
&\langle\{\mathtt{s}'\}, F, \{e \mid \exists\langle \mathtt{s},\, k\rangle \in F : \langle \mathtt{s},\, k\rangle \succ^* e \\
= &\qquad\qquad \vee\ \exists e' \in \bigcup_{\substack{\{e'\}\subseteq E \\ w\in\mu_c(\mathtt{t},E)}} \{e'[\mathtt{x}\mapsto w]\} : e' \succ^* e\}\rangle && \text{(assumption on } F) \\
\subseteq_\times &\ \langle\{\mathtt{s}'\},\ F,\ E \cup \{e'[\mathtt{x}\mapsto w] \mid e' \in E, w \in \mu_c(\mathtt{t},E)\}\rangle && \text{(First obs.)}
\end{aligned}
$$

Thirdly, observe that:

$$
\begin{aligned}
&\alpha_\Pi(E \cup \{e'[\mathtt{x}\mapsto w] \mid e' \in E, w \in \mu_c(\mathtt{t},E)\}) \\
&= \alpha_\Pi(E) \,\dot\cup\, \alpha_\Pi(\{e'[\mathtt{x}\mapsto w] \mid e' \in E, w \in \mu_c(\mathtt{t},E)\}) && (\alpha_\Pi \text{ a CJM}) \\
&= \alpha_\Pi(E) \,\dot\cup\, \alpha_\Pi(\{\lambda\mathtt{y}.\,\text{if } \mathtt{y}=\mathtt{x} \text{ then } w \text{ else } e'(\mathtt{y}) \mid e' \in E, \\
&\qquad\qquad w \in \mu_c(\mathtt{t},E)\}) && \text{(def. extend)} \\
&= \alpha_\Pi(E) \,\dot\cup\, \lambda\mathtt{y}.\,\text{if } \mathtt{y}=\mathtt{x} \text{ then } \alpha_@(\{w \mid w \in \mu_c(\mathtt{t},E)\}) \\
&\qquad\qquad \text{else } \alpha_@(\{e'(\mathtt{y}) \mid e' \in E\}) && \text{(def. } \alpha_\Pi) \\
&= \alpha_\Pi(E) \,\dot\cup\, \lambda\mathtt{y}.\,\text{if } \mathtt{y}=\mathtt{x} \text{ then } \alpha_@(\mu_c(\mathtt{t},E)) \text{ else } \alpha_\Pi(E)(\mathtt{y}) && \text{(def. } \alpha_\Pi) \\
&= \alpha_\Pi(E) \,\dot\cup\, \lambda\mathtt{y}.\,\text{if } \mathtt{y}=\mathtt{x} \text{ then } \mu^\sharp(\mathtt{t},\alpha_\Pi(E)) \text{ else } \alpha_\Pi(E)(\mathtt{y}) && \text{(by Lemma 4.8)} \\
&= \alpha_\Pi(E) \,\dot\cup\, \alpha_\Pi(E)[\mathtt{x}\mapsto \mu^\sharp(\mathtt{t},\alpha_\Pi(E))] && \text{(def. extend)} \\
&= \alpha_\Pi(E) \,\dot\cup\, [\mathtt{x}\mapsto \mu^\sharp(\mathtt{t},\alpha_\Pi(E))] && \text{(def. } \dot\cup)
\end{aligned}
$$

where we have written $[\mathtt{x}\mapsto\ldots]$ as shorthand for $\lambda_.\,\emptyset[\mathtt{x}\mapsto\ldots]$. Now we can calculate the analysis:

$$
\begin{aligned}
&\alpha_\otimes\Big(\bigcup_{\substack{\langle\{\mathtt{t}\},\{\langle \mathtt{t},[\mathtt{x},\mathtt{s}',e']::k'\rangle\},\{e\}\rangle\subseteq_\times\langle C,F,E\rangle \\ w\in\mu_c(\mathtt{t},\{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\ \{\langle \mathtt{s}',\, k'\rangle\},\ \{e'[\mathtt{x}\mapsto w]\}\rangle)\Big) \\
&= \alpha_\otimes\Big(\bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{\langle \mathtt{t},[\mathtt{x},\mathtt{s}',e']::k'\rangle\}\subseteq F \\ \{e\}\subseteq E \\ w\in\mu_c(\mathtt{t},\{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\ \{\langle \mathtt{s}',\, k'\rangle\},\ \{e'[\mathtt{x}\mapsto w]\}\rangle)\Big) && \text{(def. } \subseteq_\times) \\
&\subseteq_\otimes \alpha_\otimes\Big(\bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\mathtt{s}']\}\subseteq\alpha_{st}(F)([\mathtt{t}]_\equiv) \\ \{\langle \mathtt{s}',\, k'\rangle\}\subseteq F \\ \{e'\}\subseteq E \quad \{e\}\subseteq E \\ w\in\mu_c(\mathtt{t},\{e\})}}^{\times} \rho(\langle\{\mathtt{s}'\},\ \{\langle \mathtt{s}',\, k'\rangle\},\ \{e'[\mathtt{x}\mapsto w]\}\rangle)\Big) && \text{(by Lemma 4.7)} \\
&\subseteq_\otimes \alpha_\otimes\Big(\bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\mathtt{s}']\}\subseteq\alpha_{st}(F)([\mathtt{t}]_\equiv)}}^{\times} \langle\{\mathtt{s}'\}, F, E \cup \{e'[\mathtt{x}\mapsto w] \mid e' \in E, \\
&\qquad\qquad w \in \mu_c(\mathtt{t},E)\}\rangle\Big) && \text{(Second obs.)} \\
&= \bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\mathtt{s}']\}\subseteq\alpha_{st}(F)([\mathtt{t}]_\equiv)}}^{\otimes} \alpha_\otimes(\langle\{\mathtt{s}'\}, F, E \cup \{e'[\mathtt{x}\mapsto w] \mid e' \in E, \\
&\qquad\qquad w \in \mu_c(\mathtt{t},E)\}\rangle) && (\alpha_\otimes \text{ a CJM})
\end{aligned}
$$

$$
\begin{aligned}
& F^\sharp : P \to \wp(C) \times (C/_\equiv \to \wp(K^\sharp)) \times Env^\sharp \\
& \qquad\qquad\qquad\qquad \to \wp(C) \times (C/_\equiv \to \wp(K^\sharp)) \times Env^\sharp \\
& F^\sharp_{\mathtt{p}}(\langle C,\, F^\sharp,\, E^\sharp\rangle) = \\
& \quad \langle \{\mathtt{p}\},\, [[\mathtt{p}]_\equiv \mapsto \{[\mathtt{x_r},\, \mathtt{x_r}]\}, [\mathtt{x_r}]_\equiv \mapsto \{\mathtt{stop}\}],\, \lambda_.\,\emptyset\rangle \\
& \quad \cup_\otimes \bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\,\mathtt{s}']\}\subseteq F^\sharp([\mathtt{t}]_\equiv)}}^{\otimes} \langle \{\mathtt{s}'\},\, F^\sharp,\, E^\sharp \,\dot\cup\, [\mathtt{x} \mapsto \mu^\sharp(\mathtt{t}, E^\sharp)]\rangle \\
& \quad \cup_\otimes \bigcup_{\substack{\{\mathtt{let\ x{=}t\ in\ s}\}\subseteq C}}^{\otimes} \langle \{\mathtt{s}\},\, F^\sharp,\, E^\sharp \,\dot\cup\, [\mathtt{x} \mapsto \mu^\sharp(\mathtt{t}, E^\sharp)]\rangle \\
& \quad \cup_\otimes \bigcup_{\substack{\{\mathtt{t_0\ t_1}\}\subseteq C \\ \{[\mathtt{fn\ x\ =>\ s}']\}\in\mu^\sharp(\mathtt{t_0}, E^\sharp)}}^{\otimes} \langle \{\mathtt{s}'\},\, F^\sharp \,\dot\cup\, [[\mathtt{s}']_\equiv \mapsto F^\sharp([\mathtt{t_0\ t_1}]_\equiv)],\, E^\sharp \,\dot\cup\, [\mathtt{x} \mapsto \mu^\sharp(\mathtt{t_1}, E^\sharp)]\rangle \\
& \quad \cup_\otimes \bigcup_{\substack{\{\mathtt{let\ x{=}t_0\ t_1\ in\ s}\}\subseteq C \\ \{[\mathtt{fn\ y\ =>\ s}']\}\in\mu^\sharp(\mathtt{t_0}, E^\sharp)}}^{\otimes} \langle \{\mathtt{s}'\},\, F^\sharp \,\dot\cup\, [[\mathtt{s}']_\equiv \mapsto \{[\mathtt{x},\, \mathtt{s}]\}],\, E^\sharp \,\dot\cup\, [\mathtt{y} \mapsto \mu^\sharp(\mathtt{t_1}, E^\sharp)]\rangle
\end{aligned}
$$

