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*Theoretical and numerical analysis of the Webster
Lokshin model*

H. Haddar — D. Matignon

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Theoretical and numerical analysis of the Webster Lokshin model

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Abstract: Acoustic waves travelling in a duct with viscothermal losses at the wall and radiating conditions at both ends obey a Webster-Lokshin model that involves fractional time-derivatives in the domain and dynamical boundary conditions. This system can be interpreted as the coupling of three subsystems: a wave equation, a diffusive realization of the pseudo-differential time-operator and a dissipative realization of the impedance, thanks to the Kalman-Yakubovich-Popov lemma.

Existence and uniqueness of strong solutions of the system are proved, using the Hille-Yosida theorem.

Moreover, numerical schemes are derived and their stability is analyzed using energy methods; many simulation results are presented, which describe the behaviour of the model for different values of the parameters.

Key-words:

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Analyse thorique et numrique du modle de Webster Lokshin

Résumé : Les ondes acoustiques qui se propagent dans un pavillon dont la paroi est le site de pertes visco-thermiques et dont les deux extrémités sont sujettes des conditions de rayonnement obissent un modèle de Webster-Lokshin, lequel fait intervenir des drives fractionnaires en temps dans le milieu et des conditions aux limites dynamiques. Ce système peut s'interpréter comme le couplage de trois sous-systèmes : une équation des ondes, une réalisation diffusive de l'opérateur pseudo-différentiel en temps, et une réalisation dissipative de l'impédance par le lemme de Kalman-Yakubovich-Popov.

En utilisant le théorème de Hille-Yosida, l'existence et l'unicité des solutions fortes de ce système sont établies.

De plus, des schémas numériques sont proposés et leur stabilité est analysée en utilisant des techniques d'énergie ; de nombreuses simulations numériques viennent illustrer le comportement du modèle pour diverses valeurs des paramètres.

Mots-clés :

Chapter 1

Introduction

The Lokshin model originally presented in [10, 11] and referred to in [5] in a half-space has then been derived in a bounded space in [17], and modified in [8].

In the case of constant-coefficients, it has been solved analytically and analyzed in both time-domain and frequency-domain in [13], while the principle of an energy analysis has been given in [14].

The problem at stake here is in a bounded domain, with non-constant coefficients (due to Webster equation for horns and space-varying coefficients for the viscothermal effects). Existence and uniqueness of strong solutions of the free evolution problem is proved in an energy space (see also [20]), and the coupling between passive subsystems is used as main method of analysis, as in [12, ch. 5].

In chapter 2, we begin the theoretical study with the formulation of the problem in §2.1; a key point is the reformulation as coupled first-order systems in §2.1.3, thanks to diffusive realizations of fractional differential operators; the analysis of the global system follows in §2.2.

The following slight extensions or wider perspectives are in view:

- use some infinite-dimensional analogue of the KYP lemma for some more realistic impedances, as in [3];
- study of the boundary-controlled equation;
- proof of the asymptotic stability of the global system, using a spectral condition on the generator of the semigroup, as in [16], rather than LaSalle's invariance principle, following [12, ch. 3], which does require the precompactness of strong trajectories in the energy space.

In chapter 3, the numerical analysis of this system is presented, based on finite difference methods with technical specificities due to diffusive representations. The properties of the continuous model are displayed and explored in the discrete domain, and illustrated on many simulation examples in § 3.3.

Chapter 2

Theoretical analysis

2.1 Mathematical formulation of the Webster Lokshin model

Consider an axi-symmetric duct between $z = 0$ and $z = 1$ with cross section radius $r(z)$ (satisfying $r \geq r_0 > 0$), then the velocity potential ϕ (with appropriate scaling) satisfies the following equation:

$$\partial_t^2 \phi + (\eta(z) \partial_t^\alpha + \varepsilon(z) \partial_t^{-\beta}) \partial_t \phi - \frac{1}{r^2(z)} \partial_z (r^2(z) \partial_z \phi) = 0, \quad (2.1)$$

for some $\alpha, \beta \in]0, 1[$ and $\varepsilon, \eta \in L^\infty(0, 1; \mathbb{R}^+)$. The terms in ∂_t^α and $\partial_t^{-\beta}$ model the effect of viscous and thermal losses at the lateral walls.

We can reformulate (2.1) as a first order system in the (p, v) variables, where $p = \partial_t \phi$ is the pressure, and $v = r^2 \partial_z \phi$ is the volume velocity:

$$\partial_t p = -\frac{1}{r^2} \partial_z v - \varepsilon \partial_t^{-\beta} p - \eta \partial_t^\alpha p, \quad (2.2)$$

$$\partial_t v = -r^2 \partial_z p, \quad (2.3)$$

To take into account the interaction with the exterior domain, one can add dynamical boundary conditions at $z = 0, 1$ that are of the type

$$\hat{p}_i(s) = \mp \mathcal{Z}_i(s) \hat{v}_i(s) \quad \text{for } i = 0, 1. \quad (2.4)$$

Conditions (2.4) are formulated in the Laplace domain, with shorthand notation $p_i(t) = p(z = i, t)$ for $i = 0, 1$; the *acoustic impedances* $\mathcal{Z}_i(s)$ are *strictly positive real* in the sense of [12, ch. 5], that is $\Re(\mathcal{Z}_i(s)) > 0, \forall s, \Re(s) \geq 0$.

The system (2.2)-(2.3)-(2.4) can be transformed into a first order system in time, using appropriate realizations for the pseudo-differential operators involved in this model:

- dissipative realizations for the positive-real impedances, using Kalman-Yakubovich-Popov lemma in finite dimension, are recalled in § 2.1.1,

- dissipative realizations for positive pseudo-differential time-operators of diffusive type, such as $\partial_t^{-\beta}$ and ∂_t^α , are presented in § 2.1.2.

2.1.1 Dissipative realizations for positive-real impedances (Kalman-Yakubovich-Popov lemma)

We restrict ourselves to impedances $\mathcal{Z}_i(s)$ of *rational* type. Thus, for a strictly positive real impedance of rational type, one can choose a *minimal* realization (A_i, B_i, C_i, d_i) with state x_i of *finite* dimension n_i ($A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, $C_i \in \mathbb{R}^{1 \times n_i}$ and $d_i \in \mathbb{R}$), such that:

$$\frac{d}{dt}x_i(t) = A_i x_i + B_i v_i(t), \quad x_i(0) = 0, \quad (2.5)$$

$$\mp p_i(t) = C_i x_i(t) + d_i v_i(t). \quad (2.6)$$

Then (following e.g. [1, 18]), there exists $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i^T > 0$, such that the following energy balance holds:

$$\mp \int_0^T p_i(t) v_i(t) dt = \frac{1}{2} (x_i^T(T) P_i x_i(T)) + \frac{1}{2} \int_0^T (x_i^T(t) \quad v_i(t)) \mathcal{M}_i \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix} dt. \quad (2.7)$$

$$\text{with } \mathcal{M}_i = \begin{pmatrix} -A_i^T P_i - P_i A_i & C_i^T - P_i B_i \\ C_i - B_i^T P_i & 2d_i \end{pmatrix} \geq 0.$$

The right hand side of (2.7) is split into two terms, a storage function evaluated at time T only, proportional to $\|x_i(T)\|^2 = x_i^T(T) P_i x_i(T)$, and a dissipated energy on the time interval $(0, T)$, which involves the non-negative symmetric matrix $\mathcal{M}_i \in \mathbb{R}^{(n_i+1) \times (n_i+1)}$. We denote $E_x(t) := \frac{1}{2} \|x_0(t)\|^2 + \frac{1}{2} \|x_1(t)\|^2 = \frac{1}{2} x_0^T(t) P_0 x_0(t) + \frac{1}{2} x_1^T(t) P_1 x_1(t)$.

