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# Termination of $\lambda$ -calculus with an extra call-by-value rule

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Notations and standard results are presented in Appendix A.  
We consider the following rule in  $\lambda$ -calculus:

$$\text{assoc} \quad (\lambda x.M) ((\lambda y.N) P) \longrightarrow (\lambda y.(\lambda x.M) N) P$$

We want to prove

**Proposition 1**  $SN^\beta \subseteq SN^{\text{assoc}\beta}$ .

**Lemma 1**  $\longrightarrow_{\text{assoc}}$  is terminating in  $\lambda$ -calculus.

**Proof:** Each application of the rule decreases by one the number of pairs of  $\lambda$  that are not nested.  $\square$

To prove Proposition 1 above, it would thus be sufficient to prove that  $\longrightarrow_{\text{assoc}}$  could be adjourned with respect to  $\longrightarrow_\beta$ , in other words that  $\longrightarrow_{\text{assoc}} \cdot \longrightarrow_\beta \subseteq \longrightarrow_\beta \cdot \longrightarrow_{\text{assoc}\beta}^*$  (the adjournment technique leads directly to the desired strong normalisation result). When trying to prove the property by induction and case analysis on the  $\beta$ -reduction following the  $\text{assoc}$ -reduction to be adjourned, all cases allow the adjournment but one, namely:

$$(\lambda x.M) ((\lambda y.N) P) \longrightarrow_{\text{assoc}} (\lambda y.(\lambda x.M) N) P \longrightarrow_\beta (\lambda y. \{ \cancel{N} / x \} M) P$$

Hence, we shall assume without loss of generality that the  $\beta$ -reduction is not of the above kind. For that we need to identify a sub-relation of  $\beta$ -reduction  $\leftrightarrow$  such that

- $\longrightarrow_{\text{assoc}}$  can now be adjourned with respect to  $\leftrightarrow$
- we can justify that there is no loss of generality.

For this we give ourselves the possibility of marking  $\lambda$ -redexes and forbid reductions under their (marked) bindings, so that, if in the  $\text{assoc}$ -reduction above we make sure that  $(\lambda y.(\lambda x.M) N) P$  is marked, the problematic  $\beta$ -reduction is forbidden.

Hence we use the usual notation for a marked redex  $(\bar{\lambda}y.Q) P$ , but we can also see it as the construct  $\text{let } y = P \text{ in } Q$  of  $\lambda_C$  [Mog88] and other works on call-by-value  $\lambda$ -calculus. We start with a reminder about marked redexes.

**Definition 1** The syntax of the  $\lambda$ -calculus is extended as follows:

$$M, N ::= x \mid \lambda x.M \mid M N \mid (\bar{\lambda}x.M) N$$

Reduction is given by the following system  $\beta_{12}$ :

$$\begin{array}{l} \beta_1 \quad (\lambda x.M) N \longrightarrow \{M/x\}N \\ \beta_2 \quad (\bar{\lambda}x.M) N \longrightarrow \{M/x\}N \end{array}$$

The forgetful projection onto  $\lambda$ -calculus is straightforward:

$$\begin{array}{l} \phi(x) \quad \quad \quad := x \\ \phi(\lambda x.M) \quad \quad := \lambda x.\phi(M) \\ \phi(M N) \quad \quad \quad := \phi(M) \phi(N) \\ \phi((\bar{\lambda}x.M) N) \quad := (\lambda x.\phi(M)) \phi(N) \end{array}$$

**Remark 2** Clearly,  $\longrightarrow_{\beta_{12}}$  strongly simulates  $\longrightarrow_{\beta}$  through  $\phi^{-1}$  and  $\longrightarrow_{\beta}$  strongly simulates  $\longrightarrow_{\beta_{12}}$  through  $\phi$ .

### Reducing under $\bar{\lambda}$ and erasing $\bar{\lambda}$ can be strongly adjourned

In this section we identify the reduction notion  $\hookrightarrow (\subseteq \longrightarrow_{\beta_{12}})$  and we argue against the loss of generality by proving that  $\longrightarrow_{\beta_{12}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot (\longrightarrow_{\beta_{12}} \cup \hookrightarrow)^+$ , a strong case of adjournment, presented in Appendix B, whose direct corollary is that, for every sequence of  $\beta_{12}$ -reduction, there is also a sequence of  $\hookrightarrow$ -reduction of the same length and starting from the same term.

We thus split the reduction system  $\beta_{12}$  into two cases depending on whether or not a reduction throws away an argument that contains some markings:

#### Definition 2

$$\begin{array}{l} \beta_{\kappa} \quad \left\{ \begin{array}{l} (\lambda x.M) P \longrightarrow M \quad \text{if } x \notin \text{FV}(M) \text{ and there is a term } (\bar{\lambda}x.N) Q \sqsubseteq P \\ (\bar{\lambda}x.M) P \longrightarrow M \quad \text{if } x \notin \text{FV}(M) \text{ and there is a term } (\bar{\lambda}x.N) Q \sqsubseteq P \end{array} \right. \\ \beta_{\bar{\kappa}} \quad \left\{ \begin{array}{l} (\lambda x.M) P \longrightarrow M \quad \text{if } x \in \text{FV}(M) \text{ or there is no term } (\bar{\lambda}x.N) Q \sqsubseteq P \\ (\bar{\lambda}x.M) P \longrightarrow M \quad \text{if } x \in \text{FV}(M) \text{ or there is no term } (\bar{\lambda}x.N) Q \sqsubseteq P \end{array} \right. \end{array}$$

**Remark 3** Clearly,  $\longrightarrow_{\beta_{12}} = \longrightarrow_{\beta_{\kappa}} \cup \longrightarrow_{\beta_{\bar{\kappa}}}$ .

