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# Pseudo-Conform Polynomial Lagrange Finite Elements on Quadrilaterals and Hexahedra 

Eric Dubach — Robert Luce — Jean-Marie Thomas

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# Pseudo-Conform Polynomial Lagrange Finite Elements on Quadrilaterals and Hexahedra 

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#### Abstract

The aim of this paper is to develop a new class of finite elements on quadrilaterals and hexahedra. The degrees of freedom are the values at the vertices and the approximation is polynomial on each element $K$. In general, with this kind of finite elements, the resolution of second order elliptic problems leads to non-conform approximations. Degrees of freedom are the same than those of isoparametric finite elements. And, in the particular case when the finite elements are parallelotopes, our method is conform and coincides with the classical finite elements on structured meshes. The convergence of the method is analysed and the theory is confirmed by some numerical results.


Key-words: nonconforming method, polynomial approximation, quadrilateral and hexahedral finite element

[^0]
## Elément finis polynomiaux Pseudo-Conformes sur des quadrilatéres et hexaédres

Résumé : Dans cet article, nous proposons une nouvelle classe d'éléments finis quadrilatéraux et hexaèdraux. Les degrés de liberté sont les valeurs aux noeuds et l'approximation est polynômiale sur chaque élément $K$. En général, avec ce type d'éléments finis, on obtient des approximations non conformes lors de la résolution de probléme elliptique du second ordre. Et, dans le cas particulier de parallélotopes, notre méthode est conforme et coïncide avec les éléments finis classiques sur maillage structuré. La convergence de la méthode est analysée et la théorie est confirmée par quelques résultats numériques.

Mots-clés : méthodes non conformes, approximation polynômiale, éléments finis quadrilatéraux et hexaèdraux

## 1 Introduction

Quadrilaterals and hexahedra are often used in meshers particularly in geophysical applications and in fluids mecanics. When the geometry and the medium are structured, regular rectangular or parallelipipedic meshes are used. Otherwise general convex quadrilaterals or hexahedra are used. Then, with isoparametric Lagrange finite elements ([1],[2], [3] , see also [4, [5]), the basis functions are built by using multilinear mappings on a reference square or a reference cube. The jacobians of these mappings leads to non polynomial basis functions on the elements of the mesh.
One way for obtaining polynomial basis functions is to cut the quadrilaterals into triangles (or hexahedra into tetrahedra) and work with macro-elements ([6], 7], [8). It is not our process. We choose to built finite elements by considering quadrilaterals and hexahedra as distortions of parallelograms and parallelepipeds.
It is important to note here that the reserved vocabulary is the one of mathematicians. therefore an hexahedron is an example of polyhedron and its faces are plane. In the literature of the mechanics, usually an hexahedron denotes the image of a cube by a $Q_{1}$ transformation; commonly, the faces of a " trilinear hexahedral element" (for instance see [9) are not plane ; they are nappes of hyperbolic paraboloids. Note that the inversibility of the transformation of a biunit cube into an hexahedron is still open (10, [11). The generalization of the pseudo-conform method to these trilinear hexahedra will be presented in a work to come. The extension to mixed finite elements will be presented in further paper.

In the presented method, the Lagrange basis functions are built under conditions of weak-continuity of the unknowns between the elements. The resulting finite elements are not conform but these conditions of weak-continuity are sufficient to ensure the expected order of convergence.
The first part is devoted to the finite elements geometry. The approach choosen allows us to describe jointly quadrilaterals and hexahedra. The second part deals with general results on local error estimates for non isoparametric Lagrange finite elements. These estimations are essential to obtain the expected order of convergence of our pseudo-conform finite elements. The third part presents the elliptic problem of order 2 and gives sufficient conditions to establish the convergence of the method. In the forth part, we built the finite elements. And in the last part, numerical results are presented.

In this paper we use the following notations:
For a vector $\mathbf{v} \in \mathbb{R}^{\mathbf{n}},|\mathbf{v}|$ is the lenght of the vector $\mathbf{v}$; in matrix notation $\mathbf{v}$ is represented by the column vector $\left(v_{1}, \ldots, v_{n}\right)^{T}$ and then $|\mathbf{v}|=\left\{\sum_{1 \leq j \leq n}\left|v_{j}\right|^{2}\right\}^{1 / 2}$, the Euclidean norm of the associated column vector. And for a square matrix $B,\|B\|$ is the spectral norm.
For a triangle or a quadrilateral $K,|K|$ is the area of $K$ and if $\gamma$ is a side of $K$, $|\gamma|$ is the lenght of $\gamma$; for a tetrahedron or an hexahedron $K,|K|$ is the volume of $K$ and if $\gamma$ is a face of $K,|\gamma|$ is the area of $\gamma$.
For a polyhedral domain $K$, we note

$$
H^{m}(K)=\left\{v \in L^{2}(K) ; \partial^{\alpha} v \in L^{2}(K), \text { for all } \alpha \text { with }|\alpha| \leq m\right\}
$$

equipped with the norm and the semi-norm

$$
\|v\|_{m, K}=\left(\sum_{|\alpha| \leq m} \int_{K}\left|\partial^{\alpha} v\right|^{2} d x\right)^{1 / 2}, \quad|v|_{m, K}=\left(\sum_{|\alpha|=m} \int_{K}\left|\partial^{\alpha} v\right|^{2} d x\right)^{1 / 2}
$$

We consider also the following norm and semi-norm

$$
\|v\|_{m, \infty, K}=\max _{|\alpha| \leq m}\left\{\operatorname{ess} \sup _{x \in K}\left|\partial^{\alpha} v\right|\right\}, \quad|v|_{m, \infty, K}=\max _{|\alpha|=m}\left\{\operatorname{ess} \sup _{x \in K}\left|\partial^{\alpha} v\right|\right\}
$$

$P(K)$ is the vectorial space $\{\mathbf{x} \in K \mapsto p(\mathbf{x}) ; p \in P\}$, where $P$ is a $N$ variables polynomial space and $K$ is a domain in $\mathbb{R}^{N}$. For any integer $k, P_{k}$ is the space of polynomial functions of degree $\leq k$, while $Q_{k}$ is the space of polyomial functions of degree $\leq k$ in each variable.
For each polyhedral $K, h_{K}$ denotes the diameter of $K$ and $\rho_{K}$ denotes the diameter of the largest ball contained in $K$.

### 1.1 The geometry; vertex and face numbering.

In $\mathbb{R}^{N}$ with $N=2$ or 3 , let $K$ be a convex nondegenerated quadrilateral when $N=2$, a convex nondegenerated hexahedron when $N=3$.
Let $\left\{\mathbf{a}_{i} \in \mathbb{R}^{N}, 1 \leq i \leq 2^{N}\right\}$ be the vertices of $K$. We use hereafter the word "face" for 2D and 3D geometries with the following vocabulary convention: for $N=2$, a face of a quadrilateral $K$ designates a side of $K$. We designate by "edge" of $K$ a side of $K$ when $N=2$, the intersection of two adjacent quadrangular faces of $K$ when $N=3$.
Two vertices which do not belong to a same face of $K$ are said opposite vertices. The numbering of the vertices is shown on Figure 1 and Figure 2. Note that this vertex numbering is such that

$$
\begin{aligned}
& \forall i=2, \ldots, 2^{N-1}, \quad\left[\mathbf{a}_{1}, \mathbf{a}_{i}\right] \text { is an edge of } K \\
& \forall i=1, \ldots, 2^{N-1}, \quad \mathbf{a}_{i} \text { and } \mathbf{a}_{2^{N}+1-i} \text { are opposite vertices of } K .
\end{aligned}
$$

The center of a polyhedral is the isobarycenter of its vertices; we note $\mathbf{a}_{0}$ the center of $K$ :

$$
\mathbf{a}_{0}=\frac{1}{2^{N}} \sum_{1 \leq i \leq 2^{N}} \mathbf{a}_{i}
$$

Let now $\left\{\gamma_{m} \sqsubset \mathbb{R}^{N}, 1 \leq m \leq 2 N\right\}$ be the set of the faces of $K$. Two faces without common vertex are said opposite faces. The face numbering is shown on Figure 1 and Figure 2 This numbering is such that

$$
\begin{aligned}
& \bigcap_{1 \leq i \leq N} \gamma_{i}=\mathbf{a}_{1} \\
& \forall m=1, \ldots, N-1, \quad \mathbf{a}_{m+1} \notin \gamma_{m} \\
& \forall m=1, \ldots, N, \quad \gamma_{m} \text { and } \gamma_{2 N+1-m} \text { are opposite faces of } K .
\end{aligned}
$$

