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Analytical Solution for Wave Propagation in Stratified Poroelastic Medium. Part II: the 3D Case

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Abstract: We are interested in the modeling of wave propagation in poroelastic media. We consider the biphasic Biot's model in an infinite bilayered medium, with a plane interface. We adopt the Cagniard-De Hoop's technique. This report is devoted to the calculation of analytical solution in three dimension.

Key-words: Biot's model, poroelastic waves, analytical solution, Cagniard-De Hoop's technique.

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Solution analytique pour la propagation d'ondes en milieu poroélastique stratifié. Partie II : en dimension 3

Résumé : Nous nous intéressons à la modélisation de la propagation d'ondes dans les milieux infinis bicouches poroélastiques. Nous considérons ici le modèle bi-phasique de Biot. Cette seconde partie est consacrée au calcul de la solution analytique en dimension trois à l'aide de la technique de Cagniard-De Hoop.

Mots-clés : Modèle de Biot, ondes poroélastiques, solution analytique, technique de Cagniard de Hoop.

Introduction

Many seismic materials cannot only be considered as solid materials. They are often porous media, i.e. media made of a solid fully saturated with a fluid: there are solid media perforated by a multitude of small holes (called pores) filled with a fluid. It is in particular often the case of the oil reservoirs. It is clear that the analysis of results by seismic methods of the exploration of such media must take to account the fact that a wave being propagated in such a medium meets a succession of phases solid and fluid: we speak about poroelastic media, and the more commonly used model is the Biot's model [1, 2, 3].

When the wavelength is large in comparison with the size of the pores, rather than regarding such a medium as an heterogeneous medium, it is legitimate to use, at least locally, the theory of homogenization [4, 13]. This leads to the Biot's model [1, 2, 3] which involves as unknown not only the displacement field in the solid but also the displacement field in the fluid. The principal characteristic of this model is that in addition to the classical P and S waves in a solid one observes a P "slow" wave, which we could also call a "fluid" wave: the denomination "slow wave" refers to the fact that in practical applications, it is slower (and probably much slower) than the other two waves.

The computation of analytical solutions for wave propagation in poroelastic media is of high importance for the validation of numerical computational codes or for a better understanding of the reflexion/transmission properties of the media. Cagniard-de Hoop method [5, 7] is a useful tool to obtain such solutions and permits to compute each type of waves (P wave, S wave, head wave...) independently. Although it was originally dedicated to the solution to elastodynamic wave propagation, it can be applied to any transient wave propagation problem in stratified medium. However, as far as we know, few works have been dedicated to the application of this method to poroelastic medium, especially in three dimensions.

In order to validate computational codes of wave propagation in poroelastic media, we have implemented the codes Gar6more 2D [11] and Gar6more 3D [12] which provide the complete solution (reflected and transmitted waves) of the propagation of wave in stratified 2D or 3D media composed of acoustic/acoustic, acoustic/elastic, acoustic/poroelastic or poroelastic/poroelastic layers. The codes are freely downloadable at

<http://www.spice-rtn.org/library/software/Garcimore2D>

and

<http://www.spice-rtn.org/library/software/Gar6more3D>.

We will focus in this paper on the 3D poroelastic case, the two dimensional and the acoustic/poroelastic cases are detailed in [8, 9, 10]. The outline of the paper is as follows: we first present the model problem we want to solve and derive the Green problem from it (section 1). Then we present the analytical solution to the wave propagation problem in a stratified 2D medium composed of an acoustic and a poroelastic layer (section 2). Finally we illustrate our results through numerical applications (section 3).

1 The model problem

We consider an infinite two dimensional medium ($\Omega = \mathbf{R}^3$) composed of two homogeneous poroelastic layers $\Omega^+ = \mathbf{R}^2 \times]-\infty, 0]$ and $\Omega^- = \mathbf{R}^2 \times [0, +\infty[$ separated by an horizontal interface Γ (see Fig. 1). We first describe the equations in the two layers (§1.1) and the transmission conditions on the interface Γ (§1.2), then we present the Green problem from which we compute the analytical solution (§1.3).

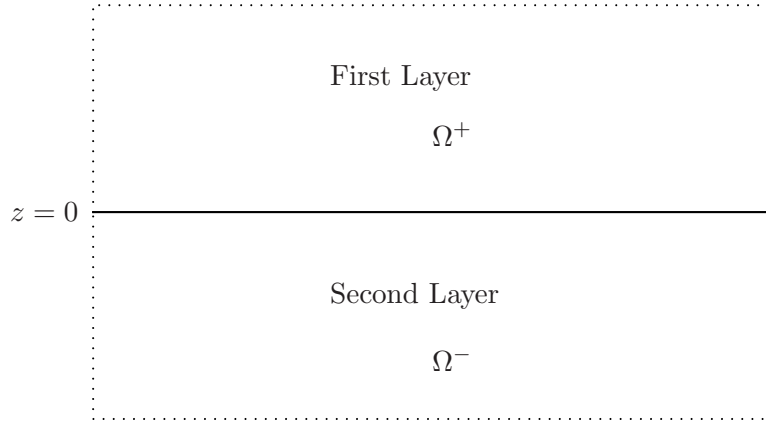


Figure 1: Configuration of the study

1.1 Poroelastic equations

We consider the second-order formulation of the poroelastic equations [1, 2, 3]:

$$\left\{ \begin{array}{ll} \rho \ddot{\mathbf{U}}_s + \rho_f \ddot{\mathbf{W}} - \nabla \cdot \Sigma = \mathbf{F}_u, & \text{in } \Omega \times]0, T], \\ \rho_f \ddot{\mathbf{U}}_s + \rho_w \ddot{\mathbf{W}} + \frac{1}{\mathcal{K}} \dot{\mathbf{W}} + \nabla P = \mathbf{F}_w, & \text{in } \Omega \times]0, T], \\ \Sigma = \lambda \nabla \cdot \mathbf{U}_s \mathbf{I}_3 + 2\mu \varepsilon(\mathbf{U}_s) - \beta P \mathbf{I}_3, & \text{in } \Omega \times]0, T], \\ \frac{1}{m} P + \beta \nabla \cdot \mathbf{U}_s + \nabla \cdot \mathbf{W} = F_p, & \text{in } \Omega \times]0, T], \\ \mathbf{U}_s(x, 0) = 0, \mathbf{W}(x, 0) = 0, & \text{in } \Omega, \\ \dot{\mathbf{U}}_s(x, 0) = 0, \dot{\mathbf{W}}(x, 0) = 0, & \text{in } \Omega, \end{array} \right. \quad (1)$$

with

$$(\nabla \cdot \Sigma)_i = \sum_{j=1}^3 \frac{\partial \Sigma_{ij}}{\partial x_j} \quad \forall i = 1, 3. \quad \text{As usual } \mathbf{I}_3 \text{ is the identity matrix of } \mathcal{M}_2(\mathbb{R}),$$

and $\varepsilon(\mathbf{U}_s)$ is the solid strain tensor defined by:

$$\varepsilon_{ij}(\mathbf{U}) = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$$

In (1), the unknowns are:

- \mathbf{U}_s the displacement field of solid particles;
- $\mathbf{W} = \phi(\mathbf{U}_f - \mathbf{U}_s)$, the relative displacement, \mathbf{U}_f being the displacement field of fluid particles and ϕ the porosity;
- P , the fluid pressure;
- Σ , the solid stress tensor.

The parameters describing the physical properties of the medium are given by:

- $\rho = \phi \rho_f + (1 - \phi) \rho_s$ is the overall density of the saturated medium, with ρ_s the density of the solid and ρ_f the density of the fluid;
- $\rho_w = a \rho_f / \phi$, where a is the tortuosity of the solid matrix;
- $\mathcal{K} = \kappa / \eta$, where κ is the permeability of the solid matrix and η is the viscosity of the fluid;
- m and β are positive physical coefficients: $\beta = 1 - K_b / K_s$ and $m = [\phi / K_f + (\beta - \phi) / K_s]^{-1}$, where K_s is the bulk modulus of the solid, K_f is the bulk modulus of the fluid and K_b is the frame bulk modulus;
- μ is the frame shear modulus, and $\lambda = K_b - 2\mu / 3$ is the Lamé constant.
- \mathbf{F}_u , \mathbf{F}_w and F_p are the force densities.

To simplify this study, we consider only the case of a compression source

$$\mathbf{F}_u(x, y, t) = f_u \nabla (\delta_x \delta_y \delta_{z-h}) f(t) \text{ and } \mathbf{F}_w(x, y, t) = f_w \nabla (\delta_x \delta_y \delta_{z-h}) f(t)$$

and a pressure source $F_p = f_p \delta_x \delta_y \delta_{z-h} f(t)$, where f_u , f_w and f_p are constant and f is a regular source function in time. We can generalize this approach for other types of punctual sources such as for instance

$$\mathbf{F}_u = f_u \nabla \times (\delta_x \delta_y \delta_{z-h} \mathbf{v}) f(t) \text{ and } \mathbf{F}_w = f_w \nabla \times (\delta_x \delta_y \delta_{z-h} \mathbf{v}) f(t)$$

where \mathbf{v} is a vector in \mathbb{R}^3 .

