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Termination of λ -calculus with an extra call-by-value rule

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Notations and standard results are presented in Appendix A.
We consider the following rule in λ -calculus:

$$\text{assoc} \quad (\lambda x.M) ((\lambda y.N) P) \longrightarrow (\lambda y.(\lambda x.M) N) P$$

We want to prove

Proposition 1 $SN^\beta \subseteq SN^{\text{assoc}\beta}$.

Lemma 1 $\longrightarrow_{\text{assoc}}$ is terminating in λ -calculus.

Proof: Each application of the rule decreases by one the number of pairs of λ that are not nested. \square

To prove Proposition 1 above, it would thus be sufficient to prove that $\longrightarrow_{\text{assoc}}$ could be adjourned with respect to \longrightarrow_β , in other words that $\longrightarrow_{\text{assoc}} \cdot \longrightarrow_\beta \subseteq \longrightarrow_\beta \cdot \longrightarrow_{\text{assoc}\beta}^*$ (the adjournment technique leads directly to the desired strong normalisation result). When trying to prove the property by induction and case analysis on the β -reduction following the assoc -reduction to be adjourned, all cases allow the adjournment but one, namely:

$$(\lambda x.M) ((\lambda y.N) P) \longrightarrow_{\text{assoc}} (\lambda y.(\lambda x.M) N) P \longrightarrow_\beta (\lambda y. \{ \cancel{x}^N \} M) P$$

Hence, we shall assume without loss of generality that the β -reduction is not of the above kind. For that we need to identify a sub-relation of β -reduction \leftrightarrow such that

- $\longrightarrow_{\text{assoc}}$ can now be adjourned with respect to \leftrightarrow
- we can justify that there is no loss of generality.

For this we give ourselves the possibility of marking λ -redexes and forbid reductions under their (marked) bindings, so that, if in the assoc -reduction above we make sure that $(\lambda y.(\lambda x.M) N) P$ is marked, the problematic β -reduction is forbidden.

Hence we use the usual notation for a marked redex $(\bar{\lambda}y.Q) P$, but we can also see it as the construct $\text{let } y = P \text{ in } Q$ of λ_C [Mog88] and other works on call-by-value λ -calculus. We start with a reminder about marked redexes.

Definition 1 The syntax of the λ -calculus is extended as follows:

$$M, N ::= x \mid \lambda x.M \mid M N \mid (\bar{\lambda}x.M) N$$

Reduction is given by the following system β_{12} :

$$\begin{array}{l} \beta_1 \quad (\lambda x.M) N \longrightarrow \{M/x\}N \\ \beta_2 \quad (\bar{\lambda}x.M) N \longrightarrow \{M/x\}N \end{array}$$

The forgetful projection onto λ -calculus is straightforward:

$$\begin{array}{l} \phi(x) \quad \quad \quad := x \\ \phi(\lambda x.M) \quad \quad := \lambda x.\phi(M) \\ \phi(M N) \quad \quad \quad := \phi(M) \phi(N) \\ \phi((\bar{\lambda}x.M) N) \quad := (\lambda x.\phi(M)) \phi(N) \end{array}$$

Remark 2 Clearly, $\longrightarrow_{\beta_{12}}$ strongly simulates \longrightarrow_{β} through ϕ^{-1} and \longrightarrow_{β} strongly simulates $\longrightarrow_{\beta_{12}}$ through ϕ .

Reducing under $\bar{\lambda}$ and erasing $\bar{\lambda}$ can be strongly adjourned

In this section we identify the reduction notion $\hookrightarrow (\subseteq \longrightarrow_{\beta_{12}})$ and we argue against the loss of generality by proving that $\longrightarrow_{\beta_{12}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot (\longrightarrow_{\beta_{12}} \cup \hookrightarrow)^+$, a strong case of adjournment, presented in Appendix B, whose direct corollary is that, for every sequence of β_{12} -reduction, there is also a sequence of \hookrightarrow -reduction of the same length and starting from the same term.

We thus split the reduction system β_{12} into two cases depending on whether or not a reduction throws away an argument that contains some markings:

Definition 2

$$\begin{array}{l} \beta_{\kappa} \quad \left\{ \begin{array}{l} (\lambda x.M) P \longrightarrow M \quad \text{if } x \notin \text{FV}(M) \text{ and there is a term } (\bar{\lambda}x.N) Q \sqsubseteq P \\ (\bar{\lambda}x.M) P \longrightarrow M \quad \text{if } x \notin \text{FV}(M) \text{ and there is a term } (\bar{\lambda}x.N) Q \sqsubseteq P \end{array} \right. \\ \beta_{\bar{\kappa}} \quad \left\{ \begin{array}{l} (\lambda x.M) P \longrightarrow M \quad \text{if } x \in \text{FV}(M) \text{ or there is no term } (\bar{\lambda}x.N) Q \sqsubseteq P \\ (\bar{\lambda}x.M) P \longrightarrow M \quad \text{if } x \in \text{FV}(M) \text{ or there is no term } (\bar{\lambda}x.N) Q \sqsubseteq P \end{array} \right. \end{array}$$

Remark 3 Clearly, $\longrightarrow_{\beta_{12}} = \longrightarrow_{\beta_{\kappa}} \cup \longrightarrow_{\beta_{\bar{\kappa}}}$.