2.1.2 Dissipative realizations for positive pseudo-differential time-operators of diffusive type

Let us recall that the operator $\partial_t^{-\beta}$ is the causal Riemann-Liouville fractional integral of order β ; it is a pseudo-differential time-operator, the symbol of which is $s^{-\beta}$, with $0 < \beta < 1$. It can also be seen as a convolution operator $\partial_t^{-\beta} p(t) = (h_\beta \star p)(t)$ with causal kernel defined by:

$$h_\beta(t) := \frac{1}{\Gamma(\beta)} t^{\beta-1} \quad \text{for } t > 0,$$

which involves the Euler Γ function. This latter kernel can be decomposed onto purely decaying exponentials as

$$h_\beta(t) = \int_0^{+\infty} e^{-\xi t} dM_\beta(\xi) \quad \text{for } t > 0,$$

where $dM_\beta(\xi) = \mu_\beta(\xi) d\xi$ with density $\mu_\beta(\xi) = \frac{\sin(\beta\pi)}{\pi} \xi^{-\beta}$.

Now ∂_t^α is the fractional derivative of order α , it is a pseudo-differential time-operator, the symbol of which is s^α , with $0 < \alpha < 1$. It can also be defined as the causal convolutive inverse of $\partial_t^{-\alpha}$; hence $\partial_t^\alpha p(t) = \frac{d}{dt}(h_{1-\alpha} \star p)(t)$, where the time derivative must be understood in the sense of distributions.

We shall now introduce the diffusive realization of the operator $\partial_t^{-\beta}$. This realization also applies to any convolution operator with kernel of the form $h(t) = \int_0^{+\infty} e^{-\xi t} dM$, where M is a positive measure on \mathbb{R}^+ , such that $\int_0^{+\infty} \frac{dM(\xi)}{1+\xi} < +\infty$. We refer to [20, § 5.] for the treatment of completely monotone kernels, and [15] and references therein for links between diffusive representations and fractional differential operators.

The following functional spaces will be of interest in the sequel: $H_\beta = L^2(\mathbb{R}^+, dM_\beta)$, $V_\beta = L^2(\mathbb{R}^+, (1+\xi) dM_\beta)$, and $\tilde{V}_\beta = L^2(\mathbb{R}^+, \xi dM_\beta)$. We also introduce the notation $c_\beta = \int_0^{+\infty} \frac{dM_\beta}{1+\xi} < +\infty$.

First diffusive representations.

Consider the dynamical system with input $p \in L^2(0, T)$ and output $\theta \in L^2(0, T)$:

$$\begin{aligned} \partial_t \varphi(\xi, t) &= -\xi \varphi(\xi, t) + p(t) \quad \text{with} \quad \varphi(\xi, 0) = 0 \quad \forall \xi \in \mathbb{R}^+, \quad (2.8) \\ \theta(t) &= \int_0^{+\infty} \varphi(\xi, t) dM_\beta(\xi). \quad (2.9) \end{aligned}$$

Then, it can easily be checked that $\theta(t) = \partial_t^{-\beta} p(t)$. The following energy balance can be proved:

$$\int_0^T p(t) \theta(t) dt = \frac{1}{2} \int_0^{+\infty} \varphi(\xi, T)^2 dM_\beta + \int_0^T \int_0^{+\infty} \xi \varphi(\xi, t)^2 dM_\beta dt. \quad (2.10)$$

Similarly to (2.7), the right hand side of (2.10) is split into two terms, a storage function evaluated at time T only, $E_\varphi(T) := \frac{1}{2} \|\varphi(T)\|_{H_\beta}^2$, and a dissipated energy on the time interval $(0, T)$.

Extended diffusive representations.

Consider now the dynamical system with input $p \in H^1(0, T)$ and output $\tilde{\theta} \in L^2(0, T)$:

$$\begin{aligned} \partial_t \tilde{\varphi}(\xi, t) &= -\xi \tilde{\varphi}(\xi, t) + p(t) \quad \text{with} \quad \tilde{\varphi}(\xi, 0) = 0 \quad \forall \xi \in \mathbb{R}^+, \quad (2.11) \\ \tilde{\theta}(t) &= \int_0^{+\infty} \partial_t \tilde{\varphi}(\xi, t) dM_{1-\alpha}(\xi) = \int_0^{+\infty} [p(t) - \xi \tilde{\varphi}(\xi, t)] dM_{1-\alpha}(\xi). \quad (2.12) \end{aligned}$$

Then, it can easily be checked that $\tilde{\theta}(t) = \partial_t^\alpha p(t)$. The following energy balance can be proved:

$$\int_0^T p(t) \tilde{\theta}(t) dt = \frac{1}{2} \int_0^{+\infty} \xi \tilde{\varphi}(\xi, T)^2 dM_{1-\alpha} + \int_0^T \int_0^{+\infty} (p - \xi \tilde{\varphi})^2 dM_{1-\alpha} dt. \quad (2.13)$$

Again, the right hand side of (2.13) is split into two terms, a storage function evaluated at time T only, $\tilde{E}_{\tilde{\varphi}}(T) := \frac{1}{2} \|\tilde{\varphi}(T)\|_{\tilde{V}_{1-\alpha}}^2$, and a dissipated energy on the time interval $(0, T)$.

2.1.3 An abstract formulation

Now, using representations (2.5)-(2.8)-(2.11), the global system (2.2)-(2.3)-(2.4) can be transformed into the first order differential equation in time

$$\frac{d}{dt}X + \mathcal{A}X = 0, \quad (2.14)$$

where $X = (x_0, x_1, p, v, \varphi, \tilde{\varphi})^T$ and

$$\mathcal{A} \begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} -A_0 x_0 - B_0 v(z=0) \\ -A_1 x_1 - B_1 v(z=1) \\ \frac{1}{r^2} \partial_z v + \varepsilon \int_0^{+\infty} \varphi dM_\beta + \eta \int_0^{+\infty} [p - \xi \tilde{\varphi}] dM_{1-\alpha} \\ r^2 \partial_z p \\ \xi \varphi - p \\ \xi \tilde{\varphi} - p \end{pmatrix}. \quad (2.15)$$

The boundary conditions $p(z=0) = -C_0 x_0 - d_0 v(z=0)$ and $p(z=1) = C_1 x_1 + d_1 v(z=1)$ must be taken into account in the functional spaces of the solutions. In the sequel, we shall analyze the well-posedness of this system. This analysis is based on the following energy balance

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_0^1 |p(z, t)|^2 r^2(z) dz + \frac{1}{2} \int_0^1 |v(z, t)|^2 r^{-2}(z) dz \right) \\ & + \frac{d}{dt} \left(E_x(t) + \int_0^1 \varepsilon(z) E_\varphi(z, t) r^2(z) dz + \int_0^1 \eta(z) \tilde{E}_{\tilde{\varphi}}(z, t) r^2(z) dz \right) \\ & = \frac{1}{2} (x_0^T \quad v(0)) \mathcal{M}_0 \begin{pmatrix} x_0 \\ v(0) \end{pmatrix} + \frac{1}{2} (x_1^T \quad v(1)) \mathcal{M}_1 \begin{pmatrix} x_1 \\ v(1) \end{pmatrix} \\ & \quad + \int_0^1 \|\varphi\|_{\tilde{V}_\beta}^2 \varepsilon r^2 dz + \int_0^1 \|p - \xi \tilde{\varphi}\|_{H_{1-\alpha}}^2 \eta r^2 dz, \end{aligned} \quad (2.16)$$

that will be proved in Theorem 2.2.1 below.

2.2 Well-posedness of the global system

We shall apply Hille-Yosida theorem in order to show existence and uniqueness of solutions to (2.14).