1.2 Transmission conditions

Let \mathbf{n} be the unitary normal vector of Γ outwardly directed to Ω^- . The transmission conditions on the interface Γ between the two poroelastic medium are [6]:

$$\begin{cases} \mathbf{U}_s^+ = \mathbf{U}_s^-, \\ \mathbf{W}^+ \cdot \mathbf{n} = \mathbf{W}^- \cdot \mathbf{n}, \\ P^+ = P^-, \\ \Sigma^+ \mathbf{n} = \Sigma^- \mathbf{n}. \end{cases} \quad (2)$$

1.3 The Green problem

We won't compute directly the solution to (2) but the solution to the following Green problem:

$$\rho^\pm \ddot{\mathbf{u}}_s^\pm + \rho_f^\pm \dot{\mathbf{w}}^\pm - \nabla \cdot \sigma^\pm = f_u \nabla (\delta_x \delta_y \delta_{z-h}) \delta_t, \quad \text{in } \Omega^\pm \times]0, T], \quad (3a)$$

$$\rho_f^\pm \ddot{\mathbf{u}}_s^\pm + \rho_w^\pm \dot{\mathbf{w}}^\pm + \frac{1}{\mathcal{K}^\pm} \dot{\mathbf{w}}^\pm + \nabla p^\pm = f_w \nabla (\delta_x \delta_y \delta_{z-h}) \delta_t, \quad \text{in } \Omega^\pm \times]0, T], \quad (3b)$$

$$\sigma^\pm = \lambda^\pm \nabla \cdot \mathbf{u}_s^\pm \mathbf{I}_3 + 2\mu^\pm \varepsilon(\mathbf{u}_s^\pm) - \beta^\pm p^\pm \mathbf{I}_3, \quad \text{in } \Omega^\pm \times]0, T], \quad (3c)$$

$$\frac{1}{m^\pm} p^\pm + \beta^\pm \nabla \cdot \mathbf{u}_s^\pm + \nabla \cdot \mathbf{w}^\pm = f_p \delta_x \delta_y \delta_{z-h} f(t), \quad \text{in } \Omega^\pm \times]0, T], \quad (3d)$$

$$\mathbf{u}_s^- = \mathbf{u}_s^+, \quad \text{on } \Gamma \times]0, T] \quad (3e)$$

$$\mathbf{w}^- \cdot \mathbf{n} = \mathbf{w}^+ \cdot \mathbf{n}, \quad \text{on } \Gamma \times]0, T] \quad (3f)$$

$$p^- = p^+, \quad \text{on } \Gamma \times]0, T] \quad (3g)$$

$$\sigma^- \mathbf{n} = \sigma^+ \mathbf{n}, \quad \text{on } \Gamma \times]0, T]. \quad (3h)$$

The solution to (1) is then computed from the solution to the Green Problem thanks to a convolution by the source function. For instance we have:

$$P^+(x, y, t) = p^+(x, y, \cdot) * f(\cdot) = \int_0^t p^+(x, y, \tau) f(t - \tau) d\tau$$

(we have similar relations for the other unknowns). We also suppose that the poroelastic medium is non dissipative, i.e the viscosity $\eta^\pm = 0$. Using the equations (3c,3d) we can eliminate σ^\pm and p^\pm in (3) and we obtain the equivalent system:

$$\begin{cases} \rho^\pm \ddot{\mathbf{u}}_s^\pm + \rho_f^\pm \dot{\mathbf{w}}^\pm - \alpha^\pm \nabla (\nabla \cdot \mathbf{u}_s^\pm) + \mu^\pm \nabla \times (\nabla \times \mathbf{u}_s^\pm) - m^\pm \beta^\pm \nabla (\nabla \cdot \mathbf{w}^\pm) \\ = (f_u - \beta^+ m^+ f_p) \nabla (\delta_x \delta_y \delta_{z-h}) \delta_t, \\ \rho_f^\pm \ddot{\mathbf{u}}_s^\pm + \rho_w^\pm \dot{\mathbf{w}}^\pm - m^\pm \beta^\pm \nabla (\nabla \cdot \mathbf{u}_s^\pm) - m^\pm \nabla (\nabla \cdot \mathbf{w}^\pm) = (f_w - m^+ f_p) \nabla (\delta_x \delta_y \delta_{z-h}) \delta_t, \end{cases} \quad (4)$$

with $\alpha^- = \lambda^- + 2\mu^- + m^- \beta^{-2}$.

And the transmission conditions on Γ are rewritten as:

$$u_{sx}^+ = u_{sx}^-, \quad (5a)$$

$$u_{sy}^+ = u_{sy}^-, \quad (5b)$$

$$u_{sz}^+ = u_{sz}^-, \quad (5c)$$

$$w_z^- = w_z^+, \quad (5d)$$

$$m^+ \beta^+ \nabla \cdot \mathbf{u}_s^+ + m^+ \nabla \cdot \mathbf{w}^+ = m^- \beta^- \nabla \cdot \mathbf{u}_s^- + m^- \nabla \cdot \mathbf{w}^-, \quad (5e)$$

$$\mu^+ (\partial_z u_{sx}^+ + \partial_x u_{sz}^+) = \mu^- (\partial_z u_{sx}^- + \partial_x u_{sz}^-), \quad (5f)$$

$$\mu^+ (\partial_z u_{sy}^+ + \partial_y u_{sz}^+) = \mu^- (\partial_z u_{sy}^- + \partial_y u_{sz}^-), \quad (5g)$$

$$(\lambda^- + m^+ \beta^{+2}) \nabla \cdot \mathbf{u}_s^+ + 2\mu^+ \partial_z u_{sz}^+ + m^+ \beta^+ \nabla \cdot \mathbf{w}^+ = \quad (5h)$$

$$(\lambda^- + m^- \beta^{-2}) \nabla \cdot \mathbf{u}_s^- + 2\mu^- \partial_z u_{sz}^- + m^- \beta^- \nabla \cdot \mathbf{w}^-.$$

We split the displacement fields \mathbf{u}_s^\pm and \mathbf{w}^\pm into irrotational and isovolumic fields (P-wave and S-wave):

$$\mathbf{u}_s^\pm = \nabla \Theta_u^\pm + \nabla \times \Psi_u^\pm ; \quad \mathbf{w}^\pm = \nabla \Theta_w^\pm + \nabla \times \Psi_w^\pm. \quad (6)$$

The vectors Ψ_u^\pm and Ψ_w^\pm are not uniquely defined since:

$$\nabla \times (\Psi_\ell^\pm + \nabla C) = \nabla \times \Psi_\ell^\pm, \quad \forall \ell \in \{u, w\}$$

for all scalar field C . To define a unique Ψ_ℓ^\pm we impose the gauge condition:

$$\nabla \cdot \Psi_\ell^\pm = 0.$$

The vectorial space of Ψ_ℓ^\pm verifying this last condition is written as:

$$\Psi_\ell^\pm = \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Psi_{\ell,1}^\pm + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2 - \partial_{yy}^2 \end{bmatrix} \Psi_{\ell,2}^\pm,$$

where $\Psi_{\ell,1}^\pm$ and $\Psi_{\ell,2}^\pm$ are two scalar fields. The displacement fields \mathbf{u}_s^\pm and \mathbf{w}^\pm are written in the form:

$$\begin{aligned} \mathbf{u}_s^\pm &= \nabla \Theta_u^\pm + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2 - \partial_{yy}^2 \end{bmatrix} \Psi_{u,1}^\pm - \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Delta \Psi_{u,2}^\pm \\ \mathbf{w}^\pm &= \nabla \Theta_w^\pm + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2 - \partial_{yy}^2 \end{bmatrix} \Psi_{w,1}^\pm - \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Delta \Psi_{w,2}^\pm. \end{aligned} \quad (7)$$

We can then rewrite system (4) in the following form:

$$\begin{cases} A^+ \ddot{\Theta}^+ - B^+ \Delta \Theta^+ = \delta_x \delta_y \delta_{z-h} \delta_t \mathbf{F}, & \text{in } \Omega^+ \times]0, T] \\ A^- \ddot{\Theta}^- - B^- \Delta \Theta^- = 0, & \text{in } \Omega^- \times]0, T] \\ \ddot{\Psi}_{u,i}^\pm - V_S^{\pm 2} \Delta \Psi_{u,i}^\pm = 0, \quad i \in \{1, 2\}, & \text{in } \Omega^\pm \times]0, T] \\ \ddot{\Psi}_w^\pm = -\frac{\rho_f^\pm}{\rho_w^\pm} \ddot{\Psi}_u^\pm, & \text{in } \Omega^\pm \times]0, T] \end{cases} \quad (8)$$

where $\Theta^\pm = (\Theta_u^-, \Theta_w^-)^t$, $\mathbf{F} = (f_u - \beta^+ m^+ f_p, f_w - m^+ f_p)^t$, A^\pm and B^\pm are 2×2 symmetric matrices:

$$A^\pm = \begin{pmatrix} \rho^\pm & \rho_f^\pm \\ \rho_f^\pm & \rho_w^\pm \end{pmatrix}; \quad B^\pm = \begin{pmatrix} \lambda^\pm + 2\mu^\pm + m^\pm (\beta^\pm)^2 & m^\pm \beta^\pm \\ m^\pm \beta^\pm & m^\pm \end{pmatrix},$$

and

$$V_S^\pm = \sqrt{\frac{\mu \rho_w^\pm}{\rho^\pm \rho_w^\pm - \rho_f^{\pm 2}}}$$

is the S-wave velocity.

We multiply the first (resp. the second) equation of system (8) by the inverse of A^+ (resp. A^-). The matrix $A^{\pm -1} B^\pm$ (resp. $A^{- -1} B^-$) is diagonalizable: $A^{\pm -1} B^\pm = \mathcal{P}^\pm D^\pm \mathcal{P}^{\pm -1}$, where \mathcal{P}^\pm is the change-of-coordinates matrix, $D^\pm = \text{diag}(V_{P_f}^{\pm 2}, V_{P_s}^{\pm 2})$ is the diagonal matrix similar to $A^{\pm -1} B^\pm$, $V_{P_f}^\pm$ and $V_{P_s}^\pm$ are respectively the fast P-wave velocity and the slow P-wave velocity ($V_{P_s}^\pm < V_{P_f}^\pm$).

Using the change of variables:

$$\Phi^\pm = (\Phi_{P_f}^\pm, \Phi_{P_s}^\pm)^t = \mathcal{P}^{\pm -1} \Theta^\pm, \quad (9)$$

we obtain the uncoupled system on fast P-waves, slow P-waves and S-waves:

$$\begin{cases} \ddot{\Phi}^+ - D^+ \Delta \Phi^+ = \delta_x \delta_y \delta_{z-h} \delta_t \mathbf{F}^+, & \text{in } \Omega^+ \times]0, T] \\ \ddot{\Phi}^- - D^- \Delta \Phi^- = 0, & \text{in } \Omega^- \times]0, T] \\ \ddot{\Psi}_{u,\ell}^\pm - V_S^{\pm 2} \Delta \Psi_{u,\ell}^\pm = 0, \quad \ell \in \{1, 2\}, & \text{in } \Omega^\pm \times]0, T] \\ \Psi_w^\pm = -\frac{\rho_f^\pm}{\rho_w^\pm} \Psi_u^\pm, & \text{in } \Omega^\pm \times]0, T] \end{cases} \quad (10)$$

with $\mathbf{F}^+ = (A^+ \mathcal{P}^+)^{-1} \mathbf{F} = (F_{P_f}^+, F_{P_s}^+)^t$.