Figure 8: The resulting analysis function

$$
\begin{aligned}
= & \bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\,\mathtt{s}']\}\subseteq \alpha_{st}(F)([\mathtt{t}]_\equiv)}}^{\otimes} \langle \{\mathtt{s}'\}, \alpha_{st}(F), \alpha_\Pi(E \cup \{e'[\mathtt{x} \mapsto w] \mid e' \in E, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad w \in \mu_c(\mathtt{t}, E)\})\rangle \qquad \text{(def. } \alpha_\otimes) \\
= & \bigcup_{\substack{\{\mathtt{t}\}\subseteq C \\ \{[\mathtt{x},\,\mathtt{s}']\}\subseteq \alpha_{st}(F)([\mathtt{t}]_\equiv)}}^{\times} \langle \{\mathtt{s}'\},\, \alpha_{st}(F),\, \alpha_\Pi(E) \,\dot\cup\, [\mathtt{x} \mapsto \mu^\sharp(\mathtt{t}, \alpha_\Pi(E))]\rangle \quad \text{(Third obs.)}
\end{aligned}
$$

The resulting analysis appears in Fig. 8. The alert reader may have noticed that this final abstraction is not *complete* in that the above equation contains an inequality. Completeness is a desirable goal in an abstract interpretation but unfortunately it is not possible in general without refining the abstract domain [Giacobazzi et al., 2000]. Consider for example the addition operator over the standard *sign-domain*: $0 = \alpha(1 + (-1)) \sqsubseteq \alpha(1) + \alpha(-1) = \top$. As traditional [Cousot, 1999], we instead limit upward judgements to a minimum.

As a corollary of the construction (modulo a monotonicity check because of the upward judgement), the analysis safely approximates the reachable states of the abstract machine.

Corollary 5.1. $\alpha_\otimes \circ \rho \circ \alpha_\times(\mathrm{lfp}\, F) \subseteq_\otimes \mathrm{lfp}\, F^\sharp$

5.1 A faster implementation

The analysis as formulated above will always terminate, as there are only a finite number of reachable expressions, variables and functions in a given program. Hence strictly speaking we do not need a widening operator. However to avoid computing redundant joins, one typically computes an equivalent sequence

sharing the same fixed point:

$$X_0 = \langle \emptyset,\ \lambda_.\,\emptyset,\ \lambda_.\,\emptyset \rangle$$
$$X_{i+1} = X_i \cup_{\otimes} F^{\sharp}(X_i)$$

where we use a join $\cup_{\otimes}$ as the widening operator.

5.2 Characteristics

First of all the analysis incorporates *reachability*: it computes an approximate set of reachable expressions and will only analyse those reachable program fragments. Reachability analyses have previously been discovered independently [Ayers, 1992, Palsberg and Schwartzbach, 1995, Biswas, 1997, Gasser et al., 1997]. In our case they arise naturally from a projecting abstraction of a reachable states collecting semantics.

Second the formulation materializes *monomorphism* into two mappings: (a) one mapping merging all bindings to the same variable, and (b) one mapping merging all calling contexts of the same function. Both characteristics are well known, but our presentation literally captures this phenomenon in two approximation functions.

Third the analysis handles returns inside-out (*"callee-restore"*), in that the called function restores control from the approximate control stack and propagates the obtained return values. This differs from the traditional presentations [Palsberg, 1995, Nielson et al., 1999] that handle returns outside-in (*"caller-restore"*) where the caller propagates the obtained return values from the body of the function to the call site (typically formulated as *conditional constraints*). Such caller-restore CFAs typically mimic the recursive nature of a corresponding interpreter, e.g., a big-step or denotational semantics. As a consequence they need not abstract the call stack. In our case the starting point was a callee-restore machine with an explicit call stack. In hindsight it is perhaps less surprising that the "abstract interpreter" inherits this callee-restore strategy.

In this presentation we did not include an explicit construct for recursive functions. Since our source language is untyped, it is possible to encode recursion though fixed-point operators. Explicit recursion is typically modelled by circular environments. The current formulation extends straight forwardly to handle those, because of our two-staged environment abstraction (closure operator and pointwise extended value abstraction).

6 Analysis equivalence

In previous work [Midtgaard and Jensen, 2008] we derived an initial CFA with reachability for a CPS language from the stack-less CE-machine [Flanagan et al., 1993]. In this section we show that the present ANF analysis achieves the same precision as obtained by first transforming a program into CPS and then using the CPS analysis. This is done by defining a relation that captures how the direct-style analysis and the CPS analysis operate in lock-step.

The grammar of CPS terms is given in Fig. 9. The grammar distinguishes variables in the original source program $\mathtt{x} \in X$, from intermediate variables $\mathtt{v} \in V$ and continuation variables $\mathtt{k} \in K$. We assume the three classes are

$$
\begin{array}{rcll}
\mathit{CProg} \ni \mathtt{p} & ::= & \mathtt{fn\ k\ =>\ e} & \text{(CPS programs)} \\
\mathit{SExp} \ni \mathtt{e} & ::= & \mathtt{t_0\ t_1\ c} \ \mid\ \mathtt{c\ t} & \text{(serious CPS expressions)} \\
\mathit{TExp} \ni \mathtt{t} & ::= & \mathtt{x} \ \mid\ \mathtt{v} \ \mid\ \mathtt{fn\ x, k\ =>\ e} & \text{(trivial CPS expressions)} \\
\mathit{CExp} \ni \mathtt{c} & ::= & \mathtt{fn\ v\ =>\ e} \ \mid\ \mathtt{k} & \text{(continuation expressions)}
\end{array}
$$

Figure 9: BNF of CPS language

non-overlapping. Their union constitute the domain of CPS variables $\mathit{Var} = X \cup V \cup K$.

6.1 CPS transformation and back again

In order to state the relation between the ANF and CPS analyses we first recall the relevant program transformations. The below presentation is based on Danvy [1991], Flanagan et al. [1993], and Sabry and Felleisen [1994].

The CPS transformation given in Fig. 10(a) is defined by two mutually recursive functions — one for serious and trivial expressions. A continuation variable `k` is provided in the initial call to $\mathcal{F}$. A fresh `k` is generated in $\mathcal{V}$'s lambda abstraction case. To ease the expression of the relation, we choose `k` unique to the serious expression `s` — $\mathtt{k_s}$. It follows that we only need one `k` per lambda abstraction in the original program + an additional `k` in the initial case.

It is immediate from the definition of $\mathcal{F}$ that the CPS transformation of a let-binding `let x=t in s` and the CPS transformation of its body `s` share the same continuation identifier — and similarly for non-tail calls. Hence we shall equate the two:

Definition 6.1. $\mathtt{k_s} \equiv \mathtt{k_{s'}}$ iff $\mathtt{s} \equiv \mathtt{s'}$

The direct-style transformation given in Fig. 10(b) is defined by two mutually recursive functions over serious and trivial CPS expressions. We define the direct-style transformation of a program `fn k => e` as the direct-style transformation of its body $\mathcal{U}[\mathtt{e}]$.

Transforming a program, a serious expression, or a trivial expression to CPS and back to direct style yields the original expression, which can be confirmed by (mutual) structural induction on trivial and serious expressions.