According to identity (2.16), the natural *energy space* for the solution X would be the following Hilbert space:

$$\mathcal{H} := \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times L_{r^2}^2 \times L_{r^{-2}}^2 \times L^2(0, 1; H_\beta; \varepsilon r^2 dz) \times L^2(0, 1; \tilde{V}_{1-\alpha}; \eta r^2 dz),$$

where $L_{r^2}^2 := L^2(0, 1; r^2(z) dz)$, $L_{r^{-2}}^2 := L^2(0, 1; r^{-2}(z) dz)$ with scalar product for $X = (x_0, x_1, p, v, \varphi, \tilde{\varphi})^T$ and $Y = (y_0, y_1, q, w, \psi, \tilde{\psi})^T$ defined by:

$$(X, Y)_{\mathcal{H}} = x_0^T P_0 y_0 + x_1^T P_1 y_1 + (p, q)_{L_{r^2}^2} + (v, w)_{L_{r^{-2}}^2} + \int_0^1 (\varphi, \psi)_{H_\beta} \varepsilon(z) r^2(z) dz + \int_0^1 (\tilde{\varphi}, \tilde{\psi})_{\tilde{V}_{1-\alpha}} \eta(z) r^2(z) dz \quad (2.17)$$

We define the Hilbert space \mathcal{V} as:

$$\mathcal{V} := \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times H_p^1 \times H_v^1 \times L^2(0, 1; V_\beta; \varepsilon r^2 dz) \times L^2(0, 1; \tilde{V}_{1-\alpha}; \eta r^2 dz),$$

where

$$H_p^1 := \left\{ p \in L_{r^2}^2, \int_0^1 [p^2 + (\partial_z p)^2] r^2(z) dz < +\infty \right\},$$

$$H_v^1 := \left\{ v \in L_{r^{-2}}^2, \int_0^1 [v^2 + (\partial_z v)^2] r^{-2}(z) dz < +\infty \right\}.$$

We set as domain of the operator \mathcal{A} , the space defined by:

$$D(\mathcal{A}) := \left\{ (x_0, x_1, p, v, \varphi, \tilde{\varphi})^T \in \mathcal{V}, \begin{cases} p(z=0) = -C_0 x_0 - d_0 v(z=0) \\ p(z=1) = C_1 x_1 + d_1 v(z=1) \\ (p - \xi \varphi) \in L^2(0, 1; H_\beta; \varepsilon r^2 dz) \\ (p - \xi \tilde{\varphi}) \in L^2(0, 1; V_{1-\alpha}; \eta r^2 dz) \end{cases} \right\}.$$

The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is well defined, it is a bounded operator, namely:

- for the first two components of equation (2.15), from trace theorem $|v(z=i)| \leq c_0 \|v\|_{H^1}$ for some positive constant c_0 , therefore $|v(z=i)| \leq c_0 \|r^2\|_{L^\infty} \|v\|_{H_v^1}$;
- for the third component, obviously $\|r^{-2} \partial_z v\|_{L_{r^2}^2} \leq \|v\|_{H_v^1}$; then using Schwarz inequality

$$\left(\int_0^\infty \varphi dM_\beta \right)^2 \leq c_\beta \int_0^\infty (1 + \xi) \varphi^2 dM_\beta;$$

hence

$$\left\| \varepsilon \int_0^\infty \varphi dM_\beta \right\|_{L_{r^2}^2}^2 \leq c_\beta \|\varepsilon\|_{L^\infty} \|\varphi\|_{L^2(0,1; V_\beta; \varepsilon r^2 dz)}^2;$$

finally, in a similar way,

$$\left\| \eta \int_0^\infty (p - \xi \tilde{\varphi}) dM_{1-\alpha} \right\|_{L_{r^2}^2}^2 \leq c_{1-\alpha} \|\eta\|_{L^\infty} \|p - \xi \tilde{\varphi}\|_{L^2(0,1; V_{1-\alpha}; \eta r^2 dz)}^2;$$

- for the fourth component, obviously, $\|r^2 \partial_z p\|_{L_{r^{-2}}^2} \leq \|p\|_{H_p^1}$;

- for the fifth component, there is nothing to prove: $(p - \xi\varphi) \in L^2(0, 1; H_\beta; \varepsilon r^2 dz)$;
- for the sixth component, since $V_{1-\alpha} \subset \tilde{V}_{1-\alpha}$, we simply have

$$\|p - \xi\tilde{\varphi}\|_{L^2(0,1;\tilde{V}_{1-\alpha};\eta r^2 dz)} \leq \|p - \xi\tilde{\varphi}\|_{L^2(0,1;V_{1-\alpha};\eta r^2 dz)}.$$

Theorem 2.2.1 *For all initial condition $X_0 \in D(\mathcal{A})$, there exists a unique solution $X \in C^1([0, +\infty[; \mathcal{H}) \cap C^0([0, +\infty[; D(\mathcal{A}))$ to*

$$\begin{cases} \frac{d}{dt}X(t) + \mathcal{A}X(t) = 0 & \forall t > 0, \\ X(0) = X_0. \end{cases}$$

This solution satisfies

$$\frac{d}{dt} \left\{ \frac{1}{2} \|X(t)\|_{\mathcal{H}}^2 \right\} = -(\mathcal{A}X(t), X(t))_{\mathcal{H}} \leq 0. \quad (2.18)$$

Proof. We shall first prove the *monotonicity* of the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$. Let $X = (x_0, x_1, p, v, \varphi, \tilde{\varphi})^T \in D(\mathcal{A})$. Integrating by parts the term $(\frac{1}{r^2}\partial_z v, v)_{L^2_{r^2}}$ in the scalar product $(\mathcal{A}X, X)_{\mathcal{H}}$, then using the boundary conditions (2.6) yields, after some algebraic manipulations on the matrices \mathcal{M}_i :

$$\begin{aligned} (\mathcal{A}X, X)_{\mathcal{H}} &= \frac{1}{2} (x_0^T \quad v(0)) \mathcal{M}_0 \begin{pmatrix} x_0 \\ v(0) \end{pmatrix} + \frac{1}{2} (x_1^T \quad v(1)) \mathcal{M}_1 \begin{pmatrix} x_1 \\ v(1) \end{pmatrix} \\ &\quad + \int_0^1 \|\varphi\|_{\tilde{V}_\beta}^2 \varepsilon r^2 dz + \int_0^1 \|p - \xi\tilde{\varphi}\|_{H_{1-\alpha}}^2 \eta r^2 dz. \end{aligned} \quad (2.19)$$

therefore $(\mathcal{A}X, X)_{\mathcal{H}} \geq 0, \forall X \in D(\mathcal{A})$ and the inequality in equation (2.18) will be fulfilled.

Now let $Y = (y_0, y_1, f, g, \chi, \tilde{\chi})^T \in \mathcal{H}$, we shall prove the existence of $X = (x_0, x_1, p, v, \varphi, \tilde{\varphi})^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})X = Y$, which proves the *maximality* of \mathcal{A} . The set of equations to solve is

- (i) $x_0 - A_0 x_0 - B_0 v(0) = y_0$,
- (ii) $x_1 - A_1 x_1 - B_1 v(1) = y_1$,
- (iii) $p + \frac{1}{r^2} \partial_z v + \varepsilon \int_0^{+\infty} \varphi dM_\beta + \eta \int_0^{+\infty} [p - \xi\tilde{\varphi}] dM_{1-\alpha} = f$,
- (iv) $v + r^2 \partial_z p = g$,
- (v) $\varphi + \xi\varphi - p = \chi$,
- (vi) $\tilde{\varphi} + \xi\tilde{\varphi} - p = \tilde{\chi}$.

We remark that one can solve the first two algebraic equations with respect to x_0, x_1 ,

$$x_i = (I_{n_i} - A_i)^{-1}(y_i + B_i v(i))$$

for $i = 0, 1$, which requires $s = 1 \notin \text{spec} A_i$. But this is indeed the case since $\mathcal{Z}_i(s) = d_i + C_i(s I_{n_i} - A_i)^{-1} B_i$ is strictly positive real and the realization is minimal: thus, all eigenvalues of A_i are poles of $\mathcal{Z}_i(s)$, with negative real parts (see e.g. [19]).