Last, let $\mathbf{b}_{m}$ be the center of the face $\gamma_{m}$, for $m=1, \ldots, 2 N$, and let us introduce the vectors $\mathbf{e}_{m} \in \mathbb{R}^{N}$ defined by

$$
\forall m=1, \ldots, N, \quad \mathbf{e}_{m}=\mathbf{a}_{0}-\mathbf{b}_{m}\left(=\mathbf{b}_{2 N+1-m}-\mathbf{a}_{0}\right)
$$

Since $K$ is assumed to be a nondegenerated polyhedron, $\left(\mathbf{e}_{1}, \ldots, \mathbf{e}_{N}\right)$ is a basis of $\mathbb{R}^{N}$ 。


Figure 1: Numerotation $(N=2)$


Figure 2: Numerotation $(N=3)$

### 1.2 Affine-equivalent elements.

Let $\widehat{K}=[-1,+1]^{N}$ be the reference square when $N=2$, the reference cube when $N=3$. The vertices of $\widehat{K}$ are denoted by $\widehat{\mathbf{a}}_{i}, 1 \leq i \leq 2^{N}$ and the faces are denoted by $\widehat{\gamma}_{m}, 1 \leq m \leq 2 N$.
We choose the following vertex numbering

$$
\begin{aligned}
& \text { for } N=2: \quad \widehat{\mathbf{a}}_{1}=\binom{-1}{-1}, \quad \widehat{\mathbf{a}}_{2}=\binom{+1}{-1} \\
& \text { for } N=3: \quad \widehat{\mathbf{a}}_{1}=\left(\begin{array}{l}
-1 \\
-1 \\
-1
\end{array}\right), \quad \widehat{\mathbf{a}}_{2}=\left(\begin{array}{l}
+1 \\
-1 \\
-1
\end{array}\right), \widehat{\mathbf{a}}_{3}=\left(\begin{array}{l}
-1 \\
+1 \\
-1
\end{array}\right), \widehat{\mathbf{a}}_{4}=\left(\begin{array}{l}
-1 \\
-1 \\
+1
\end{array}\right)
\end{aligned}
$$

and

$$
\text { for } N=2 \text { and } N=3: \quad \widehat{\mathbf{a}}_{i}=-\widehat{\mathbf{a}}_{1+2^{N}-i}, \quad 1+2^{N-1} \leq i \leq 2^{N}
$$

The face numbering is defined by

$$
\begin{aligned}
& \forall m=1, \ldots, \quad N, \quad \widehat{\gamma}_{m}=\left\{\widehat{\mathbf{x}}=\left(\widehat{x}_{1}, \ldots, \widehat{x}_{N}\right)^{T} \in \widehat{K} ; \widehat{x}_{m}=-1\right\} \\
& \forall m=1, \ldots, \quad N, \quad \widehat{\gamma}_{2 N+1-m}=\widehat{\gamma}_{m}
\end{aligned}
$$

Let $\widehat{\mathbf{b}}_{m}$ be the center of the face $\widehat{\gamma}_{m}$, for $m=1, \ldots, 2 N$. The canonical basis $\left(\widehat{\mathbf{e}}_{1}, \ldots, \widehat{\mathbf{e}}_{N}\right)$ of $\mathbf{R}^{N}$ can be simply express with the vectors $\widehat{\mathbf{b}}_{m}$

$$
\widehat{\mathbf{e}}_{m}=-\widehat{\mathbf{b}}_{m}\left(=\widehat{\mathbf{b}}_{2 N+1-m}\right), \quad 1 \leq m \leq N
$$

Let $B_{K}$ be the change of basis matrix given by

$$
B_{K} \widehat{\mathbf{e}}_{m}=\mathbf{e}_{m}, \quad 1 \leq m \leq N
$$

and $F_{K}^{\sharp}$ be the invertible affine mapping

$$
F_{K}^{\sharp}: \widehat{\mathbf{x}} \in \mathbf{R}^{N} \rightarrow F_{K}^{\sharp}(\widehat{\mathbf{x}})=\mathbf{a}_{0}+B_{K} \widehat{\mathbf{x}}
$$

This mapping $F_{K}^{\sharp}$ is the unique affine mapping such that

$$
F_{K}^{\sharp}\left(\widehat{\mathbf{b}}_{m}\right)=\mathbf{b}_{m}, \quad 1 \leq m \leq N .
$$

It is a bijection between $\widehat{K}$ and its image

$$
K^{\sharp}=F_{K}^{\sharp}(\widehat{K}) .
$$

As image of the reference parallelotope by an inversible affine mapping, $K^{\sharp}$ is a parallelotope. The associated parallelotope of $K$ being by definition the parallelotope which has the same face centers than $K$. We see that $K^{\sharp}$ is the associated parallelotope of $K$ and we have $K^{\sharp}=K$ if and only if $K$ is a parallelotope. Let

$$
K^{\vee}=\left(F_{K}^{\sharp}\right)^{-1}(K) .
$$

The parallelotope associated to the polyhedron $K^{\vee}$ is the reference parallelotope $\widehat{K}$. For the analysis of quadrangular and hexahedral finite element, it is useful to precise how the element $K$ is distorted.

### 1.3 Distortion parameters

When $N=2$, let $\mathbf{d}$ be the vector of $\mathbb{R}^{2}$ given by

$$
\begin{equation*}
\mathbf{d}=\frac{1}{4}\left(\mathbf{a}_{1}-\mathbf{a}_{2}-\mathbf{a}_{3}+\mathbf{a}_{4}\right) . \tag{1}
\end{equation*}
$$

We can interpret $\mathbf{2 d}$ as the vector whose extremities are the mid-points of the diagonals of the quadrilateral $K$. This means that the quadrilateral $K$ is a parallelogram if and only if $\mathbf{d}=0$. It is easy to see that the vertices of $K^{\sharp}$ (the parallelogram associated to the quadrilateral $K$ ), are given by

$$
\mathbf{a}_{i}^{\sharp}=\mathbf{a}_{i}-s_{i} \mathbf{d}, \quad 1 \leq i \leq 4
$$

where

$$
\begin{equation*}
s_{1}=s_{4}=+1, \quad s_{2}=s_{3}=-1 \tag{2}
\end{equation*}
$$

When $N=3$, let $\mathbf{d}_{0}, \mathbf{d}_{1}, \mathbf{d}_{2}$ and $\mathbf{d}_{3}$ be the vectors of $\mathbb{R}^{3}$ given by

$$
\left\{\begin{array}{l}
\mathbf{d}_{0}=\frac{1}{8}\left(\mathbf{a}_{1}-\mathbf{a}_{2}-\mathbf{a}_{3}-\mathbf{a}_{4}+\mathbf{a}_{5}+\mathbf{a}_{6}+\mathbf{a}_{7}-\mathbf{a}_{8}\right)  \tag{3}\\
\mathbf{d}_{1}=\frac{1}{8}\left(\mathbf{a}_{1}+\mathbf{a}_{2}-\mathbf{a}_{3}-\mathbf{a}_{4}-\mathbf{a}_{5}-\mathbf{a}_{6}+\mathbf{a}_{7}+\mathbf{a}_{8}\right) \\
\mathbf{d}_{2}=\frac{1}{8}\left(\mathbf{a}_{1}-\mathbf{a}_{2}+\mathbf{a}_{3}-\mathbf{a}_{4}-\mathbf{a}_{5}+\mathbf{a}_{6}-\mathbf{a}_{7}+\mathbf{a}_{8}\right) \\
\mathbf{d}_{3}=\frac{1}{8}\left(\mathbf{a}_{1}-\mathbf{a}_{2}-\mathbf{a}_{3}+\mathbf{a}_{4}+\mathbf{a}_{5}-\mathbf{a}_{6}-\mathbf{a}_{7}+\mathbf{a}_{8}\right)
\end{array}\right.
$$