Lemma 6.1. $\mathcal{D}[\mathcal{C}[\mathtt{p}]] = \mathtt{p} \ \wedge\ \mathcal{U}[\mathcal{F}_{\mathtt{k}}[\mathtt{s}]] = \mathtt{s} \ \wedge\ \mathcal{P}[\mathcal{V}[\mathtt{t}]] = \mathtt{t}$

6.2 CPS analysis

We recall the CPS analysis of Midtgaard and Jensen [2008] in Fig. 11. It is defined as the least fixed point of a program specific transfer function $T^\sharp_{\mathtt{p}}$. The definition relies on two helper functions $\mu^\sharp_t$ and $\mu^\sharp_c$ for trivial and continuation expressions, respectively. The analysis computes a pair consisting of (a) a set of serious expressions (the reachable expressions) and (b) an abstract environment. Abstract environments map variables to abstract values. Abstract values

$$\mathcal{C} : P \to \mathit{CProg}$$
$$\mathcal{C}[\mathtt{p}] = \mathtt{fn}\ \mathtt{k_p}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_p}}[\mathtt{p}]$$

$$\mathcal{F} : K \to C \to \mathit{SExp}$$
$$\mathcal{F}_{\mathtt{k}}[\mathtt{t}] = \mathtt{k}\ \mathcal{V}[\mathtt{t}]$$
$$\mathcal{F}_{\mathtt{k}}[\mathtt{let\ x{=}t\ in\ s}] = (\mathtt{fn\ x\ =>}\ \mathcal{F}_{\mathtt{k}}[\mathtt{s}])\ \mathcal{V}[\mathtt{t}]$$
$$\mathcal{F}_{\mathtt{k}}[\mathtt{t_0\ t_1}] = \mathcal{V}[\mathtt{t_0}]\ \mathcal{V}[\mathtt{t_1}]\ \mathtt{k}$$
$$\mathcal{F}_{\mathtt{k}}[\mathtt{let\ x{=}t_0\ t_1\ in\ s}] = \mathcal{V}[\mathtt{t_0}]\ \mathcal{V}[\mathtt{t_1}]\ (\mathtt{fn\ x\ =>}\ \mathcal{F}_{\mathtt{k}}[\mathtt{s}])$$

$$\mathcal{V} : T \to \mathit{TExp}$$
$$\mathcal{V}[\mathtt{x}] = \mathtt{x}$$
$$\mathcal{V}[\mathtt{fn\ x\ =>\ s}] = \mathtt{fn\ x, k_s\ =>}\ \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]$$

(a) CPS transformation

$$\mathcal{D} : \mathit{CProg} \to P$$
$$\mathcal{D}[\mathtt{fn\ k\ =>\ e}] = \mathcal{U}[\mathtt{e}]$$

$$\mathcal{U} : \mathit{SExp} \to C$$
$$\mathcal{U}[\mathtt{k\ t}] = \mathcal{P}[\mathtt{t}]$$
$$\mathcal{U}[(\mathtt{fn\ v\ =>\ e})\ \mathtt{t}] = \mathtt{let\ v{=}}\mathcal{P}[\mathtt{t}]\ \mathtt{in}\ \mathcal{U}[\mathtt{e}]$$
$$\mathcal{U}[\mathtt{t_0\ t_1\ k}] = \mathcal{P}[\mathtt{t_0}]\ \mathcal{P}[\mathtt{t_1}]$$
$$\mathcal{U}[\mathtt{t_0\ t_1}\ (\mathtt{fn\ v\ =>\ e})] = \mathtt{let\ v{=}}\mathcal{P}[\mathtt{t_0}]\ \mathcal{P}[\mathtt{t_1}]\ \mathtt{in}\ \mathcal{U}[\mathtt{e}]$$

$$\mathcal{P} : \mathit{TExp} \to T$$
$$\mathcal{P}[\mathtt{x}] = \mathtt{x}$$
$$\mathcal{P}[\mathtt{v}] = \mathtt{v}$$
$$\mathcal{P}[\mathtt{fn\ x, k\ =>\ e}] = \mathtt{fn\ x\ =>}\ \mathcal{U}[\mathtt{e}]$$

(b) Direct-style transformation

Figure 10: Transformations to and from CPS

$$Env^\sharp = Var \to \wp(Val^\sharp) \qquad \text{(abstract environment)}$$
$$Val^\sharp \ni w^\sharp ::= \texttt{stop} \mid [\texttt{fn x}, \texttt{k => e}] \mid [\texttt{fn v => e}] \qquad \text{(abstract values)}$$

(a) Abstract domains

$$\begin{aligned}
\mu_t^\sharp &: TExp \times Env^\sharp \to \wp(Val^\sharp) \\
\mu_t^\sharp(\texttt{x}, R^\sharp) &= R^\sharp(\texttt{x}) \\
\mu_t^\sharp(\texttt{v}, R^\sharp) &= R^\sharp(\texttt{v}) \\
\mu_t^\sharp(\texttt{fn x}, \texttt{k => e}, R^\sharp) &= \{[\texttt{fn x}, \texttt{k => e}]\}
\end{aligned}$$

$$\begin{aligned}
\mu_c^\sharp &: CExp \times Env^\sharp \to \wp(Val^\sharp) \\
\mu_c^\sharp(\texttt{k}, R^\sharp) &= R^\sharp(\texttt{k}) \\
\mu_c^\sharp(\texttt{fn v => e}, R^\sharp) &= \{[\texttt{fn v => e}]\}
\end{aligned}$$

(b) Abstract helper functions

$$T^\sharp : CProg \to \wp(SExp) \times Env^\sharp \to \wp(SExp) \times Env^\sharp$$

$$\begin{aligned}
&T^\sharp_{\texttt{fn k => e}}(\langle Q^\sharp,\ R^\sharp\rangle) = \\
&\quad \langle \{\texttt{e}\},\ [\texttt{k}_r \mapsto \{\texttt{stop}\}, \texttt{k} \mapsto \{[\texttt{fn v}_r \texttt{ => k}_r\ \texttt{v}_r]\}]\rangle \\
&\quad \cup_\otimes \bigcup_{\substack{\texttt{t}_0\ \texttt{t}_1\ \texttt{c} \in Q^\sharp \\ [\texttt{fn x},\texttt{k}' \texttt{ => e}'] \in \mu_t^\sharp(\texttt{t}_0, R^\sharp)}}^{\otimes} \langle \{\texttt{e}'\},\ R^\sharp \ \dot\cup\ [\texttt{x} \mapsto \mu_t^\sharp(\texttt{t}_1, R^\sharp), \texttt{k}' \mapsto \mu_c^\sharp(\texttt{c}, R^\sharp)]\rangle \\
&\quad \cup_\otimes \bigcup_{\substack{\texttt{c}\ \texttt{t} \in Q^\sharp \\ [\texttt{fn v => e}'] \in \mu_c^\sharp(\texttt{c}, R^\sharp)}}^{\otimes} \langle \{\texttt{e}'\},\ R^\sharp \ \dot\cup\ [\texttt{v} \mapsto \mu_t^\sharp(\texttt{t}, R^\sharp)]\rangle
\end{aligned}$$

(c) Abstract transition function

Figure 11: CPS analysis

can be either the initial continuation `stop`, function closures [`fn x`, `k => e`], or continuation closures [`fn v => e`].

The definition relies on two special variables $\texttt{k}_r$ and $\texttt{v}_r$, the first of which names the initial continuation and the second of which names the result of the program. To ensure the most precise analysis result, variables in the source program can be renamed to be distinct as is traditional in control-flow analysis [Nielson et al., 1999].

6.3 Analysis equivalence

Before formally stating the equivalence of the two analyses we will study an example run. As our example we use the ANF program:

$$\texttt{let f=fn x => x in let a}_1\texttt{=f cn1 in let a}_2\texttt{=f cn2 in a}_2$$

taken from Sabry and Felleisen [1994] where we have Church encoded the integer literals. We write `cn1` for the Church numeral `fn s => fn z => s z` and `cn2` for

the Church numeral `fn s => fn z => let` $\mathtt{t_1}$`=s z in s` $\mathtt{t_1}$. The analysis trace appears in the left column of Table 1.

Similarly we study the CPS analysis of the CPS transformed program. The analysis trace appears in the right column of Table 1 where we have written `ccn1` for $\mathcal{V}[\mathtt{cn1}]$ and `ccn2` for $\mathcal{V}[\mathtt{cn2}]$. Contrary to Sabry and Felleisen [1994] both the ANF and the CPS analyses achieve the same precision on the example, determining that $\mathtt{a_1}$ will be bound to one of the two integer literals.

We are now in position to state our main theorem relating the ANF analysis to the CPS analysis. Intuitively the theorem relates:

- reachability in ANF to CPS reachability
- abstract stacks in ANF to CPS continuation closures
- abstract stack bottom in ANF to CPS initial continuation
- ANF closures to CPS function closures

Theorem 6.1. Let `p` be given. Let $\langle C,\, F^\sharp,\, E^\sharp\rangle = \mathrm{lfp}\, F^\sharp_{\mathtt{p}}$ and $\langle Q^\sharp,\, R^\sharp\rangle = \mathrm{lfp}\, T^\sharp_{\mathcal{C}[\mathtt{p}]}$. Then

$$\begin{aligned}
\mathtt{s} \in C &\iff \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}] \in Q^\sharp \;\wedge \\
[\mathtt{x},\, \mathtt{s}'] \in F^\sharp([\mathtt{s}]_\equiv) &\iff [\mathtt{fn\ x\ =>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp(\mathtt{k_s}) \;\wedge \\
\mathtt{stop} \in F^\sharp([\mathtt{s}]_\equiv) &\iff \mathtt{stop} \in R^\sharp(\mathtt{k_s}) \;\wedge \\
[\mathtt{fn\ x\ =>\ s}] \in E^\sharp(\mathtt{y}) &\iff [\mathtt{fn\ x}, \mathtt{k_s}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp(\mathtt{y})
\end{aligned}$$

For the purpose of the equivalence we equate the special variables $\mathtt{x_r}$ and $\mathtt{v}_r$ both naming the result of the computations. We prove the theorem by combining an implication in each direction with the identity from Lemma 6.1. We formulate both implications as relations and prove that both relations are preserved by the transfer functions.