No we distinguish whether or not a reduction occurs underneath a marked redex, via the following rule and the following notion of contextual closure:

Definition 3

$$\bar{\beta} \quad (\bar{\lambda}x.M) P \longrightarrow (\bar{\lambda}x.N) P \quad \text{if } M \longrightarrow_{\beta_{12}} N$$

Now we define a weak notion of contextual closure for a rewriting system i :

$$\begin{array}{c} \frac{i : M \longrightarrow N}{M \rightarrow_i N} \quad \frac{M \rightarrow_i N}{\lambda x.M \rightarrow_i \lambda x.N} \quad \frac{M \rightarrow_i N}{M P \rightarrow_i N P} \quad \frac{M \rightarrow_i N}{P M \rightarrow_i P N} \\ \\ \frac{M \rightarrow_i N}{(\bar{\lambda}x.P) M \rightarrow_i (\bar{\lambda}x.P) N} \end{array}$$

Finally we use the following abbreviations:

Definition 4 Let $\hookrightarrow := \rightarrow_{\beta_{\bar{\kappa}}}$ and $\rightsquigarrow_1 := \rightarrow_{\beta_{\kappa}}$ and $\rightsquigarrow_2 := \rightarrow_{\bar{\beta}}$.

Remark 4 Clearly, $\longrightarrow_{\beta_{12}} = \hookrightarrow \cup \rightsquigarrow_1 \cup \rightsquigarrow_2$.

Lemma 5 If $(\bar{\lambda}x.N) Q \sqsubseteq P$, then there is P' such that $P \hookrightarrow P'$.

Proof: By induction on P

- The case $P = y$ is vacuous.
- For $P = \lambda y.M$, we have $(\overline{\lambda x}.N) Q \sqsubseteq M$ and the induction hypothesis provides $M \hookrightarrow M'$, so $\lambda y.M \hookrightarrow \lambda y.M'$.
- For $P = M_1 M_2$, we have either $(\overline{\lambda x}.N) Q \sqsubseteq M_1$ or $(\overline{\lambda x}.N) Q \sqsubseteq M_2$. In the former case the induction hypothesis provides $M_1 \hookrightarrow M'_1$, so $M_1 M_2 \hookrightarrow M'_1 M_2$. The latter case is similar.
- Suppose $P = (\overline{\lambda y}.M_1) M_2$. If there is a term $(\overline{\lambda x'}.N') Q' \sqsubseteq M_2$, the induction hypothesis provides $M_2 \hookrightarrow M'_2$, so $(\overline{\lambda y}.M_1) M_2 \hookrightarrow (\overline{\lambda y}.M_1) M'_2$. If there is no such term $(\overline{\lambda x'}.N') Q' \sqsubseteq M_2$, we have $(\overline{\lambda y}.M_1) M_2 \hookrightarrow \{M_2/y\}M_1$.

□

Lemma 6 $\rightsquigarrow_1 \sqsubseteq \hookrightarrow \rightsquigarrow_1$

Proof: By induction on the reduction step \rightsquigarrow_1 .

For the base cases $(\lambda x.M) P \rightarrow_{\beta\kappa} M$ or $(\overline{\lambda x}.M) P \rightarrow_{\beta\kappa} M$ with $x \notin \text{FV}(M)$ and $(\overline{\lambda y}.N) Q \sqsubseteq P$, Lemma 5 provides the reduction $P \hookrightarrow P'$, so $(\lambda x.M) P \hookrightarrow (\lambda x.M) P' \rightsquigarrow_1 M$ and $(\overline{\lambda x}.M) P \hookrightarrow (\overline{\lambda x}.M) P' \rightsquigarrow_1 M$.

The induction step is straightforward as the same contextual closure is used on both sides (namely, the weak one). □

Lemma 7 $\rightsquigarrow_2 \cdot \hookrightarrow \sqsubseteq \hookrightarrow \cdot \rightarrow_{\beta_{12}}^+$

Proof: By induction on the reduction step \hookrightarrow . See appendix C. □

Corollary 8 $\rightarrow_{\beta_{12}}$ can be strongly adjourned with respect to \hookrightarrow .

Proof: Straightforward from the last two theorems, and Remark 4. □

assoc-reduction

We introduce two new rules in the marked λ -calculus to simulate **assoc**:

$$\begin{array}{lcl} \overline{\text{assoc}} & (\overline{\lambda x}.M) (\overline{\lambda y}.N) P & \longrightarrow (\overline{\lambda y}.(\overline{\lambda x}.M) N) P \\ \text{act} & (\lambda x.M) N & \longrightarrow (\overline{\lambda x}.M) N \end{array}$$

Remark 9 Clearly, $\rightarrow_{\overline{\text{assoc}}\text{act}}$ strongly simulates $\rightarrow_{\text{assoc}}$ through ϕ^{-1} .