These four vectors $\mathbf{d}_{m}$ are chosen for the hexahedron $K$ to be a parallelepiped if and only if $\mathbf{d}_{0}=\mathbf{d}_{1}=\mathbf{d}_{2}=\mathbf{d}_{3}=0$. The vertices of $K^{\sharp}$ (the parallelepiped associated to the hexahedron $K$ ) are

$$
\mathbf{a}_{i}^{\sharp}=\mathbf{a}_{i}-\sum_{0 \leq m \leq 3} s_{i, m} \mathbf{d}_{m}, \quad \text { for } 1 \leq i \leq 8
$$

where

$$
\left(s_{i, m}\right)=\left(\begin{array}{cccc}
+1 & +1 & +1 & +1  \tag{4}\\
-1 & +1 & -1 & -1 \\
-1 & -1 & +1 & -1 \\
-1 & -1 & -1 & +1 \\
+1 & -1 & -1 & +1 \\
+1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & +1
\end{array}\right)
$$

¿From the equality

$$
\sum_{1 \leq i \leq 8}\left|\sum_{0 \leq m \leq 3} s_{i, m} \mathbf{d}_{m}\right|^{2}=8 \sum_{0 \leq m \leq 3}\left|\mathbf{d}_{m}\right|^{2}
$$

we deduce that:

$$
\sum_{0 \leq m \leq 3} s_{i m} \mathbf{d}_{m}=0 \text { for } 1 \leq i \leq 8 \Longrightarrow \mathbf{d}_{m}=0 \text { for } 0 \leq m \leq 3
$$

Thus as announced, the hexadron $K$ is a parallelepiped if and only if $\mathbf{d}_{m}=0$ for $0 \leq m \leq 3$. We resume the results for 2D and 3D geometries in the following proposition:

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Proposition 1 Let $N^{*}=\left(2^{N}-N-1\right) N$; there exist a vector $\mathbf{d} \in \mathbb{R}^{N^{*}}$ and matrices $S_{i}, 1 \leq i \leq 2^{N}$, with $N$ rows and $N^{*}$ columns and which entries are $\pm 1$, such that the vertex $\mathbf{a}_{i}$ of $K$ and the vertex $\mathbf{a}_{i}^{\sharp}$ of $K^{\sharp}$ are linked by the relation

$$
\begin{equation*}
\mathbf{a}_{i}=\mathbf{a}_{i}^{\sharp}+S_{i} \mathbf{d}, \quad 1 \leq i \leq 2^{N} . \tag{5}
\end{equation*}
$$

More precisely, for $N=2$, $\mathbf{d} \in \mathbb{R}^{2}$ is given by (1) and the matrices $S_{i}$ are square matrices of order 2 satisfying $S_{i}=s_{i} I$, where $s_{i}$ are scalars given by (2). For $N=3, \mathbf{d}=\left(\mathbf{d}_{0}^{T}, \mathbf{d}_{1}^{T}, \mathbf{d}_{2}^{T}, \mathbf{d}_{3}^{T}\right)^{T}$ is identified to a vector of $\mathbb{R}^{12}$, its coordinates $\mathbf{d}_{m}$ are given by (3) and the matrices $S_{i}:=\left(s_{0 i} I, s_{1 i} I, s_{2 i} I, s_{3 i} I\right)$ are 3 rows and 12 columns matrices. The scalars $s_{i, m}$ are given by (4).
¿From (5) we deduce

$$
\mathbf{d}=\frac{1}{4^{N-2}} S_{i}^{T}\left(\mathbf{a}_{i}-\mathbf{a}_{i}^{\sharp}\right), \quad 1 \leq i \leq 2^{N} .
$$

We have $K=K^{\sharp}$ if and only if $\mathbf{d}=0$ in $\mathbb{R}^{N^{*}}$.
Definition 1 The vector $\mathbf{d}$ is named the distortion vector of $K$.
Let us now introduce the $N^{*}$ numbers $\delta_{m}$ defined by:

$$
\begin{array}{ll}
\mathbf{d}=\sum_{1 \leq m \leq 2} \delta_{m} \mathbf{e}_{m} & \text { when } N=2 \\
\mathbf{d}_{l}=\sum_{1 \leq m \leq 3} \delta_{m+3 l} \mathbf{e}_{m}, \quad 0 \leq l \leq 3 & \text { when } N=3 \tag{6}
\end{array}
$$

where we recall that $\mathbf{e}_{m}$ is given by $\mathbf{e}_{m}=\mathbf{a}_{0}-\mathbf{b}_{m}$. These parameters are invariant by affine mapping; in particular for the distortion vector $\mathbf{d}^{\vee}$ of $K^{\vee}$ we have if $N=2$

$$
\mathbf{d}^{\vee}=\sum_{1 \leq m \leq 2} \delta_{m} \widehat{\mathbf{e}}_{m}
$$

and if $N=3$

$$
\mathbf{d}_{l}^{\vee}=\sum_{1 \leq m \leq 3} \delta_{m+3 l} \widehat{\mathbf{e}}_{m}, \quad 0 \leq l \leq 3
$$

Definition 2 The numbers $\left(\delta_{m}\right)_{1 \leq m \leq N^{*}}$ are named the distortion parameters of $K$.

Since the mapping $F_{K}^{\sharp}$ is invertible affine, $K$ is a convex polyhedron if and only if $K^{\vee}$ is a convex polyhedron. So we see that the convexity of K and the face planarity when $N=3$ can be expressed by a set of constraints on the distortion parameters only. For $N=2$, it is easy to show that $K$ is a convex quadrilateral if and only if we have

$$
\left|\delta_{1}\right|+\left|\delta_{2}\right|<1
$$

For $N=3$, we can write a set of 6 equations and 18 inequations on the 12 distortion parameters which means that K is a convex hexahedron; but we cannot use this set of nonlinear constraints.
¿From now on, we shall assume for $N=3$ as for $N=2$ that

$$
\begin{equation*}
\sum_{1 \leq m \leq N^{*}}\left|\delta_{m}\right|<1 \tag{7}
\end{equation*}
$$

holds . Then $K^{\vee}$ contains $B(\mathbf{0}, 1 / \sqrt{N})$ the ball centered at the origin and of radius $1 / \sqrt{N}$ and $K^{\vee}$ is contained in the cube $[-2,+2]^{N}$. The polyhedron $K$ is contained in the parallelotope

$$
K^{2 \sharp}=F_{K}^{\sharp}\left([-2,+2]^{N}\right)
$$

This element $K^{2 \sharp}$ is homothetic to $K^{\sharp}$ with a ratio equal to 2 . Then, we have the inequality

$$
h_{K} \leq 2 h_{K^{\sharp}} .
$$

Last, we note that the Euclidean norm of the distortion vector of $K$ satisfies

$$
\frac{1}{2 N}\left(\sum_{1 \leq m \leq N^{\sharp}}\left|\delta_{m}\right|\right) \rho_{K^{\sharp}} \leq|\mathbf{d}| \leq \frac{1}{2}\left(\sum_{1 \leq m \leq N^{\sharp}}\left|\delta_{m}\right|\right) h_{K^{\sharp}} .
$$

## 2 Local error estimates

### 2.1 Lagrange interpolation error estimates

Let $P_{K}$ be a finite dimensional vectorial space of polynomial functions defined over the quadrilateral or hexahedron $K$. We assume that the set $\left\{\mathbf{a}_{i}, 1 \leq i \leq 2^{N}\right\}$ is $P_{K}$-unisolvent. Then necessarly, $\operatorname{dim}\left(P_{K}\right)=2^{N}$. The basis functions of the Lagrange finite element $\left(K, P_{K},\left\{\mathbf{a}_{i}, 1 \leq i \leq 2^{N}\right\}\right)$ are noted $p_{i, K}$ and the $P_{K}$-Lagrange interpolation operator is noted $\Pi_{K}$ : for every function $u$ defined on the vertices of $K$,

$$
\Pi_{K} u=\sum_{1 \leq i \leq 2^{N}} u\left(\mathbf{a}_{i}\right) p_{i, K} .
$$

The basis functions $p_{i, K}$ are functions defined by definition on $K$; in fact, since they are polynomial, we consider them as functions defined on $K^{2 \sharp}$.