Using the transmission conditions (5a)-(5b), (5f)-(5g) and the change of variables (7), we

obtain:

$$\partial_x \Theta_u^+ + \partial_{xz}^2 \psi_{u,1}^+ - \partial_y (\Delta \psi_{u,2}^+) = \partial_x \Theta_u^- + \partial_{xz}^2 \psi_{u,1}^- - \partial_y (\Delta \psi_{u,2}^-), \quad \text{on } \Gamma, \quad (11a)$$

$$\partial_y \Theta_u^+ + \partial_{yz}^2 \psi_{u,1}^+ + \partial_x (\Delta \psi_{u,2}^+) = \partial_y \Theta_u^- + \partial_{yz}^2 \psi_{u,1}^- + \partial_x (\Delta \psi_{u,2}^-), \quad \text{on } \Gamma, \quad (11b)$$

$$\begin{aligned} \mu^+ \left(2\partial_{xz}^2 \Theta_u^+ + \partial_x (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^+ - \partial_{yz}^2 \Delta \Psi_{u,2}^+ \right) = \\ \mu^- \left(2\partial_{xz}^2 \Theta_u^- + \partial_x (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^- - \partial_{yz}^2 \Delta \Psi_{u,2}^- \right), \end{aligned} \quad \text{on } \Gamma, \quad (11c)$$

$$\begin{aligned} \mu^+ \left(2\partial_{yz}^2 \Theta_u^+ + \partial_y (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^+ + \partial_{xz}^2 \Delta \Psi_{u,2}^+ \right) = \\ \mu^- \left(2\partial_{yz}^2 \Theta_u^- + \partial_y (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^- + \partial_{xz}^2 \Delta \Psi_{u,2}^- \right) \end{aligned} \quad \text{on } \Gamma, \quad (11d)$$

with $\Delta_\perp = \partial_{xx}^2 + \partial_{yy}^2$. Applying the derivative ∂_y to the equation (11a) (resp. (11c)), ∂_x to the equation (11b) (resp. (11d)) and subtracting the first (resp. the third) obtained equation from the second (resp. the fourth) one, we get:

$$\Delta_\perp (\Delta \psi_{u,2}^+) = \Delta_\perp (\Delta \psi_{u,2}^-), \quad \text{on } \Gamma, \quad (12a)$$

$$\mu^+ (\partial_z \Delta_\perp) \Delta \Psi_{u,2}^+ = \mu^- (\partial_z \Delta_\perp) \Delta \Psi_{u,2}^-, \quad \text{on } \Gamma, \quad (12b)$$

moreover, using the third equation of (10), we have $\Psi_{u,2}^\pm$ satisfies the wave equation:

$$\ddot{\Psi}_{u,2}^\pm - V_S^{\pm 2} \Delta \Psi_{u,2}^\pm = 0, \quad \text{in } \Omega^\pm \times]0, T]$$

and, since \mathbf{u}_s^\pm and \mathbf{w}^\pm satisfy, at $t = 0$, $\mathbf{u}_s^\pm = \dot{\mathbf{u}}_s^\pm = \mathbf{w}^\pm = \dot{\mathbf{w}}^\pm = 0$, we obtain:

$$\Psi_{u,2}^\pm = 0, \quad \text{in } \Omega^\pm \times]0, T], \quad (13)$$

and from (11a)-(11b) we deduce the transmission condition equivalent to (5a) and (5b):

$$\partial_x \Theta_u^+ + \partial_{xz}^2 \psi_{u,1}^+ = \partial_x \Theta_u^- + \partial_{xz}^2 \psi_{u,1}^-, \quad \text{on } \Gamma. \quad (14)$$

In the same way, using the equality (13), we can show that the two transmission conditions (11c) and (11d) are equivalent, which gives us:

$$\mu^+ \left(2\partial_{xz}^2 \Theta_u^+ + \partial_x (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^+ \right) = \mu^- \left(2\partial_{xz}^2 \Theta_u^- + \partial_x (\partial_{zz}^2 - \Delta_\perp) \Psi_{u,1}^- \right), \quad \text{on } \Gamma. \quad (15)$$

We can then reduce the transmission conditions (5) to 6 equations: (14, 5c, 5d, 5e, 15, 5h).

Finally, we obtain the Green problem equivalent to (3):

$$\text{RR n}^\circ 6596 \quad \left\{ \begin{array}{ll} \ddot{\Phi}_i^+ - V_i^{+2} \Delta \Phi_i^+ = \delta_x \delta_y \delta_{z-h} \delta_t F_i^+, & i \in \{Pf, Ps\} \quad z > 0 \\ \ddot{\Phi}_S^+ - V_S^{+2} \Delta \Phi_S^+ = 0 & z > 0 \\ \ddot{\Phi}_i^- - V_i^{-2} \Delta \Phi_i^- = 0, & i \in \{Pf, Ps, S\} \quad z < 0 \\ \mathcal{B}(\Phi_{Pf}^+, \Phi_{Ps}^+, \Phi_S^+, \Phi_{Pf}^-, \Phi_{Ps}^-, \Phi_S^-) = 0, & z = 0 \end{array} \right. \quad (16)$$

where we have set $\Phi_S^\pm = \Psi_{u,1}^\pm$ in order to have similar notations for the Pf , Ps and S waves. The operator \mathcal{B} represents the transmission conditions on Γ :

$$\mathcal{B} \begin{pmatrix} \Phi_{Pf}^+ \\ \Phi_{Ps}^+ \\ \Phi_S^+ \\ \Phi_{Pf}^- \\ \Phi_{Ps}^- \\ \Phi_S^- \end{pmatrix} = \begin{bmatrix} \mathcal{P}_{11}^+ \partial_x & \mathcal{P}_{12}^+ \partial_x & \partial_{xz}^2 & -\mathcal{P}_{11}^- \partial_x & -\mathcal{P}_{12}^- \partial_x & -\partial_{xz}^2 \\ \mathcal{P}_{11}^+ \partial_z & \mathcal{P}_{12}^+ \partial_z & -\Delta_\perp & -\mathcal{P}_{11}^- \partial_z & -\mathcal{P}_{12}^- \partial_z & \Delta_\perp \\ \mathcal{P}_{21}^+ \partial_z & \mathcal{P}_{22}^+ \partial_z & \frac{\rho_f^+}{\rho_w^+} \Delta_\perp & -\mathcal{P}_{21}^- \partial_z & -\mathcal{P}_{22}^- \partial_z & -\frac{\rho_f^-}{\rho_w^-} \Delta_\perp \\ \mathcal{B}_{41} & \mathcal{B}_{42} & 0 & \mathcal{B}_{44} & \mathcal{B}_{45} & 0 \\ \mathcal{B}_{51} & \mathcal{B}_{52} & \mathcal{B}_{53} & \mathcal{B}_{54} & \mathcal{B}_{55} & \mathcal{B}_{56} \\ \mathcal{B}_{61} & \mathcal{B}_{62} & -2\mu^+ \partial_z(\Delta_\perp) & \mathcal{B}_{64} & \mathcal{B}_{65} & 2\mu^- \partial_z(\Delta_\perp) \end{bmatrix} \begin{bmatrix} \Phi_{Pf}^+ \\ \Phi_{Ps}^+ \\ \Phi_S^+ \\ \Phi_{Pf}^- \\ \Phi_{Ps}^- \\ \Phi_S^- \end{bmatrix}$$

where \mathcal{P}_{ij}^\pm , $i, j = 1, 2$ are the components of the change-of-coordinates matrix \mathcal{P}^\pm and

$$\begin{aligned} \mathcal{B}_{41} &= \frac{m^+(\beta^+ \mathcal{P}_{11}^+ + \mathcal{P}_{21}^+)}{V_{Pf}^{+2}} \partial_{tt}^2; \quad \mathcal{B}_{42} = \frac{m^+(\beta^+ \mathcal{P}_{12}^+ + \mathcal{P}_{22}^+)}{V_{Ps}^{+2}} \partial_{tt}^2; \\ \mathcal{B}_{44} &= -\frac{m^-(\beta^- \mathcal{P}_{11}^- + \mathcal{P}_{21}^-)}{V_{Pf}^{-2}} \partial_{tt}^2; \quad \mathcal{B}_{45} = -\frac{m^-(\beta^- \mathcal{P}_{12}^- + \mathcal{P}_{22}^-)}{V_{Ps}^{-2}} \partial_{tt}^2; \\ \mathcal{B}_{51} &= 2\mu^+ \mathcal{P}_{11}^+ \partial_{xz}^2; \quad \mathcal{B}_{52} = 2\mu^+ \mathcal{P}_{12}^+ \partial_{xz}^2; \quad \mathcal{B}_{53} = \mu^+ \partial_x(\partial_{zz}^2 - \Delta_\perp); \\ \mathcal{B}_{54} &= -2\mu^- \mathcal{P}_{11}^- \partial_{xz}^2; \quad \mathcal{B}_{55} = -2\mu^- \mathcal{P}_{12}^- \partial_{xz}^2; \quad \mathcal{B}_{56} = -\mu^- \partial_x(\partial_{zz}^2 - \Delta_\perp); \\ \mathcal{B}_{61} &= \frac{(\lambda^+ + m^+ \beta^{+2}) \mathcal{P}_{11}^+ + m^+ \beta^+ \mathcal{P}_{21}^+}{V_{Pf}^{+2}} \partial_{tt}^2 + 2\mu^+ \mathcal{P}_{11}^+ \partial_{zz}^2; \\ \mathcal{B}_{62} &= \frac{(\lambda^+ + m^+ \beta^{+2}) \mathcal{P}_{12}^+ + m^+ \beta^+ \mathcal{P}_{22}^+}{V_{Ps}^{+2}} \partial_{tt}^2 + 2\mu^+ \mathcal{P}_{12}^+ \partial_{zz}^2; \\ \mathcal{B}_{64} &= \frac{(\lambda^- + m^- \beta^{-2}) \mathcal{P}_{11}^- + m^- \beta^- \mathcal{P}_{21}^-}{V_{Pf}^{-2}} \partial_{tt}^2 + 2\mu^- \mathcal{P}_{11}^- \partial_{zz}^2; \\ \mathcal{B}_{65} &= \frac{(\lambda^- + m^- \beta^{-2}) \mathcal{P}_{12}^- + m^- \beta^- \mathcal{P}_{22}^-}{V_{Ps}^{-2}} \partial_{tt}^2 + 2\mu^- \mathcal{P}_{12}^- \partial_{zz}^2. \end{aligned}$$

To obtain this operator we have used the reduced transmission conditions (14, 5c, 5d, 5e, 15, 5h), the change of variables (6,9) and the uncoupled system (10).

Moreover, from the unknowns Φ_{Pf}^\pm , Φ_{Ps}^\pm and Φ_S^\pm we can determine the solid displacement \mathbf{u}_s^\pm and the relative displacement \mathbf{w}^\pm by using the change of variables presented below.