Notice that with the **let** = **in**-notation, $\overline{\text{assoc}}$ and **act** are simply the rules of λ_C

$$\begin{array}{lcl} \overline{\text{assoc}} & \text{let } x = (\text{let } y = P \text{ in } N) \text{ in } M & \longrightarrow \text{let } y = P \text{ in let } x = N \text{ in } M \\ \text{act} & (\lambda x.M) N & \longrightarrow \text{let } x = N \text{ in } M \end{array}$$

Lemma 10 $\rightarrow_{\overline{\text{assoc}}\text{act}} \cdot \hookrightarrow \sqsubseteq \hookrightarrow \cdot \rightarrow_{\overline{\text{assoc}}\text{act}}^*$

Proof: By induction on the reduction step \hookrightarrow . See appendix C. □

Lemma 11 $\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$ can be strongly adjourned with respect to \hookrightarrow .

Proof: We prove that $\forall k, \longrightarrow_{\text{assoc,act}}^k \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$ by induction on k .

- For $k = 0$, this is Corollary 8.
- Suppose it is true for k . By the induction hypothesis we get

$$\longrightarrow_{\text{assoc,act}} \cdot \longrightarrow_{\text{assoc,act}}^k \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \longrightarrow_{\text{assoc,act}} \cdot \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

Then by Lemma 10 we get

$$\longrightarrow_{\text{assoc,act}} \cdot \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12} \subseteq \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}} \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

□

Remark 12 Note from Lemma 5 that $\text{nf}^{\hookrightarrow} \subseteq \text{nf}^{\hookrightarrow_1 \cup \hookrightarrow_2} \subseteq \text{nf}^{\longrightarrow_{\beta 12}} \subseteq \text{nf}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$.

Theorem 13 $\text{BN}^{\hookrightarrow} \subseteq \text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$

Proof: We apply Theorem 28, since $\text{nf}^{\hookrightarrow} \subseteq \text{nf}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$ and clearly

$$(\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}) \cup \hookrightarrow = \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

□

Theorem 14 $\text{BN}^{\beta} \subseteq \text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$

Proof: Since \longrightarrow_{β} strongly simulates \hookrightarrow through ϕ , we have $\phi^{-1}(\text{BN}^{\beta}) \subseteq \text{BN}^{\hookrightarrow} \subseteq \text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$. Hence $\phi(\phi^{-1}(\text{BN}^{\beta})) \subseteq \phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}})$. Since ϕ is surjective, $\text{BN}^{\beta} = \phi(\phi^{-1}(\text{BN}^{\beta}))$. Hence $\text{BN}^{\beta} \subseteq \phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}})$. Also, $\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$ strongly simulates $\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}$ through ϕ^{-1} , so $\phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}) \subseteq \text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$. □

Theorem 15 $\text{SN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$

Proof: First, from Lemma 19, $\text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}} \subseteq \text{SN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$. Then from Lemma 1, $\longrightarrow_{\text{assoc}}$ is terminating and hence SN^{assoc} is stable under \longrightarrow_{β} . Hence we can apply Lemma 24 to get $\text{SN}^{\text{assoc}\beta} = \text{SN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$. From the previous theorem we thus have $\text{BN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$. Now, noticing that β -reduction in λ -calculus is finitely branching, Lemma 18 gives $\text{BN}^{\beta} = \text{SN}^{\beta}$ and thus $\text{SN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$. □

References

[Mog88] E. Moggi. Computational lambda-calculus and monads. Report ECS-LFCS-88-66, University of Edinburgh, Edinburgh, Scotland, October 1988.

A Reminder: Notations, Definitions and Basic Results

Definition 5 (Relations)

- We denote the composition of relations by \cdot , the identity relation by Id , and the inverse of a relation by $^{-1}$.
- If $\mathcal{D} \subseteq \mathcal{A}$, we write $\mathcal{R}(\mathcal{D})$ for $\{M \in \mathcal{B} \mid \exists N \in \mathcal{D}, N\mathcal{R}M\}$, or equivalently $\bigcup_{N \in \mathcal{D}} \{M \in \mathcal{B} \mid N\mathcal{R}M\}$. When \mathcal{D} is the singleton $\{M\}$, we write $\mathcal{R}(M)$ for $\mathcal{R}(\{M\})$.
- We say that a relation $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ is *total* if $\mathcal{R}^{-1}(\mathcal{B}) = \mathcal{A}$.

Remark 16 Composition is associative, and identity relations are neutral for the composition operation.

