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► **To cite this version:**

Antoine Lejay. On rough differential equations. *Electronic Journal of Probability*, 2008. inria-00278246v1

**HAL Id: inria-00278246**

**<https://inria.hal.science/inria-00278246v1>**

Submitted on 9 May 2008 (v1), last revised 3 Feb 2009 (v3)

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# On rough differential equations

Antoine Lejay\*

March 19, 2008

## Abstract

We prove that the Itô map, that is the map that gives the solution of a differential equation controlled by a rough path of finite  $p$ -variation with  $p \in [2, 3)$  is locally Lipschitz continuous in all its arguments and could be extended to vector fields that have only a linear growth.

**Keywords:** Rough paths, Itô map, controlled differential equations.

**AMS classification:** 60H10 (34F05, 34G05).

## 1 Introduction

The theory of rough paths allows one to properly defined solutions of controlled differential equations driven by irregular paths and asserts the continuity — in the right topology — of the map the send the control to the solution [7, 8, 10, 12, 13]. As this theory was initially designed to overcome the problem of defining stochastic differential equations in a pathwise way, this map is called the *Itô map*. As Brownian paths are  $\alpha$ -Hölder continuous for any  $\alpha < 1/2$ , the theory of rough paths asserts that one can define in a pathwise way solution of stochastic differential equations as soon as one shows that the iterated integrals  $t \mapsto \int_0^t (B_s - B_0) \otimes \circ dB_s$  have the right regularity property, and then, that the Brownian path can be embedded in an enhanced Brownian path with values in a non-commutative Lie group given by a tensor algebra. The theory of rough path can indeed be used for many stochastic processes (See for example the book [5]).

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The goal of this article is then to bring some precisions on the Itô map, which is that map that transform a rough path  $x$  of finite  $p$ -variation with  $p < 3$  into the solution  $z$  of

$$z_t = z_0 + \int_0^t f(z_s) dx_s$$

for a vector field  $f$  which is smooth enough. In particular, we show that the assumptions on the boundedness on the vector field can be drop to assume that  $f$  has a linear growth (note that however, existence of the differential of the derivative with respect to the starting point, which gives rise to a linear equation, has been studied in [1, 11]). In addition, we show that the Itô map  $(z_0, f, x) \mapsto z$  is locally Lipschitz in all its arguments (a slightly weaker statement on the continuity with respect to the vector field was proved in [2] and [9], and on the Lipschitz continuity with respect to  $x$  in [1, 12]) under the more stringent assumption that  $f$  is twice differentiable with  $\gamma$ -Hölder continuous second-order derivative with  $2 + \gamma > p$ . The proof relies on some estimates on the distance between two solutions of differential equations, which can be also used as *a priori* estimates. From this, we deduce an alternative proof of the uniqueness of the solution of a differential equation controlled by a rough paths of finite  $p$ -variation with  $p < 3$ .

The Lipschitz continuity of the Itô map can be important for applications, since it allows one to deduce for example rates of convergence of approximates solutions of differential equations from the approximation of the driving signal (see for example [14] for a practical application to functional quantization).

We have to note that we were not able to relax the assumption for uniqueness that  $f$  shall have a  $\gamma$ -Hölder continuous second order derivative with  $2 + \gamma > p$ , to get results similar to the ones of E.M. Davies [3] (see also [4]) that deal with the case where  $2 + \gamma = p$ .

We also believe that all the results of this article may be extended to deal with  $(p, q)$ -rough paths [6], although it leads to much more complicated computations.

## 2 Notations

Let  $U$  and  $V$  be some finite-dimensional Banach spaces (we have to note however that the dimension does not play a role here when we consider functions that are  $(2 + \gamma)$ -Lipschitz continuous, but not  $(1 + \gamma)$ -Lipschitz continuous, where the solutions of the rough differential equations are defined using a non-contractive fixed point theorem). The tensor product between

two such spaces will be denoted by  $U \otimes V$ , and such a space is equipped with a norm  $|\cdot|$  such that  $|x \otimes y|_{U \otimes V} \leq |x|_U \times |y|_V$ . To simplify the notations, the norm on a Banach space  $X$  is then denote by  $|\cdot|$  instead of  $|\cdot|_X$ , as there is no ambiguity. For a Banach space  $X$ , we choose a norm on the tensor space  $X \otimes X$  such that  $|a \otimes b| \leq |a| \cdot |b|$  for  $a, b$  in  $X$ . For a Banach space  $X = Y \oplus Z$  with a sub-space  $Y$ , we denote by  $\pi_Y(x)$  the projection of  $x$  onto the sub-space  $Y$ . In addition, we define on  $X \oplus (X \otimes X)$  a norm  $|\cdot|$  by  $|a| = \max\{|\pi_X(a)|, |\pi_{X \otimes X}(a)|\}$ .

Let  $x$  be a  $U$ -valued rough path on  $[0, T]$  with  $p$ -variation controlled by  $\omega$  with  $p < 3$ , that is  $x$  is a function from  $[0, T]$  to the Lie group  $(T^2(U), \otimes)$  defined by the tensor space  $T^2(U) = \{1\} \oplus U \oplus (U \otimes U)$  such that

$$\|x\|_{p,\omega} = \sup_{(s,t) \in \Delta_+^2} \max \left\{ \frac{|\pi_U(x_{s,t})|}{\omega(s,t)^{1/p}}, \frac{|\pi_{U \otimes U}(x_{s,t})|}{\omega(s,t)^{2/p}} \right\},$$

where  $\Delta_+^2 = \{(s,t) \in [0, T]^2 \mid 0 \leq s \leq t \leq T\}$ ,  $x_{s,t} = x_s^{-1} \otimes x_t$  and  $\omega$  is a function from  $\Delta_+^2$  to  $\mathbb{R}$  which is continuous near its diagonal with  $\omega(s,s) = 0$  and which is such that

$$\omega(s,r) + \omega(r,t) \leq \omega(s,t).$$

Let  $L(U, V)$  be the space of linear maps from  $U$  to  $V$ .

For  $\gamma \in (0, 1]$ , we denote by  $\text{Lip}_{\text{LG}}(1+\gamma)$  the class of continuous functions  $f : V \rightarrow L(U, V)$  for which there exists a bounded, continuous function  $\nabla f : V \rightarrow L(V \otimes U, V)$  such that

$$f(z)x - f(y)x = \int_0^1 \nabla f(y + \tau(z-y))(z-y) \otimes x \, d\tau$$

and,  $\nabla f$  is  $\gamma$ -Hölder continuous: for some constant  $C > 0$ ,

$$\|\nabla f(z) - \nabla f(y)\|_{L(U \otimes V, V)} \leq C|z-y|^\gamma, \quad \forall z, y \in V.$$

In other words, if  $U = \mathbb{R}^d$  and  $V = \mathbb{R}^m$ , this means that  $f = (f_1, \dots, f_d)$  has bounded derivatives  $\nabla f_i$  which are  $\gamma$ -Hölder continuous from  $\mathbb{R}^m$  to  $\mathbb{R}$ . Note that here,  $f$  is not necessarily bounded, but it grows at most linearly.

We denote by  $\text{Lip}(1+\gamma)$  the class of functions in  $\text{Lip}_{\text{LG}}(1+\gamma)$  which are bounded.

We also denote by  $\text{Lip}_{\text{LG}}(2+\gamma)$  the class of continuous functions  $f : V \rightarrow L(U, V)$  with bounded, continuous functions  $\nabla f : V \rightarrow L(U, V \otimes U)$  that are  $\text{Lip}(1+\gamma)$ .

