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## An impossibility result for process discrimination

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# Discrimination between B-processes is impossible.

Daniil Ryabko \*

## Abstract

Two series of binary observations  $x_1, x_1, \dots$  and  $y_1, y_2, \dots$  are presented: at each time  $n \in \mathbb{N}$  we are given  $x_n$  and  $y_n$ . It is assumed that the sequences are generated independently of each other by two B-processes. We are interested in the question of whether the sequences represent a typical realization of two different processes or of the same one. We demonstrate that this is impossible to decide, in the sense that every discrimination procedure is bound to err with non-negligible frequency when presented with sequences from some B-processes. This contrasts earlier positive results on B-processes, in particular those showing that there are consistent  $\bar{d}$ -distance estimates for this class of processes.

*Keywords:* Process discrimination, B-processes, stationary ergodic processes, time series, homogeneity testing

## 1 Introduction

Two series of binary observations  $x_1, x_1, \dots$  and  $y_1, y_2, \dots$  are presented sequentially. A *discrimination procedure*  $D$  is a family of mappings  $D_n : X^n \times X^n \rightarrow \{0, 1\}$ ,  $n \in \mathbb{N}$ , that maps a pair of samples  $(x_1, \dots, x_n), (y_1, \dots, y_n)$  into a binary (“yes” or “no”) answer: the samples are generated by different distributions, or they are generated by the same distribution.

A discrimination procedure  $D$  is *asymptotically correct* for a set  $\mathcal{C}$  of process distributions if for any two distributions  $\rho_x, \rho_y \in \mathcal{C}$  independently generating the sequences  $x_1, x_2, \dots$  and  $y_1, y_2, \dots$  correspondingly the expected output converges to the correct answer: the following limit exists and the equality holds

$$\lim_{n \rightarrow \infty} \mathbf{E} D_n((x_1, \dots, x_n), (y_1, \dots, y_n)) = \begin{cases} 0 & \text{if } \rho_x = \rho_y \\ 1 & \text{otherwise} \end{cases} .$$

Note that one can consider other notions of asymptotic correctness, for example one can require the output to stabilize on the correct answer with probability 1. The notion of correctness that we consider is perhaps one of the weakest. Clearly, asymptotically correct discriminating procedures exist for many classes of processes, for example for the class of all i.i.d. processes.

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Ornstein and Weiss [7] and Ornstein and Shields [6] show that consistent estimates of  $\bar{d}$ -distance for  $B$ -processes (see definitions below) exist, while it is impossible to estimate this distance outside this class (i.e. in general for stationary ergodic processes). We show that discrimination between  $B$ -processes is impossible, in the sense that any discrimination procedure is bound to err on some processes (the expected answer does not converge to the correct one). This demonstrates that discrimination is harder than distance estimation. This also complements earlier negative results on  $B$ -processes, such as [11] that shows that upper and lower divergence rates need not be the same for  $B$ -processes, and on stationary ergodic processes, such as [8, 2, 1], that establish negative results concerning prediction and density estimation.

The class of  $B$ -processes is sufficiently wide to include, for example,  $k$ -order Markov processes and functions of them, but, on the other hand, it is a strict subset of the set of stationary ergodic processes.  $B$ -processes play important role in such fields as information theory and ergodic theory [12, 13, 5]. Discrimination procedures for smaller classes of processes, such as the set of i.i.d. processes or various parametric families, exist and are widely studied (see e.g. [3]); some positive results on hypothesis testing for stationary ergodic process can be found in [4, 9, 10].

Next we define the  $\bar{d}$  distance and  $B$ -processes, mainly following [7] in our formulations. For two finite-valued stationary processes  $\rho_x$  and  $\rho_y$  the  $\bar{d}$ -distance  $\bar{d}(\rho_x, \rho_y)$  is said to be less than  $\varepsilon$  if there exists a single stationary process  $\nu_{xy}$  on pairs  $(x_n, y_n)$ ,  $n \in \mathbb{N}$ , such that  $x_n$ ,  $n \in \mathbb{N}$  are distributed according to  $\rho_x$  and  $y_n$  are distributed according to  $\rho_y$  while

$$\nu_{xy}(x_1 \neq y_1) \leq \varepsilon. \tag{1}$$

The infimum of the  $\varepsilon$ 's for which a coupling can be found such that (1) is satisfied is taken to be the  $\bar{d}$ -distance between  $\rho_x$  and  $\rho_y$ .

A process is called a  $B$ -process (or a Bernoulli process) if it is in the  $\bar{d}$ -closure of the set of all aperiodic stationary ergodic  $k$ -step Markov processes, where  $k \in \mathbb{N}$ . For more information on  $\bar{d}$ -distance and  $B$ -processes (including a more conventional ergodic-theory definition and its equivalence to the one above) the reader is referred to [5].

## 2 The main result

The main result of this work is the following theorem; the construction used in the proof is based on the same ideas as the construction used in [8] to demonstrate that consistent prediction for stationary ergodic processes is impossible, and to its modification in [2].