We also can rewrite the last two algebraic equations equivalently as follows

$$\begin{aligned} \varphi &= \frac{1}{1+\xi} p + \frac{1}{1+\xi} \chi, \\ \tilde{\varphi} &= \frac{1}{1+\xi} p + \frac{1}{1+\xi} \tilde{\chi}. \end{aligned} \tag{2.20}$$

System of equations (i)-(vi) can therefore be reduced to seeking $(p, v) \in H_p^1 \times H_v^1$ such that

$$\begin{cases} (1 + c_\beta \varepsilon + c_{1-\alpha} \eta) p + \frac{1}{r^2} \partial_z v = h & \text{on } (0, 1), \\ v + r^2 \partial_z p = g & \text{on } (0, 1), \\ p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(s=1) v(i) & \text{for } i = 0, 1; \end{cases} \tag{2.21}$$

where we have set

$$h(z) := f(z) - \varepsilon(z) \int_0^\infty \frac{1}{1+\xi} \chi(z, \xi) dM_\beta + \eta(z) \int_0^\infty \frac{1}{1+\xi} \tilde{\chi}(z, \xi) \xi dM_{1-\alpha}.$$

Using Cauchy-Schwarz inequality (with $\chi \in H_\beta$ locally, and $\tilde{\chi} \in \tilde{V}_{1-\alpha}$ locally) and using the boundness of ε and η , one can easily check that $h \in L_{r^2}^2$.

We now solve (2.21) with respect to (p, v) thanks to a variational formulation in p that is derived as follows:

1. take the $L_{r^2}^2$ inner product of the first equation of (2.21) with some $q \in H_p^1$,
2. take the $L_{r^2}^2$ inner product of the second equation of (2.21) with some $w \in L_{r^2}^2$,
3. set $w := r^2 \partial_z q$, add the results of the preceding steps, (note that $\int_0^1 (q \partial_z v + v \partial_z q) dz = v(1) q(1) - v(0) q(0)$, since $q \in H_p^1 \subset H^1$, $v \in H_v^1 \subset H^1$), and use the third equation of (2.21) to write the problem in the standard form:

$$\begin{cases} \text{Find } p \in H_p^1 \text{ such that:} \\ a(p, q) = l(q), \quad \forall q \in H_p^1, \end{cases} \tag{2.22}$$

where

$$\begin{aligned} a(p, q) &:= \int_0^1 [\omega(z)pq + \partial_z p \partial_z q] r^2 dz + \frac{1}{\mathcal{Z}_1(1)} p(1) q(1) + \frac{1}{\mathcal{Z}_0(1)} p(0) q(0), \\ l(q) &:= \int_0^1 h q r^2 dz + \int_0^1 g \partial_z q dz + \lambda_1 q(1) - \lambda_0 q(0); \end{aligned}$$

with $\omega(z) := 1 + c_\beta \varepsilon(z) + c_{1-\alpha} \eta(z) \geq 1 > 0$ and $\lambda_i := \frac{1}{\mathcal{Z}_i(1)} C_i (I_{n_i} - A_i)^{-1} y_i$.

Note that $\mathcal{Z}_i(s=1) > 0$ since the acoustic impedances are strictly positive real, hence, together with standard trace theorem, it is straightforward to prove that the bilinear form a is continuous and coercive on H_p^1 . On the other hand, thanks to $h \in L_{r^2}^2$, $g \in L_{r^{-2}}^2 (\Leftrightarrow r^{-2} g \in L_{r^2}^2)$, and the same trace theorem, it is straightforward to see that the linear form l is continuous on H_p^1 . We therefore can apply Lax-Milgram theorem (see e.g. [2, ch. VIII]) to prove existence and uniqueness of $p \in H_p^1$, such that (2.22) holds, $\forall q \in H_p^1$.

We finally conclude the proof of maximality by showing that this unique $p \in H_p^1$, solution of (2.22), enables to define all the state variables $(x_0, x_1, p, v, \varphi, \tilde{\varphi})$ which are needed, and that the state X belongs to $D(\mathcal{A})$. This will be done in four steps.

1. Set $v := g - r^2 \partial_z p$. It belongs to $L_{r^{-2}}^2$ a priori. The variational formulation (2.22) yields,

$$\partial_z v = (h - \omega p) r^2$$

in the distribution sense. Thanks to $p \in H_p^1 \subset L_{r^2}^2$ and $h \in L_{r^2}^2$, we get $\partial_z v \in L_{r^{-2}}^2$, and therefore $v \in H_v^1$. The first equation of (2.21) is thus recovered, together with the second.

2. Now, in order to recover the boundary conditions, choose any $q \in H_p^1$, then use (2.22) to compute:

$$\begin{aligned} \left[\lambda_1 - \frac{1}{\mathcal{Z}_1(1)} p(1) \right] q(1) &- \left[\lambda_0 - \frac{1}{\mathcal{Z}_0(1)} p(0) \right] q(0) \\ &= \int_0^1 (\omega p - h) q r^2 dz - \int_0^1 (g - r^2 \partial_z p) \partial_z q dz, \\ &= - \int_0^1 \partial_z v q dz - \int_0^1 v \partial_z q dz, \\ &= v(0) q(0) - v(1) q(1). \end{aligned}$$

Since this equality is valid for any $q(0), q(1)$, recalling the value of λ_i , we get for $i = 0, 1$, $p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(1) v(i)$, which is the third equation of (2.21).

3. In order to check that the unique solution X belongs to \mathcal{V}_2 , we need to prove that, in (2.20), $\varphi \in L^2(0, 1; V_\beta; \varepsilon r^2 dz)$ and $\tilde{\varphi} \in L^2(0, 1; \tilde{V}_{1-\alpha}; \eta r^2 dz)$, using $p \in H_p^1 \subset L^2_{r^2}$, $\chi \in L^2(0, 1; H_\beta; \varepsilon r^2 dz)$ and $\tilde{\chi} \in L^2(0, 1; \tilde{V}_{1-\alpha}; \eta r^2 dz)$. Recall $\varphi = \frac{1}{1+\xi} p + \frac{1}{1+\xi} \chi$. On the first hand, since, $\|\frac{1}{1+\xi}\|_{V_\beta}^2 = c_\beta$ one has

$$\left\| \frac{1}{1+\xi} p \right\|_{L^2(0,1;V_\beta;\varepsilon r^2 dz)}^2 \leq c_\beta \|\varepsilon\|_\infty \|p\|_{L^2_{r^2}}^2;$$

on the other hand, since, $\|\frac{1}{1+\xi} \chi\|_{V_\beta}^2 = \|\frac{1}{\sqrt{1+\xi}} \chi\|_{H_\beta}^2$, then one has

$$\left\| \frac{1}{1+\xi} \chi \right\|_{L^2(0,1;V_\beta;\varepsilon r^2 dz)}^2 \leq \|\chi\|_{L^2(0,1;H_\beta;\varepsilon r^2 dz)}^2.$$

Similar considerations apply to $\tilde{\varphi} = \frac{1}{1+\xi} p + \frac{1}{1+\xi} \tilde{\chi}$: $\|\frac{1}{1+\xi}\|_{\tilde{V}_{1-\alpha}}^2 \leq c_{1-\alpha}$ implies

$$\left\| \frac{1}{1+\xi} p \right\|_{L^2(0,1;\tilde{V}_{1-\alpha};\eta r^2 dz)}^2 \leq c_{1-\alpha} \|\eta\|_\infty \|p\|_{L^2_{r^2}}^2;$$

whereas we trivially have

$$\left\| \frac{1}{1+\xi} \tilde{\chi} \right\|_{L^2(0,1;\tilde{V}_{1-\alpha};\eta r^2 dz)}^2 \leq \|\tilde{\chi}\|_{L^2(0,1;\tilde{V}_{1-\alpha};\eta r^2 dz)}^2.$$

4. Finally, in order to check that the unique solution X belongs to $D(\mathcal{A})$, we need to prove that $p(z=0) = -C_0 x_0 - d_0 v(z=0)$, $p(z=1) = C_1 x_1 + d_1 v(z=1)$, $(p - \xi\varphi) \in L^2(0, 1; H_\beta; \varepsilon r^2 dz)$ and $(p - \xi\tilde{\varphi}) \in L^2(0, 1; V_{1-\alpha}; \eta r^2 dz)$.

Since $p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(1) v(i)$, $x_i = (I_{n_i} - A_i)^{-1} (y_i + B_i v(i))$ and recalling $\mathcal{Z}_i(1) = d_i + C_i (I_{n_i} - A_i)^{-1} B_i$, we easily get: $p(i) = \mp C_i x_i \mp d_i v(i)$.