We denote by  $\text{Lip}(2+\gamma)$  the class of functions in  $\text{Lip}_{\text{LG}}(2+\gamma)$  which are bounded.

For a  $\gamma$ -Hölder continuous function, we denote by  $H_\gamma(f)$  its Hölder norm. We also set  $N(f) = H_\gamma(f) + \|f\|_\infty$ , which is a norm on the space of  $\gamma$ -Hölder continuous function.

### 3 The main results

Let  $z$  be a solution to the rough differential equation

$$z_t = z_0 + \int_0^t f(z_s) dx_s. \quad (1)$$

By this, we mean a rough path in  $T(\mathbb{U} \times \mathbb{V})$  such that  $\pi_{T(\mathbb{U})}(z) = x$  and which is such that  $z_t = z_0 + \int_0^t \mathfrak{D}(f)(z_s) dz_s$  where  $\mathfrak{D}(f)$  is the differential form on  $\mathbb{U} \times \mathbb{V}$  defined by  $\mathfrak{D}(x, z) = 1 dx + f(z) dz$ , where 1 is the map from  $\mathbb{U} \times \mathbb{V}$  to  $L(\mathbb{U}, \mathbb{U})$  defined by  $1(x, z)a = a$  for all  $(x, z, a) \in \mathbb{U} \times \mathbb{V} \times \mathbb{U}$ .

We recall one of the main result in the theory of rough paths.

**Theorem 1** ([6, 12, 13]). *Let  $\gamma \in (0, 1]$  and  $p \in [2, 3)$  such that  $2 + \gamma > p$ . If  $f$  belongs to  $\text{Lip}(1 + \gamma)$  and  $x$  is of  $p$ -finite variation controlled by  $\omega$  with  $p \in [2, 3)$  then there exists a solution to (1) (this solution is not necessarily unique). If moreover,  $f$  belongs to  $\text{Lip}(2 + \gamma)$ , then the solution is unique. In this case,  $x \mapsto z$  is continuous from the space of  $p$ -rough paths from  $[0, T]$  to  $T_2(\mathbb{U})$  controlled by  $\omega$  to the space of  $p$ -rough paths from  $[0, T]$  to  $T_2(\mathbb{U} \times \mathbb{V})$  controlled by  $\omega$ .*

This result can be shown using a contractive fixed point theorem if  $f$  belongs to  $\text{Lip}(2 + \gamma)$  and by a Schauder fixed point theorem if  $f$  belongs to  $\text{Lip}(1 + \gamma)$ . It can also be deduced from the following result, the existence of a unique solution if  $x$  is smooth and the continuity result. The next Proposition is deduced from the proof of Theorem 1.

**Proposition 1.** *If  $x$  is a rough path of  $p$ -variation controlled by  $\omega$  and  $f$  belongs to  $\text{Lip}(1 + \gamma)$  with  $3 \geq 2 + \gamma > p \geq 2$ , then there is a  $\rho > 0$  depending only on  $\omega$ ,  $N(f)$ ,  $T$ ,  $p$  and  $\gamma$  such that any solution  $z$  to (1) satisfies  $\|z\|_{p, \omega} \leq \rho$ .*

*Remark 1.* Of course, from this, we deduce that  $t \mapsto z_t$  remains bounded in  $T^2(\mathbb{U} \times \mathbb{V})$ .

*Remark 2.* We can use this estimate as some *a priori* estimate, and use the fact that for a geometric rough path  $x$ , there exists a sequence of smooth rough paths  $x^n$  such that  $x^n$  converges to  $x$ . Thus, the solutions  $z^n$  of the corresponding equations driven by  $x^n$  are uniformly bounded with respect to  $\|\cdot\|_{p, \omega}$ , and if  $\mathbb{V}$  is a finite-dimensional space, it follows from the Ascoli-Arzelà theorem that  $(z^n)_{n \in \mathbb{N}}$  has a convergent subsequence both for the uniform norm and for the norm  $\|\cdot\|_{q, \omega}$  for any  $q > p$ . From the continuity of  $z \in T(\mathbb{U} \times \mathbb{V}) \mapsto \int g(z_s) dx_s$ , one gets that any limit of  $(z^n)_{n \in \mathbb{N}}$  is solution to (1) and is of finite  $p$ -variation controlled by  $\omega$ .

We will then extend Theorem 1 to the case of functions with a linear growth, as in the case of ordinary differential equations.

**Theorem 2.** *If  $x$  is a  $p$ -rough path with  $p$ -variation controlled by  $\omega$ ,  $f$  is a  $\text{Lip}_{\text{LG}}(1 + \gamma)$ -vector field with  $3 > 2 + \gamma > p \geq 2$ , then there exists a solution  $z$  to (1) in the sense of rough path. In addition, the  $p$ -variation of  $z$  is controlled by  $C\omega$  for some  $C$  that depends only on  $T$ ,  $\omega$ ,  $N(\nabla f)$ ,  $p$  and  $\kappa$ . If  $f$  is a vector field in  $\text{Lip}_{\text{LG}}(2 + \gamma)$ , then the solution is unique.*

From now, we denote by  $\mathfrak{I}$  the map  $(z_0, x, f) \mapsto z$ , which we call the *Itô map*.

For two elements  $z$  and  $\widehat{z}$  in  $V$ , we set  $\delta(z, \widehat{z}) = |\widehat{z} - z|$ . For two  $p$ -rough paths  $x$  and  $\widehat{x}$  of finite  $p$ -variation controlled by  $\omega$ , we set

$$\delta(x, \widehat{x}) = \sup_{(s,t) \in \Delta_+^2} \max \left\{ \frac{|\pi_{\text{U}}(x_{s,t} - \widehat{x}_{s,t})|}{\omega(s,t)^{1/p}}, \frac{|\pi_{\text{U} \otimes \text{U}}(x_{s,t} - \widehat{x}_{s,t})|}{\omega(s,t)^{2/p}} \right\}.$$

Finally, for  $f$  and  $\widehat{f}$  in  $\text{Lip}_{\text{LG}}(2 + \kappa)$  and  $\rho$  fixed, then we define  $\delta_\rho(f, \widehat{f})$  by

$$\delta_\rho(f, \widehat{f}) = \sup_{z \in B_{\text{V}}(\rho)} |f(z) - \widehat{f}(z)| + \sup_{z \in B_{\text{V}}(\rho)} |\nabla f(z) - \nabla \widehat{f}(z)|_{L(\text{U}, \text{V})},$$

where  $B_{\text{W}}(\rho) = \{z \in \text{W} \mid |z| \leq \rho\}$  for a Banach space  $\text{W}$ .

We now prove the continuity result, which improves the results on [6, 12, 13] for the continuity with respect to the signal, and the results from [2, 9] on the continuity with respect to the vector fields.

**Theorem 3.** *Let  $f$  and  $\widehat{f}$  be two  $\text{Lip}_{\text{LG}}(2 + \kappa)$ -vector fields and  $x, \widehat{x}$  be two paths of finite  $p$ -variation controlled by  $\omega$ , with  $2 \leq p < 2 + \kappa \leq 3$ . Denote by  $z$  and  $\widehat{z}$  the solutions to  $z = z_0 + \int_0^\cdot f(z_s) dx_s$  and  $\widehat{z} = \widehat{z}_0 + \int_0^\cdot f(\widehat{z}_s) d\widehat{x}_s$ . Assume that  $z$  and  $\widehat{z}$  belongs both to  $B_{T^2(\text{U} \times \text{V})}(\rho)$  and  $\|z\|_{p,\omega} + \|\widehat{z}\|_{p,\omega} \leq \rho$ . Then*

$$\delta(z, \widehat{z}) \leq C(\delta_\rho(f, \widehat{f}) + \delta(z_0, \widehat{z}_0) + \delta(x, \widehat{x})).$$

where  $C$  depends only on  $\rho$ ,  $T$ ,  $\omega$ ,  $p$ ,  $\kappa$ ,  $N(\nabla f)$  and  $N(\nabla \widehat{f})$ .