No we distinguish whether or not a reduction occurs underneath a marked redex, via the following rule and the following notion of contextual closure:

#### Definition 3

$$\bar{\beta} \quad (\bar{\lambda}x.M) P \longrightarrow (\bar{\lambda}x.N) P \quad \text{if } M \longrightarrow_{\beta_{12}} N$$

Now we define a weak notion of contextual closure for a rewriting system  $i$ :

$$\begin{array}{c} \frac{i : M \longrightarrow N}{M \rightarrow_i N} \quad \frac{M \rightarrow_i N}{\lambda x.M \rightarrow_i \lambda x.N} \quad \frac{M \rightarrow_i N}{M P \rightarrow_i N P} \quad \frac{M \rightarrow_i N}{P M \rightarrow_i P N} \\ \\ \frac{M \rightarrow_i N}{(\bar{\lambda}x.P) M \rightarrow_i (\bar{\lambda}x.P) N} \end{array}$$

Finally we use the following abbreviations:

**Definition 4** Let  $\hookrightarrow := \rightarrow_{\beta_{\bar{\kappa}}}$  and  $\rightsquigarrow_1 := \rightarrow_{\beta_{\kappa}}$  and  $\rightsquigarrow_2 := \rightarrow_{\bar{\beta}}$ .

**Remark 4** Clearly,  $\longrightarrow_{\beta_{12}} = \hookrightarrow \cup \rightsquigarrow_1 \cup \rightsquigarrow_2$ .

**Lemma 5** If  $(\bar{\lambda}x.N) Q \sqsubseteq P$ , then there is  $P'$  such that  $P \hookrightarrow P'$ .

**Proof:** By induction on  $P$

- The case  $P = y$  is vacuous.
- For  $P = \lambda y.M$ , we have  $(\overline{\lambda x}.N) Q \sqsubseteq M$  and the induction hypothesis provides  $M \hookrightarrow M'$ , so  $\lambda y.M \hookrightarrow \lambda y.M'$ .
- For  $P = M_1 M_2$ , we have either  $(\overline{\lambda x}.N) Q \sqsubseteq M_1$  or  $(\overline{\lambda x}.N) Q \sqsubseteq M_2$ . In the former case the induction hypothesis provides  $M_1 \hookrightarrow M'_1$ , so  $M_1 M_2 \hookrightarrow M'_1 M_2$ . The latter case is similar.
- Suppose  $P = (\overline{\lambda y}.M_1) M_2$ . If there is a term  $(\overline{\lambda x'}.N') Q' \sqsubseteq M_2$ , the induction hypothesis provides  $M_2 \hookrightarrow M'_2$ , so  $(\overline{\lambda y}.M_1) M_2 \hookrightarrow (\overline{\lambda y}.M_1) M'_2$ . If there is no such term  $(\overline{\lambda x'}.N') Q' \sqsubseteq M_2$ , we have  $(\overline{\lambda y}.M_1) M_2 \hookrightarrow \{M_2/y\}M_1$ .

□

**Lemma 6**  $\rightsquigarrow_1 \sqsubseteq \hookrightarrow \rightsquigarrow_1$

**Proof:** By induction on the reduction step  $\rightsquigarrow_1$ .

For the base cases  $(\lambda x.M) P \rightarrow_{\beta\kappa} M$  or  $(\overline{\lambda x}.M) P \rightarrow_{\beta\kappa} M$  with  $x \notin \text{FV}(M)$  and  $(\overline{\lambda y}.N) Q \sqsubseteq P$ , Lemma 5 provides the reduction  $P \hookrightarrow P'$ , so  $(\lambda x.M) P \hookrightarrow (\lambda x.M) P' \rightsquigarrow_1 M$  and  $(\overline{\lambda x}.M) P \hookrightarrow (\overline{\lambda x}.M) P' \rightsquigarrow_1 M$ .

The induction step is straightforward as the same contextual closure is used on both sides (namely, the weak one). □

**Lemma 7**  $\rightsquigarrow_2 \cdot \hookrightarrow \sqsubseteq \hookrightarrow \cdot \rightarrow_{\beta_{12}}^+$

**Proof:** By induction on the reduction step  $\hookrightarrow$ . See appendix C. □

**Corollary 8**  $\rightarrow_{\beta_{12}}$  can be strongly adjourned with respect to  $\hookrightarrow$ .

**Proof:** Straightforward from the last two theorems, and Remark 4. □

## assoc-reduction

We introduce two new rules in the marked  $\lambda$ -calculus to simulate **assoc**:

$$\begin{array}{lcl} \overline{\text{assoc}} & (\overline{\lambda x}.M) (\overline{\lambda y}.N) P & \longrightarrow (\overline{\lambda y}.(\overline{\lambda x}.M) N) P \\ \text{act} & (\lambda x.M) N & \longrightarrow (\overline{\lambda x}.M) N \end{array}$$

**Remark 9** Clearly,  $\rightarrow_{\text{assocact}}$  strongly simulates  $\rightarrow_{\text{assoc}}$  through  $\phi^{-1}$ .