From $\xi\varphi - p = -\frac{1}{1+\xi} p + \frac{\xi}{1+\xi} \chi$, one easily deduces that $(p - \xi\varphi) \in L^2(0, 1; H_\beta; \varepsilon r^2 dz)$, since

$$\left\| \frac{1}{1+\xi} p \right\|_{L^2(0,1;H_\beta;\varepsilon r^2 dz)}^2 \leq c_\beta \|\varepsilon\|_\infty \|p\|_{L^2_{r^2}}^2,$$

and using $\frac{\xi}{1+\xi} \leq 1$,

$$\left\| \frac{\xi}{1+\xi} \chi \right\|_{L^2(0,1;H_\beta;\varepsilon r^2 dz)}^2 \leq \|\chi\|_{L^2(0,1;H_\beta;\varepsilon r^2 dz)}^2.$$

One checks that $\xi\tilde{\varphi} - p = -\frac{1}{1+\xi} p + \frac{\xi}{1+\xi} \tilde{\chi} \in L^2(0, 1; V_{1-\alpha}; \eta r^2 dz)$ by firstly noting that

$$\left\| \frac{1}{1+\xi} p \right\|_{L^2(0,1;V_{1-\alpha};\eta r^2 dz)}^2 \leq c_{1-\alpha} \|\eta\|_\infty \|p\|_{L^2_{r^2}}^2,$$

and secondly using $\|\frac{\xi}{1+\xi} \tilde{\chi}\|_{\tilde{V}_{1-\alpha}}^2 = \|\sqrt{\frac{\xi}{1+\xi}} \tilde{\chi}\|_{\tilde{V}_{1-\alpha}}^2 \leq \|\tilde{\chi}\|_{\tilde{V}_{1-\alpha}}^2$ to deduce

$$\|\frac{\xi}{1+\xi} \tilde{\chi}\|_{L^2(0,1;V_{1-\alpha};\eta r^2 dz)}^2 \leq \|\tilde{\chi}\|_{L^2(0,1;\tilde{V}_{1-\alpha};\eta r^2 dz)}^2.$$

From the monotonicity and maximality of operator \mathcal{A} , one concludes by applying Lümer-Phillips theorem (see e.g. [12, theorem 2.27]). \square

Remark 2.2.1 *For the diffusive realizations of $\partial_t^{-\beta} p$ and $\partial_t^\alpha p$, we have introduced two state variables, namely φ and $\tilde{\varphi}$; but it is easy to notice that they both fulfill $\varphi(\xi, z, t) = \tilde{\varphi}(\xi, z, t) = \int_0^t e^{-\xi \tau} p(t-\tau, z) d\tau$: only the functional spaces are different. Hence, only one state variable is needed, which belongs to a smaller functional space as follows:*

$$\varphi = \tilde{\varphi} \in L^2(0, 1; H_\beta; \varepsilon r^2 dz) \cap L^2(0, 1; \tilde{V}_{1-\alpha}; \eta r^2 dz) .$$

This point can be used in deriving a numerical scheme to reduce the number of state variables, and it is all the more useful from a computational point of view, than these unknowns depend upon three variables!

Chapter 3

Numerical analysis

3.1 A finite differences scheme to solve the Webster Lokshin equations

We recall here the equations of this model, where $(z, t, \xi) \in [0, 1] \times (0, \infty) \times (0, \infty)$,

- (i) $\partial_t p(z, t) = -\frac{1}{r^2(z)} \partial_z v(z, t) - \varepsilon \theta(z, t) - \eta \tilde{\theta}(z, t),$
- (ii) $\partial_t v(z, t) = -r^2(z) \partial_z p(z, t),$
- (iii) $\theta(z, t) = \int_0^{+\infty} \varphi(\xi, z, t) dM_\beta(\xi)$ and $\tilde{\theta}(z, t) = \int_0^{+\infty} [p(z, t) - \xi \varphi(\xi, z, t)] dM_{1-\alpha}(\xi),$
- (iv) $\partial_t \varphi(\xi, z, t) = -\xi \varphi(\xi, z, t) + p(z, t),$
- (v) $p(0, t) = 0,$
- (vi) $p(1, t) = Cx(t) + dv(1, t),$
- (vii) $\frac{dx}{dt}(t) = Ax(t) + Bv(1, t),$

coupled with initial conditions

$$p(z, 0) = p_0(z), \quad v(z, 0) = v_0(z), \quad \varphi(\xi, z, 0) = 0, \quad x(0) = 0.$$

Remark that the impedance boundary condition at $z = 0$ has been replaced by a Dirichlet boundary condition. This does not affect the generality of our results since the treatment of an impedance boundary condition will be explained at $z = 1$. We recall that $x(t) \in \mathbb{R}^{n_1}$, $A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times 1}$, $C \in \mathbb{R}^{1 \times n_1}$ and $d \in \mathbb{R}$ for some integer n_1 .

Let Δt and $h = 1/N$ respectively be the time and space steps where N is the number of discretization points of $[0, 1]$. We set $z_i = ih$, $z_{i+\frac{1}{2}} = (i + \frac{1}{2})h$ and denote

$$p_i^n \approx p(ih, n\Delta t); v_{i+\frac{1}{2}}^{n+\frac{1}{2}} \approx v((i+\frac{1}{2})h, (n+\frac{1}{2})\Delta t); \theta_i^n \approx \theta(ih, n\Delta t); \tilde{\theta}_i^n \approx \tilde{\theta}(ih, n\Delta t)$$

Then a second order centered explicit scheme associated with (i) and (ii) can be written as

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = -\frac{1}{r^2(z_i)} \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} - \varepsilon(z_i) \frac{\theta_i^{n+1} + \theta_i^n}{2} - \eta(z_i) \frac{\tilde{\theta}_i^{n+1} + \tilde{\theta}_i^n}{2} \quad (3.1)$$

for $n > 0$ and $0 < i \leq N$ and

$$\frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} = -r^2(z_{i+\frac{1}{2}}) \frac{p_{i+1}^n - p_i^n}{h} \quad (3.2)$$

for $n > 0$ and $0 \leq i < N$. The discrete version of (vi) and (vii) can be written as follows

$$\begin{cases} \frac{p_N^{n+1} + p_N^n}{2} = C \frac{x^{n+1} + x^n}{2} + d \frac{v_{N-\frac{1}{2}}^{n+\frac{1}{2}} + v_{N+\frac{1}{2}}^{n+\frac{1}{2}}}{2} \\ \frac{x^{n+1} - x^n}{\Delta t} = A \frac{x^{n+1} + x^n}{2} + B \frac{v_{N-\frac{1}{2}}^{n+\frac{1}{2}} + v_{N+\frac{1}{2}}^{n+\frac{1}{2}}}{2}. \end{cases} \quad (3.3)$$

Note that we have introduced in (3.1) and (3.3) (as it is classically done for convenience in writing the discretized scheme at the boundary) a fictitious node at $(N + \frac{1}{2})h$ for the variable v .