*Remark 3.* Let us note that this theorem implies also the uniqueness of the solution to (1) for a vector field in  $\text{Lip}_{\text{LG}}(2 + \gamma)$ .

## 4 Preliminary computations

We fix  $T > 0$  and we define  $\Delta_+^2 = \{(s, t) \in [0, T]^2 \mid s \leq t\}$  and  $\Delta_+^3 = \{(s, r, t) \in [0, T]^3 \mid s \leq r \leq t\}$ . Fix  $p \in (2, 3]$ .

Let  $W$  be a Banach space. For  $(y_{s,t})_{(s,t) \in \Delta_+^2}$  with  $y_{s,t}$  in  $T^2(W) = 1 \oplus W \oplus (W \otimes W)$ , define

$$\|y\|_{p,\omega} = \sup_{(s,t) \in \Delta_+^2} \max \left\{ \frac{|y_{s,t}^1|}{\omega(s,t)^{1/p}}, \frac{|y_{s,t}^2|}{\omega(s,t)^{2/p}} \right\}$$

when this quantity is finite. We have already seen that a rough path of finite  $p$ -variation controlled by  $\omega$  is by definition a function  $(x_s)_{s \in \Delta_+^1}$  with values in the Lie group  $(T^2(W), \otimes)$  to which one can associate a family  $(x_{s,t})_{(s,t) \in \Delta_+^2}$  by  $x_{s,t} = x_s^{-1} \otimes x_t$  such that  $\|x\|_{p,\omega}$  is finite.

We set  $y_{s,r,t} = y_{s,t} - y_{s,r} \otimes y_{r,t}$ . An *almost rough path* is a family  $(y_{s,t})_{(s,t) \in \Delta_+^2}$  such that  $\|y\|_{p,\omega}$  is finite and for some  $\theta > 1$  and some  $C > 0$

$$|y_{s,r,t}| \leq C\omega(s,t)^\theta. \quad (2)$$

Let us recall the following results on the construction of a rough path from an almost rough path (see for example [7, 8, 12, 13]).

**Lemma 1.** *If  $y$  is an almost rough path satisfying (2), then there exists a rough path  $x$  of finite  $p$ -variation such that for some universal constants  $K_1, K_2, K_3$  and  $K_4$  (depending only on  $\omega(0, T)$ ,  $\theta$  and  $p$ ),*

$$|y_{s,t}^1 - x_{s,t}^1| \leq K_1 C \omega(s,t)^\theta, \quad (3)$$

$$|y_{s,t}^2 - x_{s,t}^2| \leq C(K_2 + (K_3 + K_4 \|y\|_{p,\omega}) \omega(0, T)^{1/p}) \omega(s,t)^\theta \quad (4)$$

and  $x$  is of  $p$ -variation controlled by  $\tilde{\omega}$  with

$$\tilde{\omega}(s,t) = \|y\|_{p,\omega} C \max\{K_1, (K_1 + (K_3 + K_4 \|y\|_{p,\omega}) \omega(0, T)^{1/p})\} \omega(0, T)^{\theta-1/p}.$$

The rough path is unique in the sense that if  $z$  is another rough path of  $p$ -variation controlled by  $\omega$  and such that  $|y_{s,t} - z_{s,t}| \leq C' \omega(s,t)^{\theta'}$  for some  $C' > 0$  and  $\theta' > 1$ , then  $z = x$ .

**Lemma 2.** *Let  $y$  and  $\hat{y}$  be two almost rough paths such that for some  $\theta > 1$  and some constant  $C$ ,*

$$\|y\|_{p,\omega} \leq C, \quad \|\hat{y}\|_{p,\omega} \leq C, \quad |y_{s,r,t}| \leq C\omega(s,t)^\theta, \quad |\hat{y}_{s,r,t}| \leq C\omega(s,t)^\theta$$

and for some  $\varepsilon > 0$ ,

$$\|y - \hat{y}\|_{p,\omega} \leq \varepsilon \quad \text{and} \quad |y_{s,r,t} - \hat{y}_{s,r,t}| \leq \varepsilon \omega(s,t)^\theta.$$

Then the rough paths  $x$  and  $\hat{x}$  associated respectively to  $y$  and  $\hat{y}$  are such that

$$|x_{s,t} - \hat{x}_{s,t}| \leq \varepsilon K \omega(s,t)^\theta$$

for some constant  $K$  that depends only on  $\omega(0, T)$ ,  $\theta$ ,  $p$  and  $C$ .

Given a solution  $z$  of (1) for a vector field  $f$  which is  $\text{Lip}(1 + \gamma)$  with  $2 + \gamma > 2$  (we know that (1) may have a solution, but which is not necessarily unique), set

$$y_{s,t} = 1 + x_{s,t} + f(z_s)x_{s,t}^1 + \nabla f(z_s)z_{s,t}^\times + f(z_s) \otimes 1 \cdot x_{s,t}^2 + 1 \otimes f(z_s) \cdot x_{s,t}^2 + f(z_s) \otimes f(z_s) \cdot x_{s,t}^2, \quad (5)$$

where  $z^\times = \pi_{V \otimes U}(z)$ ,  $x^1 = \pi_U(x)$  and  $x^2 = \pi_{U \otimes U}(x)$ . In (5), if  $a$  (resp.  $b$ ) is a linear form from a Banach space  $X$  to a Banach space  $X'$  (resp. from  $Y$  to  $Y'$ ), and  $x$  belongs to  $X$  (resp.  $Y$ ), then we denote by  $a \otimes b$  the bilinear form from  $X \otimes Y$  to  $X' \otimes Y'$  defined by  $a \otimes b \cdot x \otimes y = a(x) \otimes b(y)$ . In the previous expression,  $f(z_s)x_{s,t}^1 + \nabla f(z_s)z_{s,t}^\times$  projects on  $V$ ,  $f(z_s) \otimes f(z_s) \cdot x_{s,t}^2$  projects on  $V \otimes V$ ,  $1 \otimes f(z_s) \cdot x_{s,t}^2$  projects on  $U \otimes V$  while  $f(z_s) \otimes 1 \cdot x_{s,t}^2$  projects on  $V \otimes U$ .

The result in the next Lemma is a direct consequence of the definition of the iterated integral. However, we give its proof, since some of the computations will be used later, and  $y$  is the main object we will work with.