Notice that with the **let** = in -notation,  $\overline{\text{assoc}}$  and **act** are simply the rules of  $\lambda_C$

$$\begin{array}{lcl} \overline{\text{assoc}} & \text{let } x = (\text{let } y = P \text{ in } N) \text{ in } M & \longrightarrow \text{let } y = P \text{ in let } x = N \text{ in } M \\ \text{act} & (\lambda x.M) N & \longrightarrow \text{let } x = N \text{ in } M \end{array}$$

**Lemma 10**  $\rightarrow_{\text{assocact}} \cdot \hookrightarrow \sqsubseteq \hookrightarrow \cdot \rightarrow_{\text{assocact}}^*$

**Proof:** By induction on the reduction step  $\hookrightarrow$ . See appendix C. □

**Lemma 11**  $\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$  can be strongly adjourned with respect to  $\hookrightarrow$ .

**Proof:** We prove that  $\forall k, \longrightarrow_{\text{assoc,act}}^k \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$  by induction on  $k$ .

- For  $k = 0$ , this is Corollary 8.
- Suppose it is true for  $k$ . By the induction hypothesis we get

$$\longrightarrow_{\text{assoc,act}} \cdot \longrightarrow_{\text{assoc,act}}^k \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \longrightarrow_{\text{assoc,act}} \cdot \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

Then by Lemma 10 we get

$$\longrightarrow_{\text{assoc,act}} \cdot \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12} \subseteq \hookrightarrow \cdot \longrightarrow_{\text{assoc,act}} \cdot \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

□

**Remark 12** Note from Lemma 5 that  $\text{nf}^{\hookrightarrow} \subseteq \text{nf}^{\hookrightarrow_1 \cup \hookrightarrow_2} \subseteq \text{nf}^{\longrightarrow_{\beta 12}} \subseteq \text{nf}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$ .

**Theorem 13**  $\text{BN}^{\hookrightarrow} \subseteq \text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$

**Proof:** We apply Theorem 28, since  $\text{nf}^{\hookrightarrow} \subseteq \text{nf}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$  and clearly

$$(\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}) \cup \hookrightarrow = \longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$$

□

**Theorem 14**  $\text{BN}^{\beta} \subseteq \text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$

**Proof:** Since  $\longrightarrow_{\beta}$  strongly simulates  $\hookrightarrow$  through  $\phi$ , we have  $\phi^{-1}(\text{BN}^{\beta}) \subseteq \text{BN}^{\hookrightarrow} \subseteq \text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}$ . Hence  $\phi(\phi^{-1}(\text{BN}^{\beta})) \subseteq \phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}})$ . Since  $\phi$  is surjective,  $\text{BN}^{\beta} = \phi(\phi^{-1}(\text{BN}^{\beta}))$ . Hence  $\text{BN}^{\beta} \subseteq \phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}})$ . Also,  $\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}$  strongly simulates  $\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}$  through  $\phi^{-1}$ , so  $\phi(\text{BN}^{\longrightarrow_{\text{assoc,act}}^* \cdot \longrightarrow_{\beta 12}}) \subseteq \text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$ . □

**Theorem 15**  $\text{SN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$

**Proof:** First, from Lemma 19,  $\text{BN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}} \subseteq \text{SN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$ . Then from Lemma 1,  $\longrightarrow_{\text{assoc}}$  is terminating and hence  $\text{SN}^{\text{assoc}}$  is stable under  $\longrightarrow_{\beta}$ . Hence we can apply Lemma 24 to get  $\text{SN}^{\text{assoc}\beta} = \text{SN}^{\longrightarrow_{\text{assoc}}^* \cdot \longrightarrow_{\beta}}$ . From the previous theorem we thus have  $\text{BN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$ . Now, noticing that  $\beta$ -reduction in  $\lambda$ -calculus is finitely branching, Lemma 18 gives  $\text{BN}^{\beta} = \text{SN}^{\beta}$  and thus  $\text{SN}^{\beta} \subseteq \text{SN}^{\text{assoc}\beta}$ . □

## References

[Mog88] E. Moggi. Computational lambda-calculus and monads. Report ECS-LFCS-88-66, University of Edinburgh, Edinburgh, Scotland, October 1988.

## A Reminder: Notations, Definitions and Basic Results

### Definition 5 (Relations)

- We denote the composition of relations by  $\cdot$ , the identity relation by  $\text{Id}$ , and the inverse of a relation by  $^{-1}$ .
- If  $\mathcal{D} \subseteq \mathcal{A}$ , we write  $\mathcal{R}(\mathcal{D})$  for  $\{M \in \mathcal{B} \mid \exists N \in \mathcal{D}, N\mathcal{R}M\}$ , or equivalently  $\bigcup_{N \in \mathcal{D}} \{M \in \mathcal{B} \mid N\mathcal{R}M\}$ . When  $\mathcal{D}$  is the singleton  $\{M\}$ , we write  $\mathcal{R}(M)$  for  $\mathcal{R}(\{M\})$ .
- We say that a relation  $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$  is *total* if  $\mathcal{R}^{-1}(\mathcal{B}) = \mathcal{A}$ .