We remark that the scheme (3.3) is implicit for x^n . The evaluation of θ_i^n and $\tilde{\theta}_i^n$ requires the evaluation of the integrals in (iv). We introduce geometric grid of the ξ axis defined by lower bound ξ_m , upper bound ξ_M and the number of points N_ξ . We define

$$\xi_j = \left(\frac{\xi_M}{\xi_m} \right)^{\frac{j-1}{N_\xi-1}} \xi_m,; j = 1, \dots, N_\xi,$$

and denote $\varphi_{i,j}^n \approx \varphi(\xi_j, z_i, n\Delta t)$. Then, the differential equation for φ is discretized, using the following unconditionally stable and second order explicit scheme

$$\varphi_{i,j}^{n+1} = e^{-\xi_j \Delta t} \varphi_{i,j}^n + \frac{1 - e^{-\xi_j \Delta t}}{\xi_j} \frac{p_i^{n+1} + p_i^n}{2} \quad (3.4)$$

which has been derived from the expression

$$\varphi(\xi, z, t) = \int_0^t e^{-\xi(t-s)} p(z, s) ds$$

by exact integration of the right hand side between $n\Delta t$ and $(n+1)\Delta t$. We deduce the expressions of θ_i^n and $\tilde{\theta}_i^n$ through an exact evaluation of

$$\theta_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} \varphi_{i,j}^n \lambda_j(\xi) dM_\beta(\xi) \quad (3.5)$$

$$\tilde{\theta}_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} (p_i^n - \xi_j \varphi_{i,j}^n) \lambda_j(\xi) dM_{1-\alpha}(\xi) \quad (3.6)$$

where λ_j is continuous, piecewise linear function that satisfies $\lambda_j(\xi_i) = \delta_{i,j}$ with $\delta_{i,j}$ denoting the Kronecker symbol. Hence

$$\begin{aligned} \theta_i^n &= \sum_{j=1}^{N_\xi} \rho_j(\beta) \varphi_{i,j}^n \\ \tilde{\theta}_i^n &= \sum_{j=1}^{N_\xi} \rho_j(1-\alpha) (p_i^n - \xi_j \varphi_{i,j}^n) \end{aligned} \quad (3.7)$$

for some non-negative quadrature weights ρ_j . Let us quote that the following analysis holds for any choice of quadrature of the form (3.7). In the present study, the choice of the nodes ξ_j is made the same for the evaluation of θ_i^n and $\tilde{\theta}_i^n$ and is independent of α . Of course, one may think that a more accurate evaluation of these quantities may require a dependence of the nodes ξ_j on the order of derivative (α or β). This change of nodes ξ_j has no influence on the validity of subsequent results as long as the form of the quadrature rule (3.7) is respected.

Remark 3.1.1 *Remark that another possibility to approximate $\tilde{\theta}$ would have been to replace (3.6) with*

$$\tilde{\theta}_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} (p_i^n - \xi \varphi_{i,j}^n) \lambda_j(\xi) dM_{1-\alpha}(\xi).$$

This will lead to a quadrature formula that is not of the form (3.7) and therefore the following analysis does not include this case. Moreover, it turns out that this discretization does not respect the discrete energy balance and seems to be numerically less accurate (especially for long time behaviour).

3.2 Stability study of the discretized model

We introduce a so-called discrete wave energy associated with our scheme as being defined by

$$E^n = \frac{h}{2} \left(\sum_{i=0}^{N-1} |r(z_i) p_i^n|^2 + \frac{1}{r(z_{i+\frac{1}{2}})^2} v_{i+\frac{1}{2}}^{n+\frac{1}{2}} v_{i+\frac{1}{2}}^{n-\frac{1}{2}} \right) + \frac{h}{4} |r(z_N) p_N^n|^2.$$

Let

$$\gamma = \max \left\{ \frac{r(z_{i+\frac{1}{2}})}{r(z_{i+1})}, \frac{r(z_{i+\frac{1}{2}})}{r(z_i)}, i = 1, \dots, N-1 \right\},$$

we first show that this energy is a norm under the CFL stability condition $\gamma\Delta t < h$.

Lemma 3.2.1 *Assume that $\gamma\Delta t < h$, then there exists a constant $C > 0$, independent of n , such that*

$$E^n \geq C \sum_{i=1}^N |r(z_i) p_i^n|^2 + \left| (v_{i-\frac{1}{2}}^{n+\frac{1}{2}} + v_{i-\frac{1}{2}}^{n-\frac{1}{2}}) / 2r(z_{i-\frac{1}{2}}) \right|^2.$$

Proof. The proof of this lemma is rather classical. We give it here for the sake of completeness. From equation (3.2) and the definition of γ we get

$$\frac{1}{r(z_{i+\frac{1}{2}})^2} |v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}}|^2 \leq 2\gamma^2 \frac{\Delta t^2}{h^2} (|r(z_{i+1}) p_{i+1}^n|^2 + |r(z_i) p_i^n|^2)$$

for $i = 0, \dots, N-1$. Using the identity

$$v_{i+\frac{1}{2}}^{n+\frac{1}{2}} v_{i+\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{4} (v_{i+\frac{1}{2}}^{n+\frac{1}{2}} + v_{i+\frac{1}{2}}^{n-\frac{1}{2}})^2 - \frac{1}{4} (v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}})^2$$

and the previous estimate yields (using $p_0^n = 0$)

$$E^n \geq \frac{h}{2} \left(\sum_{i=0}^{N-1} \left(1 - \frac{\gamma^2 \Delta t^2}{h^2} \right) |r(z_i) p_i^n|^2 + \left| \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} + v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{2r(z_{i+\frac{1}{2}})} \right|^2 \right) + \frac{h}{4} \left(1 - \frac{\gamma^2 \Delta t^2}{h^2} \right) |r(z_N) p_i^N|^2,$$

whence the desired estimate under the CFL condition. \square

We shall prove that this energy is uniformly bounded with respect to n which yields the stability of our scheme under the CFL condition of Lemma 3.2.1. For the sake of clarity we will study first the three cases

1. $(\varepsilon, \eta) = (0, 0)$: stability of the coupling with the discretized impedance boundary condition,
2. $\eta = 0, \varepsilon > 0$: stability of the coupling with the discretized $\partial_t^{-\beta}$,
3. $\varepsilon = 0, \eta > 0$: stability of the coupling with the discretized ∂_t^α ,

separately then deduce the stability result for the general case.

3.2.1 Case $(\varepsilon, \eta) = (0, 0)$

We shall concentrate in this section on the stability of the coupling between the interior scheme with the discretized impedance boundary condition. We recall that there exists $P \in \mathbb{R}^{n_1 \times n_1}$, $P = P^T > 0$, such that

$$\mathcal{M} = \begin{pmatrix} -A^T P - PA & C^T - PB \\ C - B^T P & 2d \end{pmatrix}$$

is a non-negative matrix of $\mathbb{R}^{(n_1+1) \times (n_1+1)}$.

Proof of the energy balance: Multiplying equation (3.1) by $h \frac{r(z_i)^2}{2} (p_i^{n+1} + p_i^n)$ yields

$$\frac{h}{2\Delta t} (|r(z_i)p_i^{n+1}|^2 - |r(z_i)p_i^n|^2) = -\frac{1}{2} (v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i-\frac{1}{2}}^{n+\frac{1}{2}}) (p_i^{n+1} + p_i^n). \quad (3.8)$$

for $i = 1, \dots, N$. Taking the difference between equation (3.2) at step $n + \frac{3}{2}$ and the same equation at step $n + \frac{1}{2}$ yields, after multiplication by $h \frac{1}{2r(z_i)^2} v_{i+\frac{1}{2}}^{n+\frac{1}{2}}$,

$$\frac{h}{2\Delta t} \left(\frac{v_{i+\frac{1}{2}}^{n+\frac{3}{2}} v_{i+\frac{1}{2}}^{n+\frac{1}{2}}}{r(z_{i+\frac{1}{2}})^2} - \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{r(z_{i+\frac{1}{2}})^2} \right) = -\frac{1}{2} v_{i+\frac{1}{2}}^{n+\frac{1}{2}} ((p_{i+1}^{n+1} + p_{i+1}^n) - (p_i^{n+1} + p_i^n)). \quad (3.9)$$

Using (3.8) and (3.9) we easily deduce the balance

$$\frac{E^{n+1} - E^n}{\Delta t} = -\frac{v_{N+\frac{1}{2}}^{n+\frac{1}{2}} + v_{N-\frac{1}{2}}^{n+\frac{1}{2}}}{2} \frac{p_N^{n+1} + p_N^n}{2}$$