6.4 ANF-CPS equivalence

We formally define a relation $\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}$ that relates ANF analysis triples to CPS analysis pairs.

Definition 6.2. $\langle C,\, F^\sharp,\, E^\sharp\rangle\ \mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\ \langle Q^\sharp,\, R^\sharp\rangle$ iff $\forall \mathtt{s}$:

$$\begin{aligned}
\mathtt{s} \in C &\implies \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}] \in Q^\sharp \;\wedge \\
[\mathtt{x},\, \mathtt{s}'] \in F^\sharp([\mathtt{s}]_\equiv) &\implies [\mathtt{fn\ x\ =>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp(\mathtt{k_s}) \;\wedge \\
\mathtt{stop} \in F^\sharp([\mathtt{s}]_\equiv) &\implies \mathtt{stop} \in R^\sharp(\mathtt{k_s}) \;\wedge \\
[\mathtt{fn\ x\ =>\ s}] \in E^\sharp(\mathtt{y}) &\implies [\mathtt{fn\ x}, \mathtt{k_s}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp(\mathtt{y})
\end{aligned}$$

First we need a small lemma relating the ANF helper function to one of the CPS helper functions.

Lemma 6.2.

$$\begin{aligned}
[\mathtt{fn\ x\ =>\ s}] \in \mu^\sharp(\mathtt{t}, E^\sharp) \;\wedge\; \langle C,\, F^\sharp,\, E^\sharp\rangle\ \mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\ \langle Q^\sharp,\, R^\sharp\rangle \\
\implies [\mathtt{fn\ x}, \mathtt{k_s}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in \mu^\sharp_t(\mathcal{V}[\mathtt{t}], R^\sharp)
\end{aligned}$$

i	ANF trace: $\langle C_i,\ F^\sharp_i,\ E^\sharp_i\rangle$	CPS trace: $\langle Q^\sharp_i,\ R^\sharp_i\rangle$
0	$\{\texttt{let f=fn x => x in let a}_1\texttt{=f cn1 in let a}_2\texttt{=f cn2 in a}_2\}$ $\left[\begin{array}{l}[\texttt{x}_\texttt{r}]_\equiv \mapsto \{\texttt{stop}\},\\ [\texttt{let f=fn x => x in let a}_1\texttt{=f cn1 in let a}_2\texttt{=f cn2 in a}_2]_\equiv \mapsto \{[\texttt{x}_\texttt{r},\ \texttt{x}_\texttt{r}]\}\end{array}\right]$ $\lambda_.\,\emptyset$	$\{(\texttt{fn f => f ccn1 (fn a}_1\texttt{ => f ccn2 (fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2))) (\texttt{fn x, k}_\texttt{x}\texttt{ => k}_\texttt{x}\ \texttt{x})\}$ $\left[\begin{array}{l}\texttt{k}_r \mapsto \{\texttt{stop}\},\\ \texttt{k}_\texttt{p} \mapsto \{[\texttt{fn v}_r\texttt{ => k}_r\ \texttt{v}_r]\}\end{array}\right]$
1	$C_0 \cup \{\texttt{let a}_1\texttt{=f cn1 in let a}_2\texttt{=f cn2 in a}_2\}$ $F^\sharp_0$ $E^\sharp_0 \ \dot\cup\ \left[\texttt{f} \mapsto \{[\texttt{fn x => x}]\}\right]$	$Q^\sharp_0 \cup \{\texttt{f ccn1 (fn a}_1\texttt{ => f ccn2 (fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2))\}$ $R^\sharp_0 \ \dot\cup\ \left[\texttt{f} \mapsto \{[\texttt{fn x, k}_\texttt{x}\texttt{ => k}_\texttt{x}\ \texttt{x}]\}\right]$
2	$C_1 \cup \{\texttt{x}\}$ $F^\sharp_1 \ \dot\cup\ \left[[\texttt{x}]_\equiv \mapsto \{[\texttt{a}_1,\ \texttt{let a}_2\texttt{=f cn2 in a}_2]\}\right]$ $E^\sharp_1 \ \dot\cup\ \left[\texttt{x} \mapsto \{\texttt{cn1}\}\right]$	$Q^\sharp_1 \cup \{\texttt{k}_\texttt{x}\ \texttt{x}\}$ $R^\sharp_1 \ \dot\cup\ \left[\begin{array}{l}\texttt{k}_\texttt{x} \mapsto \{[\texttt{fn a}_1\texttt{ => f ccn2 (fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2)]\}\\ \texttt{x} \mapsto \{\texttt{ccn1}\}\end{array}\right]$
3	$C_2 \cup \{\texttt{let a}_2\texttt{=f cn2 in a}_2\}$ $F^\sharp_2$ $E^\sharp_2 \ \dot\cup\ \left[\texttt{a}_1 \mapsto \{\texttt{cn1}\}\right]$	$Q^\sharp_2 \cup \{\texttt{f ccn2 (fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2)\}$ $R^\sharp_2 \ \dot\cup\ \left[\texttt{a}_1 \mapsto \{\texttt{ccn1}\}\right]$
4	C_3 $F^\sharp_3 \ \dot\cup\ \left[[\texttt{x}]_\equiv \mapsto \{[\texttt{a}_1,\ \texttt{let a}_2\texttt{=f cn2 in a}_2], [\texttt{a}_2,\ \texttt{a}_2]\}\right]$ $E^\sharp_3 \ \dot\cup\ \left[\texttt{x} \mapsto \{\texttt{cn1}, \texttt{cn2}\}\right]$	$Q^\sharp_3$ $R^\sharp_3 \ \dot\cup\ \left[\begin{array}{l}\texttt{k}_\texttt{x} \mapsto \{[\texttt{fn a}_1\texttt{ => f ccn2 (fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2)], [\texttt{fn a}_2\texttt{ => k}_\texttt{p}\ \texttt{a}_2]\}\\ \texttt{x} \mapsto \{\texttt{ccn1}, \texttt{ccn2}\}\end{array}\right]$
5	$C_4 \cup \{\texttt{a}_2\}$ $F^\sharp_4$ $E^\sharp_4 \ \dot\cup\ \left[\begin{array}{l}\texttt{a}_1 \mapsto \{\texttt{cn1}, \texttt{cn2}\}\\ \texttt{a}_2 \mapsto \{\texttt{cn1}, \texttt{cn2}\}\end{array}\right]$	$Q^\sharp_4 \cup \{\texttt{k}_\texttt{p}\ \texttt{a}_2\}$ $R^\sharp_4 \ \dot\cup\ \left[\begin{array}{l}\texttt{a}_1 \mapsto \{\texttt{ccn1}, \texttt{ccn2}\}\\ \texttt{a}_2 \mapsto \{\texttt{ccn1}, \texttt{ccn2}\}\end{array}\right]$
6	$C_5 \cup \{\texttt{x}_\texttt{r}\}$ $F^\sharp_5$ $E^\sharp_5 \ \dot\cup\ \left[\texttt{x}_\texttt{r} \mapsto \{\texttt{cn1}, \texttt{cn2}\}\right]$	$Q^\sharp_5 \cup \{\texttt{k}_r\ \texttt{v}_r\}$ $R^\sharp_5 \ \dot\cup\ \left[\texttt{v}_r \mapsto \{\texttt{ccn1}, \texttt{ccn2}\}\right]$
7	$C_6 \quad F^\sharp_6 \quad E^\sharp_6$	$Q^\sharp_6 \quad R^\sharp_6$

Table 1: Analysis traces of `let f=fn x => x in let a1=f cn1 in let a2=f cn2 in a2` and its CPS transformed counterpart

The relation is preserved by the transfer functions.