**Theorem 1.** *There is no asymptotically correct discrimination procedure for the set of all  $B$ -processes.*

*Proof.* We will assume that such a procedure  $D$  exists and will construct a  $B$ -process  $\rho$  such that if both sequences  $x_i$  and  $y_i$ ,  $i \in \mathbb{N}$  are generated by  $\rho$  then

$\mathbf{E}D_n$  diverges; this contradiction will prove the theorem.

The scheme of the proof is as follows. On Step 1 we construct a sequence of  $B$ -processes  $\rho_{2k}$ ,  $\rho_{u2k+1}$ , and  $\rho_{d2k+1}$ , where  $k = 0, 1, \dots$ . On Step 2 we construct a  $B$ -process  $\rho$ . On Step 3 we show that two independent runs of the process  $\rho$  have a property that (with high probability) they first behave like two runs of a single process  $\rho_0$ , then like two runs of two different processes  $\rho_{u1}$  and  $\rho_{d1}$ , then like two runs of a single process  $\rho_2$ , and so on, thereby showing that the test  $D$  diverges and obtaining the desired contradiction.

Each processes that we construct has the form of a stationary Markov chain with a countably infinite set of states, with a (deterministic) function mapping each state to  $\{0, 1\}$ . In other words, the constructions are based on partially observable Markov processes, where the observed variables are from  $\{0, 1\}$  and the (non-observable) states are from the set  $\mathbb{N}$ .

The construction of each of the processes  $\rho_{2k}$ ,  $\rho_{u2k+1}$ ,  $\rho_{d2k+1}$ , for  $k > 0$ , and of the process  $\rho$  is broken into two parts: first it is given in terms of a chain on which Markov property is violated, and then, in order to define the initial distribution on this chain, and to show that the resulting process is stationary ergodic (and a  $B$ -process), we show that this chain is equivalent to a Markov chain, which has a stationary distribution with positive probabilities of all states.

Assume that there exists an asymptotically correct discriminating procedure  $D$ . Fix some  $\varepsilon > 0$  and  $\delta \in [1/2, 1)$ , to be defined on Step 3.

*Step 1.* We will construct the sequence of process  $\rho_{2k}$ ,  $\rho_{u2k+1}$ , and  $\rho_{d2k+1}$ , where  $k = 0, 1, \dots$ . Step 1 consists of sub-steps 1.0, on which  $\rho_0$  is constructed, 1.1, which constructs  $\rho_{u1}$  and  $\rho_{d1}$ , followed by 1.2 with  $\rho_2$ , and the step 1. $k$  with  $\rho_{u3}$  and  $\rho_{d3}$ ,  $\rho_4$  and a general scheme of constructing the rest of the sequence.

*Step 1.0.* Construct the process  $\rho_0$  as follows. A Markov chain  $m_0$  is defined on the set  $\mathbb{N}$  of states. From each state  $i \in \mathbb{N}$  the chain passes to the state 0 with probability  $\delta$  and to the state  $i + 1$  with probability  $1 - \delta$ . With transition probabilities so defined, the chain possesses a unique stationary distribution  $M_0$  on the set  $\mathbb{N}$  of states such that  $M_0(i) > 0$  for all  $i \in \mathbb{N}$  (see e.g. [14]). Take this distribution as the initial distribution over the states.

The function  $f_0$  maps the states to the output alphabet  $\{0, 1\}$  as follows:  $f_0(i) = 1$  for every  $i \in \mathbb{N}$ . Let  $s_t$  be the state of the chain at time  $t$ . The process  $\rho_0$  is defined as  $\rho_0 = f_0(s_t)$  for  $t \in \mathbb{N}$ . As a result of this definition, the process  $\rho_0$  simply outputs 1 with probability 1 on every time step (however by using different functions  $f$  we will have less trivial processes in the sequel). Clearly, the constructed process is stationary ergodic and a  $B$ -process. So, we have defined the chain  $m_0$  (and the process  $\rho_0$ ) up to a parameter  $\delta$ .

*Step 1.1.* We begin with the process  $\rho_0$  and the chain  $m_0$  of the previous step. Since the test  $D$  is asymptotically correct we will have

$$\mathbf{E}_{\rho_0 \times \rho_0} D_{t_0}((x_1, \dots, x_{t_0}), (y_1, \dots, y_{t_0})) < \varepsilon,$$

for some  $t_0$ , where both samples  $x_i$  and  $y_i$  are generated by  $\rho_0$  (that is, both samples consist of 1s only). Let  $k_0$  be such an index that the chain  $m_0$  starting from the state 0 with probability 1 does not reach the state  $k_0 - 1$  by time  $t_0$  (we can take  $k_0 = t_0 + 2$ ).