**Remark 16** Composition is associative, and identity relations are neutral for the composition operation.

### Definition 6 (Reduction relation)

- A *reduction relation* on  $\mathcal{A}$  is a relation from  $\mathcal{A}$  to  $\mathcal{A}$ .
- Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$ , we define the set of  $\rightarrow$ -*reducible forms* (or just *reducible forms* when the relation is clear) as  $\text{rf}^{\rightarrow} := \{M \in \mathcal{A} \mid \exists N \in \mathcal{A}, M \rightarrow N\}$ . We define the set of *normal forms* as  $\text{nf}^{\rightarrow} := \{M \in \mathcal{A} \mid \nexists N \in \mathcal{A}, M \rightarrow N\}$ .
- Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$ , we write  $\leftarrow$  for  $\rightarrow^{-1}$ , and we define  $\rightarrow^n$  by induction on the natural number  $n$  as follows:
  - $\rightarrow^0 := \text{Id}$
  - $\rightarrow^{n+1} := \rightarrow \cdot \rightarrow^n (= \rightarrow^n \cdot \rightarrow)$
  - $\rightarrow^+$  denotes the transitive closure of  $\rightarrow$  (i.e.  $\rightarrow^+ := \bigcup_{n \geq 1} \rightarrow^n$ ).
  - $\rightarrow^*$  denotes the transitive and reflexive closure of  $\rightarrow$  (i.e.  $\rightarrow^* := \bigcup_{n \geq 0} \rightarrow^n$ ).
  - $\leftrightarrow$  denotes the symmetric closure of  $\rightarrow$  (i.e.  $\leftrightarrow := \leftarrow \cup \rightarrow$ ).
  - $\leftrightarrow^*$  denotes the transitive, reflexive and symmetric closure of  $\rightarrow$ .
- An *equivalence relation* on  $\mathcal{A}$  is a transitive, reflexive and symmetric reduction relation on  $\mathcal{A}$ , i.e. a relation  $\rightarrow = \leftrightarrow^*$ , hence denoted more often by  $\sim, \equiv, \dots$
- Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$  and a subset  $\mathcal{B} \subseteq \mathcal{A}$ , the *closure of  $\mathcal{B}$  under  $\rightarrow$*  is  $\rightarrow^*(\mathcal{B})$ .

**Definition 7 (Finitely branching relation)** A reduction relation  $\rightarrow$  on  $\mathcal{A}$  is *finitely branching* if  $\forall M \in \mathcal{A}, \rightarrow(M)$  is finite.

**Definition 8 (Stability)** Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$ , we say that a subset  $\mathcal{T}$  of  $\mathcal{A}$  is  $\rightarrow$ -*stable* (or *stable under  $\rightarrow$* ) if  $\rightarrow(\mathcal{T}) \subseteq \mathcal{T}$ .

### Definition 9 (Strong simulation)

Let  $\mathcal{R}$  be a relation between two sets  $\mathcal{A}$  and  $\mathcal{B}$ , respectively equipped with the reduction relations  $\rightarrow_{\mathcal{A}}$  and  $\rightarrow_{\mathcal{B}}$ .

$\rightarrow_{\mathcal{B}}$  *strongly simulates*  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$  if  $(\mathcal{R}^{-1} \cdot \rightarrow_{\mathcal{A}}) \subseteq (\rightarrow_{\mathcal{B}}^+ \cdot \mathcal{R}^{-1})$ .

### Remark 17

1. If  $\rightarrow_{\mathcal{B}}$  strongly simulates  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ , and if  $\rightarrow_{\mathcal{B}} \subseteq \rightarrow'_{\mathcal{B}}$  and  $\rightarrow'_{\mathcal{A}} \subseteq \rightarrow_{\mathcal{A}}$ , then  $\rightarrow'_{\mathcal{B}}$  strongly simulates  $\rightarrow'_{\mathcal{A}}$  through  $\mathcal{R}$ .

2. If  $\rightarrow_{\mathcal{B}}$  strongly simulates  $\rightarrow_{\mathcal{A}}$  and  $\rightarrow'_{\mathcal{A}}$  through  $\mathcal{R}$ , then it also strongly simulates  $\rightarrow_{\mathcal{A}} \cdot \rightarrow'_{\mathcal{A}}$  through  $\mathcal{R}$ .
3. Hence, if  $\rightarrow_{\mathcal{B}}$  strongly simulates  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ , then it also strongly simulates  $\rightarrow_{\mathcal{A}}^+$  through  $\mathcal{R}$ .

**Definition 10 (Patriarchal)** Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$ , we say that

- a subset  $\mathcal{T}$  of  $\mathcal{A}$  is  $\rightarrow$ -patriarchal (or just *patriarchal* when the relation is clear) if  $\forall N \in \mathcal{A}, \rightarrow(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$ .
- a predicate  $P$  on  $\mathcal{A}$  is *patriarchal* if  $\{M \in \mathcal{A} \mid P(M)\}$  is *patriarchal*.

**Definition 11 (Normalising elements)** Given a reduction relation  $\rightarrow$  on  $\mathcal{A}$ , the set of  $\rightarrow$ -strongly normalising elements is

$$SN^{\rightarrow} := \bigcap_{\mathcal{T} \text{ is patriarchal}} \mathcal{T}$$

**Definition 12 (Bounded elements)** The set of  $\rightarrow$ -bounded elements is defined as

$$BN^{\rightarrow} := \bigcup_{n \geq 0} BN_n^{\rightarrow}$$

where  $BN_n^{\rightarrow}$  is defined by induction on the natural number  $n$  as follows:

$$\begin{aligned} BN_0^{\rightarrow} &:= nf^{\rightarrow} \\ BN_{n+1}^{\rightarrow} &:= \{M \in \mathcal{A} \mid \exists n' \leq n, \rightarrow(M) \subseteq BN_{n'}^{\rightarrow}\} \end{aligned}$$

**Lemma 18** *If  $\rightarrow$  is finitely branching, then  $BN^{\rightarrow}$  is patriarchal. As a consequence,  $BN^{\rightarrow} = SN^{\rightarrow}$ .*

**Lemma 19**

1. If  $n < n'$  then  $BN_n^{\rightarrow} \subseteq BN_{n'}^{\rightarrow} \subseteq BN^{\rightarrow}$ . In particular,  $nf^{\rightarrow} \subseteq BN_n^{\rightarrow} \subseteq BN^{\rightarrow}$ .
2.  $BN^{\rightarrow} \subseteq SN^{\rightarrow}$ .