Theorem 6.2.

$$\langle C,\, F^\sharp,\, E^\sharp\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp,\, R^\sharp\rangle \implies F^\sharp_{\mathtt{p}}(\langle C,\, F^\sharp,\, E^\sharp\rangle)\; \mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; T^\sharp_{\mathcal{C}[\mathtt{p}]}(\langle Q^\sharp,\, R^\sharp\rangle)$$

Proof. First we name the individual triples of the union in the function body of $F^\sharp$. We name the first triple of results as initial:

$$\langle C_I,\, F^\sharp_I,\, E^\sharp_I\rangle = \langle\{\mathtt{p}\},\, [[\mathtt{p}]_\equiv \mapsto \{[\mathtt{x_r},\, \mathtt{x_r}]\}, [\mathtt{x_r}]_\equiv \mapsto \{\mathtt{stop}\}],\, \lambda_.\,\emptyset\rangle$$

The results of the second, third, fourth, and fifth joined triples corresponding to return, binding, tail call, and non-tail call are named $\langle C_{ret},\, F^\sharp_{ret},\, E^\sharp_{ret}\rangle$, $\langle C_{bind},\, F^\sharp_{bind},\, E^\sharp_{bind}\rangle$, $\langle C_{tc},\, F^\sharp_{tc},\, E^\sharp_{tc}\rangle$ and $\langle C_{ntc},\, F^\sharp_{ntc},\, E^\sharp_{ntc}\rangle$, respectively. Similarly we name the first result pair in the function body of the CPS analysis as initial: $\langle Q^\sharp_I,\, R^\sharp_I\rangle = \langle\{\mathtt{e}\},\, [\mathtt{k}_r \mapsto \{\mathtt{stop}\}, \mathtt{k} \mapsto \{[\mathtt{fn}\ \mathtt{v}_r\ \mathtt{=>}\ \mathtt{k}_r\ \mathtt{v}_r]\}]\rangle$. The results of the second and third joined pair corresponding to call and return are named $\langle Q^\sharp_{call},\, R^\sharp_{call}\rangle$ and $\langle Q^\sharp_{ret},\, R^\sharp_{ret}\rangle$, respectively.

The proof proceeds by verifying five relations:

$$\langle C_I,\, F^\sharp_I,\, E^\sharp_I\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_I,\, R^\sharp_I\rangle \tag{1}$$

$$\langle C_{ret},\, F^\sharp_{ret},\, E^\sharp_{ret}\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_{ret},\, R^\sharp_{ret}\rangle \tag{2}$$

$$\langle C_{bind},\, F^\sharp_{bind},\, E^\sharp_{bind}\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_{ret},\, R^\sharp_{ret}\rangle \tag{3}$$

$$\langle C_{tc},\, F^\sharp_{tc},\, E^\sharp_{tc}\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_{call},\, R^\sharp_{call}\rangle \tag{4}$$

$$\langle C_{ntc},\, F^\sharp_{ntc},\, E^\sharp_{ntc}\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_{call},\, R^\sharp_{call}\rangle \tag{5}$$

We now prove the return case relation (2): $\langle C_{ret},\, F^\sharp_{ret},\, E^\sharp_{ret}\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp_{ret},\, R^\sharp_{ret}\rangle$. The remaining cases follow by similar reasoning.

Let $\mathtt{s}$ be given.

(2a) Assume $\mathtt{s} \in C_{ret}$. Hence there exists $\mathtt{x}, \mathtt{s}', \mathtt{t}$ such that $\mathtt{s} = \mathtt{s}'$, $\{\mathtt{t}\} \subseteq C$, and $\{[\mathtt{x},\, \mathtt{s}']\} \subseteq F^\sharp([\mathtt{t}]_\equiv)$.

From the $\langle C,\, F^\sharp,\, E^\sharp\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp,\, R^\sharp\rangle$ assumption we have $\mathcal{F}_{\mathtt{k_t}}[\mathtt{t}] \in Q^\sharp$ and $[\mathtt{fn}\ \mathtt{x}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp(\mathtt{k_t})$.

Hence $\mathtt{k_t}\ \mathcal{V}[\mathtt{t}] \in Q^\sharp$ and $[\mathtt{fn}\ \mathtt{x}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in \mu^\sharp_c(\mathtt{k_t}, R^\sharp)$. As a consequence $\mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}'] \in Q^\sharp_{ret}$.

(2b) Assume $[\mathtt{x},\, \mathtt{s}'] \in F^\sharp_{ret}([\mathtt{s}]_\equiv)$. Hence there exists $\mathtt{x}'', \mathtt{s}'', \mathtt{t}$ such that $\{\mathtt{t}\} \subseteq C$, $\{[\mathtt{x}'',\, \mathtt{s}'']\} \subseteq F^\sharp([\mathtt{t}]_\equiv)$, and $[\mathtt{x},\, \mathtt{s}'] \in F^\sharp_{ret}([\mathtt{s}]_\equiv) = F^\sharp([\mathtt{s}]_\equiv)$.

From the $\langle C,\, F^\sharp,\, E^\sharp\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp,\, R^\sharp\rangle$ assumption we have $\mathcal{F}_{\mathtt{k_t}}[\mathtt{t}] \in Q^\sharp$, $[\mathtt{fn}\ \mathtt{x}''\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s''}}}[\mathtt{s}'']] \in R^\sharp(\mathtt{k_t})$, and $[\mathtt{fn}\ \mathtt{x}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp(\mathtt{k_s})$.

Hence $\mathtt{k_t}\ \mathcal{V}[\mathtt{t}] \in Q^\sharp$, $[\mathtt{fn}\ \mathtt{x}''\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s''}}}[\mathtt{s}'']] \in \mu^\sharp_c(\mathtt{k_t}, R^\sharp)$, and $[\mathtt{fn}\ \mathtt{x}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp(\mathtt{k_s})$. Since $R^\sharp \dot{\subseteq} R^\sharp_{ret}$ we have $[\mathtt{fn}\ \mathtt{x}\ \mathtt{=>}\ \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s}']] \in R^\sharp_{ret}(\mathtt{k_s})$.

(2c) Assume $\mathtt{stop} \in F^\sharp_{ret}([\mathtt{s}]_\equiv)$. Hence there exists $\mathtt{x}'', \mathtt{s}'', \mathtt{t}$ such that $\{\mathtt{t}\} \subseteq C$, $\{[\mathtt{x}'',\, \mathtt{s}'']\} \subseteq F^\sharp([\mathtt{t}]_\equiv)$, and $\mathtt{stop} \in F^\sharp_{ret}([\mathtt{s}]_\equiv) = F^\sharp([\mathtt{s}]_\equiv)$.

From the $\langle C,\, F^\sharp,\, E^\sharp\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp,\, R^\sharp\rangle$ assumption we have $\mathcal{F}_{\mathtt{k_t}}[\mathtt{t}] \in Q^\sharp$, $[\mathtt{fn\ x''} \Rightarrow \mathcal{F}_{\mathtt{k_{s''}}}[\mathtt{s''}]] \in R^\sharp(\mathtt{k_t})$, and $\mathtt{stop} \in R^\sharp(\mathtt{k_s})$.

Hence $\mathtt{k_t}\ \mathcal{V}[\mathtt{t}] \in Q^\sharp$, $[\mathtt{fn\ x''} \Rightarrow \mathcal{F}_{\mathtt{k_{s''}}}[\mathtt{s''}]] \in \mu^\sharp_c(\mathtt{k_t}, R^\sharp)$, and $\mathtt{stop} \in R^\sharp(\mathtt{k_s})$. Since $R^\sharp \dot{\subseteq} R^\sharp_{ret}$ we have $\mathtt{stop} \in R^\sharp_{ret}(\mathtt{k_s})$.

(2d) Assume $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in E^\sharp_{ret}(\mathtt{y})$. Hence there exists $\mathtt{x'}, \mathtt{s'}, \mathtt{t}$ such that $\{\mathtt{t}\} \subseteq C$, $\{[\mathtt{x'},\, \mathtt{s'}]\} \subseteq F^\sharp([\mathtt{t}]_\equiv)$, and $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in (E^\sharp \dot{\cup} [\mathtt{x'} \mapsto \mu^\sharp(\mathtt{t}, E^\sharp)])(\mathtt{y})$.

From the $\langle C,\, F^\sharp,\, E^\sharp\rangle \;\mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\; \langle Q^\sharp,\, R^\sharp\rangle$ assumption we have $\mathcal{F}_{\mathtt{k_t}}[\mathtt{t}] \in Q^\sharp$ and $[\mathtt{fn\ x'} \Rightarrow \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s'}]] \in R^\sharp(\mathtt{k_t})$.