Definition 6 (Reduction relation)

- A *reduction relation* on \mathcal{A} is a relation from \mathcal{A} to \mathcal{A} .
- Given a reduction relation \rightarrow on \mathcal{A} , we define the set of \rightarrow -*reducible forms* (or just *reducible forms* when the relation is clear) as $\text{rf}^{\rightarrow} := \{M \in \mathcal{A} \mid \exists N \in \mathcal{A}, M \rightarrow N\}$. We define the set of *normal forms* as $\text{nf}^{\rightarrow} := \{M \in \mathcal{A} \mid \nexists N \in \mathcal{A}, M \rightarrow N\}$.
- Given a reduction relation \rightarrow on \mathcal{A} , we write \leftarrow for \rightarrow^{-1} , and we define \rightarrow^n by induction on the natural number n as follows:
 - $\rightarrow^0 := \text{Id}$
 - $\rightarrow^{n+1} := \rightarrow \cdot \rightarrow^n (= \rightarrow^n \cdot \rightarrow)$
 - \rightarrow^+ denotes the transitive closure of \rightarrow (i.e. $\rightarrow^+ := \bigcup_{n \geq 1} \rightarrow^n$).
 - \rightarrow^* denotes the transitive and reflexive closure of \rightarrow (i.e. $\rightarrow^* := \bigcup_{n \geq 0} \rightarrow^n$).
 - \leftrightarrow denotes the symmetric closure of \rightarrow (i.e. $\leftrightarrow := \leftarrow \cup \rightarrow$).
 - \leftrightarrow^* denotes the transitive, reflexive and symmetric closure of \rightarrow .
- An *equivalence relation* on \mathcal{A} is a transitive, reflexive and symmetric reduction relation on \mathcal{A} , i.e. a relation $\rightarrow = \leftrightarrow^*$, hence denoted more often by \sim, \equiv, \dots
- Given a reduction relation \rightarrow on \mathcal{A} and a subset $\mathcal{B} \subseteq \mathcal{A}$, the *closure of \mathcal{B} under \rightarrow* is $\rightarrow^*(\mathcal{B})$.

Definition 7 (Finitely branching relation) A reduction relation \rightarrow on \mathcal{A} is *finitely branching* if $\forall M \in \mathcal{A}, \rightarrow(M)$ is finite.

Definition 8 (Stability) Given a reduction relation \rightarrow on \mathcal{A} , we say that a subset \mathcal{T} of \mathcal{A} is \rightarrow -*stable* (or *stable under \rightarrow*) if $\rightarrow(\mathcal{T}) \subseteq \mathcal{T}$.

Definition 9 (Strong simulation)

Let \mathcal{R} be a relation between two sets \mathcal{A} and \mathcal{B} , respectively equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.

$\rightarrow_{\mathcal{B}}$ *strongly simulates* $\rightarrow_{\mathcal{A}}$ through \mathcal{R} if $(\mathcal{R}^{-1} \cdot \rightarrow_{\mathcal{A}}) \subseteq (\rightarrow_{\mathcal{B}}^+ \cdot \mathcal{R}^{-1})$.

Remark 17

1. If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through \mathcal{R} , and if $\rightarrow_{\mathcal{B}} \subseteq \rightarrow'_{\mathcal{B}}$ and $\rightarrow'_{\mathcal{A}} \subseteq \rightarrow_{\mathcal{A}}$, then $\rightarrow'_{\mathcal{B}}$ strongly simulates $\rightarrow'_{\mathcal{A}}$ through \mathcal{R} .

2. If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ and $\rightarrow'_{\mathcal{A}}$ through \mathcal{R} , then it also strongly simulates $\rightarrow_{\mathcal{A}} \cdot \rightarrow'_{\mathcal{A}}$ through \mathcal{R} .
3. Hence, if $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through \mathcal{R} , then it also strongly simulates $\rightarrow_{\mathcal{A}}^+$ through \mathcal{R} .

Definition 10 (Patriarchal) Given a reduction relation \rightarrow on \mathcal{A} , we say that

- a subset \mathcal{T} of \mathcal{A} is \rightarrow -*patriarchal* (or just *patriarchal* when the relation is clear) if $\forall N \in \mathcal{A}, \rightarrow(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$.
- a predicate P on \mathcal{A} is *patriarchal* if $\{M \in \mathcal{A} \mid P(M)\}$ is *patriarchal*.

Definition 11 (Normalising elements) Given a reduction relation \rightarrow on \mathcal{A} , the set of \rightarrow -*strongly normalising* elements is

$$SN^{\rightarrow} := \bigcap_{\mathcal{T} \text{ is patriarchal}} \mathcal{T}$$

Definition 12 (Bounded elements) The set of \rightarrow -*bounded* elements is defined as

$$BN^{\rightarrow} := \bigcup_{n \geq 0} BN_n^{\rightarrow}$$

where BN_n^{\rightarrow} is defined by induction on the natural number n as follows:

$$\begin{aligned} BN_0^{\rightarrow} &:= nf^{\rightarrow} \\ BN_{n+1}^{\rightarrow} &:= \{M \in \mathcal{A} \mid \exists n' \leq n, \rightarrow(M) \subseteq BN_{n'}^{\rightarrow}\} \end{aligned}$$

Lemma 18 *If \rightarrow is finitely branching, then BN^{\rightarrow} is patriarchal. As a consequence, $BN^{\rightarrow} = SN^{\rightarrow}$.*

Lemma 19

1. If $n < n'$ then $BN_n^{\rightarrow} \subseteq BN_{n'}^{\rightarrow} \subseteq BN^{\rightarrow}$. In particular, $nf^{\rightarrow} \subseteq BN_n^{\rightarrow} \subseteq BN^{\rightarrow}$.
2. $BN^{\rightarrow} \subseteq SN^{\rightarrow}$.