Construct two processes  $\rho_{u1}$  and  $\rho_{d1}$  as follows. They are also based on the Markov chain  $m_0$ , but the functions  $f$  are different. The function  $f_{u1} : \mathbb{N} \rightarrow \{0, 1\}$  is defined as follows:  $f_{u1}(i) = f_0(i) = 1$  for  $i \leq k_0$  and  $f_{u1}(i) = 0$  for  $i > k_0$ . The function  $f_{d1}$  is identically 1 ( $f_{d1}(i) = 1, i \in \mathbb{N}$ ). The processes  $\rho_{u1}$  and  $\rho_{d1}$  are defined as  $\rho_{u1} = f_{u1}(s_t)$  and  $\rho_{d1} = f_{d1}(s_t)$  for  $t \in \mathbb{N}$ . Thus the process  $\rho_{d1}$  will again produce only 1s, but the process  $\rho_{u1}$  will occasionally produce 0s. It is easy to check that the processes  $\rho_{u1}$  and  $\rho_{d1}$  are  $B$ -processes (cf. Step 2c below, where it is shown that the process  $\rho$  is a  $B$ -process).

*Step 1.2.* Being run on two samples generated by  $\rho_{u1}$  and  $\rho_{d1}$  the test  $D_n$  on the first  $t_0$  steps with high probability (that is, at least if both processes start at the state 0) produces many 0s, since on these first  $k_0$  states all the functions  $f, f_{u1}$  and  $f_{d1}$  coincide. However since the processes are different and the test is asymptotically correct (by assumption), the test starts producing 1s, until by a certain time step  $t_1$  almost all answers are 1s. Next we will construct the process  $\rho_2$  by “gluing” together  $\rho_{u1}$  and  $\rho_{d1}$  and continuing them in such a way that, being run on two samples produced by  $\rho_2$  the test first produces 0s (as if the samples were drawn from  $\rho_0$ ), then, with probability close to 1/2 it will produce many 1s (as if the samples were from  $\rho_{u1}$  and  $\rho_{d1}$ ) and then again 0s.

The process  $\rho_2$  is constructed in two steps. On step 1.2a the construction is given in the form of a chain on which Markov property is violated, and on step 1.2b we show that the construction is equivalent to a Markov chain. The stationary distribution on this chain will be used to finish the construction of  $\rho_2$ .

*Step 1.2a.* Let  $t_1 > t_0$  be such a time index that

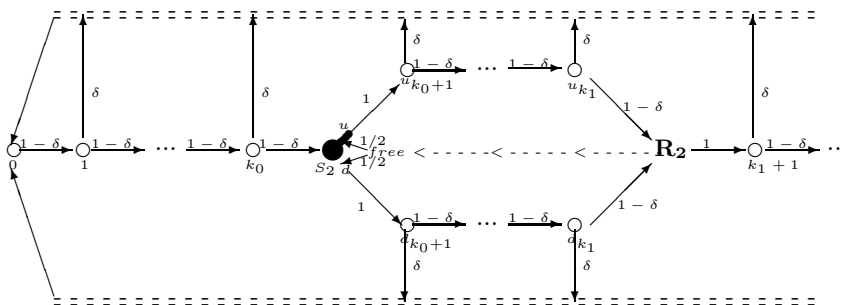
$$\mathbf{E}_{\rho_{u1} \times \rho_{d1}} D_k((x_1, \dots, x_{t_1}), (y_1, \dots, y_{t_1})) > 1 - \varepsilon,$$

where the samples  $x_i$  and  $y_i$  are generated by  $\rho_{u1}$  and  $\rho_{d1}$  correspondingly (the samples are generated independently; that is, the process are based on two independent copies of the Markov chain  $m$ ). Let  $k_1$  be such an index that the chain  $m$  starting from the state 0 with probability 1 does not reach the state  $k_1 - 1$  by time  $t_1$ .

Construct the process  $\rho_2$  as follows (see fig. 1). It is based on a chain  $m_2$  on which Markov assumption is violated. The transition probabilities on states  $0, \dots, k_0$  are the same as for the Markov chain  $m$  (from each state return to 0 with probability  $\delta$  or go to the next state with probability  $1 - \delta$ ).

There are two “special” states: the “switch”  $S_2$  and the “reset”  $R_2$ . From the state  $k_0$  the chain passes with probability  $1 - \delta$  to the “switch” state  $S_2$ . The switch  $S_2$  can itself have 3 values:  $u, d$  or  $free$ . If  $S_2$  has the value  $u$  then from  $S_2$  the chain passes to the state  $u_{k_0+1}$ , while if  $S_2 = d$  the chain goes to  $d_{k_0+1}$ , with probability 1. In these cases  $S_2$  does not change its value. If  $S_2 = free$  then  $S_2$  takes the value  $u$  or  $d$  with equal probabilities and passes either to  $u_{k_0+1}$  or to  $d_{k_0+1}$  accordingly. If the chain reaches the state  $R_2$  then the value of  $S_2$  is set to  $free$ . For now assume that the initial value of  $S_2$  is  $free$ . In other words, the first transition from  $S_2$  is random (either to  $u_{k_0+1}$  or to  $d_{k_0+1}$  with equal probabilities) and then this decision is remembered until the “reset” state  $R_2$  is visited.

Figure 1: The processes  $m_2$  and  $\rho_2$ . The states are depicted as circles, the arrows symbolize transition probabilities: from every state the process returns to 0 with probability  $\delta$  or goes to the next state with probability  $1 - \delta$ . The function  $f_2$  is 1 on all states except  $u_{k_0+1}, \dots, u_{k_1}$  where it is 0;  $f_2$  applied to the states output by  $m_2$  defines  $\rho_2$ .