**Lemma 3.** *For a rough path  $x$  of finite  $p$ -variation controlled by  $\omega$  and  $f$  in  $\text{Lip}(1 + \gamma)$  with  $2 + \gamma > p$ , the family  $(y_{s,t})_{(s,t) \in \Delta_+^2}$  defined by (5) for a solution  $z$  to (1) is an almost rough path whose associated rough path is  $z$ .*

*Proof.* In the proof, we assume that  $z$  is controlled by  $\omega$ . Since  $x_{s,t} = x_{s,r} \otimes x_{r,t}$ , note that

$$y_{s,r,t} := y_{s,t} - y_{s,r} \otimes y_{r,t} \quad (6a)$$

$$= (\nabla f(z_s) - \nabla f(z_r))z_{r,t}^\times \quad (6b)$$

$$+ (f(z_s) - f(z_r))x_{r,t}^1 + \nabla f(z_s)(z_{s,t}^\times - z_{s,r}^\times - z_{r,t}^\times) \quad (6c)$$

$$+ (f(z_s) - f(z_r)) \otimes 1 \cdot x_{r,t}^2 + 1 \otimes (f(z_s) - f(z_r)) \cdot x_{r,t}^2 \quad (6d)$$

$$+ (f(z_s) \otimes f(z_s) - f(z_r) \otimes f(z_r)) \cdot x_{r,t}^2 \quad (6e)$$

$$+ (f(z_s) \otimes (f(z_r) - f(z_s))) \cdot x_{s,r}^1 \otimes x_{r,t}^1 \quad (6f)$$

$$+ (1 \otimes (f(z_s) - f(z_r))) \cdot x_{s,r}^1 \otimes x_{r,t}^1. \quad (6g)$$

We denote by  $L_{(i)}$  the quantity of Line (i).

Since  $z_{s,t} = z_{s,r} \otimes z_{r,t}$ , we get that  $z_{s,t}^\times = z_{s,r}^\times + z_{r,t}^\times + z_{s,r}^1 \otimes x_{r,t}^1$  and then that Line (6c) is equal to

$$L_{(6c)} := (f(z_s) - f(z_r))x_{r,t}^1 + \nabla f(z_s)z_{s,r}^1 \otimes x_{r,t}^1.$$



Since  $f$  is in  $\text{Lip}(1 + \gamma)$ ,

$$\begin{aligned} |L_{(6c)}| &:= \left| \int_0^1 \nabla f(z_s + \tau z_{s,r}^1) z_{s,r}^1 d\tau - \nabla f(z_s) z_{s,r}^1 \right| \cdot |x_{r,t}^1| \\ &\leq C_1 \omega(s, t)^{(2+\gamma)/p} \text{ with } C_1 \leq H_\gamma(\nabla f). \end{aligned}$$

Similarly, since  $|x_{s,r}^1 \otimes x_{r,t}^1| \leq \omega(s, t)^{2/p}$ ,

$$\begin{aligned} |L_{(6b)}| &\leq C_2 \omega(s, t)^{(2+\gamma)/p}, \quad |L_{(6d)}| \leq C_3 \omega(s, t)^{3/p}, \quad |L_{(6e)}| \leq C_4 \omega(s, t)^{3/p} \\ |L_{(6f)}| &\leq C_5 \omega(s, t)^{3/p} \text{ and } |L_{(6g)}| \leq C_6 \omega(s, t)^{3/p} \end{aligned}$$

with  $C_2 \leq H_\gamma(\nabla f)$ ,  $C_3 \leq \|\nabla f\|_\infty$ ,  $C_4 \leq 2\|f\|_\infty \|\nabla f\|_\infty$ ,  $C_5 \leq \|f\|_\infty \|\nabla f\|_\infty$  and  $C_6 \leq \|\nabla f\|_\infty$ .

Thus,  $|y_{s,r,t}| \leq C_7 \omega(s, t)^{2+\gamma}(s, t)$ . In addition, one gets easily that

$$\|y\|_{p,\omega} \leq \max\{\|f\|_\infty + \|\nabla f\|_\infty \omega(0, T)^{1/p}, \|f\|_\infty^2\}.$$

This implies that  $y$  is an almost rough path. Of course, the rough path associated to  $y$  is  $z$  from the very definition of the integral of a one-form along the rough path  $z$ .  $\square$

The proof of the following Lemma is immediate and will be used to localize.

**Lemma 4.** *Let us assume that  $z$  is a rough path of finite  $p$ -variations in  $T(U \times V)$  such for some  $\rho > 0$ ,  $|z_t| \leq \rho$  for  $t \in [0, T]$  and let us consider two vector fields  $f$  and  $g$  in  $\text{Lip}(1 + \gamma)$  with  $f(\pi_V(x)) = g(\pi_V(x))$  and  $\nabla f(\pi_V(x)) = \nabla g(\pi_V(x))$  for  $|x| \leq \rho$ . Then  $\int_0^t \mathfrak{D}(f)(z_s) dz_s = \int_0^t \mathfrak{D}(g)(z_s) dz_s$  for all  $t \in [0, T]$ .*

## 5 Proof of Theorem 2 on the linear growth

For a vector field  $f \in \text{Lip}(1 + \gamma)$ , a rough path  $x$  in  $T(U)$  of finite  $p$ -variation controlled by  $\omega$  and an initial point  $z_0$  in  $T(U \times V)$ , let us consider the sequence of rough paths with values in  $T(U \times V)$  defined by  $z_t^0 = (z_0, x_t)$  and

$$z_t^{n+1} = z_0 + \int_0^t g(z_s^n) dz_s^n,$$

where  $g = \mathfrak{D}(f)$ . In order to simplify the notations, we extend  $g$  to  $T(U \times V)$  by setting  $g(z) = g(\pi_{U \times V}(z))$  and  $g$  will be confounded as a function on  $T(U \times V)$ . For  $b \in T(U \times V)$ , we denote by  $|g(b)|$  the operator norm of

$g(b)$  which satisfies  $|g(b)a|_{U \times V} \leq |g(z_0)| \cdot |a|$  for all  $a \in U \times V$ . Thus, we this convention,  $\pi_{T(U)}(z^n) = x$ , and this procedure corresponds in defining a Picard sequence.

Set  $\tilde{z}^n = \pi_{T(U) \oplus V \oplus (U \otimes V)}(z^n)$ .

We denote by  $B(\tilde{z}_0, \rho)$  the ball of radius  $\rho$  in  $T(U) \oplus V \oplus (U \otimes V)$  centered on  $\tilde{z}_0$ . Our aim is to show that for a given  $\rho$  that can be chosen independently from  $\|f\|_\infty$ , if  $\tilde{z}_t^n$  belongs to  $B(\tilde{z}_0, \rho)$  for  $t \in [0, T]$ , then  $\tilde{z}^{n+1}$  stays in  $B(\tilde{z}_0, \rho)$ .

If  $f$  is a  $(2 + \gamma)$ -Lipschitz vector field, then it is now a classical result that  $(z^n)_{n \in \mathbb{N}}$  converges to the solution  $z$  to  $z = z_0 + \int_0^\cdot f(z_s) dx_s$ , thanks to a contractive fixed point theorem. If  $f$  is only a  $(1 + \gamma)$ -Lipschitz vector field, then, for  $T$  small enough, then there exists some  $R$  that depends on  $\omega$ ,  $p$ ,  $\|f\|_\infty$ ,  $\|\nabla f\|_\infty$  and  $H_\gamma(\nabla f)$  such that  $\|\tilde{z}^n\|_\infty + \|\tilde{z}^n\|_{p,\omega} \leq R$  for any  $n \in \mathbb{N}$ . The projection  $\tilde{z}$  on  $T(U) \oplus V \oplus (U \otimes V)$  of a solution  $z$  to (1) is then a limit of a subsequence of  $(z^n)_{n \in \mathbb{N}}$ .