Proposition 2 Let us assume that the distortion parameters of $K$ satisfy (7), that the set $\left\{\mathbf{a}_{i}, 1 \leq i \leq 2^{N}\right\}$ is $P_{K}$-unisolvent and that the inclusion $P_{1}(K) \subset$ $P_{K}$ holds. Then, for every $u \in H^{2}(K)$,

$$
\begin{equation*}
\left\|u-\Pi_{K} u\right\|_{0, K} \leq 4 h_{K^{\sharp}}^{2}\left(\sum_{1 \leq i \leq 2^{N}}\left\|p_{i, K}\right\|_{0, \infty, K^{2 \sharp}}\right)|u|_{2, K} . \tag{8}
\end{equation*}
$$

Moreover, let $r$ be an integer sufficiently large for the the inclusion $P_{K} \subseteq P_{r}(K)$ to hold. Then there exists a constant $c_{r}$, which depends only on $r$, such that for every $u \in H^{2}(K)$,

$$
\begin{equation*}
\left\|u-\Pi_{K} u\right\|_{1, K} \leq c_{r} \frac{h_{K^{\sharp}}^{2}}{\rho_{K^{\sharp}}}\left(\sum_{1 \leq i \leq 2^{N}}\left\|p_{i, K}\right\|_{0, \infty, K^{2 \sharp}}\right)|u|_{2, K} . \tag{9}
\end{equation*}
$$

Proof 1 By the Taylor's formula with integral remainder, we have for all $\mathbf{x} \in$ K,

$$
\begin{aligned}
u\left(\mathbf{a}_{i}\right)= & u(\mathbf{x})+\left(\mathbf{a}_{i}-\mathbf{x}\right)^{T} \operatorname{grad} u(\mathbf{x}) \\
& +\int_{0}^{1}(1-\theta)\left(\mathbf{a}_{i}-\mathbf{x}\right)^{T} D^{2} u\left(\mathbf{x}+\theta\left(\mathbf{a}_{i}-\mathbf{x}\right)\right)\left(\mathbf{a}_{i}-\mathbf{x}\right) d \theta
\end{aligned}
$$

RR n ${ }^{\circ} 6572$
where $D^{2} u$ denotes the hessian matrix of $u$. Reporting that in the expression of $\Pi_{K} u$, and using the assumption that the inclusion $P_{1}(K) \subset P(K)$ holds, we obtain

$$
\begin{aligned}
& \left(u-\Pi_{K} u\right)(\mathbf{x})= \\
& \quad-\sum_{1 \leq i \leq 2^{N}}\left\{\int_{0}^{1}(1-\theta)\left(\mathbf{a}_{i}-\mathbf{x}\right)^{T} D^{2} u\left(\mathbf{x}+\theta\left(\mathbf{a}_{i}-\mathbf{x}\right)\right)\left(\mathbf{a}_{i}-\mathbf{x}\right) d \theta\right\} p_{i, K}(\mathbf{x})
\end{aligned}
$$

and then

$$
\begin{aligned}
& \left|\left(u-\Pi_{K} u\right)(\mathbf{x})\right| \leq \\
& \quad h_{K}^{2} \sum_{1 \leq i \leq 2^{N}}\left\{\int_{0}^{1}(1-\theta)\left\|D^{2} u\left(\mathbf{x}+\theta\left(\mathbf{a}_{i}-\mathbf{x}\right)\right)\right\| d \theta\right\}\left|p_{i, K}(\mathbf{x})\right|
\end{aligned}
$$

where $\left\|D^{2} u\right\|$ is the spectral norm of the matrix $D^{2} u$.
Now, we note that on a domain $K$ which is star-shaped with respect to $\mathbf{a}_{i}$, if $|g(\mathbf{x})| \leq \int_{0}^{1}(1-\theta) G\left(\mathbf{x}+\theta\left(\mathbf{a}_{i}-\mathbf{x}\right)\right) d \theta$ holds $\mathbf{x}$ a.e. in $K$, then $\|g\|_{0, K} \leq$ $\|G\|_{0, K}$. Therefore, we obtain

$$
\left\|u-\Pi_{K} u\right\|_{0, K} \leq h_{K}^{2}\left(\sum_{1 \leq i \leq 2^{N}}\left\|p_{i, K}\right\|_{0, \infty, K}\right)|u|_{2, K}
$$

and a fortiori (8).
In a same way, we obtain for $\mathbf{x}$ a.e. in $K$,

$$
\begin{aligned}
& \operatorname{grad} \Pi_{K} u(\mathbf{x})=\sum_{1 \leq i \leq 2^{N}} u\left(\mathbf{a}_{i}\right) \operatorname{grad} p_{i, K}(\mathbf{x}) \\
& \operatorname{grad}\left(u-\Pi_{K} u\right)(\mathbf{x})= \\
& -\sum_{1 \leq i \leq 2^{N}}\left\{\int_{0}^{1}(1-\theta)\left(\mathbf{a}_{i}-\mathbf{x}\right)^{T} D^{2} u\left(\mathbf{x}+\theta\left(\mathbf{a}_{i}-\mathbf{x}\right)\right)\left(\mathbf{a}_{i}-\mathbf{x}\right) d x\right\} \operatorname{grad} p_{i, K}(\mathbf{x})
\end{aligned}
$$

and as before, we deduce from that

$$
\left|u-\Pi_{K} u\right|_{1, K} \leq h_{K}^{2}\left(\sum_{1 \leq i \leq 2^{N}}\left|p_{i, K}\right|_{1, \infty, K}\right)|u|_{2, K}
$$

and a fortiori

$$
\left|u-\Pi_{K} u\right|_{1, K} \leq 4 h_{K^{\sharp}}^{2}\left(\sum_{1 \leq i \leq 2^{N}}\left|p_{i, K}\right|_{1, \infty, K^{2 \sharp}}\right)|u|_{2, K} .
$$

Now, we can note that there exists for every integer $r$ a constant $\widehat{c}_{r}$ which depends only on $r$, such that

$$
\forall p \in P_{r}(\widehat{K}), \quad|p|_{1, \infty, \widehat{K}} \leq \widehat{c}_{r}\|p\|_{0, \infty, \widehat{K}}
$$

which is an "inverse inequality" on the biunit cube. Using an affine invertible mapping of $\widehat{K}$ onto $K^{2 \sharp}$, we obtain the inequality

$$
\forall p \in P_{r}\left(K^{2 \sharp}\right), \quad|p|_{1, \infty, K^{2 \sharp}} \leq \widehat{c}_{r} \frac{2 \sqrt{N}}{\rho_{K^{\sharp}}}\|p\|_{0, \infty, K^{2 \sharp}} .
$$

Since by assumption, the basis functions $p_{i, K}$ belong to $P_{r}\left(K^{2 \sharp}\right)$, that leads to (9) with a constant $c_{r}=8 \sqrt{N} \widehat{c}_{r}$.

### 2.2 Face error estimates

Let $\gamma_{m}^{\vee}=\left(F_{K}^{\sharp}\right)^{-1}\left(\gamma_{m}\right)$ be a face of $K^{\vee}$. We begin by bounding, independently of the distortion parameters, the continuous linear trace operator which sends $H^{1}\left(K^{\vee}\right)$ into $L^{2}\left(\gamma_{m}^{\vee}\right)$.

Lemma 1 Assume (7); then there exists a constant $C$, independent of the distortion parameters, such that for every $u \in H^{1}\left(K^{\vee}\right)$ ) and every $m$ with $1 \leq m \leq 2 N$

$$
\|u\|_{0, \gamma_{m}^{\vee}} \leq C\|u\|_{1, K^{\vee}}
$$

Proof 2 Consider first the case $N=2$ : let $T_{m}^{\vee}$ be the triangle which vertices are the two extremities of $\gamma^{\vee}$ and $(0,0)^{T}$, the center of $K^{\vee}$. Let now $F_{m}^{\vee}$ be an invertible linear mapping such that the reciprocal image of the triangle $T_{m}^{\vee}$ by $F_{m}^{\vee}$ is the unit triangle $\widehat{T}$ which vertices are $(0,1)^{T},(1,0)^{T}$ and $(0,0)^{T}$. As we have already noticed, (7) implies that the quadrilateral $K^{\vee}$ contains a ball of radius greater than $1 / \sqrt{2}$, from where we deduce

$$
\left|T_{m}^{\vee}\right| \geq \frac{1}{2 \sqrt{2}}\left|\gamma_{m}^{\vee}\right|
$$