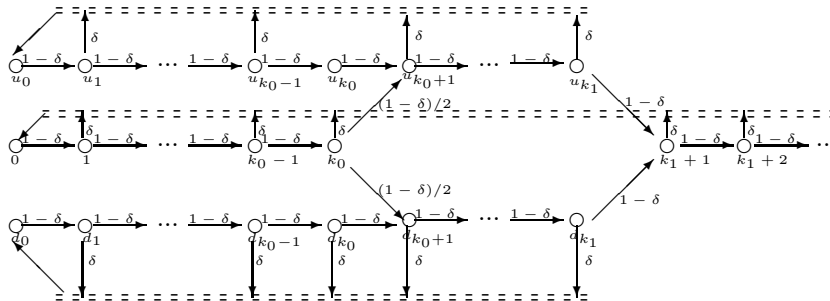


The rest of the transitions are as follows. From each state  $u_i$ ,  $k_0 \leq i \leq k_1$  the chain passes to the state 0 with probability  $\delta$  and to the next state  $u_{i+1}$  with probability  $1 - \delta$ . From the state  $u_{k_1}$  the process goes with probability  $\delta$  to 0 and with probability  $1 - \delta$  to the “reset” state  $R_2$ . The same with states  $d_i$ : for  $k_0 < i \leq k_1$  the process returns to 0 with probability  $\delta$  or goes to the next state  $d_{i+1}$  with probability  $1 - \delta$ , where the next state for  $d_{k_1}$  is the “reset” state  $R_2$ . From  $R_2$  the process goes with probability 1 to the state  $k_1 + 1$  where from the chain continues ad infinitum: to the state 0 with probability  $\delta$  or to the next state  $k_1 + 2$  etc. with probability  $1 - \delta$ .

The function  $f_2$  is defined as follows:  $f_2(i) = 1$  for  $0 \leq i \leq k_0$  and  $i > k_1$  (before the switch and after the reset);  $f_2(u_i) = 0$  for all  $i$ ,  $k_0 < i \leq k_1$  and  $f_2(d_i) = 1$  for all  $i$ ,  $k_0 < i \leq k_1$ . The function  $f_2$  is undefined on  $S_2$  and  $R_2$ , therefore there is no output on these states (we also assume that passing through  $S_2$  and  $R_2$  does not increment time). As before, the process  $\rho_2$  is defined as  $\rho_2 = f_2(s_t)$  where  $s_t$  is the state of  $m_2$  at time  $t$ , omitting the states  $S_2$  and  $R_2$ . The resulting process is illustrated on fig. 1.

*Step 1.2b.* To define the initial distribution on the states of the process  $m_2$ , we first show that it is equivalent to a Markov chain. Indeed, construct the Markov chain  $m'_2$  as follows (see fig. 2). This chain has states  $0, \dots, k_0, k_1 + 1, \dots$  and also  $u_0, \dots, u_{k_0}, u_{k_0+1}, \dots, u_{k_1}$  and  $d_0, \dots, d_{k_0}, d_{k_0+1}, \dots, d_{k_1}$ . Transitions from the states 0 to  $k_0 - 1$  are defined in the same way as for all the chains described before. From the state  $k_0$  the chain passes with probability  $(1 - \delta)/2$  to the state  $u_{k_0+1}$  and with probability  $(1 - \delta)/2$  to the state  $d_{k_0+1}$ , while with probability  $\delta$  it returns to 0; thus the state  $k_0$  corresponds to the *free* state of the switch  $S_2$ . From the states  $u_i$ ,  $i = 0, \dots, k_1$  the chain passes with probability  $1 - \delta$  to the next state  $u_{i+1}$ , where the next state for  $u_{k_1}$  is  $k_1 + 1$  and with probability  $\delta$  returns to the state  $u_0$  (and not to the state 0). Transitions for the state  $d_0, \dots, d_{k_1-1}$  are defined analogously. Thus the state

Figure 2: The process  $m'_2$ . The function  $f_2$  is 1 everywhere except the states  $d_{k_0+1}, \dots, d_{k_1}$  where it is 0.



$u_{k_0}$  corresponds to the state  $u$  of the switch  $S_2$  and the state  $d_{k_0}$  — to the state  $d$  of the switch. Transitions for the states  $k+1, k+2, \dots$  are defined as before: with probability  $\delta$  to 0 and with probability  $1-\delta$  to the next state. Thus the state  $k_1+1$  corresponds to the reset  $R_2$ . Clearly, the chain  $m'_2$  as defined possesses a unique stationary distribution  $M_2$  over the set of states and  $M_2(i) > 0$  for every state  $i$ ; take this distribution as its initial distribution. The resulting process  $m'_2$  is stationary ergodic. The initial distribution  $M_2$  can be used to define the initial distribution on the original (non-Markov) process  $m_2$ , with the initial value of the switch  $S_2$  random: if  $m'_2$  starts at a state  $u_i$  where  $i \leq k_0$  this corresponds to the process  $m'_2$  starting at the state  $i$  with  $S_2$  set to  $u$ , and analogously for  $d$ . If  $m'_2$  starts at a state  $j$  where  $j \leq k_0$  or  $j > k_1$  this corresponds to the process  $m_2$  starting at the state  $j$  with  $S_2$  set to *free*; the initial probabilities for  $S_2$  and  $R_2$  themselves are 0. Thus, we have defined (in two equivalent ways) the process  $\rho_2$ , which is stationary ergodic, and Bernoulli.