Let us set

$$y_{s,t}^{n+1} = 1 + x_{s,t} + g(z_s^n)x_{s,t}^1 + \nabla g(z_s^n)z_{s,t}^{\times,n} + g(z_s^n) \otimes 1 \cdot x_{s,t}^2.$$

As in the proof of Lemma 3, for  $n = 0, 1, 2, \dots$ ,  $y^{n+1}$  is an almost rough path for  $n \geq 1$ . Let us note that knowing  $\tilde{z}^n$  is sufficient to compute  $y^{n+1}$ , and to indeed reconstruct  $z^{n+1}$  with the full expression (5). As in the proof of Lemma 3, but keeping only the relevant terms,

$$|y_{s,t}^{n+1} - y_{s,r}^{n+1} \otimes y_{r,t}^{n+1}| \leq C_8 \omega(s, t)^{(2+\gamma)/p}$$

where  $C_8$  depends on  $\|\nabla g\|_\infty$  and  $H_\gamma(\nabla g)$  (this is why we have considered to work with  $\tilde{z}^n$  instead to  $z^n$ , in order to get rid of the terms in  $V \otimes V$ ).

Let us assume that for some  $\tau \in [0, T]$  and some  $\rho \geq 1$ ,  $\tilde{z}_t^n$  belongs to  $B(\tilde{z}_0, \rho)$  for any  $t \in [0, \tau]$ . Then

$$\begin{aligned} |y_{s,t}^{1,n+1}| &\leq (|g(z_0)| + \|\nabla g\|_\infty \rho \omega(0, T)^{1/p}) \omega(s, t)^{1/p} + \|\nabla g\|_\infty \omega(s, t)^{2/p}, \\ |y_{s,t}^{2,n+1}| &\leq (|g(z_0)| + \|\nabla g\|_\infty \rho \omega(0, T)^{1/p}) \omega(s, t)^{2/p}. \end{aligned}$$

As  $\rho \geq 1$ , this implies that

$$\|y^{n+1}\|_{p,\omega} \leq (|g(z_0)| + \rho C_9) \text{ with } C_9 = 2\|\nabla g\|_\infty \omega(0, T)^{1/p}. \quad (7)$$

We have seen that there exist some universal constants  $K_1$ ,  $K_2$ ,  $K_3$  and  $K_4$  such that

$$\begin{aligned} |\tilde{z}_{s,t}^{1,n+1} - y_{s,t}^{1,n+1}| &\leq K_1 C_8 \omega(s, t)^{(2+\gamma)/p}, \\ |\tilde{z}_{s,t}^{2,n+1} - y_{s,t}^{2,n+1}| &\leq C_8 (K_2 + \omega(0, T)^{1/p} (K_3 + K_4 \|y^{n+1}\|_{p,\omega})) \omega(s, t)^{(2+\gamma)/p}. \end{aligned}$$

With (7),

$$\|\tilde{z}^{n+1}\|_{p,\omega} \leq \Gamma(\rho) := C_{10}(1 + |g(z_0)| + \rho C_9).$$

Since  $\tilde{z}^{n+1}$  is of finite  $p$ -variation,

$$|\tilde{z}_t^{n+1} - \tilde{z}_0| \leq \Gamma(\rho)\alpha_{0,t} \text{ with } \alpha_{0,t} = \max\{\omega(0,t)^{1/p}, \omega(0,t)^{2/p}\}.$$

It is then possible to choose  $\tau$  small enough so that

$$\alpha_{0,\tau}\Gamma(\rho) \leq \rho$$

to get that  $\tilde{z}^{n+1}$  remains in  $B(\tilde{z}_0, \rho)$  for  $t \in [0, \tau]$ .

Now, if  $z^0 = z_0$ , then

$$\begin{aligned} \pi_{T(U)}(\tilde{z}^1) &= x, \\ \pi_V(\tilde{z}_t^1) &= \pi_V(z_0) + g(z_0)x_{0,t}^1, \\ \pi_{V \otimes U}(\tilde{z}_t^1) &= \pi_{V \otimes U}(z_0) + \nabla g(z_0)x_{0,t}^2. \end{aligned}$$

Setting

$$\rho = \max\{1, |g(z_0)|, \|\nabla g(z)\|_\infty\} \|x\|_{p,\omega} \alpha_{0,T}$$

we get that  $\tilde{z}_t^1$  belongs to  $B(\tilde{z}_0, \rho)$  for  $t \leq T$ .

Then  $\tilde{z}_t = \pi_{U \oplus V \oplus V \otimes U}(z_t)$  also belongs to  $B(\tilde{z}_0, \rho)$  for  $t \in [0, \tau]$ , since  $\tilde{z}_t^n$  belongs to  $B(\tilde{z}_0, \rho)$  for  $t \in [0, \tau]$  with  $\tau$  such that

$$\alpha_{0,\tau} \leq \frac{C_{10}^{-1} \rho}{1 + |g(z_0)| + \rho C_9}.$$

From now, we set  $\tau_0 = 0$  and  $\rho_0 = 1$ . We set recursively

$$\rho_i := \max\{\rho_{i-1}, \max\{1, |g(z_{\tau_i})|, \|\nabla g\|_\infty\} \|x\|_{p,\omega} \alpha_{0,T}\}$$

and find  $\tau_{i+1} \in [\tau_i, T]$  such that

$$\alpha_{\tau_i, \tau_{i+1}} \leq \frac{C_{10}^{-1} \rho_i}{1 + |g(z_{\tau_i})| + \rho_i C_9} < \alpha_{\tau_i, \tau_{i+1} +}$$

unless

$$\alpha_{\tau_i, T} > \frac{C_{10}^{-1} \rho_i}{1 + |g(z_{\tau_i})| + \rho_i C_9},$$

in which case we set  $\tau_{i+1} = T$ . On  $[\tau_i, \tau_{i+1}]$ ,  $\tilde{z}$  belongs to  $B(\tilde{z}_{\tau_i}, \rho_i)$ . The question is now whether or not  $\tau_{i_0} = T$  for some integer  $i_0$ . If it is the case, then one can find a finite  $\rho$  such that  $\tilde{z}_t$  belongs to  $B(\tilde{z}, \rho)$  for  $t \in [0, T]$ .

Of course, if  $\alpha_{\tau_i, \tau_{i+1}} > c > 0$  for some constant  $c$ , then number of intervals  $[\tau_i, \tau_{i+1}]$  to cover  $[0, T]$  is finite.

If the sequence  $(\alpha_{\tau_i, \tau_{i+1}})_{i \in \mathbb{N}}$  is not bounded from below, then  $\tau_{i+1} - \tau_i$  decreases to 0 and since  $\omega$  is continuous near its diagonal, we may assume that  $\alpha_{\tau_i, \tau_{i+1}} = \alpha_{\tau_i, \tau_{i+1}+}$  after a given rank  $i_1$ .

