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# A NECESSARY CONDITION FOR DYNAMIC EQUIVALENCE

JEAN-BAPTISTE POMET\*

**Abstract.** If two control systems on manifolds of the same dimension are dynamic equivalent, we prove that either they are static equivalent –*i.e.* equivalent via a classical diffeomorphism– or they are both ruled; for systems of different dimensions, the one of higher dimension must be ruled. A ruled system is one whose equations define at each point in the state manifold, a ruled submanifold of the tangent space. Dynamic equivalence is also known as equivalence by endogenous dynamic feedback, or by a Lie-Bäcklund transformation when control systems are viewed as underdetermined systems of ordinary differential equations; it is very close to absolute equivalence for Pfaffian systems.

It was already known that a differentially flat system must be ruled; this is a particular case of the present result, in which one of the systems is assumed to be “trivial” (or linear controllable).

**Key words.** Control systems, ordinary differential equations, underdetermined systems, dynamic equivalence, absolute equivalence, ruled submanifolds.

**AMS subject classifications.** 34C41, 34L30, 93B17, 93B29.

**1. Introduction.** We consider time-invariant control systems, or underdetermined systems of ordinary differential equations (ODEs) where the independent variable is time. *Static equivalence* refers to equivalence via a diffeomorphism in the variables of the equation, or in the state and control variables, with a triangular structure that induces a diffeomorphism (preserving time) in the state variables too. It is also known as “feedback equivalence”. *Dynamic equivalence* refers to equivalence via invertible transformations in jet spaces that do not induce any diffeomorphism in a finite number of variables, except when it coincides with static equivalence; these transformations are also known as endogenous dynamic feedback [16, 6], or Lie-Bäcklund transformations [1, 6, 17], although this terminology is more common for systems of partial differential equations (PDEs); dynamic equivalence is very close to absolute equivalence [4, 19, 20].

The literature on classification and invariants for static equivalence is too large to be quoted here; let us only recall that as evidenced by all detailed studies, and mentioned in [22], each equivalence class (within control systems on the same manifold, or germs of control systems) is very very thin, indeed it has infinite co-dimension except in trivial cases. For instance, the class of systems that are static equivalent to a linear controllable systems, characterized in [13, 10], is very small.

Since dynamic equivalence is a priori more general, it is natural to ask how more general it is. Systems on manifolds of different dimension may be dynamic equivalent, but obviously not static equivalent. Restricting our attention to systems on the same manifold and considering dynamic equivalence instead of static, how bigger are the equivalence classes ?

Relevant to the class containing linear controllable systems (or a “trivial” systems) are the literature on dynamic feedback linearization [11, 5] and differential flatness [6, 16] (first in a differential algebraic framework) and the one pointing to absolute equivalence [19]; the authors of [6, 16] made the link with deep differential geometric questions dating back to [9, 4], see for instance [2] for a recent overview; despite these efforts, no characterization is available except for systems with one control (*i.e.* systems whose general solution depends on one function of one variable),

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and there are many systems that one suspects to be non-flat while no proof is available, see (3.20) below. There is however one powerful necessary condition [18, 21]: a flat system must be ruled, *i.e.* its equations must define a ruled submanifold in each tangent space. As pointed out in [18], this proves that the equivalence class of linear systems for dynamic equivalence, although bigger than for static equivalence, still has infinite co-dimension.

Deciding whether two general systems are dynamic equivalent is obviously at least as difficult; there is no method to prove that two systems are not dynamic equivalent. The contribution of this paper (Theorem 4.2) is a necessary condition for two systems on manifolds of the same dimension to be dynamic equivalent: either they are static equivalent or they are both ruled; this is a generalization of [18, 21]. Besides being useful to prove that some pairs of systems are not dynamic equivalent, it also implies that “generic” equivalence classes for dynamic equivalence are the same as for static equivalence, see Corollary 4.4..

*Outline.* Notations on jet bundles and differential operators are recalled in Section 2; the notions of systems, “ruled” systems, dynamic and static equivalence are precisely defined in Section 3. Our main result is stated and commented in Section 4, and proved in Section 5.

**2. Miscellaneous notations.** Let  $M$  be an  $n$ -dimensional manifold, either  $\mathbf{C}^\infty$  (infinitely differentiable) or  $\mathbf{C}^\omega$  (real analytic).