Moreover, the diameter of $T_{m}^{\vee}$ is bounded by 4. $\widehat{\gamma}$ denoting the side of $\widehat{T}$ which extremities are $(0,1)^{T}$ and $(1,0)^{T}$, we obtain for some constants $c_{1}, c_{2}$ and $c_{3}$ which are independent of the geometry

$$
\begin{aligned}
\|u\|_{0, \gamma_{m}^{\vee}} & \leq c_{1}\left|\gamma_{m}^{\vee}\right|^{1 / 2}\left\|u \circ F_{m}^{\vee}\right\|_{0, \widehat{\gamma}} \\
& \leq c_{1} c_{2}\left|\gamma_{m}^{\vee}\right|^{1 / 2}\left\|u \circ F_{m}^{\vee}\right\|_{1, \widehat{T}} \\
& \leq c_{1} c_{2} c_{3}\left(\frac{\left|\gamma_{m}^{\vee}\right|}{\left|T_{m}^{\vee}\right|}\right)^{1 / 2}\left\|u \circ F_{m}^{\vee}\right\|_{1, T_{m}^{\vee}}
\end{aligned}
$$

which proves the Lemma with $C=(2 \sqrt{2})^{1 / 2} c_{1} c_{2} c_{3}$.
For the case $N=3$, we begin by decomposing a quadrilateral face $\gamma_{m}^{\vee}$ in two triangles $\gamma_{1, m}^{\vee}$ and $\gamma_{2, m}^{\vee}$; in the same way as in the case $N=2$, we obtain for $i=1$ and $i=2$ the inequality

$$
\|u\|_{0, \gamma_{i, m}^{\vee}} \leq C\|u\|_{1, T_{i, m}^{\vee}}
$$

and we conclude easily.
Proposition 3 Assume that the distortion parameters of $K$ satisfy (7). Then there exists a constant $C$, independent of the geometry of $K$, such that: $\forall u \in$ $H^{1}(K)$ and $\forall m$ with $1 \leq m \leq 2 N$, we have

$$
\begin{equation*}
\left\|u-\bar{u}^{\gamma_{m}}\right\|_{0, \gamma_{m}} \leq C h_{K^{\sharp}}^{1 / 2}\left(\frac{h_{K^{\sharp}}}{\rho_{K^{\sharp}}}\right)^{(N-1) / 2}|u|_{1, K} \tag{10}
\end{equation*}
$$

where $\bar{u}^{\gamma_{m}}$ denotes the mean value of $u$ on the face $\gamma_{m}$ of $K$.
Let $u$ be in $H^{1}(K)$ and $\gamma_{m}$ be a face of $K$. The best approximation of the trace on $\gamma_{m}$ of $u$ in $L^{2}\left(\gamma_{m}\right)$ is the mean value of $u$ on $\gamma_{m}$

## Proof 3

$$
\left\|u-\bar{u}^{\gamma_{m}}\right\|_{0, \gamma_{m}}=\inf _{\chi \in \mathbb{R}}\|u-\chi\|_{0, \gamma_{m}}
$$

We note $\gamma_{m}^{\vee}$ the reciprocal image of $\gamma_{m}$ by the application $F_{K}^{\sharp}$. Since for each constant $\chi$ we have $(u-\chi) \circ F_{K}^{\sharp}=u \circ F_{K}^{\sharp}-\chi$, we deduce that

$$
\|u-\chi\|_{0, \gamma_{m}} \leq\left\|B_{K^{\sharp}}^{-1}\right\|^{(N-1) / 2}\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{0, \gamma_{m}^{\vee}}
$$

Using the lemma (1), there exists a constant $C$ such that

$$
\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{0, \gamma_{m}^{\vee}} \leq C\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{1, K^{\vee}}
$$

Therefore, we have

$$
\inf _{\chi \in \mathbb{R}}\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{0, \gamma_{m}^{\vee}} \leq C \inf _{\chi \in \mathbb{R}}\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{1, K^{\vee}}
$$

and

$$
\inf _{\chi \in \mathbb{R}}\left\|u \circ F_{K}^{\sharp}-\chi\right\|_{0, \gamma_{m}^{\vee}} \leq C\left\|u \circ F_{K}^{\sharp}-\bar{u}^{K}\right\|_{1, K^{\vee}}
$$

Where $\bar{u}^{K}$ is the mean value of $u$ on $K$ and also the mean value of $u \circ F_{K}^{\sharp}$ on $K^{\vee}$. Moreover, we have for $x$ a.e. on $K^{\vee}$

$$
\left(u \circ F_{K}^{\sharp}-\bar{u}^{K}\right)(\mathbf{x})=\frac{1}{\left|K^{\vee}\right|} \int_{K^{\vee}}\left\{\int_{0}^{1}(\mathbf{x}-\mathbf{y})^{T} \operatorname{grad}(u(\mathbf{x}+\theta(\mathbf{y}-\mathbf{x})) d \theta\} d \mathbf{y}\right.
$$

Consequently

$$
\begin{gathered}
\left\|u \circ F_{K}^{\sharp}-\bar{u}^{K}\right\|_{0, K^{\vee}} \leq 4 \sqrt{2}\left|u \circ F_{K}^{\sharp}\right|_{1, K^{\vee}} \\
\left\|u \circ F_{K}^{\sharp}-\bar{u}^{K}\right\|_{1, K^{\vee}} \leq \sqrt{33}\left|u \circ F_{K}^{\sharp}\right|_{1, K^{\vee}} \\
\left|u \circ F_{K}^{\sharp}\right|_{1, K^{\vee}} \leq\left\|B_{K}\right\|^{N / 2}|u|_{1, K}
\end{gathered}
$$

¿From all the previous inequalities, we have

$$
\left\|u-\bar{u}^{\gamma_{m}}\right\|_{0, \gamma_{m}} \leq C \sqrt{33}\left\|B_{K}^{-1}\right\|^{(N-1) / 2}\left\|B_{K}\right\|^{N / 2}|u|_{1, K^{\vee}}
$$

Using (7), it remains to notice that

$$
\left\|B_{K}\right\| \leq \frac{1}{2} h_{K^{\sharp}} \quad \text { et } \quad\left\|B_{K}^{-1}\right\| \leq \frac{2 \sqrt{N}}{\rho_{K^{\sharp}}} .
$$

And the proof of the proposition is achieved.

## 3 The model problem

We consider the second order elliptic model problem:

$$
\begin{align*}
-\operatorname{div}(A \nabla u) & =f & & \text { in } \Omega  \tag{11}\\
u & =0 & & \text { on } \Gamma
\end{align*}
$$

where $A=\left(a_{i, j}\right)$ a symmetric matrix satisfying

$$
\forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^{N}, \quad c \sum_{i=1}^{N} \xi_{i}^{2} \leq \sum_{i, j=1}^{N} a_{i, j}(x) \xi_{i} \xi_{j} \leq c^{-1} \sum_{i=1}^{N} \xi_{i}^{2}
$$

and $\Gamma:=\partial \Omega$ is the boundary of a polyhedral domain $\Omega \subset \mathbb{R}^{N}(N=2,3)$. The variational problem associated to (11) is: find $u \in H_{0}^{1}(\Omega)$ such that

$$
\begin{equation*}
\forall v \in H_{0}^{1}(\Omega) \quad \int_{\Omega} A \nabla u \nabla v d x=\int_{\Omega} f v d x \tag{12}
\end{equation*}
$$

Let $\mathcal{T}_{h}$ be a triangulation of $\Omega$ into quadrilaterals $(N=2)$ or hexahedra $(N=3)$. Let $\partial \mathcal{T}_{h}$ denotes the set of the faces of the elements of $\mathcal{T}_{h}$ and $\partial \mathcal{T}_{h} \backslash \partial \Omega$ denotes the set of interior faces. For each element $\gamma$ of $\partial \mathcal{T}_{h} \backslash \partial \Omega$, there exist $K^{+}$and $K^{-}$ in $\mathcal{T}_{h}$ such that $\bar{K}^{+} \cap \bar{K}^{-}=\gamma$. The unitary outward normal of $K^{+}$is noted $\mathbf{n}^{+}$and the normal of a face is defined by $\mathbf{n}=\mathbf{n}^{+}$. For each subset $\gamma$ of $\partial \Omega, \mathbf{n}$ denotes the unitary outward normal of $\Omega$.
We consider

$$
V_{\mathcal{T}_{h}}=\left\{v \in L^{2}(\Omega) ;\left.v\right|_{K} \in H^{1}(K) \text { for each } K \in \mathcal{T}_{h}\right\}
$$

and

$$
V_{h}=\left\{v_{h} \in L^{2}(\Omega) ;\left.v_{h}\right|_{K} \in P_{K} \text { for each } K \in \mathcal{T}_{h}\right\}
$$