*Step 1.k.* As before, we can continue the construction of the processes  $\rho_{u3}$  and  $\rho_{d3}$ , that start with a segment of  $\rho_2$ . Let  $t_2 > t_1$  be a time index such that

$$\mathbf{E}_{\rho_2 \times \rho_2} D_{t_2} < \varepsilon,$$

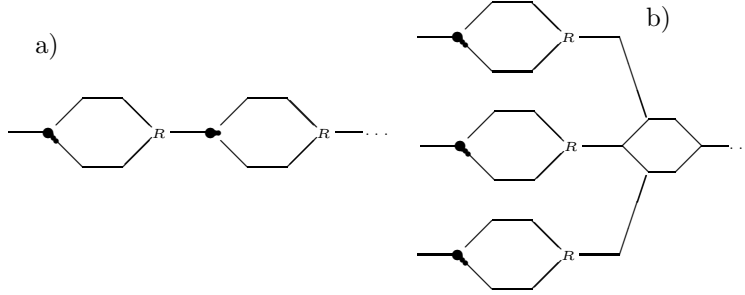
where both samples are generated by  $\rho_2$ . Let  $k_2$  be such an index that when starting from the state 0 the process  $m_2$  with probability 1 does not reach  $k_2-1$  by time  $t_2$  (equivalently: the process  $m'_2$  does not reach  $k_2-1$  when starting from either 0,  $u_0$  or  $d_0$ ). The processes  $\rho_{u3}$  and  $\rho_{d3}$  are based on the same process  $m_2$  as  $\rho_2$ . The functions  $f_{u3}$  and  $f_{d3}$  coincide with  $f_2$  on all states up to the state  $k_2$  (including the states  $u_i$  and  $d_i$ ,  $k_0 < i \leq k_1$ ). After  $k_2$  the function  $f_{u3}$  outputs 0s while  $f_{d3}$  outputs 1s:  $f_{u3}(i) = 0$ ,  $f_{d3}(i) = 1$  for  $i > k_2$ .

Furthermore, we find a time  $t_3 > t_2$  by which we have  $\mathbf{E}_{\rho_{u3} \times \rho_{d3}} D_{t_3} > 1 - \varepsilon$ , where the samples are generated by  $\rho_{u3}$  and  $\rho_{d3}$ , which is possible since  $D$  is consistent. Next, find an index  $k_3$  such that the process  $m_2$  does not reach  $k_3-1$  with probability 1 if the processes  $\rho_{u3}$  and  $\rho_{d3}$  are used to produce two independent sequences. We then construct the process  $\rho_4$  based on a (non-Markovian) process  $m_4$  by “gluing” together  $\rho_{u3}$  and  $\rho_{d3}$  after the step  $k_3$  with

a switch  $S_4$  and a reset  $R_4$  exactly as was done when constructing the process  $\rho_2$ . The process  $m_4$  is illustrated on fig. 3 a).

The process  $m_4$  can be shown to be equivalent to a Markov chain  $m'_4$  (cf. fig. 3 b)), which consists of 3 copies of the process  $m_2$  (which was shown to be equivalent to a Markov chain) truncated at step  $k_2$  and linked as follows. From the step  $k_2$  the process passes with probability  $1/2$  to the state  $u_{k_2+1}$  with probability  $(1 - \delta)/2$  and to the state  $d_{k_2+1}$  with the same probability, while with probability  $\delta$  it goes to the origin 0. From states  $u_i$ ,  $k_2 < i \leq k_3$  the process returns to  $u_0$  with probability  $\delta$  or goes to the next state  $u_i + 1$  with probability  $1 - \delta$ , and analogously for  $d_i$ . For the states  $u_{k_3}$  and  $d_{k_3}$  the next state is  $k_3 + 1$ , from which the process returns to the state 0 with probability  $\delta$  or continues to the next state with probability  $1 - \delta$ . The definition of  $f_4$  is analogous to the previous definitions of  $f_i$ .