Its very convenient to use the notation

$$v_N^{n+\frac{1}{2}} = \frac{v_{N+\frac{1}{2}}^{n+\frac{1}{2}} + v_{N-\frac{1}{2}}^{n+\frac{1}{2}}}{2} \text{ and } p_N^{n+\frac{1}{2}} = \frac{p_N^{n+1} + p_N^n}{2} \quad (3.10)$$

so that the previous balance can be written in the form

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}}. \quad (3.11)$$

From the first equation of (3.3) it is clear that

$$v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} = v_N^{n+\frac{1}{2}} C x^{n+\frac{1}{2}} + d |v_N^{n+\frac{1}{2}}|^2 \quad (3.12)$$

where $x^{n+\frac{1}{2}} := \frac{x^{n+1} + x^n}{2}$. Let us set

$$E_x^n = \frac{1}{2} x^n \cdot P x^n \quad (3.13)$$

as being the energy associated with the boundary variable x^n . The second equation of (3.3) implies after taking the sum of a multiplication by $Px^{n+\frac{1}{2}}$ and a multiplication by $(x^{n+\frac{1}{2}})^T P$,

$$2 \frac{E_x^{n+1} - E_x^n}{\Delta t} = x^{n+\frac{1}{2}} \cdot (A^T P + PA)x^{n+\frac{1}{2}} + v_N^{n+\frac{1}{2}} (Px^{n+\frac{1}{2}} \cdot B + x^{n+\frac{1}{2}} \cdot PB). \quad (3.14)$$

Combining (3.12) and (3.14) yields

$$v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} = \frac{E_x^{n+1} - E_x^n}{\Delta t} + \frac{1}{2} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \quad (3.15)$$

Let us define the total energy as

$$\mathcal{E}^n = E^n + E_x^n, \quad (3.16)$$

we deduce from (3.11) and (3.15) that

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = -\frac{1}{2} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix}$$

which shows that the this energy is non increasing with n . Hence it is uniformly bounded with respect to n . Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.2 Case $\eta = 0$

We start by rewriting in a suitable form equation (3.4). Introducing

$$\omega_j = \frac{\xi_j \Delta t}{2} \frac{1 + e^{-\xi_j \Delta t}}{1 - e^{-\xi_j \Delta t}}$$

we see that (3.4) is equivalent to

$$\omega_j \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\Delta t} = -\xi_j \frac{\varphi_{i,j}^{n+1} + \varphi_{i,j}^n}{2} + \frac{p_i^{n+1} + p_i^n}{2}. \quad (3.17)$$

Multiplying this equation by $\frac{1}{2} (\varphi_{i,j}^{n+1} + \varphi_{i,j}^n)$ yields

$$\frac{\varphi_{i,j}^{n+1} + \varphi_{i,j}^n}{2} \frac{p_i^{n+1} + p_i^n}{2} = \frac{1}{2} \omega_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^n|^2}{\Delta t} + \xi_j \left| \frac{\varphi_{i,j}^{n+1} + \varphi_{i,j}^n}{2} \right|^2.$$

Therefore, from (3.7),

$$\frac{\theta_i^{n+1} + \theta_i^n}{2} \frac{p_i^{n+1} + p_i^n}{2} = \sum_{j=1}^{N_\xi} \frac{1}{2} \rho_j(\beta) \omega_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^n|^2}{\Delta t} + \rho_j(\beta) \xi_j \left| \frac{\varphi_{i,j}^{n+1} + \varphi_{i,j}^n}{2} \right|^2. \quad (3.18)$$

Following the same steps as in the proof of (3.11) yields in the present case to

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \sum_{i=1}^{N-1} \varepsilon(z_i) \theta_i^{n+\frac{1}{2}} p_i^{n+\frac{1}{2}} - \frac{1}{2} \varepsilon(z_N) \theta_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} \quad (3.19)$$

with short-hand notation

$$\theta_i^{n+\frac{1}{2}} := \frac{\theta_i^{n+1} + \theta_i^n}{2} \quad \text{and} \quad p_i^{n+\frac{1}{2}} := \frac{p_i^{n+1} + p_i^n}{2}. \quad (3.20)$$

Let us introduce now

$$E_\varphi^n := \frac{1}{2} \sum_{j=1}^{N_\xi} \rho_j(\beta) \omega_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) |\varphi_{i,j}^n|^2 + \frac{1}{2} \varepsilon(z_N) |\varphi_{N,j}^n|^2 \right) \quad (3.21)$$

as being the energy associated with the antiderivative term $\partial_t^{-\beta}$. It is clear from (3.19) and (3.18) that we have the following balance

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \frac{E_\varphi^{n+1} - E_\varphi^n}{\Delta t} - \sum_{j=1}^{N_\xi} \rho_j(\beta) \xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \quad (3.22)$$

where we have set

$$\varphi_{i,j}^{n+\frac{1}{2}} := \frac{\varphi_{i,j}^{n+1} + \varphi_{i,j}^n}{2}. \quad (3.23)$$

Define the total energy in the present case as

$$\mathcal{E}^n = E^n + E_x^n + E_\varphi^n \quad (3.24)$$

we deduce from (3.15) and (3.22) that

$$\begin{aligned} \frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} &= -\frac{1}{2} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(\beta) \xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \end{aligned}$$

which shows that the this energy is non increasing with n . Hence it is uniformly bounded with respect to n . Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.3 Case $\varepsilon = 0$

We study here the stability of the coupling with the discretization of the fractional derivative ∂_t^α . Using equation (3.17) and shorthand notation (3.20) and (3.23), we have the identity,

$$p_i^{n+\frac{1}{2}} \left(p_i^{n+\frac{1}{2}} - \xi_j \varphi_{i,j}^{n+\frac{1}{2}} \right) = \omega_j^2 \left| \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\Delta t} \right|^2 + \frac{1}{2} \omega_j \xi_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^n|^2}{\Delta t}$$

Therefore, from (3.7),

$$\tilde{\theta}_i^{n+\frac{1}{2}} p_i^{n+\frac{1}{2}} = \sum_{j=1}^{N_\xi} \frac{1}{2} \rho_j (1-\alpha) \omega_j \xi_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^n|^2}{\Delta t} + \rho_j (1-\alpha) \left| \omega_j \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2. \quad (3.25)$$

with short-hand notation

$$\tilde{\theta}_i^{n+\frac{1}{2}} := \frac{\tilde{\theta}_i^{n+1} + \tilde{\theta}_i^n}{2} \quad \text{and} \quad \dot{\varphi}_{i,j}^{n+\frac{1}{2}} := \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\Delta t}.$$

On the other hand, following the same steps as in the proof of (3.11) yields in the present case to

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \sum_{i=1}^{N-1} \eta(z_i) \tilde{\theta}_i^{n+\frac{1}{2}} p_i^{n+\frac{1}{2}} - \frac{1}{2} \eta(z_N) \tilde{\theta}_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}}. \quad (3.26)$$

Let us introduce now

$$\tilde{E}_\varphi^n := \frac{1}{2} \sum_{j=1}^{N_\xi} \rho_j (1-\alpha) \omega_j \xi_j \left(\sum_{i=1}^{N-1} \eta(z_i) |\varphi_{i,j}^n|^2 + \frac{1}{2} \eta(z_N) |\varphi_{N,j}^n|^2 \right) \quad (3.27)$$

as being the energy associated with the derivative term ∂_t^α . It is clear from (3.26) and (3.25) that we have the following balance

$$\begin{aligned} \frac{E^{n+1} - E^n}{\Delta t} &= -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \frac{\tilde{E}_\varphi^{n+1} - \tilde{E}_\varphi^n}{\Delta t} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j (1-\alpha) \omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i) \left| \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \eta(z_N) \left| \dot{\varphi}_{N,j}^{n+\frac{1}{2}} \right|^2 \right). \end{aligned} \quad (3.28)$$

Define the total energy in the present case as

$$\mathcal{E}^n = E^n + E_x^n + \tilde{E}_\varphi^n, \quad (3.29)$$

then it follows from (3.15) and (3.28) that

$$\begin{aligned} \frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} &= -\frac{1}{2} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j (1-\alpha) \omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i) \left| \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \eta(z_N) \left| \dot{\varphi}_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \end{aligned}$$

which shows that this energy is non increasing with n . Hence it is uniformly bounded with respect to n . Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.4 The general case

From the results of previous subsections it is clear now that if we define the total energy as

$$\mathcal{E}^n = E^n + E_x^n + E_\varphi^n + \tilde{E}_\varphi^n,$$

then one has the following balance,

$$\begin{aligned} \frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} &= -\frac{1}{2} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(\beta) \xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(1-\alpha) \omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i) \left| \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \eta(z_N) \left| \dot{\varphi}_{N,j}^{n+\frac{1}{2}} \right|^2 \right). \end{aligned}$$

Combining this with Lemma 3.2.1 we deduce:

Lemma 3.2.2 *Assume that $\gamma\Delta t < h$, then there exists a constant $C > 0$, independent of n , such that*

$$\sum_{i=1}^N |r(z_i) p_i^n|^2 + \left| (v_{i-\frac{1}{2}}^{n+\frac{1}{2}} + v_{i-\frac{1}{2}}^{n-\frac{1}{2}}) / 2r(z_{i-\frac{1}{2}}) \right|^2 + E_x^n + E_\varphi^n + \tilde{E}_\varphi^n \leq C.$$

This shows the L^2 stability of our scheme.