2 Expression of the analytical solution

Since the problem is invariant by a rotation around the z -axis, we will only consider the case $y = 0$ and $x > 0$, so that the y -component of all the displacements are zero. The solution for

$y \neq 0$ or $x \leq 0$ is deduced from the solution for $y = 0$ by the relations

$$u_{sx}^\pm(x, y, z, t) = \frac{x}{\sqrt{x^2 + y^2}} u_{sx}^\pm(\sqrt{x^2 + y^2}, 0, z, t) \quad (17)$$

$$u_{sy}^\pm(x, y, z, t) = \frac{y}{\sqrt{x^2 + y^2}} u_{sx}^\pm(\sqrt{x^2 + y^2}, 0, z, t) \quad (18)$$

$$u_{sz}^\pm(x, y, z, t) = u_{sz}^\pm(\sqrt{x^2 + y^2}, 0, z, t) \quad (19)$$

To state our results, we need the following notations and definitions:

1. **Definition of the complex square root.** For $q \in \mathbb{C} \setminus \mathbb{R}^-$, we use the following definition of the square root $g(q) = q^{1/2}$:

$$g(q)^2 = q \quad \text{and} \quad \Re[g(q)] > 0.$$

The branch cut of $g(q)$ in the complex plane will thus be the half-line defined by $\{q \in \mathbb{R}^-\}$ (see Fig. 2). In the following, we use the abuse of notation $g(q) = i\sqrt{-q}$ for $q \in \mathbb{R}^-$.

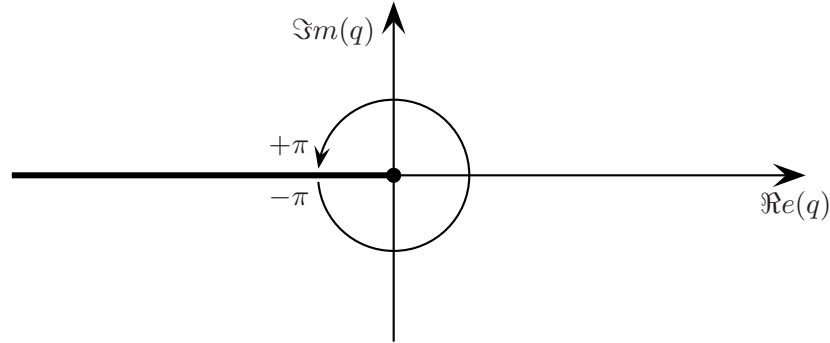


Figure 2: Definition of the function $x \mapsto (x)^{1/2}$

2. **Definition of the fictitious velocities** For a given $q \in \mathbb{R}$, we define the fictitious velocities $\mathcal{V}_i^\pm(q)$ for $i \in \{Pf, Ps, S\}$ by

$$\mathcal{V}_i^\pm := \mathcal{V}_i^\pm(q) = V_i^\pm \sqrt{\frac{1}{1 + V_i^{\pm 2} q^2}}.$$

These fictitious velocities will be helpful to turn the 3D-problem into the sum of 2D-problems indexed by the variable q . Note that $\mathcal{V}_i^\pm(0)$ correspond to the real velocities V_i^\pm .

3. **Definition of the functions κ_i^\pm .** For $i \in \{Pf, Ps, S\}$ and $(q_x, q_y) \in \mathbb{C} \times \mathbb{R}$, we define the functions

$$\kappa_i^\pm := \kappa_i^\pm(q_x, q_y) = \left(\frac{1}{V_i^{\pm 2}} + q_x^2 + q_y^2 \right)^{1/2} = \left(\frac{1}{\mathcal{V}_i^{\pm 2}(q_y)} + q_x^2 \right)^{1/2}.$$

4. **Definition of the reflection and transmission coefficients.** For a given $\mathbf{q} = (q_x, q_y) \in \mathbb{C} \times \mathbb{R}$, we denote by $\mathcal{R}_{P_f P_f}(\mathbf{q})$, $\mathcal{R}_{P_f P_s}(\mathbf{q})$, $\mathcal{R}_{P_f S}(\mathbf{q})$, $\mathcal{T}_{P_f P_f}(\mathbf{q})$, $\mathcal{T}_{P_f P_s}(\mathbf{q})$, and $\mathcal{T}_{P_f S}(\mathbf{q})$ the solution to the linear system

$$\mathcal{A}(\mathbf{q}) \begin{bmatrix} \mathcal{R}_{P_f P_f}(\mathbf{q}) \\ \mathcal{R}_{P_f P_s}(\mathbf{q}) \\ \mathcal{R}_{P_f S}(\mathbf{q}) \\ \mathcal{T}_{P_f P_f}(\mathbf{q}) \\ \mathcal{T}_{P_f P_s}(\mathbf{q}) \\ \mathcal{T}_{P_f S}(\mathbf{q}) \end{bmatrix} = \frac{1}{2\kappa_{P_f}^+(\mathbf{q})V_{P_f}^{+2}} \begin{bmatrix} i q_x \mathcal{P}_{11}^+ \\ -\kappa_{P_f}^+(\mathbf{q})\mathcal{P}_{11}^+ \\ -\kappa_{P_f}^+(\mathbf{q})\mathcal{P}_{21}^+ \\ -\frac{m^+}{V_{P_f}^{+2}}(\beta^+ \mathcal{P}_{11}^+ + \mathcal{P}_{21}^+) \\ 2i q_x \mu^+ \kappa_{P_f}^+(\mathbf{q})\mathcal{P}_{11}^+ \\ -\frac{(\lambda^+ + m^+ \beta^+) \mathcal{P}_{11}^+ + m^+ \beta^+ \mathcal{P}_{21}^+}{V_{P_f}^{+2}} - 2\mu^+ \mathcal{P}_{11}^+ \kappa_{P_f}^{+2}(\mathbf{q}) \end{bmatrix}$$

and by $\mathcal{R}_{P_s P_f}(\mathbf{q})$, $\mathcal{R}_{P_s P_s}(\mathbf{q})$, $\mathcal{R}_{P_s S}(\mathbf{q})$, $\mathcal{T}_{P_s P_f}(\mathbf{q})$, $\mathcal{T}_{P_s P_s}(\mathbf{q})$ and $\mathcal{T}_{P_s S}(\mathbf{q})$ the solution to the linear system

$$\mathcal{A}(\mathbf{q}) \begin{bmatrix} \mathcal{R}_{P_s P_f}(\mathbf{q}) \\ \mathcal{R}_{P_s P_s}(\mathbf{q}) \\ \mathcal{R}_{P_s S}(\mathbf{q}) \\ \mathcal{T}_{P_s P_f}(\mathbf{q}) \\ \mathcal{T}_{P_s P_s}(\mathbf{q}) \\ \mathcal{T}_{P_s S}(\mathbf{q}) \end{bmatrix} = \frac{1}{2\kappa_{P_s}^+(\mathbf{q})V_{P_s}^{+2}} \begin{bmatrix} i q_x \mathcal{P}_{12}^+ \\ -\kappa_{P_s}^+(\mathbf{q})\mathcal{P}_{12}^+ \\ -\kappa_{P_s}^+(\mathbf{q})\mathcal{P}_{22}^+ \\ -\frac{m^+}{V_{P_s}^{+2}}(\beta^+ \mathcal{P}_{12}^+ + \mathcal{P}_{22}^+) \\ 2i q_x \mu^+ \kappa_{P_s}^+(\mathbf{q})\mathcal{P}_{12}^+ \\ -\frac{(\lambda^+ + m^+ \beta^+) \mathcal{P}_{12}^+ + m^+ \beta^+ \mathcal{P}_{22}^+}{V_{P_s}^{+2}} - 2\mu^+ \mathcal{P}_{12}^+ \kappa_{P_s}^{+2}(\mathbf{q}) \end{bmatrix},$$

where the matrix $\mathcal{A}(\mathbf{q})$ is defined for $\mathbf{q} = (q_x, q_y) \in \mathbb{C} \times \mathbb{R}$ by:

$$\mathcal{A}(\mathbf{q}) = \begin{bmatrix} -i q_x \mathcal{P}_{11}^+ & -i q_x \mathcal{P}_{12}^+ & i q_x \kappa_S^+(\mathbf{q}) & i q_x \mathcal{P}_{11}^- & i q_x \mathcal{P}_{12}^- & i q_x \kappa_S^-(\mathbf{q}) \\ -\kappa_{P_f}^+(\mathbf{q})\mathcal{P}_{11}^+ & -\kappa_{P_s}^+(\mathbf{q})\mathcal{P}_{12}^+ & q_x^2 + q_y^2 & -\kappa_{P_f}^-(\mathbf{q})\mathcal{P}_{11}^- & -\kappa_{P_s}^-(\mathbf{q})\mathcal{P}_{12}^- & -q_x^2 - q_y^2 \\ -\kappa_{P_f}^+(\mathbf{q})\mathcal{P}_{21}^+ & -\kappa_{P_s}^+(\mathbf{q})\mathcal{P}_{22}^+ & -(q_x^2 + q_y^2) \frac{\rho_f^+}{\rho_w} & -\kappa_{P_f}^-(\mathbf{q})\mathcal{P}_{21}^- & -\kappa_{P_s}^-(\mathbf{q})\mathcal{P}_{22}^- & (q_x^2 + q_y^2) \frac{\rho_f^-}{\rho_w} \\ \mathcal{A}_{41}(\mathbf{q}) & \mathcal{A}_{42}(\mathbf{q}) & 0 & \mathcal{A}_{44}(\mathbf{q}) & \mathcal{A}_{45}(\mathbf{q}) & 0 \\ \mathcal{A}_{51}(\mathbf{q}) & \mathcal{A}_{52}(\mathbf{q}) & \mathcal{A}_{53}(\mathbf{q}) & \mathcal{A}_{54}(\mathbf{q}) & \mathcal{A}_{55}(\mathbf{q}) & \mathcal{A}_{56}(\mathbf{q}) \\ \mathcal{A}_{61}(\mathbf{q}) & \mathcal{A}_{62}(\mathbf{q}) & \mathcal{A}_{63}(\mathbf{q}) & \mathcal{A}_{64}(\mathbf{q}) & \mathcal{A}_{65}(\mathbf{q}) & \mathcal{A}_{66}(\mathbf{q}) \end{bmatrix},$$