Figure 3: The processes  $m_4$  and  $m'_4$



Proceeding this way we can construct the processes  $\rho_{2j}$ ,  $\rho_{u_{2j+1}}$  and  $\rho_{d_{2j+1}}$ ,  $j \in \mathbb{N}$  choosing the time steps  $t_j > t_{j-1}$  so that the test converges to 0 by  $t_j$  being run on two samples produced by  $\rho_j$  for even  $j$ , and converges to 1 by  $t_j$  being run on samples produced by  $\rho_{u_j}$  and  $\rho_{d_j}$  for odd  $j$ :

$$\mathbf{E}_{\rho_{2j} \times \rho_{2j}} D_{t_{2j}} < \varepsilon \quad (2)$$

and

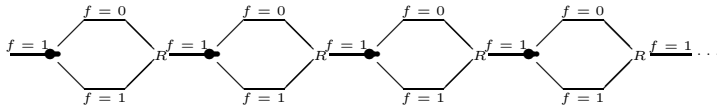
$$\mathbf{E}_{\rho_{2j+1} \times \rho_{2j+1}} D_{t_{2j+1}} > (1 - \varepsilon). \quad (3)$$

For each  $j$  the number  $k_j$  is selected in a such a way that the state  $k_j - 1$  is not reached (with probability 1) by the time  $t_j$  when starting from the state 0.

*Step 2.* Having defined  $k_j$ ,  $j \in \mathbb{N}$  we can define the process  $\rho$ , illustrated on fig. 4. The construction of the process is described in Step 2a up to an initial distribution on the states, while on Step 2b we show that the process is equivalent to a Markov chain, which we use to define the initial distribution for  $\rho$ . On Step 2c we show that the process  $\rho$  is a B-process. *Step 2a.* The construction is based on the process  $m_\rho$  that has states  $0, \dots, k_0, k_{2j+1} + 1, \dots, k_{2(j+1)}, u_{k_{2j}+1}, \dots, u_{k_{2j+1}}$  and  $d_{k_{2j}+1}, \dots, d_{k_{2j+1}}$  for  $j \in \mathbb{N}$ , along with switch states  $S_{2j}$  and reset states  $R_{2j}$ . Each switch  $S_{2j}$  diverts the process to the state  $u_{k_{2j}+1}$



Figure 4: The processes  $m_\rho$  and  $\rho$ . The states are on horizontal lines. The function  $f$  that defines the process  $\rho$  takes value 0 on the states on the upper lines and 1 on the rest of the states.

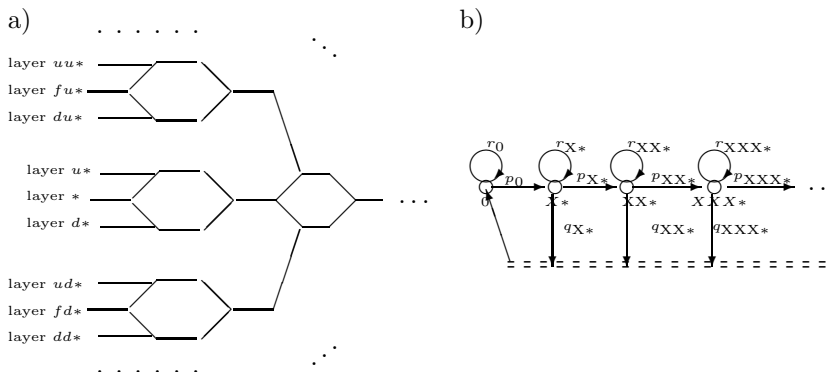


if the switch has value  $u$  and to  $d_{k_{2j}+1}$  if it has the value  $d$ . If the switch has the value *free* it assumes one of the values  $u$  or  $d$  with equal probabilities. The reset  $R_{2j}$  sets  $S_{2j}$  to *free*. From each state that is neither a reset nor a switch, the process goes to the next state with probability  $1 - \delta$  and returns to the state 0 with probability  $\delta$  (cf. Step 1*k*). The function  $f$  is defined as 1 everywhere except for the states  $u_j$  (for all  $j \in \mathbb{N}$  for which  $u_j$  is defined) on which  $f$  takes the value 0. The process  $\rho$  is defined at time  $t$  as  $f(s_t)$ , where  $s_t$  is the state of  $m_\rho$  at time  $t$ .

*Step 2b.* As before, we can show that the process  $m_\rho$  is equivalent to a Markov chain  $m'_\rho$ . Indeed, the corresponding Markov chain can be constructed inductively, by replacing each switch-reset pair with two branches of a tree (see fig. 5a) as follows. To get rid of the switch  $S_2$  and reset  $R_2$  we proceed as in constructing the process  $\rho_2$  on Step 1.2. That is, we introduce two copies  $u_0, \dots, u_{k_0}$  and  $d_0, \dots, d_{k_0}$  of the states  $0, \dots, k_0$ . Call each of these copies a layer, the upper one we call layer  $u^*$ , the lower one  $d^*$ , and the central one (states  $0, \dots, k_0, k_1 + 1, \dots, k_2, \dots$ ) we call  $*$ . Change the transition probabilities of the states  $u_i$  and  $d_i$ ,  $i = k_0, \dots, k_1$  as in the Step 1.2, that is, the return state is not 0 but  $u_0$  for  $u_i$  and  $d_0$  for  $d_i$ . To get rid of the switch  $S_4$  and reset  $R_4$ , introduce two more copies of all the preceding states (three layers each), one copy corresponding to the state  $u$  of the switch  $S_4$  and the other to the state  $d$  of the switch  $S_4$ , while the central copy corresponds to the state *free* (cf. the construction of the process  $\rho_4$  and fig 3 b). Proceed analogously with the rest of the switches. We will have infinitely many layers in the process, each corresponding to some combination of the states of first  $n$  switches, for  $n \in \mathbb{N}$ . Let the layer  $*$  correspond to all switches at the state *free*, the layer  $u^*$  to the first switch at  $u$  and the rest *free*, layer  $d^*$  to the first switch  $d$  and the rest *free*, layer  $uu^*$  to the first two switches  $u$  and the rest *free*, layer  $fu^*$  to the first switch *free*, second  $u$  and the rest *free*, and so on (cf. fig 5a); thus  $u$ ,  $d$ , or  $f$  mean the the corresponding switch in the sequence has the value  $u$ ,  $d$ , or *free*, and the symbol  $*$  means that all the rest of the switches are set to *free*.