Lemma 20

1. SN^{\rightarrow} is patriarchal.
2. If $M \in BN^{\rightarrow}$ then $\rightarrow(M) \subseteq BN^{\rightarrow}$.
If $M \in SN^{\rightarrow}$ then $\rightarrow(M) \subseteq SN^{\rightarrow}$.

Theorem 21 (Induction principle) *Given a predicate P on \mathcal{A} , suppose $\forall M \in SN^{\rightarrow}, (\forall N \in \rightarrow(M), P(N)) \Rightarrow P(M)$. Then $\forall M \in SN^{\rightarrow}, P(M)$.*

When we use this theorem to prove a statement $P(M)$ for all M in SN^{\rightarrow} , we just add $(\forall N \in \rightarrow(M), P(N))$ to the assumptions, which we call the induction hypothesis.

We say that we prove the statement by induction in SN^{\rightarrow} .

Lemma 22

1. If $\rightarrow_1 \subseteq \rightarrow_2$, then $nf^{\rightarrow_1} \supseteq nf^{\rightarrow_2}$, $SN^{\rightarrow_1} \supseteq SN^{\rightarrow_2}$,
and for all n , $BN_n^{\rightarrow_1} \supseteq BN_n^{\rightarrow_2}$.

2. $nf^{\rightarrow} = nf^{\rightarrow+}$, $SN^{\rightarrow} = SN^{\rightarrow+}$, and for all n , $BN_n^{\rightarrow+} = BN_n^{\rightarrow}$.

Notice that this result enables us to use a stronger induction principle: in order to prove $\forall M \in SN^{\rightarrow}, P(M)$, it now suffices to prove

$$\forall M \in SN^{\rightarrow}, (\forall N \in \rightarrow^+(M), P(N)) \Rightarrow P(M)$$

This induction principle is called the *transitive induction in SN^{\rightarrow}* .

Theorem 23 (Strong normalisation by strong simulation) *Let \mathcal{R} be a relation between \mathcal{A} and \mathcal{B} , equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through \mathcal{R} , then $\mathcal{R}^{-1}(SN^{\rightarrow_{\mathcal{B}}}) \subseteq SN^{\rightarrow_{\mathcal{A}}}$.

Lemma 24 *Given two reduction relations $\rightarrow_1, \rightarrow_2$, suppose that SN^{\rightarrow_1} is stable under \rightarrow_2 . Then $SN^{\rightarrow_1 \cup \rightarrow_2} = SN^{\rightarrow_1^* \rightarrow_2} \cap SN^{\rightarrow_1}$.*

B Strong adjournment

Definition 13 Suppose $\rightarrow_{\mathcal{A}}$ is a reduction relation on \mathcal{A} , $\rightarrow_{\mathcal{B}}$ is a reduction relation on \mathcal{B} , \mathcal{R} is a relation from \mathcal{A} to \mathcal{B} .

$\rightarrow_{\mathcal{B}}$ *simulates the reduction lengths of $\rightarrow_{\mathcal{A}}$ through \mathcal{R} if*

$$\forall k, \forall M, N \in \mathcal{A}, \forall P \in \mathcal{B}, M \rightarrow_{\mathcal{A}}^k N \wedge M \mathcal{R} P \Rightarrow \exists Q \in \mathcal{B}, P \rightarrow_{\mathcal{B}}^k Q$$

Lemma 25 *Suppose $\rightarrow_{\mathcal{A}}$ is a reduction relation on \mathcal{A} , $\rightarrow_{\mathcal{B}}$ is a reduction relation on \mathcal{B} , \mathcal{R} is a relation from \mathcal{A} to \mathcal{B} .*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through \mathcal{R} , then $\rightarrow_{\mathcal{B}}$ simulates the reduction lengths of $\rightarrow_{\mathcal{A}}$ through \mathcal{R} .

Proof: We prove by induction on k that $\forall k, \forall M, N \in \mathcal{A}^2, \forall P \in \mathcal{B}, M \rightarrow_{\mathcal{A}}^k N \wedge M \mathcal{R} P \Rightarrow \exists Q, P \rightarrow_{\mathcal{B}}^k Q$.