with

$$\begin{aligned}
\mathcal{A}_{41}(\mathbf{q}) &= \frac{m^+}{V_{Pf}^{+2}} [\beta^+ \mathcal{P}_{11}^+ + \mathcal{P}_{21}^+]; & \mathcal{A}_{44}(\mathbf{q}) &= -\frac{m^-}{V_{Pf}^{-2}} [\beta^- \mathcal{P}_{11}^- + \mathcal{P}_{21}^-]; \\
\mathcal{A}_{42}(\mathbf{q}) &= \frac{m^+}{V_{Ps}^{+2}} [\beta^+ \mathcal{P}_{12}^+ + \mathcal{P}_{22}^+]; & \mathcal{A}_{45}(\mathbf{q}) &= -\frac{m^-}{V_{Ps}^{-2}} [\beta^- \mathcal{P}_{12}^- + \mathcal{P}_{22}^-]; \\
\mathcal{A}_{51}(\mathbf{q}) &= 2i q_x \mu^+ \kappa_{Pf}^+(\mathbf{q}) \mathcal{P}_{11}^+; & \mathcal{A}_{54}(\mathbf{q}) &= 2i q_x \mu^- \kappa_{Pf}^-(\mathbf{q}) \mathcal{P}_{11}^-; \\
\mathcal{A}_{52}(\mathbf{q}) &= 2i q_x \mu^+ \kappa_{Ps}^+(\mathbf{q}) \mathcal{P}_{12}^+; & \mathcal{A}_{55}(\mathbf{q}) &= 2i q_x \mu^- \kappa_{Ps}^-(\mathbf{q}) \mathcal{P}_{12}^-; \\
\mathcal{A}_{53}(\mathbf{q}) &= -i \mu^+ q_x (\kappa_S^{+2}(\mathbf{q}) + q_x^2 + q_y^2); & \mathcal{A}_{56}(\mathbf{q}) &= i \mu^- q_x (\kappa_S^{-2}(\mathbf{q}) + q_x^2 + q_y^2); \\
\mathcal{A}_{61}(\mathbf{q}) &= \frac{(\lambda^+ + m^+ \beta^{+2}) \mathcal{P}_{11}^+ + m^+ \beta^+ \mathcal{P}_{21}^+}{V_{Pf}^{+2}} + 2\mu^+ \kappa_{Pf}^{+2}(\mathbf{q}) \mathcal{P}_{11}^+; \\
\mathcal{A}_{62}(\mathbf{q}) &= \frac{(\lambda^+ + m^+ \beta^{+2}) \mathcal{P}_{12}^+ + m^+ \beta^+ \mathcal{P}_{22}^+}{V_{Ps}^{+2}} + 2\mu^+ \kappa_{Ps}^{+2}(\mathbf{q}) \mathcal{P}_{12}^+; \\
\mathcal{A}_{63}(\mathbf{q}) &= -2(q_x^2 + q_y^2) \mu^+ \kappa_S^+(\mathbf{q}); \\
\mathcal{A}_{64}(\mathbf{q}) &= -\frac{(\lambda^- + m^- \beta^{-2}) \mathcal{P}_{11}^- + m^- \beta^- \mathcal{P}_{21}^-}{V_{Pf}^{-2}} - 2\mu^- \kappa_{Pf}^{-2}(\mathbf{q}) \mathcal{P}_{11}^-; \\
\mathcal{A}_{65}(\mathbf{q}) &= -\frac{(\lambda^- + m^- \beta^{-2}) \mathcal{P}_{12}^- + m^- \beta^- \mathcal{P}_{22}^-}{V_{Ps}^{-2}} - 2\mu^- \kappa_{Ps}^{-2}(\mathbf{q}) \mathcal{P}_{12}^-; \\
\mathcal{A}_{66}(\mathbf{q}) &= -2(q_x^2 + q_y^2) \mu^- \kappa_S^-(\mathbf{q}).
\end{aligned}$$

We also denote by V_{\max} the greatest velocity in the medium:

$$V_{\max} = \max(V_{Pf}^+, V_{Ps}^+, V_S^+, V_{Pf}^-, V_{Ps}^-, V_S^-).$$

We can now present the expression of the solution to the Green Problem:

Theorem 2.1. *The solid displacement in the top medium is given by*

$$\begin{aligned}
\mathbf{u}_s^+(x, 0, z, t) &= \mathbf{u}_{Pf}^+(x, z, t) + \mathbf{u}_{PfPf}^+(x, z, t) + \mathbf{u}_{PfPs}^+(x, z, t) + \mathbf{u}_{PfS}^+(x, z, t) \\
&+ \mathbf{u}_{Ps}^+(x, z, t) + \mathbf{u}_{PsPf}^+(x, z, t) + \mathbf{u}_{PsPs}^+(x, z, t) + \mathbf{u}_{PsS}^+(x, z, t)
\end{aligned}$$

and the solid displacement in the bottom medium is given by

$$\begin{aligned}
\mathbf{u}_s^-(x, 0, z, t) &= \mathbf{u}_{PfPf}^+(x, z, t) + \mathbf{u}_{PfPs}^+(x, z, t) + \mathbf{u}_{PfS}^+(x, z, t) \\
&+ \mathbf{u}_{PsPf}^+(x, z, t) + \mathbf{u}_{PsPs}^+(x, z, t) + \mathbf{u}_{PsS}^+(x, z, t),
\end{aligned}$$

where

- \mathbf{u}_{Pf}^+ is the solid displacement of the incident Pf wave and satisfies:

$$\begin{cases} u_{Pf,x}^+(x, z, t) & := -\frac{\mathcal{P}_{11}^+ F_{Pf}^+}{V_{Pf}^{+2}} \frac{xtH(t-t_0)}{4\pi r^3} \\ u_{Pf,z}^+(x, z, t) & := -\frac{\mathcal{P}_{11}^+ F_{Pf}^+}{V_{Pf}^{+2}} \frac{(z-h)tH(t-t_0)}{4\pi r^3} \end{cases},$$

where H denotes the usual Heaviside function. Moreover we set $r = (x^2 + (z-h)^2)^{1/2}$ and $t_0 = r/V_{Pf}^+$ denotes the time arrival of the incident Pf wave at point $(x, 0, z)$.

- \mathbf{u}_{Ps}^+ is the solid displacement of the incident Ps wave and satisfies:

$$\begin{cases} u_{Ps,x}^+(x, z, t) & := -\frac{\mathcal{P}_{12}^+ F_{Ps}^+}{V_{Ps}^{+2}} \frac{xtH(t-t_0)}{4\pi r^3} \\ u_{Ps,z}^+(x, z, t) & := -\frac{\mathcal{P}_{12}^+ F_{Ps}^+}{V_{Ps}^{+2}} \frac{(z-h)tH(t-t_0)}{4\pi r^3}. \end{cases}$$

We set here $r = (x^2 + (z-h)^2)^{1/2}$ and $t_0 = r/V_{Ps}^+$ denotes the time arrival of the incident Ps wave at point $(x, 0, z)$.

- \mathbf{u}_{PfPf}^+ is the solid displacement of the reflected PfPf wave (the Pf reflected wave generated by the Pf incident wave) and satisfies:

$$\begin{cases} u_{PfPf,x}^+(x, z, t) & = \mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_1(t)} \frac{\Im m \left[i v(t, q) \kappa_{Pf}^+(v(t, q)) \mathcal{R}_{PfPf}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 + q_0^2(t)}} dq, \\ u_{PfPf,z}^+(x, z, t) & = \mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_1(t)} \frac{\Im m \left[\kappa_{Pf}^{+2}(v(t, q)) \mathcal{R}_{PfPf}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 + q_0^2(t)}} dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $\frac{x}{r} > \frac{V_{Pf}^+}{V_{\max}}$,

$$\begin{cases} u_{PfPf,x}^+(x, z, t) & = \mathcal{P}_{11}^+ F_{Pf}^+ \int_{q_0(t)}^{q_1(t)} \frac{\Im m \left[(i v(t, q) \kappa_{Pf}^+(v(t, q)) \mathcal{R}_{PfPf}(v(t, q))) \right]}{\pi^2 r \sqrt{q^2 - q_0^2(t)}} dq \\ & - \mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_0(t)} \frac{\Re e \left[i \gamma(t, q) \kappa_{Pf}^+(\gamma(t, q)) \mathcal{R}_{PfPf}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \\ u_{PfPf,z}^+(x, z, t) & = \mathcal{P}_{11}^+ F_{Pf}^+ \int_{q_0(t)}^{q_1(t)} \frac{\Im m \left[\kappa_{Pf}^{+2}(v(t, q)) \mathcal{R}_{PfPf}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 - q_0^2(t)}} dq \\ & - \mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_0(t)} \frac{\Re e \left[\kappa_{Pf}^{+2}(\gamma(t, q)) \mathcal{R}_{PfPf}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \end{cases} \quad \text{INRIA}$$

if $t_0 < t \leq t_{h_2}$ and $\frac{x}{r} > \frac{V_{Pf}^+}{V_{\max}}$,

$$\begin{cases} u_{PfPf,x}^+(x, z, t) = -\mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_0(t)} \frac{\Re \left[i \gamma(t, q) \kappa_{Pf}^+(\gamma(t, q)) \mathcal{R}_{PfPf}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \\ u_{PfPf,z}^+(x, z, t) = -\mathcal{P}_{11}^+ F_{Pf}^+ \int_0^{q_0(t)} \frac{\Re \left[\kappa_{Pf}^+{}^2(\gamma(t, q)) \mathcal{R}_{PfPf}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \end{cases}$$

if $t_{h_2} < t$ and $\frac{x}{r} > \frac{V_{Pf}^+}{V_{\max}}$ or if $t_0 < t$ and $\frac{x}{r} \leq \frac{V_{Pf}^+}{V_{\max}}$ and

$$\mathbf{u}_{PfPf}(x, z, t) = 0 \text{ else .}$$

We set here $r = (x^2 + (z + h)^2)^{1/2}$ and $t_0 = r/V_{Pf}^+$ denotes the arrival time of the reflected PfPf volume wave at point $(x, 0, z)$,

$$t_{h_1} = (z + h) \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (20)$$

denotes the arrival time of the reflected PfPf head-wave at point $(x, 0, z)$ and

$$t_{h_2} = \frac{r}{z + h} \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \quad (21)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, (contrary to the 2D case, this time does not coincide with the arrival time of the volume wave). We also define the functions γ , v , q_0 and q_1 by

$$\gamma : \{t \in \mathbb{R} \mid t > t_0\} \times \mathbb{R} \mapsto \mathbb{C} := \gamma(t, q_y) = i \frac{xt}{r^2} + \frac{z + h}{r} \sqrt{\frac{t^2}{r^2} - \frac{1}{\mathcal{V}_{Pf}^+{}^2(q_y)}}$$

$$v : \{t \in \mathbb{R} \mid t_{h_1} < t < t_{h_2}\} \times \mathbb{R} \mapsto \mathbb{C} := v(t, q_y) = -i \left(\frac{z + h}{r} - \sqrt{\frac{1}{\mathcal{V}_{Pf}^+{}^2(q_y)} - \frac{t^2}{r^2}} + \frac{x}{r^2} t \right),$$

$$q_0 : \mathbb{R} \rightarrow \mathbb{R} := q_0(t) = \sqrt{\left| \frac{t^2}{r^2} - \frac{1}{V_{Pf}^+{}^2} \right|}$$

and

$$q_1 : \mathbb{R} \rightarrow \mathbb{R} := q_1(t) = \sqrt{\frac{1}{x^2} \left(t - (z + h) \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

• \mathbf{u}_{PfPs}^+ is the solid displacement of the reflected PfPs wave and satisfies:

$$\begin{cases} u_{PfPs,x}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[i v(t, q) \mathcal{R}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfPs,z}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[\kappa_{Ps}^+(v(t, q)) \mathcal{R}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{PfPs,x}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{R}_{PfPs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \mathcal{R}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfPs,z}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Ps}^+(\gamma(t, q)) \mathcal{R}_{PfPs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[\kappa_{Ps}^+(v(t, q)) \mathcal{R}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{PfPs,x}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{R}_{PfPs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PfPs,z}^+(x, z, t) = -\frac{\mathcal{P}_{12}^+ F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Ps}^+(\gamma(t, q)) \mathcal{R}_{PfPs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$
and $u_{PfPs}^+(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the reflected $PfPs$ volume wave at point $(x, 0, z)$ (its calculation is similar to the calculation of the arrival time of the transmitted wave, see the appendix of [9]),

$$t_{h_1} = h \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} + z \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (22)$$

denotes the arrival time of the reflected $PfPs$ head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 + hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} + \frac{z}{c_2}} \quad (23)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted $PfPs$ volume

wave, propagating at a velocity $\mathcal{V}_{Pf}^+(q)$ from the source to the interface and at velocity $\mathcal{V}_{Ps}^+(q)$ from the interface to point $(x, 0, z)$ (we refer again to [9] for details on its calculation).