Since  $|g(z_{\tau_i}) - g(z_{\tau_{i-1}})| \leq \rho_{i-1} \|\nabla g(z)\|_\infty \cdot \alpha_{\tau_{i-1}, \tau_i}$  and  $\alpha_{\tau_{i-1}, \tau_i} \leq \alpha_{0, T}$ ,

$$\rho_{i+1} \leq \rho_i + \rho_i \|\nabla g(z)\|_\infty \|x\|_{p, \omega} \alpha_{0, T} \leq L \rho_i \text{ with } L = 1 + \|\nabla g\| \cdot \|x\|_{p, \omega} \alpha_{0, T}^2.$$

On the other hand, if  $\tau_i < T$ ,  $\alpha_{\tau_{i-1}, \tau_i} > 0$  and for  $i \geq i_1$ ,

$$\begin{aligned} 1 + |g(z_{\tau_i})| + \rho_i C_9 &\leq 1 + |g(z_{\tau_{i-1}})| + (\rho_{i-1} + \rho_i) C_9 \\ &\leq \frac{1}{\alpha_{\tau_{i-1}, \tau_i}} \frac{\rho_{i-1}}{C_{10}} + \frac{L_{i-1} \rho_{i-1}}{C_{10}} C_9. \end{aligned}$$

Thus, since  $\rho_i \geq \rho_{i-1}$ ,

$$\frac{1}{\alpha_{\tau_i, \tau_{i+1}}} \leq \frac{1}{\alpha_{\tau_{i-1}, \tau_i}} + L \rho_1 C_9.$$

and then

$$\alpha_{\tau_i, \tau_{i+1}} \geq \frac{1}{\frac{1}{\alpha_{0, \tau_1}} + i C_9 L}. \quad (8)$$

Let us remark that

$$\sum_{i=0}^n \omega(\tau_i, \tau_{i+1})^{1/p} \leq \omega(0, T)^{1/p}, \quad \forall n \in \mathbb{N},$$

and then, since  $\alpha_{s, t} \leq \omega(s, t)^{1/p} \max\{1, \omega(0, T)^{1/p}\}$ ,

$$\sum_{i=0}^n \alpha_{\tau_i, \tau_{i+1}} \leq \max\{1, \omega(0, T)^{1/p}\} \omega(0, \tau)^{1/p}, \quad \forall n \in \mathbb{N}.$$

Thus, if  $\alpha_{\tau_i, \tau_{i+1}} > 0$  for all  $i \geq 0$ , the series  $\sum_{i=0}^{+\infty} \alpha_{\tau_i, \tau_{i+1}}$  is finite. Because of (8), this cannot happen. In particular, there exists a rank  $i_0$  such that  $\tau_{i_0} = T$ .

Then there exists a finite value  $\rho$  such that  $\tilde{z}_t$  belongs to  $B(\tilde{z}_0, \rho)$  for all  $t \in [0, T]$ .

We now define  $y$  with the help of (5). Thus, for  $(s, r, t) \in \Delta_+^3$ ,

$$\begin{aligned} |y_{s, t} - y_{s, r, t}| &\leq C_{11} \max\{1, \|g(z)\|_\infty, \|\nabla g(z)\|_\infty, H_\gamma(g)\}^2 \omega(s, t)^{(2+\gamma)/p}, \\ \text{and } \|y_{s, t}\|_{p, \omega} &\leq \max\{\|g(z)\|_\infty, \|g(z)\|_\infty^2\}. \end{aligned}$$

We then deduce that there exists a  $\rho' > 0$  such that  $z_t$  belongs to  $B(z_0, \rho')$  for all  $t \in [0, T]$ .

Now, let us note that the choice of the  $\rho_i$ 's depends only on  $|g(z_0)|$ ,  $\|\nabla g\|_\infty$ , and  $x$ .

Given  $f$  in  $\text{Lip}_{\text{LG}}(1 + \gamma)$ , one can choose a  $\rho$  large enough in function of  $z_0$ ,  $f(z_0)$ ,  $\|\nabla f\|_\infty$  and select a vector field  $h$  in  $\text{Lip}(1 + \gamma)$  such that  $\|\nabla h\|_\infty \leq \|\nabla f\|_\infty$ , the solution to  $z_t = z_0 + \int_0^t h(z_s) dx_s$  belongs to  $B(z_0, \rho)$  and  $f = h$  on  $B(z_0, \rho)$ . With Lemma 4, this proves the result.

## 6 Proof of Theorem 3 on the Lipschitz continuity of the Itô map

In order to understand how we get the Lipschitz continuity under the assumption that  $f$  and  $\hat{f}$  belongs to  $\text{Lip}_{\text{LG}}(2 + \gamma)$  with  $2 + \gamma > p$ , we evaluate first the distance between to almost rough paths associated to solution of controlled differential equations when the vector field only belongs to  $\text{Lip}_{\text{LG}}(1 + \gamma)$ . Without loss of generality, with Proposition 1 and Lemma 4, we assume that indeed  $f$  and  $\hat{f}$  are bounded and belongs to  $\text{Lip}(1 + \gamma)$ .

### 6.1 On the distance between the almost rough paths associated to controlled differential equations

For a rough path  $\hat{x}$  of finite  $p$ -variation, and let  $\hat{z}$  be the solution to  $\hat{z}_t = \hat{z}_0 + \int_0^t \hat{f}(\hat{z}_s) d\hat{x}_s$ . Define  $\hat{y}$  with the same formula as in (5) by replacing  $z$  (resp.  $f$ ,  $x$ ) by  $\hat{z}$  (resp.  $\hat{f}$ ,  $\hat{x}$ ).

We set  $z^1 = \pi_V(z)$  and  $\hat{z}^1 = \pi_V(\hat{z})$ , and we set  $\|z_s\|_{\infty, [s, t]} := \sup_{r \in [s, t]} |z_r|$ .

We set also  $\delta(f, \hat{f}) = \|f - \hat{f}\|_\infty$  for two functions  $f$  and  $\hat{f}$ .

Let us note that

$$\begin{aligned} |y_{s,t}^1 - \hat{y}_{s,t}^1| &\leq |(f(z_s) - f(\hat{z}_s))x_{s,t}^1| + |(f(\hat{z}_s) - \hat{f}(\hat{z}_s))x_{s,t}^1| + |\hat{f}(\hat{z}_s)(x_{s,t}^1 - \hat{x}_{s,t}^1)| \\ &\quad + |(\nabla f(z_s) - \nabla f(\hat{z}_s))z_{s,t}^\times| + |(\nabla f(\hat{z}_s) - \nabla \hat{f}(\hat{z}_s))z_{s,t}^\times| \\ &\quad + |\nabla \hat{f}(\hat{z}_s)(z_{s,t}^\times - \hat{z}_{s,t}^\times)| + |x_{s,t}^1 - \hat{x}_{s,t}^1|. \end{aligned}$$

It follows that

$$\begin{aligned} |y_{s,t}^1 - \hat{y}_{s,t}^1| &\leq C_{12}(\|z^1 - \hat{z}^1\|_{\infty, [s, t]} + \delta(f, \hat{f}) + \delta(x, \hat{x}))\omega(s, t)^{1/p} \\ &\quad + H_\gamma(\nabla f)(\|z^1 - \hat{z}^1\|_{\infty, [s, t]}^\gamma + \delta(\nabla f, \nabla \hat{f}))\|z\|_{p, \omega}\|x\|_{p, \omega}\omega(s, t)^{2/p} \\ &\quad + \|\nabla \hat{f}\|_\infty \delta(z, \hat{z})\|x\|_{p, \omega}\omega(s, t)^{2/p} \quad (9) \end{aligned}$$

with

$$C_{12} \leq \max\{\|x\|_{p,\omega}, \|\widehat{x}\|_{p,\omega}\} \max\{1, \|\widehat{f}\|_\infty, \|\nabla f\|_\infty\}.$$

With similar computations,

$$|y_{s,t}^2 - \widehat{y}_{s,t}^2| \leq C_{13}(\delta(f, \widehat{f}) + \delta(x, \widehat{x}) + \|z^1 - \widehat{z}^1\|_{\infty,[s,t]})\omega(s,t)^{2/p}. \quad (10)$$