Let us show that the Markov chain  $m'_\rho$  has a stationary distribution over the states which assigns a positive probability to every state. To do this, consider a simpler Markov chain  $\mu'_\rho$ , constructed by grouping the states of  $m'_\rho$  as follows: unite all the states  $0, \dots, k_0$  of  $m_\rho$  into a state 0 of  $\mu'_\rho$ . Unite all the states of  $m_\rho$  which are between  $k_0$  and  $k_2$ , including the layers  $u^*, d^*$ , into the state  $X^*$  of  $\mu'_\rho$ . That is, the state  $X^*$  corresponds to all the states that are past the first

Figure 5: a) The process  $m'_\rho$ , expanded. There are infinitely many layers above and below, corresponding to different combinations of switches. b) The process  $\mu'_\rho$ : the states of the process  $m'_\rho$  are combined.



switch ( $S_2$ ) but before the second ( $S_4$ ). Unite all the states of  $m'_\rho$  which are between  $k_2$  and  $k_4$ , including the layers layers  $X_1X_2*$  of  $m'_\rho$ , where  $X_1$  and  $X_2$  take any value in  $u, d$ , or  $f$ , into the state  $XX^*$  of  $\mu'_\rho$ , and so on: the states between  $k_{2j}$  and  $k_{2(j+1)}$ , including the layers  $X_1..X_j*$  (each  $X_i$  is either  $u, d$ , or  $f$ ) are grouped into the state  $X..X^*$  ( $j$  symbols  $X$ ) of  $\mu'_\rho$ . In other words, the state  $X..X^*$  ( $j$  symbols  $X$ ) corresponds to the process  $m'_\rho$  passing the switch  $S_{2j}$  but not reaching the next switch. From each of the resulting states  $i$  the chain proceeds to the next state with probability  $p_i$ , remains in the state  $i$  with probability  $r_i$ , and returns to the state 0 with probability  $q_i$ . The values of  $p_i$ ,  $q_i$ , and  $r_i$  are taken from the groups of states of the original Markov chain  $m_\rho$ , for example  $p_0 = (1 - \delta)^{k_0}$ . Since  $\delta \geq 1/2$ , it is easy to see that for every state  $i$  we have  $q_i \geq 2p_i$ . Therefore, the Markov chain  $\mu'_\rho$  possesses a stationary distribution with positive probabilities of all states, as can be seen, for example, by checking the conditions given in [14, p. 582]. From this it follows that the original Markov chain  $m'_\rho$  is recurrent and the state 0 has a positive probability, which implies (since all the other states are connected with the state 0) that all states have positive probability. Finally, the stationary distribution on the states of  $m'_\rho$  defines a stationary distribution on the states of  $m_\rho$ , and a distribution on the values of the switches. We take this distribution as the initial one. Clearly, the process  $\rho$  is stationary ergodic.

*Step 2c.* To show that  $\rho$  constructed is a  $B$ -processes, observe that it is obtained by applying the function  $f$  to the states of a chain  $m_\rho$ . Since this chain has infinitely many states, for each  $k$  we can find a state  $n_k$  such that the sum of probabilities (in the initial stationary distribution) of all of the states that follow the state  $n_k$  is not greater than  $2^{-k}$ . Let  $g_k$  be a function that coincides with  $f$  on all states up to  $n_k$  and is 0 for all states that follow  $n_k$ . The process  $\mu_k$  obtained by applying  $g_k$  to the states of  $m_\rho$  is equivalent (that is, the distributions are the same) to that which would be obtained by applying

$g_k$  to a  $n_k + 1$ -order Markov chain constructed by replacing all the states of  $m'_\rho$  greater than  $n_k$  by a single state, from which the chain passes to the state 0 with probability  $\delta$  and with probability  $1 - \delta$  remains in this state. Therefore, each  $g_k$  is a function of an aperiodic stationary ergodic  $n_k$ -state Markov chain, and hence it is a  $B$ -process (see e.g. [5]). Moreover,  $\rho$  is a limit in  $\bar{d}$  distance of the processes  $\mu_k$ ; indeed, if we couple  $\rho$  with  $\mu_k$  in an obvious way by requiring that the underlying Markov chains always take state transitions together, then the probability of observing a different output is not greater than the probability that the chains are in one of the states that follow the state  $n_k$ , which is bounded by  $2^{-k}$ . Since the set of  $B$ -processes is closed in the  $\bar{d}$ -distance, the process  $\rho$  is a  $B$ -process.