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t - z \sqrt{\frac{1}{V_{Ps}^+{}^2} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = z \left(\frac{1}{\mathcal{V}_{Ps}^+{}^2(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = z \left(\frac{1}{\mathcal{V}_{Ps}^+{}^2(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PfS}^+ is the solid displacement of the reflected Pfs wave and satisfies:

$$\begin{cases} u_{PfS,x}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^+(v(t, q)) \mathcal{R}_{Pfs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfS,z}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{R}_{Pfs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PfS,x}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \kappa_S^+(\gamma(t, q)) \mathcal{R}_{Pfs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^+(v(t, q)) \mathcal{R}_{Pfs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfS,z}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2) \mathcal{R}_{Pfs}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{R}_{Pfs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PfS,x}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \kappa_S^+(\gamma(t, q)) \mathcal{R}_{PfS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PfS,z}^+(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2) \mathcal{R}_{PfS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $\mathbf{u}_{PfS}^+(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the reflected PfS volume wave at point $(x, 0, z)$ (its calculation is similar to the calculation of the arrival time of the transmitted wave, see the appendix of [9]),

$$t_{h_1} = h \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} + z \sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (24)$$

denotes the arrival time of the reflected PfS head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 + hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} + \frac{z}{c_2}} \quad (25)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PfS volume wave, propagating at a velocity $\mathcal{V}_{Pf}^+(q)$ from the source to the interface and at velocity $\mathcal{V}_S^+(q)$ from the interface to point $(x, 0, z)$ (we refer again to [9] for details on its calculation).

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t - z \sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = z \left(\frac{1}{\mathcal{V}_S^{+2}(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = z \left(\frac{1}{\mathcal{V}_S^{+2}(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^{+2}(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PsPf}^+ is the solid displacement of the reflected PsPf wave and satisfies:

$$\begin{cases} u_{PsPf,x}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \mathcal{R}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[\kappa_{Pf}^+(v(t, q)) \mathcal{R}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsPf,x}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{R}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \mathcal{R}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Pf}^+(\gamma(t, q)) \mathcal{R}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[\kappa_{Pf}^+(v(t, q)) \mathcal{R}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsPf,x}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{R}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^+(x, z, t) = -\frac{\mathcal{P}_{11}^+ F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Pf}^+(\gamma(t, q)) \mathcal{R}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m [\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$

and $\mathbf{u}_{PsPf}^+(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the reflected PsPf volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} + z \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (26)$$

denotes the arrival time of the reflected PsPf head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 + hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} + \frac{z}{c_2}} \quad (27)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PsPf volume wave, propagating at a velocity $\mathcal{V}_{Ps}^+(q)$ from the source to the interface and at velocity $\mathcal{V}_{Pf}^+(q)$ from the interface to point $(x, 0, z)$.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t - z \sqrt{\frac{1}{V_{Pf}^{+2}} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = z \left(\frac{1}{\mathcal{V}_{Pf}^{+2}(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = z \left(\frac{1}{\mathcal{V}_{Pf}^{+2}(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PsPs}^+ is the solid displacement of the reflected PsPs wave and satisfies:

$$\begin{cases} u_{PsPs,x}^+(x, z, t) = \mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_1(t)} \frac{\Im m \left[i v(t, q) \kappa_{Ps}^+(v(t, q)) \mathcal{R}_{PsPs}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 + q_0^2(t)}} dq, \\ u_{PsPs,z}^+(x, z, t) = \mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_1(t)} \frac{\Im m \left[\kappa_{Ps}^+{}^2(v(t, q)) \mathcal{R}_{PsPs}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 + q_0^2(t)}} dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $\frac{x}{r} > \frac{V_{Ps}^+}{V_{\max}}$,

$$\begin{cases} u_{PsPs,x}^+(x, z, t) = \mathcal{P}_{12}^+ F_{Ps}^+ \int_{q_0(t)}^{q_1(t)} \frac{\Im m \left[(i v(t, q) \kappa_{Ps}^+(v(t, q)) \mathcal{R}_{PsPs}(v(t, q))) \right]}{\pi^2 r \sqrt{q^2 - q_0^2(t)}} dq \\ - \mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_0(t)} \frac{\Re e \left[i \gamma(t, q) \kappa_{Ps}^+(\gamma(t, q)) \mathcal{R}_{PsPs}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \\ u_{PsPs,y}^+(x, z, t) = \mathcal{P}_{12}^+ F_{Ps}^+ \int_{q_0(t)}^{q_1(t)} \frac{\Im m \left[\kappa_{Ps}^+{}^2(v(t, q)) \mathcal{R}_{PsPs}(v(t, q)) \right]}{\pi^2 r \sqrt{q^2 - q_0^2(t)}} dq \\ - \mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_0(t)} \frac{\Re e \left[\kappa_{Ps}^+{}^2(\gamma(t, q)) \mathcal{R}_{PsPs}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $\frac{x}{r} > \frac{V_{Ps}^+}{V_{\max}}$,

$$\begin{cases} u_{PsPs,x}^+(x, z, t) = -\mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_0(t)} \frac{\Re e \left[i \gamma(t, q) \kappa_{Ps}^+(\gamma(t, q)) \mathcal{R}_{PsPs}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \\ u_{PsPs,y}^+(x, z, t) = -\mathcal{P}_{12}^+ F_{Ps}^+ \int_0^{q_0(t)} \frac{\Re e \left[\kappa_{Ps}^+{}^2(\gamma(t, q)) \mathcal{R}_{PsPs}(\gamma(t, q)) \right]}{\pi^2 r \sqrt{q_0^2(t) - q^2}} dq, \end{cases}$$

if $t_{h_2} < t$ and $\frac{x}{r} > \frac{V_{Ps}^+}{V_{\max}}$ or if $t_0 < t$ and $\frac{x}{r} \leq \frac{V_{Ps}^+}{V_{\max}}$ and

$$\mathbf{u}_{PsPs}(x, z, t) = 0 \text{ else .}$$

We set here $r = (x^2 + (z + h)^2)^{1/2}$ and $t_0 = r/V_{Ps}^+$ denotes the arrival time of the reflected PsPs volume wave at point $(x, 0, z)$,

$$t_{h_1} = (z + h) \sqrt{\frac{1}{V_{Ps}^+{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (28)$$

denotes the arrival time of the reflected PsPs head-wave at point $(x, 0, z)$ and

$$t_{h_2} = \frac{r}{z + h} \sqrt{\frac{1}{V_{Ps}^+{}^2} - \frac{1}{V_{\max}^2}} \quad (29)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, (contrary to the 2D case, this time does not coincide with the arrival time of the volume wave). We also define the functions γ , v , q_0 and q_1 by

$$\gamma : \{t \in \mathbb{R} \mid t > t_0\} \times \mathbb{R} \mapsto \mathbb{C} := \gamma(t, q_y) = i \frac{xt}{r^2} + \frac{z+h}{r} \sqrt{\frac{t^2}{r^2} - \frac{1}{V_{Ps}^{+2}(q_y)}}$$

$$v : \{t \in \mathbb{R} \mid t_{h_1} < t < t_{h_2}\} \times \mathbb{R} \mapsto \mathbb{C} := v(t, q_y) = -i \left(\frac{z+h}{r} - \sqrt{\frac{1}{V_{Ps}^{+2}(q_y)} - \frac{t^2}{r^2}} + \frac{x}{r^2} t \right),$$

$$q_0 : \mathbb{R} \rightarrow \mathbb{R} := q_0(t) = \sqrt{\left| \frac{t^2}{r^2} - \frac{1}{V_{Ps}^{+2}} \right|}$$

and

$$q_1 : \mathbb{R} \rightarrow \mathbb{R} := q_1(t) = \sqrt{\frac{1}{x^2} \left(t - (z+h) \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

- \mathbf{u}_{PsS}^+ is the solid displacement of the reflected PsS wave and satisfies:

$$\begin{cases} u_{PsS,x}^+(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[i v(t, q) \kappa_S^+(v(t, q)) \mathcal{R}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^+(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[(v^2(t, q) + q^2) \mathcal{R}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsS,x}^+(x, z, t) = \frac{\mathcal{F}_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[i \gamma(t, q) \kappa_S^+(\gamma(t, q)) \mathcal{R}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{F}_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re \left[i v(t, q) \kappa_S^+(v(t, q)) \mathcal{R}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^+(x, z, t) = \frac{\mathcal{F}_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[(\gamma^2(t, q) + q^2) \mathcal{R}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{F}_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re \left[(v^2(t, q) + q^2) \mathcal{R}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsS,x}^+(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[i \gamma(t, q) \kappa_S^+(\gamma(t, q)) \mathcal{R}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^+(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[(\gamma^2(t, q) + q^2) \mathcal{R}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $\mathbf{u}_{PsS}^+(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the reflected PsS volume wave at point $(x, 0, z)$ (its calculation is similar to the calculation of the arrival time of the transmitted wave, see the appendix of [9]),

$$t_{h_1} = h\sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} + z\sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (30)$$

denotes the arrival time of the reflected PsS head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 + hz\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) + x^2}{\frac{h}{c_1} + \frac{z}{c_2}} \quad (31)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PsS volume wave, propagating at a velocity $\mathcal{V}_{Ps}^+(q)$ from the source to the interface and at velocity $\mathcal{V}_S^+(q)$ from the interface to point $(x, 0, z)$ (we refer again to [9] for details on its calculation).