- For $k = 0$: take $Q := M = N$.
- Suppose it is true for k and take $M \rightarrow_{\mathcal{A}} M' \rightarrow_{\mathcal{A}}^k N$. The strong simulation gives P' such that $P \rightarrow_{\mathcal{B}}^+ P'$ and $M' \mathcal{R} P'$. The induction hypothesis gives Q' such that $P' \rightarrow_{\mathcal{B}}^k Q'$. Then it suffices to take the prefix $P \rightarrow_{\mathcal{B}}^{k+1} Q$ (of length $k + 1$) of $P \rightarrow_{\mathcal{B}}^+ P' \rightarrow_{\mathcal{B}}^k Q'$.

□

Lemma 26 $\forall n, \forall M, (\forall k, \forall N, M \rightarrow^k N \Rightarrow k \leq n) \iff M \in BN_n^{\rightarrow}$

Proof: By transitive induction on n .

- For $n = 0$: clearly both sides are equivalent to $M \in nf^{\rightarrow}$.
- Suppose it is true for all $i \leq n$.

Suppose $\forall k, \forall N, M \rightarrow^k N \Rightarrow k \leq n + 1$. Then take $M \rightarrow M'$ and assume $M' \rightarrow^{k'} N'$. We have $M \rightarrow^{k'+1} N'$ so from the hypothesis we derive $k' + 1 \leq n + 1$, i.e. $k' \leq n$. We apply the induction hypothesis on M' and get $M' \in BN_{n+1}^{\rightarrow}$. By definition of BN_{n+1}^{\rightarrow} we get $M \in BN_{n+1}^{\rightarrow}$.

Conversely, suppose $M \in BN_{n+1}^{\rightarrow}$ and $M \rightarrow^k N$. We must prove that $k \leq n + 1$. If $k = 0$ we are done. If $k = k' + 1$ we have $M \rightarrow M' \rightarrow^{k'} N$; by definition of BN_{n+1}^{\rightarrow} there is $i \leq n$ such that $M' \in BN_i^{\rightarrow}$, and by induction hypothesis we have $k' \leq i$; hence $k = k' + 1 \leq i + 1 \leq n + 1$.

□

Theorem 27 Suppose $\rightarrow_{\mathcal{A}}$ is a reduction relation on \mathcal{A} , $\rightarrow_{\mathcal{B}}$ is a reduction relation on \mathcal{B} , \mathcal{R} is a relation from \mathcal{A} to \mathcal{B} .

If $\rightarrow_{\mathcal{B}}$ simulates the reduction lengths of $\rightarrow_{\mathcal{A}}$ through \mathcal{R} , then

$$\forall n, \mathcal{R}^{-1}(BN_n^{\rightarrow_{\mathcal{B}}}) \subseteq BN_n^{\rightarrow_{\mathcal{A}}} \quad (\subseteq SN^{\rightarrow_{\mathcal{A}}})$$

Proof: Suppose $N \in BN_n^{\rightarrow_{\mathcal{B}}}$ and $M\mathcal{R}N$. If $M \xrightarrow{\mathcal{A}}^k M'$ then by simulation $N \xrightarrow{\mathcal{B}}^k N'$ so by Lemma 26 we have $k \leq n$. Hence by (the other direction of) Lemma 26 we have $M \in BN_n^{\rightarrow_{\mathcal{A}}}$. \square

Definition 14 Let \rightarrow_1 and \rightarrow_2 be two reduction relations on \mathcal{A} .

The relation \rightarrow_1 can be *strongly adjourned with respect to* \rightarrow_2 if

whenever $M \rightarrow_1 N \rightarrow_2 P$ there exists Q such that $M \rightarrow_2 Q(\rightarrow_1 \cup \rightarrow_2)^+ P$.

Theorem 28 Let \rightarrow_1 and \rightarrow_2 be two reduction relations on \mathcal{A} . If $\text{nf}^{\rightarrow_2} \subseteq \text{nf}^{\rightarrow_1}$ and \rightarrow_1 can be strongly adjourned with respect to \rightarrow_2 then $BN^{\rightarrow_2} \subseteq BN^{\rightarrow_1 \cup \rightarrow_2}$.

Proof: From Theorem 27, it suffices to show that \rightarrow_2 simulates the reduction lengths of $\rightarrow_1 \cup \rightarrow_2$ through the identity. We show by induction on k that

$$\forall k, \forall M, N, M(\rightarrow_1 \cup \rightarrow_2)^k N \Rightarrow \exists Q, M \rightarrow_2^k Q$$

- For $k = 0$: take $Q := M$
- For $k = 1$: If $M \rightarrow_2 N$ take $Q := N$; if $M \rightarrow_1 N$ use the hypothesis $\text{nf}^{\rightarrow_2} \subseteq \text{nf}^{\rightarrow_1}$ to produce Q such that $M \rightarrow_2 Q$.
- Suppose it is true for $k + 1$ and take $M(\rightarrow_1 \cup \rightarrow_2)P(\rightarrow_1 \cup \rightarrow_2)^{k+1}N$.

The induction hypothesis provides T such that $P \rightarrow_2^{k+1} T$, in other words $P \rightarrow_2 S \rightarrow_2^k T$.