*Step 3.* Finally, we will show that the expected output of test  $D$  diverges if the test is run on two independent samples produced by  $\rho$ .

Let  $M_{2j}$  denote the initial state distribution of the process  $m_{2j}$ ,  $j \in \mathbb{N}$ , and  $M$  that of the process  $m_\rho$ . Since each of the processes  $m_{2j}$ ,  $j \in \mathbb{N}$  and the process  $m_\rho$  from each state returns to the state 0 with probability  $\delta$ , the limiting (and hence initial) probability of this state is  $\delta$ :  $M_{2j}(0) = M(0) = \delta$ . By construction, if the process  $m_\rho$  starts at the state 0 then up to time  $k_{2j}$  it behaves exactly as  $\rho_{2j}$  that has started at state 0. In symbols, we have

$$E_{\rho \times \rho}(D_{t_{2j}} | s_0^x = 0, s_0^y = 0) = E_{\rho_{2j} \times \rho_{2j}}(D_{t_{2j}} | s_0^x = 0, s_0^y = 0) \quad (4)$$

for  $j \in \mathbb{N}$ , where  $s_0^x$  and  $s_0^y$  denote the states of the processes generating the samples  $x$  and  $y$  correspondingly.

For each of the considered processes, the probability to start from the state 0 is high enough to ignore the behaviour of the tests if the processes start in other states. More formally, we use the following simple decomposition

$$\mathbf{E}(D_{t_j}) = \delta^2 \mathbf{E}(D_{t_j} | s_0^x = 0, s_0^y = 0) + (1 - \delta^2) \mathbf{E}(D_{t_j} | s_0^x \neq 0 \text{ or } s_0^y \neq 0), \quad (5)$$

(4), and (2) we have

$$\begin{aligned} \mathbf{E}_{\rho \times \rho}(D_{t_{2j}}) &\leq \delta^2 \mathbf{E}_{\rho \times \rho}(D_{t_{2j}} | s_0^x = 0, s_0^y = 0) + (1 - \delta^2) \\ &= \delta^2 \mathbf{E}_{\rho_{2j} \times \rho_{2j}}(D_{t_{2j}} | s_0^x = 0, s_0^y = 0) + (1 - \delta^2) \\ &\leq \mathbf{E}_{\rho_{2j} \times \rho_{2j}} + (1 - \delta^2) < \varepsilon + (1 - \delta^2). \end{aligned} \quad (6)$$

For odd indices, if the process  $\rho$  starts at the state 0 then (from the definition of  $t_{2j+1}$ ) by the time  $t_{2j+1}$  it will not reach reset  $R_{2j}$ ; therefore, the value of the switch  $S_{2j}$  either did not change up to the time  $t_{2j+1}$ , or has changed once from *free* to either *u* or *d*. Since the definition of  $m_\rho$  is symmetric with respect to the values *u* and *d* of each switch, the probability that two samples  $x_1, \dots, x_{t_{2j+1}}$  and  $y_1, \dots, y_{t_{2j+1}}$  generated independently by (two runs of) the process  $\rho$  produced different values of the switch  $S_{2j}$  when passing through it for the first time is  $1/2$ . In other words, with probability  $1/2$  two samples generated by  $\rho$  starting at the state 0 will look by the time  $t_{2j+1}$  as two samples generated by  $\rho_{u_{2j+1}}$

and  $\rho_{d2j+1}$  that has started at state 0. Thus

$$E_{\rho \times \rho}(D_{t_{2j+1}} | s_0^x = 0, s_0^y = 0) \geq \frac{1}{2} E_{\rho_{a2j+1} \times \rho_{d2j+1}}(D_{t_{2j+1}} | s_0^x = 0, s_0^y = 0) \quad (7)$$

for  $j \in \mathbb{N}$ . Using this, (5), and (3) we obtain

$$\begin{aligned} \mathbf{E}_{\rho \times \rho}(D_{t_{2j+1}}) &\geq \delta^2 \mathbf{E}_{\rho \times \rho}(D_{t_{2j+1}} | s_0^x = 0, s_0^y = 0) \\ &\geq \frac{1}{2} \delta^2 \mathbf{E}_{\rho_{2j+1} \times \rho_{2j+1}}(D_{t_{2j+1}} | s_0^x = 0, s_0^y = 0) \\ &\geq \frac{1}{2} (\mathbf{E}_{\rho_{2j+1} \times \rho_{2j+1}}(D_{t_{2j+1}}) - (1 - \delta^2)) > \frac{1}{2} (\delta^2 - \varepsilon). \end{aligned} \quad (8)$$

Taking  $\delta$  large and  $\varepsilon$  small (e.g.  $\delta = 0.9$  and  $\varepsilon = 0.1$ ), we can make the bound (6) close to 0 and the bound (8) close to  $1/2$ , and the expected output of the test will cross these values infinitely often. Therefore, we have shown that the expected output of the test  $D$  diverges on two independent runs of the process  $\rho$ , contradicting the consistency of  $D$ .  $\square$

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