**Lemma 20**

1.  $SN^{\rightarrow}$  is patriarchal.
2. If  $M \in BN^{\rightarrow}$  then  $\rightarrow(M) \subseteq BN^{\rightarrow}$ .  
If  $M \in SN^{\rightarrow}$  then  $\rightarrow(M) \subseteq SN^{\rightarrow}$ .

**Theorem 21 (Induction principle)** *Given a predicate  $P$  on  $\mathcal{A}$ , suppose  $\forall M \in SN^{\rightarrow}, (\forall N \in \rightarrow(M), P(N)) \Rightarrow P(M)$ . Then  $\forall M \in SN^{\rightarrow}, P(M)$ .*

*When we use this theorem to prove a statement  $P(M)$  for all  $M$  in  $SN^{\rightarrow}$ , we just add  $(\forall N \in \rightarrow(M), P(N))$  to the assumptions, which we call the induction hypothesis.*

*We say that we prove the statement by induction in  $SN^{\rightarrow}$ .*

**Lemma 22**

1. If  $\rightarrow_1 \subseteq \rightarrow_2$ , then  $nf^{\rightarrow_1} \supseteq nf^{\rightarrow_2}$ ,  $SN^{\rightarrow_1} \supseteq SN^{\rightarrow_2}$ ,  
and for all  $n$ ,  $BN_n^{\rightarrow_1} \supseteq BN_n^{\rightarrow_2}$ .

2.  $nf^{\rightarrow} = nf^{\rightarrow+}$ ,  $SN^{\rightarrow} = SN^{\rightarrow+}$ , and for all  $n$ ,  $BN_n^{\rightarrow+} = BN_n^{\rightarrow}$ .

Notice that this result enables us to use a stronger induction principle: in order to prove  $\forall M \in SN^{\rightarrow}, P(M)$ , it now suffices to prove

$$\forall M \in SN^{\rightarrow}, (\forall N \in \rightarrow^+(M), P(N)) \Rightarrow P(M)$$

This induction principle is called the *transitive induction in  $SN^{\rightarrow}$* .

**Theorem 23 (Strong normalisation by strong simulation)** *Let  $\mathcal{R}$  be a relation between  $\mathcal{A}$  and  $\mathcal{B}$ , equipped with the reduction relations  $\rightarrow_{\mathcal{A}}$  and  $\rightarrow_{\mathcal{B}}$ .*

*If  $\rightarrow_{\mathcal{B}}$  strongly simulates  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ , then  $\mathcal{R}^{-1}(SN^{\rightarrow_{\mathcal{B}}}) \subseteq SN^{\rightarrow_{\mathcal{A}}}$ .*

**Lemma 24** *Given two reduction relations  $\rightarrow_1, \rightarrow_2$ , suppose that  $SN^{\rightarrow_1}$  is stable under  $\rightarrow_2$ . Then  $SN^{\rightarrow_1 \cup \rightarrow_2} = SN^{\rightarrow_1} \cap SN^{\rightarrow_2}$ .*

## B Strong adjournment

**Definition 13** Suppose  $\rightarrow_{\mathcal{A}}$  is a reduction relation on  $\mathcal{A}$ ,  $\rightarrow_{\mathcal{B}}$  is a reduction relation on  $\mathcal{B}$ ,  $\mathcal{R}$  is a relation from  $\mathcal{A}$  to  $\mathcal{B}$ .

$\rightarrow_{\mathcal{B}}$  simulates the reduction lengths of  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$  if

$$\forall k, \forall M, N \in \mathcal{A}, \forall P \in \mathcal{B}, M \rightarrow_{\mathcal{A}}^k N \wedge M \mathcal{R} P \Rightarrow \exists Q \in \mathcal{B}, P \rightarrow_{\mathcal{B}}^k Q$$

**Lemma 25** *Suppose  $\rightarrow_{\mathcal{A}}$  is a reduction relation on  $\mathcal{A}$ ,  $\rightarrow_{\mathcal{B}}$  is a reduction relation on  $\mathcal{B}$ ,  $\mathcal{R}$  is a relation from  $\mathcal{A}$  to  $\mathcal{B}$ .*

*If  $\rightarrow_{\mathcal{B}}$  strongly simulates  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ , then  $\rightarrow_{\mathcal{B}}$  simulates the reduction lengths of  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ .*

**Proof:** We prove by induction on  $k$  that  $\forall k, \forall M, N \in \mathcal{A}^2, \forall P \in \mathcal{B}, M \rightarrow_{\mathcal{A}}^k N \wedge M \mathcal{R} P \Rightarrow \exists Q, P \rightarrow_{\mathcal{B}}^k Q$ .

- For  $k = 0$ : take  $Q := M = N$ .
- Suppose it is true for  $k$  and take  $M \rightarrow_{\mathcal{A}} M' \rightarrow_{\mathcal{A}}^k N$ . The strong simulation gives  $P'$  such that  $P \rightarrow_{\mathcal{B}}^+ P'$  and  $M' \mathcal{R} P'$ . The induction hypothesis gives  $Q'$  such that  $P' \rightarrow_{\mathcal{B}}^k Q'$ . Then it suffices to take the prefix  $P \rightarrow_{\mathcal{B}}^{k+1} Q$  (of length  $k + 1$ ) of  $P \rightarrow_{\mathcal{B}}^+ P' \rightarrow_{\mathcal{B}}^k Q'$ .