Where $P_{K}$ is a polynomial space. A non conforming finite element method for problem (12) is: find $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
\forall v_{h} \in V_{h}, \quad \sum_{K \in \mathcal{T}_{h}} \int_{K} A \nabla u_{h} \nabla v_{h} d x=\int_{\Omega} f v_{h} d x \tag{13}
\end{equation*}
$$

For $v, w \in H_{0}^{1}(\Omega)+V_{\mathcal{T}_{h}}$ we define

$$
a_{h}(v, w)=\sum_{K \in \mathcal{T}_{h}} \int_{K} A \nabla v \nabla w d x
$$

and

$$
\|v\|_{1, h}=\left(\sum_{K \in \mathcal{T}_{h}}\|v\|_{1, K}^{2}\right)^{1 / 2},|v|_{1, h}=\left(\sum_{K \in \mathcal{T}_{h}}|v|_{1, K}^{2}\right)^{1 / 2}
$$

Let us define the jump of $w_{h} \in V_{h}$ on $\gamma$. If $\gamma$ is in $\partial \mathcal{T}_{h} \backslash \partial \Omega$ then $\left[w_{h}\right]=w_{h}^{+}-w_{h}^{-}$ where $w_{h}^{ \pm}$is the trace on $\gamma$ of $w_{h}^{ \pm} \in H^{1}\left(K^{ \pm}\right)$else $\left[w_{h}\right]$ represents the trace on $\gamma$ of $w_{h} \in H^{1}(K)$.
Since $a_{h}(.,$.$) is uniformly V_{\mathcal{T}_{h}}$ elliptic, the following basic error estimate (approximation and consistancy error) hold (see [1], 3], 4] ).

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{1, h} \leq c\left(\inf _{w_{h} \in V_{h}}\left\|u-w_{h}\right\|_{1, h}+\sup _{0 \neq w_{h} \in V_{h}} \frac{\left|a_{h}\left(u, w_{h}\right)-\left(f, w_{h}\right)\right|}{\left\|w_{h}\right\|_{1, h}}\right) \tag{14}
\end{equation*}
$$

with

$$
a_{h}\left(u, w_{h}\right)-\left(f, w_{h}\right)=\sum_{\gamma \in \mathcal{T}_{h}} \int_{\gamma} \frac{\partial u}{\partial n}\left[w_{h}\right] d s
$$

Now, we give the definition of afamily of regular meshes.

Definition 3 Regular Mesh
A family of quadrangular meshes is regular if and only if for each $K$

- the distorsion parameters satisfy (7),
- $\exists \sigma>0 \quad \frac{h_{K \#}}{\rho_{K} \#} \leq \sigma$.

We can notice that this definition corresponds to the classical definition given by P.G. Ciarlet ([1]) when the $K$ 's are parallelotopes.

Proposition 4 Convergence of the method
We suppose that the solution $u$ of (11) is in $H^{2}(\Omega)$ and the mesh is regular. And we assume the following assumptions.
i) Approximation properties:

- $\forall K$, the set $\left\{a_{i}, 1 \leq i \leq 4\right\}$ is $P_{K}$-unisolvent.
- $\exists r>0 \forall K P_{1}(K) \subset P_{K} \subset P_{r}(K)$.
- The basis functions $p_{i, K}$ of $P_{K}$ satisfy $\exists C>0\left\|p_{i, K}\right\|_{0, \infty, K^{2} \#}<C$.
ii) Patch Test
- $\forall w_{h} \in V_{h}, \quad \forall \gamma \in \partial \mathcal{T}_{h} \quad \int_{\gamma}\left[w_{h}\right] d s=0$.

Under these assumptions we have

$$
\left\|u-u_{h}\right\|_{1, h} \leq C h|u|_{2, \Omega}
$$

with $C$ a constant independent of the mesh.
Proof 4 Using the first three assumptions and Proposition (2, we have

$$
\left(\sum_{K \in \mathcal{T}_{h}}\left\|u-\Pi_{K} u\right\|_{1, h}\right)^{1 / 2} \leq C h|u|_{2, \Omega}
$$

where $C$ is a constant independent of the mesh.
So, it remains to prove that

$$
\left|a_{h}\left(u, w_{h}\right)-\left(f, w_{h}\right)\right| \leq C h|u|_{2, \Omega}\left\|w_{h}\right\|_{1, h}
$$

Let $\gamma$ be in $\partial \mathcal{T}_{h} \backslash \partial \Omega$, we have $\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}}\left[w_{h}\right] d \sigma=\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}^{+}}\left(w_{h}^{+}-w_{h}^{-}\right) d \sigma$ where $w_{h}$ satisfies $\int_{\gamma} w_{h^{+}} d \sigma=\int_{\gamma} w_{h^{-}} d \sigma$. Therefore, dividing by $|\gamma|$, we have ${\overline{w_{h}^{+}}}^{\gamma}={\overline{w_{h}^{-}}}^{\gamma}$. Thus, we obtain

$$
\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}}\left[w_{h}\right] d \sigma=\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}^{+}}\left(w_{h}^{+}-{\overline{w_{h}^{+}}}^{\gamma}\right) d \sigma-\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}^{-}}\left(w_{h}^{-}-{\overline{w_{h}^{-}}}^{\gamma}\right) d \sigma
$$

Obviously, for each constant c we have

$$
\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}^{+}}\left(w_{h^{+}}-{\overline{w_{h^{+}}}}^{\gamma}\right) d \sigma=\int_{\gamma}\left(\operatorname{grad} u \cdot \mathbf{n}^{+}-c\right)\left(w_{h^{+}}-{\overline{w_{h^{+}}}}^{\gamma}\right) d \sigma
$$

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As $\gamma$ is flat face, $\mathbf{n}^{+}$is a constant vector on $\gamma$ and it can be extented to $K^{+}$. Since $u \in H^{2}(\Omega), \operatorname{grad} u \cdot \mathbf{n}^{+} \in H^{1}\left(K^{+}\right)$. From (10) we have

$$
\begin{aligned}
& \left|\int_{\gamma}\left(\operatorname{grad} u \cdot \mathbf{n}^{+}-c\right)\left(w_{h^{+}}-\overline{w_{h^{+}}}{ }^{\gamma}\right) d \sigma\right| \\
& \quad \leq C h\left|\operatorname{grad} u \cdot \mathbf{n}^{+}\right|_{1, K^{+}}\left|w_{h}\right|_{1, K^{+}} \leq C h|u|_{2, K^{+}}\left|w_{h}\right|_{1, K^{+}}
\end{aligned}
$$

and finally

$$
\left|\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}}\left[w_{h}\right] d \sigma\right| \leq C h\left(|u|_{2, K^{+}}\left|w_{h}\right|_{1, K^{+}}+|u|_{2, K^{-}}\left|w_{h}\right|_{1, K^{-}}\right) .
$$

The result, for each $\gamma \subset \partial \Omega$, is similar

$$
\left|\int_{\gamma} \frac{\partial u}{\partial \mathbf{n}}\left[w_{h}\right] d \sigma\right| \leq C h|u|_{2, K}\left|w_{h}\right|_{1, K}
$$

We sum on all the faces $\gamma_{m}$. In the right hand side of the inequality an element $K$ appears at most 4 times when $N=2$ and 6 times when $N=3$. So the expected result holds.

## 4 Lagrange polynomial finite element

This section is devoted to building of finite elements ( $K, P_{K}, S_{K}$ ) where the set of degrees of freedom $S_{K}$ is the vertices set of $K$, and $P_{K}$ is a polynomial space. Note that $P_{k}=\left\{q=q^{\vee} \circ\left(F_{K}^{\sharp}\right)^{-1} ; q^{\vee} \in P_{k}\right\}$. The same property is not true for the space $Q_{k}$. Therefore we introduce the space:

$$
Q_{k}^{K}=\left\{q^{\vee} \circ\left(F_{K}^{\sharp}\right)^{-1} ; q^{\vee} \in Q_{k}\right\}
$$

which is a subspace of $P_{N k}$.
If we choose $P_{K}=Q_{1}^{K}$ then $\left(K, P_{K}, S_{K}\right)$ is a finite element, but the approximation $u_{h}$ of the solution of (11) obtained with this element does not converge without additional assumptions (see the numerical results in the next section). Indeed the basic functions of the space $V_{\mathcal{T}_{h}}$ are discontinuous on the faces of the elements, and we loose the order of convergence on the consistancy error term. Therefore our goal is to build polynomial finite elements on quadrilaterals and hexahedra satisfying the assumptions of Proposition 4