We are now willing to estimate  $|y_{s,r,t} - \widehat{y}_{s,r,t}|$ . By watching at the expressions (6b)–(6e), we get

$$|y_{s,r,t} - \widehat{y}_{s,r,t}| \leq C_{14}(\delta(f, \widehat{f}) + \delta(\nabla f, \nabla \widehat{f}) + \delta(x, \widehat{x}))\omega(s,t)^{(2+\gamma)/p} \quad (11a)$$

$$+ |(f(z_s) - f(z_r))x_{r,t}^1 + \nabla f(z_s)z_{s,r}^1 \otimes x_{r,t}^1 \quad (11b)$$

$$- (f(\widehat{z}_s) - f(\widehat{z}_r))x_{r,t}^1 - \nabla f(\widehat{z}_s)\widehat{z}_{s,r}^1 \otimes x_{r,t}^1| \quad (11c)$$

$$+ |(f(z_s) - f(z_r)) \otimes 1 \cdot x_{r,t}^2 + 1 \otimes (f(z_s) - f(z_r)) \cdot x_{r,t}^2 \quad (11d)$$

$$- (f(\widehat{z}_s) - f(\widehat{z}_r)) \otimes 1 \cdot x_{r,t}^2 + 1 \otimes (f(\widehat{z}_s) - f(\widehat{z}_r)) \cdot x_{r,t}^2| \quad (11e)$$

$$+ |(f(z_s) \otimes f(z_s) - f(z_r) \otimes f(z_r)) \cdot x_{r,t}^2 \quad (11f)$$

$$- (f(\widehat{z}_s) \otimes f(\widehat{z}_s) - f(\widehat{z}_r) \otimes f(\widehat{z}_r)) \cdot x_{r,t}^2| \quad (11g)$$

$$+ |f(z_s) \otimes (f(z_s) - f(z_r)) \cdot x_{s,r}^1 \otimes x_{r,t}^1 \quad (11h)$$

$$- f(\widehat{z}_r) \otimes (f(\widehat{z}_s) - f(\widehat{z}_r)) \cdot x_{s,r}^1 \otimes x_{r,t}^1| \quad (11i)$$

$$+ |1 \otimes (f(z_s) - f(z_r) - f(\widehat{z}_s) + f(\widehat{z}_r)) \cdot x_{s,r}^1 \otimes x_{r,t}^1|. \quad (11j)$$

First we consider Lines (11d)–(11j). For this, let us note that

$$\begin{aligned} |(f(z_r) - f(z_s)) - (f(\widehat{z}_r) - f(\widehat{z}_s))| &= \left| \int_0^1 \nabla f(z_s + \tau(\widehat{z}_s - z_s))(\widehat{z}_s - z_s) d\tau \right. \\ &\quad \left. - \int_0^1 \nabla f(z_r + \tau(\widehat{z}_r - z_r))(\widehat{z}_r - z_r) d\tau \right| \\ &\leq \left| \int_0^1 (\nabla f(z_s + \tau(\widehat{z}_s - z_s)) - \nabla f(z_r + \tau(\widehat{z}_r - z_r)))(\widehat{z}_r - z_r) d\tau \right| \\ &\quad + \left| \int_0^1 \nabla f(z_r + \tau(\widehat{z}_r - z_r))(\widehat{z}_{s,r}^1 - z_{s,r}^1) d\tau \right| \\ &\leq \|z^1 - \widehat{z}^1\|_{\infty,[s,t]} \int_0^1 |\tau(z_s^1 - z_r^1) + (1 - \tau)(\widehat{z}_s^1 - \widehat{z}_r^1)|^\gamma d\tau \\ &\quad + C_{16}\delta(z, \widehat{z})\omega(s,t)^{1/p} \\ &\leq C_{15}\omega(s,t)^\gamma \|z^1 - \widehat{z}^1\|_{\infty,[s,t]} (\|z\|_{p,\omega}^\gamma + \|\widehat{z}\|_{p,\omega}^\gamma) \\ &\quad + C_{16}\delta(z, \widehat{z})\omega(s,t)^{1/p}. \quad (12) \end{aligned}$$

With this, we get a control over the lines (11d)-(11e), (11f)-(11g), (11h)-(11i) and (11j) of type

$$\begin{aligned} & |L_{(11d)-(11e)} + L_{(11f)-(11g)} + L_{(11h)-(11i)} + L_{(11j)}| \\ & \leq C_{17}\omega(s, t)^{(2+\gamma)/p} \|z^1 - \widehat{z}^1\|_{\infty, [s, t]} (\|z\|_{p, \omega}^\gamma + \|\widehat{z}\|_{p, \omega}^\gamma) \\ & \quad + C_{18}\delta(z, \widehat{z})\omega(s, t)^{3/p}. \end{aligned} \quad (13)$$

For (11b)-(11c), we use the same kind of computation as we did for  $L_{(6c)}$  and then,

$$\begin{aligned} & |L_{(11b)-(11c)}| \\ & \leq \omega(s, t)^{1/p} \left| \int_0^1 (\nabla f(z_s + \tau z_{s,r}^1) - \nabla f(z_s)) z_{s,r}^1 d\tau \right. \\ & \quad \left. - \int_0^1 (\nabla f(\widehat{z}_s + \tau \widehat{z}_{s,r}^1) - \nabla f(\widehat{z}_s)) \widehat{z}_{s,r}^1 d\tau \right| \\ & \leq \omega(s, t)^{1/p} \left| \int_0^1 ((\nabla f(z_s + \tau z_{s,r}^1) - \nabla f(z_s)) - (\nabla f(\widehat{z}_s + \tau \widehat{z}_{s,r}^1) - \nabla f(\widehat{z}_s))) z_{s,r}^1 d\tau \right| \\ & \quad + \omega(s, t)^{1/p} \left| \int_0^1 (\nabla f(\widehat{z}_s + \tau \widehat{z}_{s,r}^1) - \nabla f(\widehat{z}_s)) (\widehat{z}_{s,r}^1 - z_{s,r}^1) d\tau \right|. \end{aligned}$$

The last term in the previous inequality is bounded by  $\omega(s, t)^{(2+\gamma)/p} \|\widehat{z}\|_{p, \omega}^\gamma \delta(z, \widehat{z})$ .

On the other hand, for  $\tau \in [0, 1]$ , if

$$\Delta := |((\nabla f(z_s + \tau z_{s,r}^1) - \nabla f(z_s)) - (\nabla f(\widehat{z}_s + \tau \widehat{z}_{s,r}^1) - \nabla f(\widehat{z}_s)))|$$

then

$$\Delta \leq \begin{cases} 2H_\gamma(\nabla f) \|z^1 - \widehat{z}^1\|_{\infty, [s, t]}^\gamma \\ 2H_\gamma(\nabla f) (\|z\|_{p, \omega} + \|\widehat{z}\|_{p, \omega}) \omega(s, t)^{\gamma/p}. \end{cases} \quad (14)$$

Thus, we may combine choose  $\eta \in (0, 1)$  and combine the two terms in (14) to obtain

$$\Delta \leq C_{19}\omega(s, t)^{\eta\gamma/p} (\|z\|_{p, \omega} + \|\widehat{z}\|_{p, \omega})^{\gamma\eta} \|z^1 - \widehat{z}^1\|_{\infty, [s, t]}^{(1-\eta)\gamma}.$$