Hence $\mathtt{k_t}\ \mathcal{V}[\mathtt{t}] \in Q^\sharp$ and $[\mathtt{fn\ x'} \Rightarrow \mathcal{F}_{\mathtt{k_{s'}}}[\mathtt{s'}]] \in \mu^\sharp_c(\mathtt{k_t}, R^\sharp)$.

There are now two subcases:

1. $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in E^\sharp(\mathtt{y})$. Hence $[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp(\mathtt{y})$. Since $R^\sharp \dot{\subseteq} R^\sharp_{ret}$ we have $[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp_{ret}(\mathtt{y})$.
2. $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in [\mathtt{x'} \mapsto \mu^\sharp(\mathtt{t}, E^\sharp)](\mathtt{y})$. If $\mathtt{y} \neq \mathtt{x'}$ our assumption reads $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in \emptyset$. Hence $[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp_{ret}(\mathtt{y})$ is trivially true.

   If $\mathtt{y} = \mathtt{x'}$ our assumption reads $[\mathtt{fn\ x} \Rightarrow \mathtt{s}] \in \mu^\sharp(\mathtt{t}, E^\sharp)$. By Lemma 6.2 it now follows that $[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in \mu^\sharp_t(\mathcal{V}[\mathtt{t}], R^\sharp)$. As a consequence $[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathcal{F}_{\mathtt{k_s}}[\mathtt{s}]] \in R^\sharp_{ret}(\mathtt{y})$.

Realizing that the union of related triples and pairs are related we obtain the desired result. □

After realizing that the bottom elements are related by the above relation, it follows by fixed point induction that their least fixed points (and hence the analyses) are related.

Corollary 6.1. $\mathrm{lfp}\ F^\sharp_{\mathtt{p}}\ \mathrm{R}^{\mathrm{ANF}}_{\mathrm{CPS}}\ \mathrm{lfp}\ T^\sharp_{\mathcal{C}[\mathtt{p}]}$

6.5 CPS-ANF equivalence

Again we formally define a relation now relating CPS analysis pairs to ANF analysis triples.

Definition 6.3. $\langle Q^\sharp,\, R^\sharp\rangle \;\mathrm{R}^{\mathrm{CPS}}_{\mathrm{ANF}}\; \langle C,\, F^\sharp,\, E^\sharp\rangle$ iff $\forall \mathtt{e}$:

$$\begin{aligned}
\mathtt{e} \in Q^\sharp &\implies \mathcal{U}[\mathtt{e}] \in C \;\wedge \\
[\mathtt{fn\ x} \Rightarrow \mathtt{e}] \in R^\sharp(\mathtt{k_s}) &\implies [\mathtt{x}, \mathcal{U}[\mathtt{e}]] \in F^\sharp([\mathtt{s}]_\equiv) \;\wedge \\
\mathtt{stop} \in R^\sharp(\mathtt{k_s}) &\implies \mathtt{stop} \in F^\sharp([\mathtt{s}]_\equiv) \;\wedge \\
[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathtt{e}] \in R^\sharp(\mathtt{y}) &\implies [\mathtt{fn\ x} \Rightarrow \mathcal{U}[\mathtt{e}]] \in E^\sharp(\mathtt{y})
\end{aligned}$$

We again need a helper lemma relating the helper functions.

Lemma 6.3.

$$\begin{aligned}
[\mathtt{fn\ x}, \mathtt{k_s} \Rightarrow \mathtt{e}] \in \mu^\sharp_t(\mathtt{t}, R^\sharp) \;\wedge\; \langle Q^\sharp,\, R^\sharp\rangle \;\mathrm{R}^{\mathrm{CPS}}_{\mathrm{ANF}}\; \langle C,\, F^\sharp,\, E^\sharp\rangle \\
\implies [\mathtt{fn\ x} \Rightarrow \mathcal{U}[\mathtt{e}]] \in \mu^\sharp(\mathcal{P}[\mathtt{t}], E^\sharp)
\end{aligned}$$

This relation is also preserved by the transfer functions.

Theorem 6.3.

$$\langle Q^\sharp,\, R^\sharp\rangle \;\mathrm{R}^{\mathrm{CPS}}_{\mathrm{ANF}}\; \langle C,\, F^\sharp,\, E^\sharp\rangle \implies T^\sharp_{\mathcal{C}[\mathtt{p}]}(\langle Q^\sharp,\, R^\sharp\rangle)\; \mathrm{R}^{\mathrm{CPS}}_{\mathrm{ANF}}\; F^\sharp_{\mathtt{p}}(\langle C,\, F^\sharp,\, E^\sharp\rangle)$$

Proof. The proof follows a similar structure to the earlier proof. □

The bottom elements are related by the relation and it follows by fixed point induction that their least fixed points (and hence the analyses) are related.

Corollary 6.2. $\mathrm{lfp}\, T^\sharp_{\mathcal{C}[\mathtt{p}]}\; \mathrm{R}^{\mathrm{CPS}}_{\mathrm{ANF}}\; \mathrm{lfp}\, F^\sharp_{\mathtt{p}}$

7 Extracting constraints

The resulting analysis may appear complex at first glance. However we can express the analysis in the popular constraint formulation, extracted from the obtained definition. The formulation shown below is in terms of program-specific conditional constraints.

Constraints have a (possibly empty) list of preconditions and a conclusion [Palsberg and Schwartzbach, 1995, Gasser et al., 1997]:

$$\{u_1\} \subseteq rhs_1 \;\wedge\; \ldots \;\wedge\; \{u_n\} \subseteq rhs_n \Rightarrow lhs \subseteq rhs$$

The constraints operate on the same three domains as the above analysis. Left-hand sides *lhs* can be of the form $\{u\}$, $F^\sharp([\mathtt{s}]_\equiv)$, or $E^\sharp(\mathtt{x})$, right-hand sides *rhs* can be of the form C, $F^\sharp([\mathtt{s}]_\equiv)$, or $E^\sharp(\mathtt{x})$, and singleton elements u can be of the form $\mathtt{s}$, c, $[\mathtt{fn\ x\ =>\ s}]$, or $[\mathtt{x},\, \mathtt{s}]$. From Fig. 8 we directly read off the following constraints.

- For the program $\mathtt{p}$:

$$\{\mathtt{p}\} \subseteq C \qquad \{[\mathtt{x_r},\, \mathtt{x_r}]\} \subseteq F^\sharp([\mathtt{p}]_\equiv) \qquad \{\mathtt{stop}\} \subseteq F^\sharp([\mathtt{x_r}]_\equiv)$$

- For each return expression $\mathtt{t}$ and non-tail call $\mathtt{let\ x{=}t_0\, t_1\ in\ s'}$ in $\mathtt{p}$:

$$\{\mathtt{t}\} \subseteq C \;\wedge\; \{[\mathtt{x},\, \mathtt{s'}]\} \subseteq F^\sharp([\mathtt{t}]_\equiv) \Rightarrow \begin{cases} \{\mathtt{s'}\} \subseteq C \;\wedge \\ \mu_{sym}(\mathtt{t}, E^\sharp) \subseteq E^\sharp(\mathtt{x}) \end{cases}$$

- For each let-binding $\mathtt{let\ x{=}t\ in\ s}$ in $\mathtt{p}$:

$$\{\mathtt{let\ x{=}t\ in\ s}\} \subseteq C \Rightarrow \begin{cases} \{\mathtt{s}\} \subseteq C \;\wedge \\ \mu_{sym}(\mathtt{t}, E^\sharp) \subseteq E^\sharp(\mathtt{x}) \end{cases}$$

- For each tail call $\mathtt{t_0\, t_1}$ and function $\mathtt{fn\ x\ =>\ s'}$ in $\mathtt{p}$:

$$\begin{array}{c} \{\mathtt{t_0\, t_1}\} \subseteq C \;\wedge \\ \{[\mathtt{fn\ x\ =>\ s'}]\} \subseteq \mu_{sym}(\mathtt{t_0}, E^\sharp) \end{array} \Rightarrow \begin{cases} \{\mathtt{s'}\} \subseteq C \;\wedge \\ F^\sharp([\mathtt{t_0\, t_1}]_\equiv) \subseteq F^\sharp([\mathtt{s'}]_\equiv) \;\wedge \\ \mu_{sym}(\mathtt{t_1}, E^\sharp) \subseteq E^\sharp(\mathtt{x}) \end{cases}$$