### 4.1 The quadrilateral case

The set of degrees of freedom is $S_{K}=\left\{a_{i} ; 1 \leq i \leq 4\right\}$
Since the trapezoidal formula is exact for each polynomial of order 1, we have

$$
\begin{equation*}
\forall q \in P_{1}, \int_{\gamma_{m}} q d \sigma=\frac{1}{2}\left|\gamma_{m}\right| \sum_{a_{i} \in \gamma_{m}} q\left(a_{i}\right), \text { for all } m=1, \ldots, 4 \tag{15}
\end{equation*}
$$

where $\left|\gamma_{m}\right|$ is the length of the edge $\gamma_{m}$.
Let $P_{K}$ be the following polynomial space:

$$
\begin{equation*}
P_{K}=\left\{q \in Q_{2}^{K} \cap P_{3} ; \int_{\gamma_{m}} q d \sigma=\frac{1}{2}\left|\gamma_{m}\right| \sum_{a_{i} \in \gamma_{m}} q\left(a_{i}\right), \text { for all } m=1, \ldots, 4\right\} \tag{16}
\end{equation*}
$$

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The Simpson formula integrates exactly the cubic functions on each edge and consequently the space $P_{K}$ can be defined as well as

$$
P_{K}=\left\{q \in Q_{2}^{K} \cap P_{3} ; q\left(b_{m}\right)=\frac{1}{2} \sum_{a_{i} \in \gamma_{m}} q\left(a_{i}\right), \text { for all } m=1, \ldots, 4\right\}
$$

Proposition 5 For any convex quadrilateral $K$, the triad $\left(K, P_{K}, S_{K}\right)$ is a Lagrange finite element.

Proof 5 Let us introduce $a_{i}^{\vee}=b_{i-4}^{\vee}$ and $\widehat{a}_{i}=\widehat{b}_{i-4}$ for $i=5, \ldots, 8$. Using the invertible affine mapping $F_{K}^{\sharp}$, we only need to prove that:
For each distortion parameters $\delta=\left(\delta_{1}, \delta_{2}\right)$ such that $\left|\delta_{1}\right|+\left|\delta_{2}\right|<1$ the unique function $q \in Q_{2}^{K} \cap P_{3}$ satisfying $\left(q^{\vee}\left(a_{i}^{\vee}\right)=0 ; 1 \leq i \leq 8\right)$ is $q \equiv 0$.
let us introduce the polynomials $r_{j} \in Q_{2} \cap P_{3}$ satisfying

$$
r_{j}\left(\widehat{a}_{i}\right)=\delta_{i, j},
$$

and the square matrix $R$ of order 8 defined by $R_{i, j}=r_{j}\left(a_{i}^{\vee}\right)$. Both symbolic calculus and an explicit calculus give

$$
\operatorname{det} R=\left(1-\delta_{1}^{2}\right)\left(1-\delta_{2}^{2}\right)\left(1-\left(\delta_{1}+\delta_{2}\right)^{2}\right)\left(1-\left(\delta_{1}-\delta_{2}\right)^{2}\right)
$$

Since $\left|\delta_{1}\right|+\left|\delta_{2}\right|<1$ then det $R>0$. Therefore $R$ is invertible and $q \equiv 0$. Note that if $\delta=0$ the $r_{j}$ 's correspond to the basis of the serendipity finite element and in this case $P_{K} \equiv Q_{1}$.

## Remarks

- The finite element basis depends on $\delta$.
- if $\delta=0$ (i.e. $K$ is a parallelogram) then $\left(K, P_{K}, S_{K}\right)$ coincides with the classical bilinear finite element.
- $P_{K}=\operatorname{span}\left(1, x_{1}, x_{2}, F_{K}^{\sharp}\left(\omega^{K^{\vee}}\right)\right)$ where $\omega^{K^{\vee}}$ can be expressed explicitly.
- Numericaly, it is more efficient to calculate the finite element basis by solving a linear system of order 8 than to the explicit basis of $P_{K}$.

Proposition 6 We assume that there exists $\alpha>0$ such that for each $K \in$ $\mathcal{T}_{h}, \quad\left|\delta_{1}\right|+\left|\delta_{2}\right| \leq 1-\alpha$. Then the assumptions of Proposition 4 are satisfied.

Proof 6 The inclusions $P_{1} \subseteq P_{K} \subseteq P_{3}$ are obvious.
Since $\left|\delta_{1}\right|+\left|\delta_{2}\right| \leq 1-\alpha, \frac{1}{|\operatorname{det} R|}$ is bounded, and consequently the $P_{i, K}$ 's are bounded on $K^{2 \#}$. Finally, by construction, the patch test is satisfied.

### 4.2 The hexahedron case

In this subsection $P_{K}$ has to be a polynomial space of dimension 6. In order to control the mean value of the functions on each face of the hexahedron, $P_{K}$ must be built from a polynomial space of dimension 14 ( 8 vertices +6 faces).

Noting that the dimension of $Q_{2} \cap P_{3}$ is 17 , we propose to consider the subspace $Z$ of $Q_{2} \cap P_{3}$ defined by

$$
\begin{array}{r}
Z=\operatorname{span}\left\{1, x_{1}^{\vee}, x_{2}^{\vee}, x_{3}^{\vee}, x_{1}^{\vee} x_{2}^{\vee}, x_{1}^{\vee} x_{3}^{\vee}, x_{2}^{\vee} x_{3}^{\vee}, x_{1}^{\vee} x_{2}^{\vee} x_{3}^{\vee}, x_{1}^{\vee}, x_{2}^{\vee}, x_{3}^{\vee}\right. \\
\\
\left.x_{1}^{\vee}\left(x_{2}^{\vee}{ }^{\vee}+x_{3}^{\vee 2}\right), x_{2}^{\vee}\left(x_{1}^{\vee 2}+x_{3}^{\vee}\right), x_{3}^{\vee}\left(x_{1}^{\vee 2}+x_{2}^{\vee 2}\right)\right\}
\end{array}
$$

which has the advantage of being invariant under any permutation of $x_{1}^{\vee}, x_{2}^{\vee}, x_{3}^{\vee}$. Therefore we introduce:

$$
Z^{K}=\left\{q^{\vee} \circ\left(F_{K}\right)^{-1} ; q^{\vee} \in Z\right\}
$$

$Z^{K}$ is a subspace of $Q_{2}^{K} \cap P_{3}$. The generalization to the 3-D case of formula (15) is to find $\omega_{m, i}$ such that for all $q \in P_{1}$ we have

$$
\int_{\gamma_{m}} q d \sigma=\frac{1}{\left|\gamma_{m}\right|} \sum_{i=1}^{4} \omega_{m, i} q\left(a_{m, i}\right), \text { for all } m=1, \ldots, 6
$$

where $\left\{a_{m, i}, i=1, \ldots, 4\right\}$ are the vertices of the face $\gamma_{m}$ and of aera $\left|T_{m, i}\right|=$ $\theta_{m, i}\left|\gamma_{m}\right|$
It is enough to restrict ourselves to $q \in P_{1}\left(\gamma_{m}\right)$. However the coefficients $\omega_{m, i}$ are not unique. A solution consists in cutting the face $\gamma_{m}$ into triangles, see Figure 3) To obtain a symetric formula, we consider the four triangles $\left\{T_{i}, i=1, \ldots, 4\right\}$ constituted with the vertices $\left\{a_{m, j}, j=1, \ldots, 4, j \neq i\right\}$.