□

**Lemma 26**  $\forall n, \forall M, (\forall k, \forall N, M \rightarrow^k N \Rightarrow k \leq n) \iff M \in BN_n^{\rightarrow}$

**Proof:** By transitive induction on  $n$ .

- For  $n = 0$ : clearly both sides are equivalent to  $M \in nf^{\rightarrow}$ .
- Suppose it is true for all  $i \leq n$ .

Suppose  $\forall k, \forall N, M \rightarrow^k N \Rightarrow k \leq n + 1$ . Then take  $M \rightarrow M'$  and assume  $M' \rightarrow^{k'} N'$ . We have  $M \rightarrow^{k'+1} N'$  so from the hypothesis we derive  $k' + 1 \leq n + 1$ , i.e.  $k' \leq n$ . We apply the induction hypothesis on  $M'$  and get  $M' \in BN_{n+1}^{\rightarrow}$ . By definition of  $BN_{n+1}^{\rightarrow}$  we get  $M \in BN_{n+1}^{\rightarrow}$ .

Conversely, suppose  $M \in BN_{n+1}^{\rightarrow}$  and  $M \rightarrow^k N$ . We must prove that  $k \leq n + 1$ . If  $k = 0$  we are done. If  $k = k' + 1$  we have  $M \rightarrow M' \rightarrow^{k'} N$ ; by definition of  $BN_{n+1}^{\rightarrow}$  there is  $i \leq n$  such that  $M' \in BN_i^{\rightarrow}$ , and by induction hypothesis we have  $k' \leq i$ ; hence  $k = k' + 1 \leq i + 1 \leq n + 1$ .

□



**Theorem 27** Suppose  $\rightarrow_{\mathcal{A}}$  is a reduction relation on  $\mathcal{A}$ ,  $\rightarrow_{\mathcal{B}}$  is a reduction relation on  $\mathcal{B}$ ,  $\mathcal{R}$  is a relation from  $\mathcal{A}$  to  $\mathcal{B}$ .

If  $\rightarrow_{\mathcal{B}}$  simulates the reduction lengths of  $\rightarrow_{\mathcal{A}}$  through  $\mathcal{R}$ , then

$$\forall n, \mathcal{R}^{-1}(BN_n^{\rightarrow_{\mathcal{B}}}) \subseteq BN_n^{\rightarrow_{\mathcal{A}}} \quad (\subseteq SN^{\rightarrow_{\mathcal{A}}})$$

**Proof:** Suppose  $N \in BN_n^{\rightarrow_{\mathcal{B}}}$  and  $M\mathcal{R}N$ . If  $M \xrightarrow{\mathcal{A}}^k M'$  then by simulation  $N \xrightarrow{\mathcal{B}}^k N'$  so by Lemma 26 we have  $k \leq n$ . Hence by (the other direction of) Lemma 26 we have  $M \in BN_n^{\rightarrow_{\mathcal{A}}}$ .  $\square$

**Definition 14** Let  $\rightarrow_1$  and  $\rightarrow_2$  be two reduction relations on  $\mathcal{A}$ .

The relation  $\rightarrow_1$  can be *strongly adjourned with respect to*  $\rightarrow_2$  if

whenever  $M \rightarrow_1 N \rightarrow_2 P$  there exists  $Q$  such that  $M \rightarrow_2 Q(\rightarrow_1 \cup \rightarrow_2)^+ P$ .

**Theorem 28** Let  $\rightarrow_1$  and  $\rightarrow_2$  be two reduction relations on  $\mathcal{A}$ . If  $\text{nf}^{\rightarrow_2} \subseteq \text{nf}^{\rightarrow_1}$  and  $\rightarrow_1$  can be strongly adjourned with respect to  $\rightarrow_2$  then  $BN^{\rightarrow_2} \subseteq BN^{\rightarrow_1 \cup \rightarrow_2}$ .

**Proof:** From Theorem 27, it suffices to show that  $\rightarrow_2$  simulates the reduction lengths of  $\rightarrow_1 \cup \rightarrow_2$  through the identity. We show by induction on  $k$  that

$$\forall k, \forall M, N, M(\rightarrow_1 \cup \rightarrow_2)^k N \Rightarrow \exists Q, M \rightarrow_2^k Q$$

- For  $k = 0$ : take  $Q := M$
- For  $k = 1$ : If  $M \rightarrow_2 N$  take  $Q := N$ ; if  $M \rightarrow_1 N$  use the hypothesis  $\text{nf}^{\rightarrow_2} \subseteq \text{nf}^{\rightarrow_1}$  to produce  $Q$  such that  $M \rightarrow_2 Q$ .
- Suppose it is true for  $k + 1$  and take  $M(\rightarrow_1 \cup \rightarrow_2)P(\rightarrow_1 \cup \rightarrow_2)^{k+1}N$ .

The induction hypothesis provides  $T$  such that  $P \rightarrow_2^{k+1} T$ , in other words  $P \rightarrow_2 S \rightarrow_2^k T$ .

If  $M \rightarrow_2 P$  we are done. If  $M \rightarrow_1 P$  we use the hypothesis of adjournment to transform  $M \rightarrow_1 P \rightarrow_2 S$  into  $M \rightarrow_2 P'(\rightarrow_1 \cup \rightarrow_2)^+ S$ . Take the prefix  $P'(\rightarrow_1 \cup \rightarrow_2)^{k+1}R$  (of length  $k + 1$ ) of  $P'(\rightarrow_1 \cup \rightarrow_2)^+ S \rightarrow_2^k T$ , and apply on this prefix the induction hypothesis to get  $P' \rightarrow_2^{k+1} R$ . We thus get  $M \rightarrow_2^{k+2} R$ .