If $M \rightarrow_2 P$ we are done. If $M \rightarrow_1 P$ we use the hypothesis of adjournment to transform $M \rightarrow_1 P \rightarrow_2 S$ into $M \rightarrow_2 P'(\rightarrow_1 \cup \rightarrow_2)^+ S$. Take the prefix $P'(\rightarrow_1 \cup \rightarrow_2)^{k+1}R$ (of length $k + 1$) of $P'(\rightarrow_1 \cup \rightarrow_2)^+ S \rightarrow_2^k T$, and apply on this prefix the induction hypothesis to get $P' \rightarrow_2^{k+1} R$. We thus get $M \rightarrow_2^{k+2} R$.

\square

C Proofs

Lemma 7 $\rightsquigarrow_2 \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \xrightarrow{\beta_{12}}^+$

Proof: By induction on the reduction step \hookrightarrow .

- For the base case where the $\beta\bar{\kappa}$ -reduction is a β_2 -reduction, we have $M \rightsquigarrow_2 (\bar{\lambda}x.N) P \hookrightarrow \{P/x\}N$ with $x \in \text{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\bar{\lambda}x.N) P$.
 If $M = (\bar{\lambda}x.N') P \rightsquigarrow_2 (\bar{\lambda}x.N) P$ because $N' \xrightarrow{\beta_{12}} N$ then $(\bar{\lambda}x.N') P \hookrightarrow \{P/x\}N' \xrightarrow{\beta_{12}} \{P/x\}N$.
 If $M = (\bar{\lambda}x.N) P' \rightsquigarrow_2 (\bar{\lambda}x.N) P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \text{FV}(N)$. Hence $(\bar{\lambda}x.N) P' \hookrightarrow \{P'/x\}N \xrightarrow{\beta_{12}}^+ \{P/x\}N$.

- For the base case where the $\beta\bar{K}$ -reduction is a $\beta 1$ -reduction, we have $M \rightsquigarrow_2 (\lambda x.N) P \hookrightarrow \{P/x\}N$ with $x \in \text{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\lambda x.N) P$.
If $M = M' P \rightsquigarrow_2 (\lambda x.N) P$ because $M' \rightsquigarrow_2 \lambda x.N$ then M' must be of the form $\lambda x.M''$ with $M'' \rightsquigarrow_2 N$. Then $(\lambda x.M'') P \hookrightarrow \{P/x\}M''$ (in case P has a marked subterm, notice that $x \in \text{FV}(N) \subseteq \text{FV}(M'')$), and $\{P/x\}M'' \xrightarrow{\beta 12} \{P/x\}N$.
If $M = (\lambda x.N) P' \rightsquigarrow_2 (\lambda x.N) P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \text{FV}(N)$. Hence $(\lambda x.N) P' \hookrightarrow \{P'/x\}N \xrightarrow{\beta 12}^+ \{P/x\}N$.
- The closure under λ is straightforward.
- For the closure under application, left-hand side, we have $M \rightsquigarrow_2 N P \hookrightarrow N' P$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N P$.
If $M = M' P \rightsquigarrow_2 N P$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N'$ and the weak contextual closure gives $M' P \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N' P$.
If $M = N P' \rightsquigarrow_2 N P$ with $P' \rightsquigarrow_2 P$, we can also derive $N P' \hookrightarrow N' P' \xrightarrow{\beta 12} N' P$.
- For the closure under application, right-hand side, we have $M \rightsquigarrow_2 N P \hookrightarrow N P'$ with $P \hookrightarrow P'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N P$.
If $M = M' P \rightsquigarrow_2 N P$ with $M' \rightsquigarrow_2 N$, we can also derive $M' P \hookrightarrow M' P' \xrightarrow{\beta 12} N P'$.
If $M = N M' \rightsquigarrow_2 N P$ with $M' \rightsquigarrow_2 P$, the induction hypothesis gives $M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ P'$ and the weak contextual closure gives $N M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N P'$.
- For the closure under marked redex we have $M \rightsquigarrow_2 (\bar{\lambda}x.P) N \hookrightarrow (\bar{\lambda}x.P) N'$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\bar{\lambda}x.P) N$.
If $M = (\bar{\lambda}x.P') N \rightsquigarrow_2 (\bar{\lambda}x.P) N$ because $P' \xrightarrow{\beta 12} P$, we can also derive $(\bar{\lambda}x.P') N \hookrightarrow (\bar{\lambda}x.P') N' \xrightarrow{\beta 12} (\bar{\lambda}x.P) N'$.
If $M = (\bar{\lambda}x.P) M' \rightsquigarrow_2 (\bar{\lambda}x.P) N$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow Q \xrightarrow{\beta 12}^+ N'$ and the weak contextual closure gives $(\bar{\lambda}x.P) M' \hookrightarrow (\bar{\lambda}x.P) Q \xrightarrow{\beta 12}^+ (\bar{\lambda}x.P) N'$.