- For each non-tail call $\texttt{let x=t}_0\ \texttt{t}_1\ \texttt{in s}$ and function $\texttt{fn y => s}'$ in $\texttt{p}$:

$$\begin{array}{l}\{\texttt{let x=t}_0\ \texttt{t}_1\ \texttt{in s}\} \subseteq C\ \wedge \\ \{[\texttt{fn y => s}']\} \subseteq \mu_{sym}(\texttt{t}_0, E^\sharp)\end{array} \Rightarrow \begin{cases}\{\texttt{s}'\} \subseteq C\ \wedge \\ \{[\texttt{x},\ \texttt{s}]\} \subseteq F^\sharp([\texttt{s}']_\equiv)\ \wedge \\ \mu_{sym}(\texttt{t}_1, E^\sharp) \subseteq E^\sharp(\texttt{y})\end{cases}$$

where we partially evaluate the helper function μ_{sym}, i.e., interpret the helper function symbolically at constraint-generation time, to generate a lookup for variables, and a singleton for constants and lambda expressions. The definition of the symbolic helper function otherwise coincides with the abstract helper function $\mu^\sharp$:

$$\begin{aligned}\mu_{sym}(c, E^\sharp) &= \{c\} \\ \mu_{sym}(\texttt{x}, E^\sharp) &= E^\sharp(\texttt{x}) \\ \mu_{sym}(\texttt{fn x => s}, E^\sharp) &= \{[\texttt{fn x => s}]\}\end{aligned}$$

We may generate constraints $\{[\texttt{fn x => s}]\} \subseteq \{[\texttt{fn y => s}']\}$ of a form not covered by the above grammar. We therefore first pre-process the constraints in linear time,

- removing vacuously true inclusions $\{[\texttt{fn x => s}]\} \subseteq \{[\texttt{fn x => s}]\}$ from each constraint, and
- removing constraints with vacuously false preconditions $\{[\texttt{fn x => s}]\} \subseteq \{w^\sharp\}$, where $[\texttt{fn y => s}'] \neq w^\sharp$.

The resulting constraint system is formally equivalent to the control flow analysis in the sense that all solutions yield correct control flow information and that the best (smallest) solution of the constraints is as precise as the information computed by the analysis. More formally:

Theorem 7.1. A solution to the CFA constraints of program $\texttt{p}$ is a safe approximation of the least fixpoint of the analysis function $F^\sharp$ induced by $\texttt{p}$. Furthermore, the least solution to the CFA constraints is equal to the least fixpoint of $F^\sharp$.

Proof. The first part of the theorem is proved by showing that a solution to the CFA constraints $\langle C,\ F,\ E\rangle$ is a post-fixpoint of $F^\sharp$, *i.e.*, that it satisfies $F^\sharp(\langle C,\ F,\ E\rangle) \subseteq_\otimes \langle C,\ F,\ E\rangle$ and then appeal to the Knaster-Tarski fixpoint theorem that the least fixpoint of a monotone operator $F^\sharp$ is the greatest lower bound of the set of post-fixpoints of $F^\sharp$. This reduces to showing that for each of the expressions defining $F^\sharp$ in Fig. 8 we have that its value is already included in the solution $\langle C,\ F,\ E\rangle$. For example, for the expression

$$\bigcup_{\substack{\{\texttt{t}\}\subseteq C \\ \{[\texttt{x},\,\texttt{s}']\}\subseteq F^\sharp([\texttt{t}]_\equiv)}}^{\otimes} \langle\{\texttt{s}'\},\ F^\sharp,\ E^\sharp \dot\cup [\texttt{x} \mapsto \mu^\sharp(\texttt{t}, E^\sharp)]\rangle$$

we must have, for all $\texttt{t}$ satisfying $\{\texttt{t}\} \subseteq C$ and $\texttt{s}'$ satisfying $\{[\texttt{x},\ \texttt{s}']\} \subseteq F^\sharp([\texttt{t}]_\equiv)$, that

$$\{\texttt{s}'\} \subseteq C \quad \text{and} \quad E^\sharp \dot\cup [\texttt{x} \mapsto \mu^\sharp(\texttt{t}, E^\sharp)] \dot\subseteq E^\sharp.$$

The latter inequality reduces to $\mu_{sym}(\mathtt{t}, E^\sharp) \subseteq E^\sharp(\mathtt{x})$. and we obtain exactly the constraints for return expressions. The other cases follow by similar reasoning.

For the equality of the least solution and the least fixpoint, it then suffices to prove that the fixpoint is a solution to the CFA constraints. The argumentation is again based on unfolding the definition of $F^\sharp$ and using reasoning similar to above. □

Implemented naively, a single constraint may take $O(n)$ space alone. However by using pointers or by labelling each sub-expression and using the pointer or label instead of the sub-expression itself, a single constraint takes only constant space. By linearly determining a representative for each sub-expression, by generating $O(n^2)$ constraints, linear post-processing, and iteratively solving them using a well-known algorithm [Palsberg and Schwartzbach, 1995, Gasser et al., 1997, Nielson et al., 1999], we can compute the analysis in worst-case $O(n^3)$ time.

The extracted constraints bear similarities to existing constraint-based analyses in the literature. Consider, e.g., calls $\mathtt{t_0}\,\mathtt{t_1}$, which usually gives rise to two conditional constraints [Palsberg, 1995, Nielson et al., 1999]: (1) $\{[\mathtt{fn\ x\ =>\ s'}]\} \subseteq \widehat{C}(\mathtt{t_0}) \Rightarrow \widehat{C}(\mathtt{t_1}) \subseteq \widehat{E}(\mathtt{x})$ and (2) $\{[\mathtt{fn\ x\ =>\ s'}]\} \subseteq \widehat{C}(\mathtt{t_0}) \Rightarrow \widehat{C}(\mathtt{s'}) \subseteq \widehat{C}(\mathtt{t_0}\,\mathtt{t_1})$. The first constraint resembles our third constraint for tail calls. The second "return constraint" differs in that it has a inside-out (or caller-restore) nature, i.e., propagation of return-flow from the function body is handled at the call-site. The extracted reachability constraints are similar to Gasser et al. [1997] (modulo an isomorphic encoding $\wp(C) \simeq C \to \wp(\{\mathbf{on}\})$ of powersets).

8 Applications of the analysis

In a compiler a 0-CFA can be used for a number of transformations and optimizations. As an example we can disregard any expression from the program which is not reachable $\mathtt{s} \notin C$. In CPS where everything is a call, 0-CFA lends itself to a number of call optimizations. Fluet and Weeks [2001] coined the term *contification* for the transformation that turns a function into a continuation. In the words of Kennedy [2007]:

> Sometimes it is the case that a function can be transformed into a continuation, a process known as *contification*. This is possible exactly when the function always returns to the same place.

This condition is exactly the property that our analysis computes!

By appealing to the ANF-CPS isomorphism [Danvy, 1994] we formulate an equivalent condition for ANF: A function always returning to the same place can be transformed into a function representing *the rest of the computation*, i.e., turning non-tail calls into tail calls. Recall the example from the introduction:

```
let g z = z in
 let f k = if b then k 1 else k 2 in
  let y = f (fn x => x) in
     g y
```

Notice that the two calls to `k` are in tail-position: when either of the two calls return, control continues by binding the intermediate result to `y` and to the outer call to `g`.

When evaluated in some environment where `b` is bound, our analysis (straight forwardly extended with conditionals) determines that $F^\sharp(\mathtt{x}) = \{[\mathtt{y},\ \mathtt{g}\ \mathtt{y}]\}$, i.e., the function `fn x => x` will always return to the same `let`-binding, and hence we can *inline* the rest of the computation in the function body:

```
let g z = z in
 let f k = if b then k 1 else k 2 in
   f (fn x => let y = x in
                g y)
```

The transformation lends itself to further optimizations: the let-binding can be eliminated, `g` can be inlined, etc.

Traditionally, a compiler may decide to inline a particular function call $\mathtt{t}_0\,\mathtt{t}_1$, if a 0-CFA can determine that only one particular lambda can be called: $\mu^\sharp(\mathtt{t}_0, E^\sharp) = \{[\mathtt{fn\ x\ =>\ s}]\}$, provided that the values of any free variables of the function are available in the lexical scope of the call-site. Dually, a compiler should be able to inline a particular function return `t`, if an analysis determines that it will always return to the same point: $F^\sharp(\mathtt{t}) = \{[\mathtt{x},\ \mathtt{s}]\}$, provided that the values of any free variables of the rest of the computation are available in the lexical scope of the function-body. This idea is precisely the higher-order, direct-style version of the contification transformation described above.