$\square$

## C Proofs

**Lemma 7**  $\rightsquigarrow_2 \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \xrightarrow{\beta_{12}}^+$

**Proof:** By induction on the reduction step  $\hookrightarrow$ .

- For the base case where the  $\beta\bar{\kappa}$ -reduction is a  $\beta_2$ -reduction, we have  $M \rightsquigarrow_2 (\bar{\lambda}x.N) P \hookrightarrow \{P/x\}N$  with  $x \in \text{FV}(N)$  or  $P$  has no marked redex as a subterm. We do a case analysis on the reduction step  $M \rightsquigarrow_2 (\bar{\lambda}x.N) P$ .  
 If  $M = (\bar{\lambda}x.N') P \rightsquigarrow_2 (\bar{\lambda}x.N) P$  because  $N' \xrightarrow{\beta_{12}} N$  then  $(\bar{\lambda}x.N') P \hookrightarrow \{P/x\}N' \xrightarrow{\beta_{12}} \{P/x\}N$ .  
 If  $M = (\bar{\lambda}x.N) P' \rightsquigarrow_2 (\bar{\lambda}x.N) P$  because  $P' \rightsquigarrow_2 P$ , then it means that  $P$  has a marked redex as a subterm, so we must have  $x \in \text{FV}(N)$ . Hence  $(\bar{\lambda}x.N) P' \hookrightarrow \{P'/x\}N \xrightarrow{\beta_{12}}^+ \{P/x\}N$ .

- For the base case where the  $\beta\bar{K}$ -reduction is a  $\beta 1$ -reduction, we have  $M \rightsquigarrow_2 (\lambda x.N) P \hookrightarrow \{P/x\}N$  with  $x \in \text{FV}(N)$  or  $P$  has no marked redex as a subterm. We do a case analysis on the reduction step  $M \rightsquigarrow_2 (\lambda x.N) P$ .  
If  $M = M' P \rightsquigarrow_2 (\lambda x.N) P$  because  $M' \rightsquigarrow_2 \lambda x.N$  then  $M'$  must be of the form  $\lambda x.M''$  with  $M'' \rightsquigarrow_2 N$ . Then  $(\lambda x.M'') P \hookrightarrow \{P/x\}M''$  (in case  $P$  has a marked subterm, notice that  $x \in \text{FV}(N) \subseteq \text{FV}(M'')$ ), and  $\{P/x\}M'' \xrightarrow{\beta 12} \{P/x\}N$ .  
If  $M = (\lambda x.N) P' \rightsquigarrow_2 (\lambda x.N) P$  because  $P' \rightsquigarrow_2 P$ , then it means that  $P$  has a marked redex as a subterm, so we must have  $x \in \text{FV}(N)$ . Hence  $(\lambda x.N) P' \hookrightarrow \{P'/x\}N \xrightarrow{\beta 12}^+ \{P/x\}N$ .
- The closure under  $\lambda$  is straightforward.
- For the closure under application, left-hand side, we have  $M \rightsquigarrow_2 N P \hookrightarrow N' P$  with  $N \hookrightarrow N'$ . We do a case analysis on the reduction step  $M \rightsquigarrow_2 N P$ .  
If  $M = M' P \rightsquigarrow_2 N P$  with  $M' \rightsquigarrow_2 N$ , the induction hypothesis gives  $M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N'$  and the weak contextual closure gives  $M' P \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N' P$ .  
If  $M = N P' \rightsquigarrow_2 N P$  with  $P' \rightsquigarrow_2 P$ , we can also derive  $N P' \hookrightarrow N' P' \xrightarrow{\beta 12} N' P$ .
- For the closure under application, right-hand side, we have  $M \rightsquigarrow_2 N P \hookrightarrow N P'$  with  $P \hookrightarrow P'$ . We do a case analysis on the reduction step  $M \rightsquigarrow_2 N P$ .  
If  $M = M' P \rightsquigarrow_2 N P$  with  $M' \rightsquigarrow_2 N$ , we can also derive  $M' P \hookrightarrow M' P' \xrightarrow{\beta 12} N P'$ .  
If  $M = N M' \rightsquigarrow_2 N P$  with  $M' \rightsquigarrow_2 P$ , the induction hypothesis gives  $M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ P'$  and the weak contextual closure gives  $N M' \hookrightarrow \cdot \xrightarrow{\beta 12}^+ N P'$ .
- For the closure under marked redex we have  $M \rightsquigarrow_2 (\bar{\lambda}x.P) N \hookrightarrow (\bar{\lambda}x.P) N'$  with  $N \hookrightarrow N'$ . We do a case analysis on the reduction step  $M \rightsquigarrow_2 (\bar{\lambda}x.P) N$ .  
If  $M = (\bar{\lambda}x.P') N \rightsquigarrow_2 (\bar{\lambda}x.P) N$  because  $P' \xrightarrow{\beta 12} P$ , we can also derive  $(\bar{\lambda}x.P') N \hookrightarrow (\bar{\lambda}x.P') N' \xrightarrow{\beta 12} (\bar{\lambda}x.P) N'$ .  
If  $M = (\bar{\lambda}x.P) M' \rightsquigarrow_2 (\bar{\lambda}x.P) N$  with  $M' \rightsquigarrow_2 N$ , the induction hypothesis gives  $M' \hookrightarrow Q \xrightarrow{\beta 12}^+ N'$  and the weak contextual closure gives  $(\bar{\lambda}x.P) M' \hookrightarrow (\bar{\lambda}x.P) Q \xrightarrow{\beta 12}^+ (\bar{\lambda}x.P) N'$ .