Figure 3:

$$
\int_{\gamma_{m}} q d \sigma=\frac{1}{2} \sum_{i=1}^{4} \int_{T_{i}} q d \sigma=\frac{\left|\gamma_{m}\right|}{6} \sum_{i=1}^{4}\left(\theta_{m, i} \sum_{j=1, j \neq i}^{4} q\left(a_{m, j}\right)\right)
$$

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Therefore

$$
\omega_{m, i}=\frac{1}{6} \sum_{j=1, j \neq i}^{4} \theta_{m, j}=\frac{1}{6}\left(2-\theta_{m, i}\right)
$$

Finally we let

$$
P_{K}=\left\{q \in Z^{K} ; \int_{\gamma_{m}} q d \sigma=\frac{\left|\gamma_{m}\right|}{6} \sum_{i=1}^{4}\left(2-\theta_{m, i}\right) q\left(a_{m, i}\right), \forall m=1, \ldots, 6\right\}
$$

It seems difficult to prove the analogous of Proposition 5 in the hexahedron case. However we formulate a sufficient condition to obtain a finite element. For any polynomial $p$ we let :

$$
\widehat{I}_{i}(p)=p\left(\widehat{a}_{i}\right) \quad 1 \leq i \leq 8, \quad \widehat{I}_{8+m}(p)=\int_{\widehat{\gamma}_{m}} p d \sigma \quad 1 \leq m \leq 6
$$

and

$$
I_{i}^{\vee}(p)=p\left(a_{i}^{\vee}\right) \quad 1 \leq i \leq 8, \quad I_{8+m}^{\vee}(p)=\int_{\gamma_{m}^{\vee}} p d \sigma \quad 1 \leq m \leq 6
$$

In a first step, we assume that $K^{\vee}=\widehat{K}$. And we define

$$
\Sigma=\left\{\widehat{a}_{i} ; 1 \leq i \leq 8, v \rightarrow \int_{\widehat{\gamma}_{m}} v d \sigma ; 1 \leq m \leq 6 .\right\}
$$

It is easy to show that the triad $(\widehat{K}, Z, \Sigma)$ is a Lagrange finite element. We denote by $r_{j}$ 's the basis functions, they satisfy $\widehat{I}_{i}\left(r_{j}\right)=\delta_{i, j}, 1 \leq i, j \leq 14$.
We assume that $K^{\vee} \neq \widehat{K}$ and we define the matrix $R$ by $R_{i, j}=I_{i}^{\vee}\left(r_{j}\right)$. So the matrix $R$ can be written $R=I-B$ where $B$ is a matrix depending continously on the $\delta_{i}$ 's parameters and $B=0$ if the $\delta_{i}$ 's vanish.
Proposition 7 For any hexahedron $K$ such that $\|B\|<1$, the triad $\left(K, P_{K}, S_{K}\right)$ is a Lagrange finite element.
Proof 7 If $\|B\|<1$ the matrix $R$ is invertible and thus $\left(K, P_{K}, S_{K}\right)$ is a Lagrange finite element.

In the previous proposition the norm $\|$.$\| can be replace by any subordinate$ norm.
Remarks: It is possible to choose another example of space $Z$. For instance, we can use the hiearchical basis proposed by Solin (see [12]). In this case

$$
\begin{aligned}
Z=Q_{1}+\operatorname{span} & \left(\left(1-x_{1}^{\vee}\right)\left(x_{2}^{\vee^{2}}-1\right)\left(x_{3}^{\vee}-1\right),\left(1+x_{1}^{\vee}\right)\left(x_{2}^{\vee} 2\right.\right. \\
V^{2} & )\left(x_{3}^{\vee^{2}}-1\right), \\
& \left(1-x_{2}^{\vee}\right)\left(x_{1}^{\vee^{2}}-1\right)\left(x_{3}^{\vee^{2}}-1\right),\left(1+x_{2}^{\vee}\right)\left(x_{1}^{\vee^{2}}-1\right)\left(x_{3}^{\vee^{2}}-1\right), \\
& \left.\left(1-x_{3}^{\vee}\right)\left(x_{2}^{\vee^{2}}-1\right)\left(x_{1}^{\vee^{2}}-1\right),\left(1+x_{3}^{\vee}\right)\left(x_{2}^{\vee^{2}}-1\right)\left(x_{1}^{\vee^{2}}-1\right)\right),
\end{aligned}
$$

but the degree of the polynomials is higher.
Proposition 8 We assume that there exists $\alpha>0$ such that for each $K \in$ $\mathcal{T}_{h},\|B\| \leq 1-\alpha$. Then the assumptions of Proposition 4 are satisfied.
Proof 8 The inclusions $P_{1} \subseteq P_{K} \subseteq P_{3}$ are obvious.
Since $\|B\| \leq 1-\alpha,\left\|R^{-1}\right\|$ is bounded, and consequently the $P_{i, K}$ 's are bounded on $K^{2 \#}$. Finally, by construction, the patch test is satisfied.

## 5 Numerical results

### 5.1 Quadrilateral case

We take $\Omega=] 0,1[\times] 0,1\left[\right.$ and the exact solution is $u\left(x_{1}, x_{2}\right)=\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right)$. We considere two types of mesh. They are composed of two patterns and their shapes are the same for each mesh used, see Figure 4 The first mesh is a mesh in chevron given in [13] and the second is a mesh in honeycomb. In the


Figure 4: meshes 1 and 2
first test, we take $P_{K}=Q_{1}^{K}$ and as expected the method does not converge on (deformed) quadrilateral meshes but converges on meshes based on squares or parallelograms, see Figure 5. In the second test, $P_{K}$ is given by (16). For the two meshes proposed, we obtain the expected order of convergence, see Figure 6

### 5.2 Hexahedral case

We take $\Omega=] 0,1[\times] 0,1[\times] 0,1\left[\right.$ and the exact solution is $u\left(x_{1}, x_{2}, x_{3}\right)=$ $\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) \sin \left(\pi x_{3}\right)$.
The first mesh used is based on truncated pyramids and the second is a generalization of the 2D Chevron mesh, see Figure 7. Figure 8 confirms the a priori error estimate.

## 6 Conclusion

One of the motivations of this work refers to the loss of convergence problem when using classical mixed finite elements on quadrilaterals and hexahedra (see for instance [14], [8]). The theoretical part of this paper allows us to built pseudo-conform finite elements of higher order without any particular difficulty. Moreover we are able to adapt the process in the mixed finite elements context.


Figure 5: Convergence curves when $P_{K}=Q_{1}^{K}$


Figure 6: Convergence curves when $P_{K}$ is given by (16)

The results will be presented in a companion paper.
The extension to generalized hexahedra (i.e. hexahedra with non-plane faces) is a work in progress.


Figure 7: 3D meshes


Figure 8: Convergence curves on mesh 3 and mesh 4

## References

[1] P.G. Ciarlet, "The Finite Element Method for Elliptic Problems," Classics in Applied Mathematics, Vol.40, SIAM, Philadelphia, 2002. First edition published by North-Holland, Amsterdam, 1978.
[2] P.G. Ciarlet, Basic error estimates for elliptic problems, in "Handbook of Numerical Analysis, Vol.II" (eds. P.G. Ciarlet and J.L. Lions), Elsevier, (1991), 17-351.
[3] G. Strang and G.J. Fix, "An Analysis of the Finite Element Method," Prentice-Hall, Englewood Cliffs, New Jersey (1973); second edition, Wellesley-

Cambridge Press, Wellesley (2008).
[4] S.C. Brenner and L.R. Scott, "The Mathematical Theory of Finite Element Methods," 2nd edition, Springer-Verlag, New York, 2002.
[5] D. Braess, "Finite Elements, Theory, Fast Solvers, and Applications in Solid Mechanics," 2nd edition, Cambridge University Press, Cambridge, 2001.
[6] Y. Kuznetsov, S. Repin, "New mixed finite element method on polygonal and polyhedral meshes", Russ. J. Numer. Anal. Math. Model. 18 (3) (2003) 261-278.
[7] Y. Kuznetsov, S. Repin, "Convergence analysis and error estimates for mixed finite element method on distorted meshes", J. Numer. Math. 13 (1) (2005) 33-51.
[8] A. Sboui, J. E. Roberts and J. Jaffré, "A Composite Mixed Finite element for general hexahedral grids for Darcy flow calculations", Preprint INRIA, submittted to SIAM
[9] T.J.R. Hughes, "The Finite Element Method, Linear Static and Dynamic Finite Element Analysis ," Prentice-Hall, Englewood Cliffs, New Jersey (1987).
[10] P. Knabner, S. Korotov and G. Summ, Conditions for the invertibility of the isoparametric mapping for hexahedral finite elements, Finite Elements in Analysis and Design, 40 (2003), 159-172.
[11] S. Zhang, "Invertible Jacobian for Hexahedral Finite Elements, Part I and II", Preprint University of Delaware (2005).
[12] P. Solin , K. Segeth and I. Dolezel, "Higher-Order Finite Element Methods" Chapman and Hall/ CRC Press, , Boca Raton, FL, (2003), 408 pp.
[13] D.N. Arnold, D. Boffi and R.S. Falk, Approximation by quadrilateral finite elements, Maths of Comp., 239 (2002) 909-922.
[14] D.N. Arnold, D. Boffi, R.S. Falk, Quadrilateral H(div) finite elements, SIAM J. Numerical Analysis, 42 (2005) 2529-2451.


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