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t - z\sqrt{\frac{1}{V_S^{+2}} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = z \left(\frac{1}{\mathcal{V}_S^{+2}(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = z \left(\frac{1}{\mathcal{V}_S^{+2}(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m[\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PfPf}^- is the solid displacement of the transmitted $PfPf$ wave (the Pf transmitted wave generated by the Pf incident wave) and satisfies:

$$\begin{cases} u_{PfPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[i v(t, q) \mathcal{T}_{PfPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re \left[\kappa_{Pf}^-(v(t, q)) \mathcal{T}_{PfPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PfPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[i \gamma(t, q) \mathcal{T}_{PfPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re \left[i v(t, q) \mathcal{T}_{PfPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[\kappa_{Pf}^-(\gamma(t, q)) \mathcal{T}_{PfPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re \left[\kappa_{Pf}^-(v(t, q)) \mathcal{T}_{PfPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PfPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[i \gamma(t, q) \mathcal{T}_{PfPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PfPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re \left[\kappa_{Pf}^-(\gamma(t, q)) \mathcal{T}_{PfPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $\mathbf{u}_{PfPf}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted $PfPf$ volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} - z \sqrt{\frac{1}{V_{Pf}^-{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (32)$$

denotes the arrival time of the transmitted $PfPf$ head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (33)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_{Pf}^-{}^2} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PfPf volume wave, propagating at a velocity $\mathcal{V}_{Pf}^+(q)$ in the top layer and at velocity $\mathcal{V}_{Pf}^-(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z \sqrt{\frac{1}{V_{Pf}^-{}^2} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{\mathcal{V}_{Pf}^-{}^2(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{\mathcal{V}_{Pf}^-{}^2(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PfPs}^- is the solid displacement of the transmitted PfPs wave and satisfies:

$$\begin{cases} u_{PfPs,x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \mathcal{T}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfPs,z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[\kappa_{Ps}^-(v(t, q)) \mathcal{T}_{PfPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{P_f P_s, x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{P_f P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \mathcal{T}_{P_f P_s}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{P_f P_s, z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{P_s}^-(\gamma(t, q)) \mathcal{T}_{P_f P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[\kappa_{P_s}^-(v(t, q)) \mathcal{T}_{P_f P_s}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{P_f P_s, x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{P_f P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{P_f P_s, z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{P_f}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{P_s}^-(\gamma(t, q)) \mathcal{T}_{P_f P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$
and $u_{P_f P_s}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted $P_f P_s$ volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{P_f}^+{}^2} - \frac{1}{V_{\max}^2}} - z \sqrt{\frac{1}{V_{P_s}^-{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (34)$$

denotes the arrival time of the transmitted $P_f P_s$ head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (35)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{P_f}^+{}^2} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_{P_s}^-{}^2} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted $P_f P_s$ volume wave, propagating at a velocity $\mathcal{V}_{P_f}^+(q)$ in the top layer and at velocity $\mathcal{V}_{P_s}^-(q)$ in the bottom layer.

The function $q_1 : [t_1 ; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z \sqrt{\frac{1}{V_{Ps}^-^2} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Pf}^+^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{V_{Ps}^-^2(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{V_{Pf}^+^2(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{V_{Ps}^-^2(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{V_{Pf}^+^2(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- u_{PfS}^- is the solid displacement of the transmitted PFS wave and satisfies:

$$\begin{cases} u_{PfS,x}^-(x, z, t) = -\frac{F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^-(v(t, q)) \mathcal{T}_{PFS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfS,z}^-(x, z, t) = \frac{F_{Pf}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{T}_{PFS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PfS,x}^-(x, z, t) = -\frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \kappa_S^-(\gamma(t, q)) \mathcal{T}_{PFS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{F_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^-(v(t, q)) \mathcal{T}_{PFS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PfS,z}^-(x, z, t) = \frac{\mathcal{F}_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2) \mathcal{T}_{PFS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{F}_{Pf}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{T}_{PFS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{Pfs,x}^-(x, z, t) = -\frac{F_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i\gamma(t, q)\kappa_S^-(\gamma(t, q))\mathcal{T}_{Pfs}(\gamma(t, q))\frac{\partial\gamma}{\partial t}(t, q) \right] dq, \\ u_{Pfs,z}^-(x, z, t) = \frac{\mathcal{F}_{Pf}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2)\mathcal{T}_{Pfs}(\gamma(t, q))\frac{\partial\gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $u_{Pfs}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted Pfs volume wave at point $(x, 0, z)$,

$$t_{h_1} = h\sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} - z\sqrt{\frac{1}{V_S^-{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (36)$$

denotes the arrival time of the transmitted Pfs head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (37)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_S^-{}^2} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted Pfs volume wave, propagating at a velocity $\mathcal{V}_{Pf}^+(q)$ in the top layer and at velocity $\mathcal{V}_S^-(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z\sqrt{\frac{1}{V_S^-{}^2} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V_{Pf}^+{}^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{\mathcal{V}_S^-{}^2(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{\mathcal{V}_S^-{}^2(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Pf}^+{}^2(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PsPf}^- is the solid displacement of the transmitted *PsPf* wave and satisfies:

$$\begin{cases} u_{PsPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \mathcal{T}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[\kappa_{Pf}^-(v(t, q)) \mathcal{T}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \mathcal{T}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Pf}^-(\gamma(t, q)) \mathcal{T}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[\kappa_{Pf}^-(v(t, q)) \mathcal{T}_{PsPf}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsPf,x}^-(x, z, t) = -\frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PsPf,z}^-(x, z, t) = \frac{\mathcal{P}_{11}^- F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{Pf}^-(\gamma(t, q)) \mathcal{T}_{PsPf}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m [\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$
and $\mathbf{u}_{PsPf}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted *PsPf* volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{Ps}^+{}^2} - \frac{1}{V_{\max}^2}} - z \sqrt{\frac{1}{V_{Pf}^-{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (38)$$

denotes the arrival time of the transmitted PsPf head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (39)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_{Pf}^{-2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PsPf volume wave, propagating at a velocity $\mathcal{V}_{Ps}^+(q)$ in the top layer and at velocity $\mathcal{V}_{Pf}^-(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z \sqrt{\frac{1}{V_{Pf}^{-2}} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{\mathcal{V}_{Pf}^{-2}(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{\mathcal{V}_{Pf}^{-2}(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- \mathbf{u}_{PsPs}^- is the solid displacement of the transmitted PsPs wave and satisfies:

$$\begin{cases} u_{PsPs,x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \mathcal{I}_{PsPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsPs,z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[\kappa_{Ps}^-(v(t, q)) \mathcal{I}_{PsPs}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{P_s P_s, x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{P_s P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \mathcal{T}_{P_s P_s}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{P_s P_s, z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{P_s}^-(\gamma(t, q)) \mathcal{T}_{P_s P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[\kappa_{P_s}^-(v(t, q)) \mathcal{T}_{P_s P_s}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\left\{ \begin{array}{l} u_{P_s P_s, x}^-(x, z, t) = -\frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \mathcal{T}_{P_s P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{P_s P_s, z}^-(x, z, t) = \frac{\mathcal{P}_{12}^- F_{P_s}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[\kappa_{P_s}^-(\gamma(t, q)) \mathcal{T}_{P_s P_s}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{array} \right.$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $\mathbf{u}_{P_s P_s}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted PsPs volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{P_s}^+{}^2} - \frac{1}{V_{\max}^2}} - z \sqrt{\frac{1}{V_{P_s}^-{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (40)$$

denotes the arrival time of the transmitted PsPs head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (41)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{P_s}^+{}^2} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_{P_s}^-{}^2} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PsPs volume wave, propagating at a velocity $\mathcal{V}_{P_s}^+(q)$ in the top layer and at velocity $\mathcal{V}_{P_s}^-(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z \sqrt{\frac{1}{\mathcal{V}_{Ps}^-^2} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{\mathcal{V}_{Ps}^+^2} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{\mathcal{V}_{Ps}^-^2(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^+^2(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{\mathcal{V}_{Ps}^-^2(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^+^2(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

- u_{PsS}^- is the solid displacement of the transmitted PsS wave and satisfies:

$$\begin{cases} u_{PsS,x}^-(x, z, t) = -\frac{F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^-(v(t, q)) \mathcal{T}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^-(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{T}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_1} < t \leq t_0$ and $|\Im m [\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsS,x}^-(x, z, t) = -\frac{F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \kappa_S^-(\gamma(t, q)) \mathcal{T}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad - \frac{F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[i v(t, q) \kappa_S^-(v(t, q)) \mathcal{T}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^-(x, z, t) = \frac{F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2) \mathcal{T}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq \\ \quad + \frac{F_{Ps}^+}{\pi^2} \int_{q_0(t)}^{q_1(t)} \Re e \left[(v^2(t, q) + q^2) \mathcal{T}_{PsS}(v(t, q)) \frac{\partial v}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$,

$$\begin{cases} u_{PsS,x}^-(x, z, t) = -\frac{F_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[i \gamma(t, q) \kappa_S^-(\gamma(t, q)) \mathcal{T}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \\ u_{PsS,z}^-(x, z, t) = \frac{\mathcal{F}_{Ps}^+}{\pi^2} \int_0^{q_0(t)} \Re e \left[(\gamma^2(t, q) + q^2) \mathcal{T}_{PsS}(\gamma(t, q)) \frac{\partial \gamma}{\partial t}(t, q) \right] dq, \end{cases}$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0, 0)]| < \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0, 0)]| \geq \frac{1}{V_{\max}}$ and $\mathbf{u}_{PsS}^-(x, z, t) = 0$ else.

t_0 denotes here the arrival time of the transmitted PsS volume wave at point $(x, 0, z)$,

$$t_{h_1} = h \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} - z \sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \quad (42)$$

denotes the arrival time of the transmitted PsS head wave at point $(x, 0, z)$,

$$t_{h_2} = \frac{h^2 + z^2 - hz \left(\frac{c_2}{c_1} + \frac{c_1}{c_2} \right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \quad (43)$$

denotes the time after which there is no longer head wave at point $(x, 0, z)$, where

$$c_1 = \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \text{ and } c_2 = \sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}}.$$

The function $q_0 : [t_0; +\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0 : \mathbb{R}^+ \mapsto [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x, 0, z)$ of the fictitious transmitted PsS volume wave, propagating at a velocity $\mathcal{V}_{Ps}^+(q)$ in the top layer and at velocity $\mathcal{V}_S^-(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2} \left(t + z \sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}} - h \sqrt{\frac{1}{V_{Ps}^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma, q, t) = -z \left(\frac{1}{\mathcal{V}_S^{-2}(q)} + \gamma^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v, q, t) = -z \left(\frac{1}{\mathcal{V}_S^{-2}(q)} + v^2 \right)^{1/2} + h \left(\frac{1}{\mathcal{V}_{Ps}^{+2}(q)} + v^2 \right)^{1/2} + ivx - t$$

such that $\Im m [\partial_t v(t, q)] < 0$, with

$$E_1 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\}$$

and

$$E_2 = \{(t, q) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\}.$$

Remark 2.1. For the practical computations of the velocities, we won't have to explicitly compute the derivatives of the displacement u , which would be rather tedious, since

$$\frac{du}{dt} * f = u * f'.$$

Therefore, we'll only have to compute the derivative of the source function f .