It follows that

$$\begin{aligned} |L_{(11b)-(11c)}| & \leq \omega(s, t)^{(2+\gamma)/p} \delta(z, \widehat{z}) \\ & \quad + C_{20} (\|z\|_{p, \omega} + \|\widehat{z}\|_{p, \omega})^{\gamma\eta} \|z^1 - \widehat{z}^1\|_{\infty, [s, t]}^{(1-\eta)\gamma} \omega(s, t)^{(2+\eta\gamma)/p}. \end{aligned} \quad (15)$$

If  $f$  belongs to  $\text{Lip}(2 + \kappa)$  with  $2 + \kappa > p$ , then one can set  $\gamma = 1$  in the previous equations. In this case, one can give a better estimate on  $L_{(11b)-(11c)}$ , which we use below to prove the Lipschitz continuity of the Itô map. In this case, we use the same computation as in (12) by replacing  $f$  by  $\nabla f$  to get

$$|L_{(11b)-(11c)}| \leq \omega(s, t)^{3/p} \delta(z, \widehat{z}) + C_{21}\omega(s, t)^{(2+\kappa)/p} \|z^1 - \widehat{z}^1\|_{\infty, [s, t]}. \quad (16)$$

**On the difference between the two cases.** In order to prove the Lipschitz continuity of the Itô map, which implies uniqueness using a regularization procedure, we will see that we will get an inequality of type  $\delta(z, \widehat{z}) \leq A(1 + C_T \delta(z, \widehat{z}))$  with  $C_T$  converges to 0, and then one can compare  $\delta(z, \widehat{z})$  with  $A$ . This is possible thanks to (16). With (15), we get an inequality of type  $\delta(z, \widehat{z}) \leq A(1 + C_T \delta(z, \widehat{z})^\lambda)$  with  $\lambda < 1$  and then it is impossible to deduce an inequality on  $\delta(z, \widehat{z})$  in function of  $A$  when  $\delta(z, \widehat{z}) \leq 1$ . Anyway, in [3], A.M. Davies showed that there could exist several solutions to a rough differential equation for  $f$  in  $\text{Lip}(1 + \gamma)$  but not in  $\text{Lip}(2 + \gamma)$ .

## 6.2 Proof of Theorem 3

Since the solutions of rough differential equations remains bounded, we may assume that  $f$  belongs to  $\text{Lip}(2 + \kappa)$  instead of  $\text{Lip}_{\text{LG}}(2 + \kappa)$ . To simplify the notations, we set  $\|f\|_\infty = \sup_{x \in B_V(\rho)} |f(x)|_{L(U, V)}$ .

We now assume that  $f$  is a  $(2 + \kappa)$ -Lipschitz vector field with  $2 \leq p \leq 2 + \kappa < 3$ .

Let us note that

$$\|z - \widehat{z}\|_\infty \leq \delta(z_0, \widehat{z}_0) + C_{0,T} \delta(z, \widehat{z}) \text{ with } C_{s,t} = \max\{\omega(s, t)^{1/p}, \omega(s, t)^{2/p}\}. \quad (17)$$

Note that  $C_{0,T}$  decreases to 0 as  $T$  converges to 0.

Since  $\gamma = 1$ , we have obtained from (16) and (17) that for all  $(s, r, t) \in \Delta_T^3$ ,

$$\begin{aligned} |y_{s,r,t} - \widehat{y}_{s,r,t}| & \leq C_{22}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0)) \\ & \quad + \omega(s, t)^{(1-\kappa)/p} \delta(z, \widehat{z}) + C_{0,T} \delta(z, \widehat{z}) \omega(s, t)^{(2+\kappa)/p}, \end{aligned} \quad (18a)$$

$$\begin{aligned} |y_{s,t}^1 - \widehat{y}_{s,t}^1| & \leq C_{23}(C_{0,T} \delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x})) \omega(s, t)^{1/p} \\ & \quad + C_{24}((1 + C_{0,T}) \delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(f, \widehat{f})) \omega(s, t)^{2/p} \end{aligned} \quad (18b)$$

$$|y_{s,t}^2 - \widehat{y}_{s,t}^2| \leq C_{25}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + C_{0,T} \delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0)) \omega(s, t)^{2/p}, \quad (18c)$$

$$|y_{s,r,t}| \leq C_{26} \omega(s, t)^{3/p} \text{ and } |\widehat{y}_{s,r,t}| \leq C_{27} \omega(s, t)^{3/p}. \quad (18d)$$

For convenience, we assume that  $\kappa < 1$ .

Since  $z$  (resp.  $\widehat{z}$ ) is the rough path associated to  $y$  (resp.  $\widehat{y}$ ), it follows from (18a)-(18d) and Lemma 1 that

$$|y_{s,t} - z_{s,t}| \leq C_{28} \omega(s, t)^{3/p}, |\widehat{y}_{s,t} - \widehat{z}_{s,t}| \leq C_{28} \omega(s, t)^{3/p}.$$

and from Lemma 2 that

$$|z_{s,t}^1 - \widehat{z}_{s,t}^1| \leq \beta(T) \omega(s, t)^{1/p} \text{ and } |z_{s,t}^2 - \widehat{z}_{s,t}^2| \leq \beta(T) \omega(s, t)^{2/p} \quad (19)$$



with

$$\begin{aligned} \beta(T) \leq & C_{29} \max\{C_{0,T}\delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}), \\ & ((1 + C_{0,T})\delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(\nabla f, \nabla \widehat{f}))\omega(0, T)^{1/p}, \\ & (\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0) \\ & + \omega(0, T)^{(1-\kappa)/p}\delta(z, \widehat{z}) + C_{0,T}\delta(z, \widehat{z}))\}. \end{aligned}$$

In particular, (19) means that  $\delta(z, \widehat{z}) \leq \beta(T)$ , but  $\beta(T)$  also depends on  $\delta(z, \widehat{z})$ . More precisely, the constant  $\beta(T)$  satisfies

$$\begin{aligned} \beta(T) \leq & C_{29}\delta(z, \widehat{z})(C_{0,T} + (1 + C_{0,T})\omega(0, T)^{1/p} + \omega(0, T)^{(1-\kappa)/p}) \\ & + C_{29}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0)). \end{aligned}$$

Now, choose  $\tau$  small enough so that for all  $s \in [0, T - \tau]$ ,

$$C_{29}(C_{s,s+\tau} + (1 + C_{s,s+\tau})\omega(s, s + \tau)^{1/p} + \omega(s, s + \tau)^{(1-\kappa)/p}) \leq \frac{1}{2}.$$

This is possible since by definition,  $(s, t) \in \Delta_+^2 \mapsto \omega(s, t)$  is continuous close to its diagonal.

Hence, one may choose  $\tau$  small enough in order to get

$$\delta(z, \widehat{z}) \leq 2C_{29}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0)). \quad (20)$$

for  $T \leq \tau$ . By a standard stacking argument where  $z_0$  (resp.  $\widehat{z}_0$ ) is replaced recursively by  $z_{k\tau}$  (resp.  $\widehat{z}_{k\tau}$ ) for  $k = 1, 2, \dots$ , we can then obtain that

$$\delta(z, \widehat{z}) \leq C_{30}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0)), \quad (21)$$

since the choice of  $\tau$  does not depend on  $z_0$ .

In other words, the Itô map  $\mathfrak{J}$  is Lipschitz continuous in all its arguments.

**Acknowledgement.** This article was started while the author was staying at the Mittag-Leffler Institute and the author wishes to express his thanks to the organizers of the semester in ‘‘Stochastic Partial Differential Equations’’ for their invitation. The author is also indebted to Laure Coutin for some interesting discussions about this article.

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