□

Lemma 10 $\xrightarrow{\text{assocact}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \xrightarrow{\text{assocact}^*}$

Proof: By induction on the reduction step \hookrightarrow .

- For the first base case, we have $M \xrightarrow{\text{assocact}} (\lambda x.N) P \hookrightarrow \{P/x\}N$ with $x \in \text{FV}(N)$ or P has no marked subterm. Since root $\overline{\text{assocact}}$ -reduction produces neither λ -abstractions nor applications at the root, note that M has to be of the form $(\lambda x.N') P'$, with either $N' \xrightarrow{\text{assocact}} N$ (and $P' = P$) or $P' \xrightarrow{\text{assocact}} P$ (and $N' = N$). In both cases, $x \in \text{FV}(N) \subseteq \text{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N') P' \hookrightarrow \{P'/x\}N' \xrightarrow{\text{assocact}^*} \{P/x\}N$.
- For the second base case, we have $M \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P \hookrightarrow \{P/x\}N$ with $x \in \text{FV}(N)$ or P has no marked subterm. We do a case analysis on $M \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P$.

If $M = (\bar{\lambda}x'.M_1) (\bar{\lambda}x.M_2) P \xrightarrow{\text{assoc}} (\bar{\lambda}x.(\bar{\lambda}x'.M_1) M_2) P$ with $N = (\bar{\lambda}x'.M_1) M_2$, we also have $M = (\bar{\lambda}x'.M_1) (\bar{\lambda}x.M_2) P \hookrightarrow (\bar{\lambda}x'.M_1) \{P/x\} M_2 = \{P/x\} N$.

If $M = (\lambda x.N) P \xrightarrow{\text{act}} (\bar{\lambda}x.N) P$ then $M \hookrightarrow \{P/x\} N$.

If $M = (\bar{\lambda}x.N') P' \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P$ with either $N' \xrightarrow{\text{assocact}} N$ (and $P' = P$) or $P' \xrightarrow{\text{assocact}} P$ (and $N' = N$), we have, in both cases, $x \in \text{FV}(N) \subseteq \text{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N') P' \hookrightarrow \{P/x\} N' \xrightarrow{\text{assocact}^*} \{P/x\} N$.

- The closure under λ is straightforward.

- For the closure under application, left-hand side, we have $Q \xrightarrow{\text{assocact}} M N \hookrightarrow M' N$ with $M \hookrightarrow M'$. We do a case analysis on $Q \xrightarrow{\text{assocact}} M N$.

If $Q = M'' N \xrightarrow{\text{assocact}} M N$ with $M'' \xrightarrow{\text{assocact}} M$, the induction hypothesis provides $M'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M'$ so $M'' N \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M' N$.

If $Q = M N' \xrightarrow{\text{assocact}} M N$ with $N' \xrightarrow{\text{assocact}} N$, we also have $M N' \hookrightarrow M' N' \xrightarrow{\text{assocact}} M' N$.

- For the closure under application, right-hand side, we have $Q \xrightarrow{\text{assocact}} M N \hookrightarrow M N'$ with $N \hookrightarrow N'$. We do a case analysis on $Q \xrightarrow{\text{assocact}} M N$.

If $Q = M' N \xrightarrow{\text{assocact}} M N$ with $M' \xrightarrow{\text{assocact}} M$, we also have $M' N \hookrightarrow M' N' \xrightarrow{\text{assocact}} M N'$.

If $Q = M N'' \xrightarrow{\text{assocact}} M N$ with $N'' \xrightarrow{\text{assocact}} N$, the induction hypothesis provides $N'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} N'$ so $M N'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M N'$.

- For the closure under marked redex, the \hookrightarrow -reduction can only come from the right-hand side because of the weak contextual closure (\hookrightarrow does not reduce under $\bar{\lambda}$), so we have $Q \xrightarrow{\text{assocact}} (\bar{\lambda}y.M) P \hookrightarrow (\bar{\lambda}y.M) P'$ with $P \hookrightarrow P'$. We do a case analysis on $Q \xrightarrow{\text{assocact}} (\bar{\lambda}y.M) P$.

If $Q = (\bar{\lambda}x.M') (\bar{\lambda}y.N) P \xrightarrow{\text{assoc}} (\bar{\lambda}y.(\bar{\lambda}x.M') N) P$ with $M = (\bar{\lambda}x.M') N$, we also have $Q = (\bar{\lambda}x.M') (\bar{\lambda}y.N) P \hookrightarrow (\bar{\lambda}x.M') (\bar{\lambda}y.N) P' \xrightarrow{\text{assoc}} (\bar{\lambda}y.(\bar{\lambda}x.M') N) P'$.

If $Q = (\lambda y.M) P \xrightarrow{\text{act}} (\bar{\lambda}y.M) P$, then we also have $Q = (\lambda y.M) P \hookrightarrow (\lambda y.M) P' \xrightarrow{\text{act}} (\bar{\lambda}y.M) P'$.

□