Determining that two expressions always agree on the values of their free variables when evaluated is itself an interesting problem. A crude but correct condition is to prohibit inlining in the presence of free variables. A better approximation would be to allow inlining only when the values of any free variables are constant, e.g., when they denote top-level functions. More powerful flow analysis techniques have been pursued by Steckler and Wand [1997] and more recently by Might and Shivers [2006].

Alternatively, if a 0-CFA determines that only one particular lambda is called at a particular call-site, the compiler can generate a *direct call*, rather than an *indirect call* to an extracted lambda-expression of a closure. Dually, if our analysis determines that a particular function will always return to the same point, the compiler can generate a *direct return*, i.e., a direct jump and a call stack pop, rather than an *indirect return* through a code pointer stored on the call stack.

Debray and Proebsting [1997] list a number of applications of CFA: most notable the creation of interprocedural control-flow graphs, which in turn enable an optimization like interprocedural unboxing. An alternative optimization enabled by CFA is interprocedural basic block fusion, which bears a strong resemblance to direct-style contification as described above.

Formulating a CFA as traditional abstract interpretation furthermore allows us to integrate the CFA-domains with other domains and analyses. Hence it should be possible to formulate interval, polyhedra, or octagon analyses of higher-order functional programs using an approach similar to Nielson et al. [1999, Ch.3].

Fluet and Weeks [2001] defined the *contification* transformation for a first-order language. Furthermore they developed an optimal algorithm based on dominators. Kennedy [2007] formulated a *local contification* transformation for a higher-order language in CPS. Debray and Proebsting [1997] studied control-flow analysis for a tail-call optimized first-order language. They showed how the problem corresponds to traditional concepts from parsing theory. In this light,

one can regard the current paper as a higher-order counterpart of Debray and Proebsting's first-order tail call-optimized 0-CFA.

9 Conclusion

We have presented a control-flow analysis determining interprocedural control-flow of both calls and returns for a direct-style language. Existing CFAs have focused on analysing which functions are called at a given call site. In contrast, the systematic derivation of our CFA has lead to an analysis that provides extra information about where a function returns to at no additional cost. In the presence of tail-call optimization, such information enables the creation of more precise call graphs.

The analysis was developed systematically using Galois connection-based abstract interpretation of a standard operational semantics for that language: the C_aEK abstract machine of Flanagan et al. In addition to being more principled, such a formulation of the analysis is pedagogically pleasing since monomorphism of the analysis is made explicit through two Galois connections: one literally merges all bindings to the same variable and one merges all calling contexts of the same function.

The analysis has been shown to provide a result equivalent to what can be obtained by first CPS transforming the program and then running a control flow analysis derived from a CPS-based operational semantics. This extends previous results obtained by Damian and Danvy, and Palsberg and Wand. The close correspondence between the way that the analyses operate (as illustrated by the analysis trace in Table 1) leads us to conjecture that such equivalence results can be obtained for other CFAs derived using abstract interpretation.

The functional, derived by abstract interpretation, that defines the analysis may appear rather complex at first glance. As a final result, we have shown how to extract from the analysis an equivalent constraint-based formulation expressed in terms of the more familiar conditional constraints. Nevertheless, we stress that the derived functional can be used directly to implement the analysis. We have developed a prototype implementation of the resulting analysis in OCaml.[2]

The analysis has been developed for a minimalistic functional language in order to be able to focus on the abstraction of the control structure induced by function calls and returns. An obvious extension is to enrich the language with numerical operators and study how our Galois connections interact with abstractions such as the interval or polyhedral abstraction of numerical entities.

The calculations involved in the derivation of a CFA are lengthy and would benefit enormously from some form of machine support. *Certified abstract interpretation* [Pichardie, 2005, Cachera et al., 2005] has so far focused on proving the correctness of the analysis inside a proof assistant by using the concretization (γ) component of the Galois connection to prove the correctness of an already defined analysis. Further work should investigate whether proof assistants such as Coq are suitable for conducting the kind of reasoning developed in this paper in a machine-checkable way.

[2] available at `http://www.brics.dk/~jmi/ANF-CFA/`

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A Underlying mathematical material

This section is based on known material from the abstract interpretation literature [Cousot and Cousot, 1979, Cousot, 1981, Cousot and Cousot, 1992b, 1994, Davey and Priestley, 2002].

A partially ordered set (poset) $\langle S; \sqsubseteq \rangle$ is a set S equipped with a partial order $\sqsubseteq$. A complete lattice is a poset $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$, such that the least upper bound $\sqcup S$ and the greatest lower bound $\sqcap S$ exists for every subset S of C. $\bot = \sqcap C$ denotes the infimum of C and $\top = \sqcup C$ denotes the supremum of C. The set of total functions $D \to C$, whose domain is a complete lattice $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$, is itself a complete lattice $\langle D \to C; \dot{\sqsubseteq}, \dot{\bot}, \dot{\top}, \dot{\sqcup}, \dot{\sqcap} \rangle$ under the pointwise ordering $f \dot{\sqsubseteq} f' \iff \forall x. f(x) \sqsubseteq f'(x)$, and with bottom, top, join, and meet extended similarly. The powersets $\wp(S)$ of a set S ordered by set inclusion is a complete lattice $\langle \wp(S); \subseteq, \emptyset, S, \cup, \cap \rangle$.

A Galois connection is a pair of functions α, γ between two posets $\langle C; \sqsubseteq \rangle$ and $\langle A; \leq \rangle$ such that for all $a \in A, c \in C : \alpha(c) \leq a \iff c \sqsubseteq \gamma(a)$. Equivalently a Galois connection can be defined as a pair of functions satisfying (a) α and γ are monotone (for all $c, c' \in C : c \sqsubseteq c' \implies \alpha(c) \leq \alpha(c')$ and for all $a, a' \in A : a \leq a' \implies \gamma(a) \sqsubseteq \gamma(a')$), (b) $\alpha \circ \gamma$ is reductive (for all $a \in A : \alpha \circ \gamma(a) \leq a$), and (c) $\gamma \circ \alpha$ is extensive (for all $c \in C : c \sqsubseteq \gamma \circ \alpha(c)$). Galois connections are typeset as $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$. We omit the orderings when they are clear from the context. For a Galois connection between two complete lattices $\langle C; \sqsubseteq, \bot_c, \top_c, \sqcup, \sqcap \rangle$ and $\langle A; \leq, \bot_a, \top_a, \vee, \wedge \rangle$, α is a complete join-morphism (CJM) (for all $S_c \subseteq C : \alpha(\sqcup S_c) = \vee \alpha(S_c) = \vee \{\alpha(c) \mid c \in S_c\}$) and γ is a complete meet morphism (for all $S_a \subseteq A : \gamma(\wedge S_a) = \sqcap \gamma(S_a) = \sqcap \{\gamma(a) \mid a \in S_a\}$). The composition of two Galois connections $\langle C; \sqsubseteq \rangle \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} \langle B; \subseteq \rangle$ and $\langle B; \subseteq \rangle \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \langle A; \leq \rangle$ is itself a Galois connection $\langle C; \sqsubseteq \rangle \underset{\alpha_2 \circ \alpha_1}{\overset{\gamma_1 \circ \gamma_2}{\leftrightarrows}} \langle A; \leq \rangle$. Galois connections in which α is surjective (or equivalently γ is injective) are typeset as: $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$. Galois connections in which γ is surjective (or equivalently α is injective) are typeset as: $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$. When both α and γ are surjective, the two domains are isomorphic.

A(n upper) closure operator ρ is map $\rho : S \to S$ on a poset $\langle S; \sqsubseteq \rangle$, that is (a) monotone: (for all $s, s' \in S : s \sqsubseteq s' \implies \rho(s) \sqsubseteq \rho(s')$), (b) extensive (for all $s \in S : s \sqsubseteq \rho(s)$), and (c) idempotent, (for all $s \in S : \rho(s) = \rho(\rho(s))$). A closure operator ρ induces a Galois connection $\langle S; \sqsubseteq \rangle \underset{\rho}{\overset{1}{\leftrightarrows}} \langle \rho(S); \sqsubseteq \rangle$, writing $\rho(S)$ for $\{\rho(s) \mid s \in S\}$ and 1 for the identity function. Furthermore the image of a complete lattice $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ by an upper closure operator is itself a complete lattice $\langle \rho(C); \sqsubseteq, \rho(\bot), \top, \lambda X.\, \rho(\sqcup X), \sqcap \rangle$.

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