3.3 Some numerical results

We shall give hereafter some numerical experiments that illustrate the behaviour of the discretized model and mimicks the energy balance. They also aim at giving some ideas on the effect of the damping introduced by fractional integral and derivative terms. Let us point out that these experiments are simply synthetic ones; the numerical values of the physical constants are completely arbitrary.

In order to produce initial conditions we put at $z = 0$ a source term

$$p(0, t) = s(t)$$

where $s(t)$ is one prescribed signal that vanishes after some time T . The signal and the time T are chosen so that an initial pulse has been produced inside the spatial domain $(0, 1)$. To concentrate on the effect of diffusive terms, we take a special case of the impedance boundary condition at $z = 1$, which produces a perfectly transparent boundary condition: it corresponds to

$$A = 0, B = 0, C = 0 \text{ and } d = 1.$$

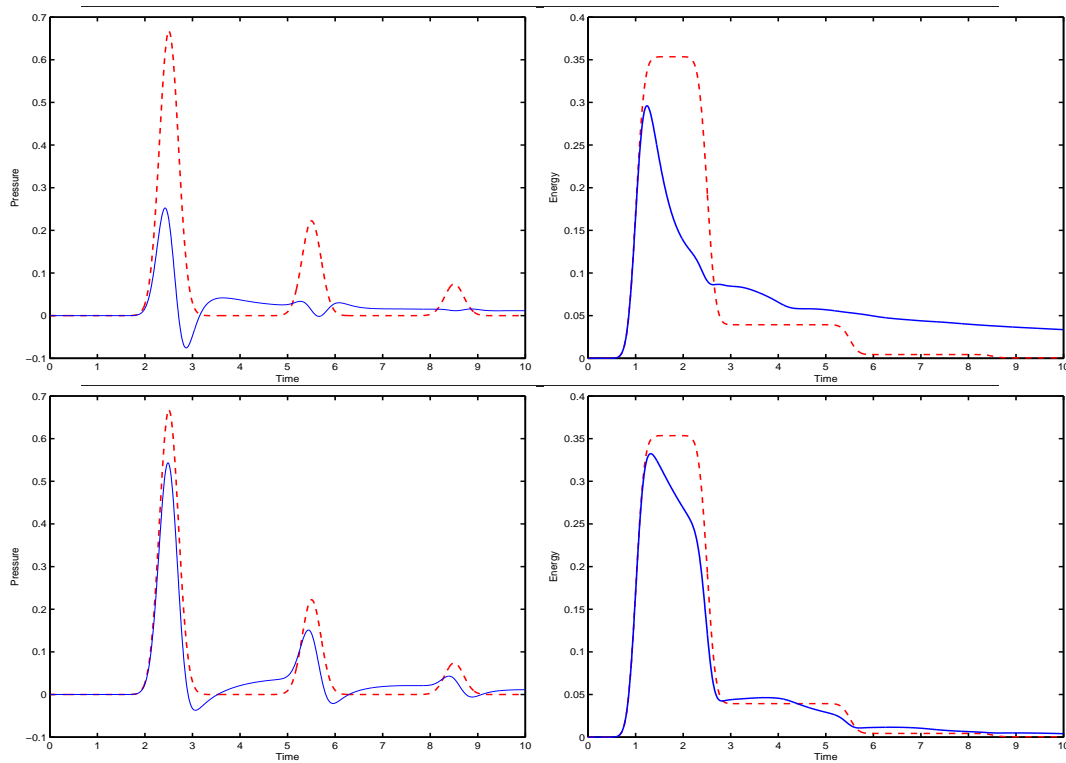


Figure 3.1: Influence of the parameter ε in a cylinder. Top: $\varepsilon = 1$ in blue plain line, compared to the reference $\varepsilon = 0$ in red dotted line. Bottom: $\varepsilon = 0.2$ in blue plain line, compared to the reference $\varepsilon = 0$ in red dotted line. The pressure signals at the output versus time are shown on the left, whereas the wave energy in the cylinder is plotted versus time on the right.

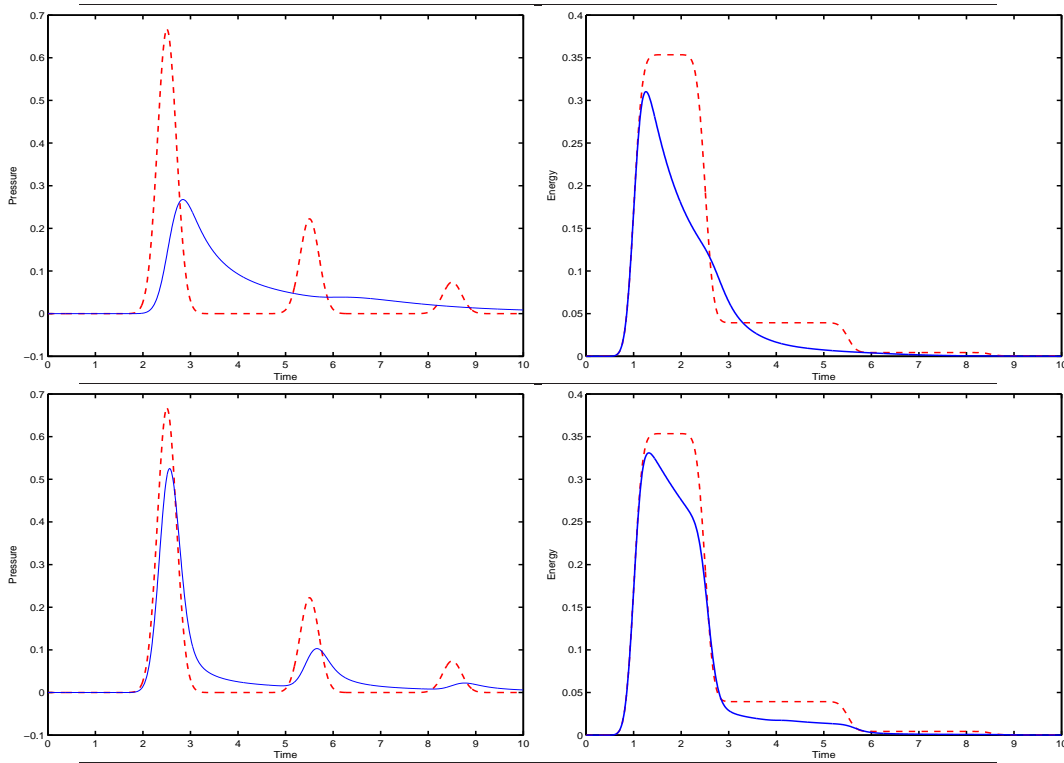


Figure 3.2: Influence of the parameter η in a cylinder. Top: $\eta = 0.5$ in blue plain line, compared to the reference $\eta = 0$ in red dotted line. Bottom: $\eta = 0.11$ in blue plain line, compared to the reference $\eta = 0$ in red dotted line. The pressure signals at the output versus time are shown on the left, whereas the wave energy in the cylinder is plotted versus time on the right.

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