3 Numerical illustration

To illustrate our results, we have computed the green function and the analytical solution to the following problem: we consider an two-layered poroelastic medium whose characteristic coefficients are

- the solid density: $\rho_s^+ = 2200 \text{ kg/m}^3$ and $\rho_s^- = 2650 \text{ kg/m}^3$;
- the fluid density: $\rho_f^+ = 950 \text{ kg/m}^3$ and $\rho_f^- = 750 \text{ kg/m}^3$;
- the porosity: $\phi^+ = 0.4$ and $\phi^- = 0.2$;
- the tortuosity: $a^+ = 2$ and $a^- = 2$;
- the solid bulk modulus: $K_s^+ = 6.9 \text{ GPa}$ and $K_s^- = 37 \text{ GPa}$;
- the fluid bulk modulus: $K_f^+ = 2 \text{ GPa}$ and $K_f^- = 1.7 \text{ GPa}$;
- the frame bulk modulus: $K_b^+ = 6.7 \text{ GPa}$ and $K_b^- = 2.2 \text{ GPa}$;
- the frame shear modulus $\mu^+ = 3 \text{ GPa}$ and $\mu^- = 4.4 \text{ GPa}$;

so that the celerity of the waves in the poroelastic medium are:

- for the fast P wave, $V_{Pf}^+ = 2692 \text{ m/s}$ and $V_{Pf}^- = 2535 \text{ m/s}$;
- for the slow P wave, $V_{Ps}^+ = 1186 \text{ m/s}$ and $V_{Ps}^- = 744 \text{ m/s}$;
- for the ψ wave, $V_S^+ = 1409 \text{ m/s}$ and $V_S^- = 1415 \text{ m/s}$.

The source is located in the top layer, at 500 m from the interface. We used two types of sources in space: the first one is a bulk source such that $f_u = f_w = -10^{10}$ and $f_p = 0$; the second one is a pressure source such that $f_u = f_w = 0$ and $f_p = 1$. In each case we used a fourth derivative of a Gaussian of dominant frequency $f_0 = 15 \text{ Hz}$:

$$f(t) = 2 \frac{\pi^2}{f_0^2} \left[3 + 12 \frac{\pi^2}{f_0^2} \left(t - \frac{1}{f_0} \right)^2 + 4 \frac{\pi^4}{f_0^4} \left(t - \frac{1}{f_0} \right)^4 \right] e^{-\frac{\pi^2}{f_0^2} \left(t - \frac{1}{f_0} \right)^2}$$

for the source in time. We compute the solution at two receivers, the first one is in the upper layer, at 533 m from the interface; the second one is in the bottom layer, at 533 m from the interface; both are located on a vertical line at 400 m from the source (see Fig. 3). We represent the z component of the green function associated to the solid displacement from $t = 0$ to $t = 1.4$ s in Fig. 4 for the bulk source and in Fig. 6 for the pressure source. In Figs. 5 and 7, we plot the solid displacement. The left pictures represents the solution at receiver 1 while the right pictures represents the solution at receiver 2. As all the types of waves are computed independently, it is easy to distinguish all of them, as it is indicated in the figures. solution.

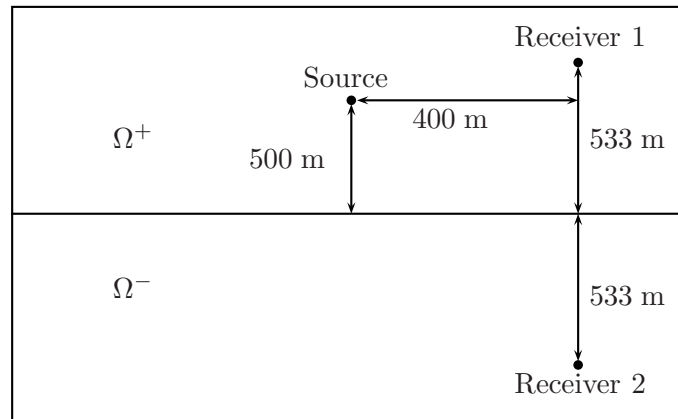


Figure 3: Configuration of the experiment

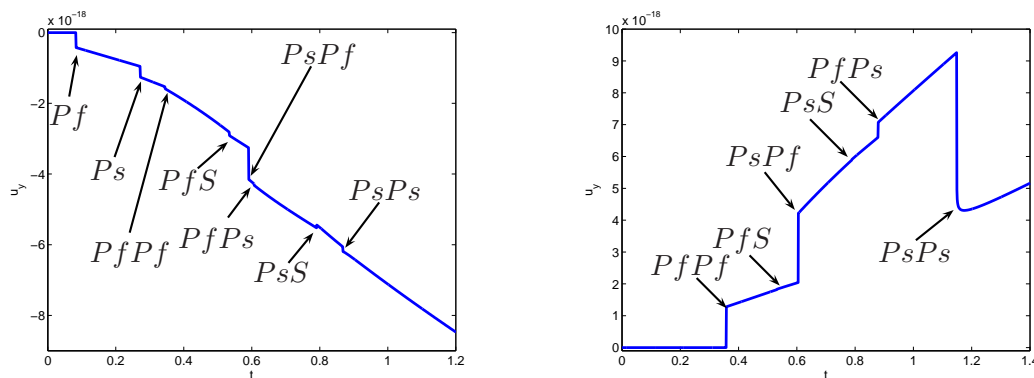


Figure 4: The z component of the green function associated to the displacement at receiver 1 (left picture) and 2 (right picture), in the case of a bulk source

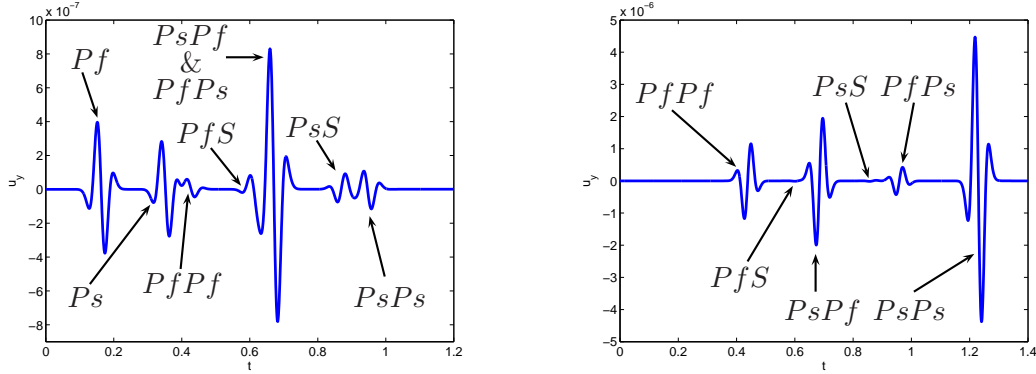


Figure 5: The z component of the displacement at receiver 1 (left picture) and 2 (right picture) in the case of a bulk source.

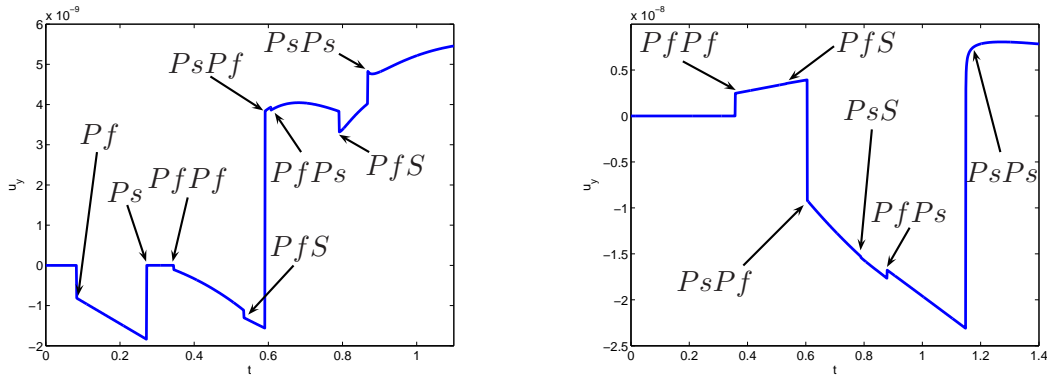


Figure 6: The z component of the green function associated to the displacement at receiver 1 (left picture) and 2 (right picture) in the case of a pressure source.

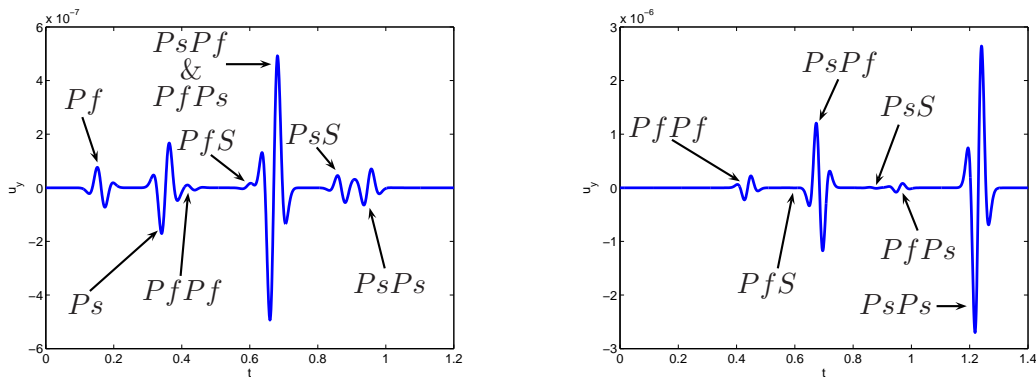


Figure 7: The z component of the displacement at receiver 1 (left picture) and 2 (right picture) in the case of a pressure source.

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