□

**Lemma 10**  $\xrightarrow{\text{assocact}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \xrightarrow{\text{assocact}^*}$

**Proof:** By induction on the reduction step  $\hookrightarrow$ .

- For the first base case, we have  $M \xrightarrow{\text{assocact}} (\lambda x.N) P \hookrightarrow \{P/x\}N$  with  $x \in \text{FV}(N)$  or  $P$  has no marked subterm. Since root  $\overline{\text{assocact}}$ -reduction produces neither  $\lambda$ -abstractions nor applications at the root, note that  $M$  has to be of the form  $(\lambda x.N') P'$ , with either  $N' \xrightarrow{\text{assocact}} N$  (and  $P' = P$ ) or  $P' \xrightarrow{\text{assocact}} P$  (and  $N' = N$ ). In both cases,  $x \in \text{FV}(N) \subseteq \text{FV}(N')$  or  $P'$  has no marked subterm, so we also have  $(\lambda x.N') P' \hookrightarrow \{P'/x\}N' \xrightarrow{\text{assocact}^*} \{P/x\}N$ .
- For the second base case, we have  $M \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P \hookrightarrow \{P/x\}N$  with  $x \in \text{FV}(N)$  or  $P$  has no marked subterm. We do a case analysis on  $M \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P$ .

If  $M = (\bar{\lambda}x'.M_1) (\bar{\lambda}x.M_2) P \xrightarrow{\text{assoc}} (\bar{\lambda}x.(\bar{\lambda}x'.M_1) M_2) P$  with  $N = (\bar{\lambda}x'.M_1) M_2$ , we also have  $M = (\bar{\lambda}x'.M_1) (\bar{\lambda}x.M_2) P \hookrightarrow (\bar{\lambda}x'.M_1) \{P/x\} M_2 = \{P/x\} N$ .

If  $M = (\lambda x.N) P \xrightarrow{\text{act}} (\bar{\lambda}x.N) P$  then  $M \hookrightarrow \{P/x\} N$ .

If  $M = (\bar{\lambda}x.N') P' \xrightarrow{\text{assocact}} (\bar{\lambda}x.N) P$  with either  $N' \xrightarrow{\text{assocact}} N$  (and  $P' = P$ ) or  $P' \xrightarrow{\text{assocact}} P$  (and  $N' = N$ ), we have, in both cases,  $x \in \text{FV}(N) \subseteq \text{FV}(N')$  or  $P'$  has no marked subterm, so we also have  $(\lambda x.N') P' \hookrightarrow \{P'/x\} N' \xrightarrow{\text{assocact}^*} \{P/x\} N$ .

- The closure under  $\lambda$  is straightforward.

- For the closure under application, left-hand side, we have  $Q \xrightarrow{\text{assocact}} M N \hookrightarrow M' N$  with  $M \hookrightarrow M'$ . We do a case analysis on  $Q \xrightarrow{\text{assocact}} M N$ .

If  $Q = M'' N \xrightarrow{\text{assocact}} M N$  with  $M'' \xrightarrow{\text{assocact}} M$ , the induction hypothesis provides  $M'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M'$  so  $M'' N \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M' N$ .

If  $Q = M N' \xrightarrow{\text{assocact}} M N$  with  $N' \xrightarrow{\text{assocact}} N$ , we also have  $M N' \hookrightarrow M' N' \xrightarrow{\text{assocact}} M' N$ .

- For the closure under application, right-hand side, we have  $Q \xrightarrow{\text{assocact}} M N \hookrightarrow M N'$  with  $N \hookrightarrow N'$ . We do a case analysis on  $Q \xrightarrow{\text{assocact}} M N$ .

If  $Q = M' N \xrightarrow{\text{assocact}} M N$  with  $M' \xrightarrow{\text{assocact}} M$ , we also have  $M' N \hookrightarrow M' N' \xrightarrow{\text{assocact}} M N'$ .

If  $Q = M N'' \xrightarrow{\text{assocact}} M N$  with  $N'' \xrightarrow{\text{assocact}} N$ , the induction hypothesis provides  $N'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} N'$  so  $M N'' \hookrightarrow \cdot \xrightarrow{\text{assocact}^*} M N'$ .

- For the closure under marked redex, the  $\hookrightarrow$ -reduction can only come from the right-hand side because of the weak contextual closure ( $\hookrightarrow$  does not reduce under  $\bar{\lambda}$ ), so we have  $Q \xrightarrow{\text{assocact}} (\bar{\lambda}y.M) P \hookrightarrow (\bar{\lambda}y.M) P'$  with  $P \hookrightarrow P'$ . We do a case analysis on  $Q \xrightarrow{\text{assocact}} (\bar{\lambda}y.M) P$ .

If  $Q = (\bar{\lambda}x.M') (\bar{\lambda}y.N) P \xrightarrow{\text{assoc}} (\bar{\lambda}y.(\bar{\lambda}x.M') N) P$  with  $M = (\bar{\lambda}x.M') N$ , we also have  $Q = (\bar{\lambda}x.M') (\bar{\lambda}y.N) P \hookrightarrow (\bar{\lambda}x.M') (\bar{\lambda}y.N) P' \xrightarrow{\text{assoc}} (\bar{\lambda}y.(\bar{\lambda}x.M') N) P'$ .

If  $Q = (\lambda y.M) P \xrightarrow{\text{act}} (\bar{\lambda}y.M) P$ , then we also have  $Q = (\lambda y.M) P \hookrightarrow (\lambda y.M) P' \xrightarrow{\text{act}} (\bar{\lambda}y.M) P'$ .

□