**2.1. Jet bundles.** Using the notations and definitions of [8, Chapter II, §2],  $J^k(\mathbb{R}, M)$  denotes the  $k^{\text{th}}$  jet bundle of maps  $\mathbb{R} \rightarrow M$ . It is a bundle both over  $\mathbb{R}$  and over  $M$ . If  $(x^1, \dots, x^n)$  is a system of coordinates on an open subset of  $M$ , coordinates on the lift of this open subset are given by  $t, x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n, \dots, (x^1)^{(k)}, \dots, (x^n)^{(k)}$  where  $t$  is the projection on  $\mathbb{R}$ .

As an additive group,  $\mathbb{R}$  acts on  $J^k(\mathbb{R}, M)$  by translation of the  $t$ -component; the quotient by this action is well defined and we denote it by

$$J^k(M) = J^k(\mathbb{R}, M) / \mathbb{R} . \quad (2.1)$$

Since we only study time-invariant systems, we prefer to work with  $J^k(M)$ . Quotienting indeed drops the  $t$  information: local coordinates on  $J^k(M)$  are given by  $x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n, \dots, (x^1)^{(k)}, \dots, (x^n)^{(k)}$ ; for short, we write  $x, \dot{x}, \dots, x^{(k)}$ . For  $\ell < k$ , there is a canonical projection

$$\pi_{k,\ell} : J^k(M) \rightarrow J^\ell(M) \quad (2.2)$$

that makes  $J^k(M)$  a bundle over  $J^\ell(M)$ ; in particular it is a bundle over  $M = J^0(M)$  and over  $TM = J^1(M)$ . In coordinates,

$$\pi_{k,\ell}(x, \dot{x}, \dots, x^{(\ell)}, \dots, x^{(k)}) = (x, \dot{x}, \dots, x^{(\ell)}) .$$

*Notation.* To a subset  $\Omega \subset J^k(M)$ , we associate, for all  $\ell$ , a subset  $\Omega_\ell \subset J^\ell(M)$  in the following manner (obviously,  $\Omega_k = \Omega$ ):

$$\Omega_\ell = \begin{cases} \pi_{k,\ell}(\Omega) & \text{if } \ell \leq k, \\ \pi_{\ell,k}^{-1}(\Omega) & \text{if } \ell \geq k. \end{cases} \quad (2.3)$$

**2.2. The  $k^{\text{th}}$  jet of a smooth ( $\mathbf{C}^\infty$ ) map  $x(\cdot) : I \rightarrow M$ .** With  $I \subset \mathbb{R}$  a time interval, it is a smooth map  $j^k(x(\cdot)) : I \rightarrow J^k(M)$  (see again [8]); in coordinates,

$$j^k(x(\cdot))(t) = (x(t), \dot{x}(t), \ddot{x}(t), \dots, x^{(k)}(t)) .$$

By a *smooth map whose  $k^{\text{th}}$  jet remains in  $\Omega$* , for some  $\Omega \subset J^k(M)$ , we mean a smooth  $x(\cdot) : I \rightarrow M$  such that  $j^k(x(\cdot))(t) \in \Omega$  for all  $t$  in  $I$ .

**2.3. Differential operators.** If  $\Omega$  is an open subset of  $J^k(M)$ , and  $M'$  is a manifold of dimension  $n'$ , a smooth ( $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$ ) map  $\Phi : \Omega \rightarrow M'$  defines the smooth differential operator of order<sup>1</sup>  $k$

$$\mathcal{D}_\Phi^k = \Phi \circ j^k . \quad (2.4)$$

Obviously,  $\mathcal{D}_\Phi^k$  sends smooth maps  $I \rightarrow M$  whose  $k^{\text{th}}$  jet remains in  $\Omega$  to smooth maps  $I \rightarrow M'$ . In coordinates, the image of  $t \mapsto x(t)$  is  $t \mapsto \Phi(x(t), \dot{x}(t), \ddot{x}(t), \dots, x^{(k)}(t))$ . Note that we do not require that  $k$  be minimal, so  $\Phi$  *might* not depend on  $x^{(k)}$ .

We call  $j^r \circ \mathcal{D}_\Phi^k$  the  $r^{\text{th}}$  *prolongation* of the differential operator  $\mathcal{D}_\Phi^k$ ; it sends smooth maps  $I \rightarrow M$  whose  $k^{\text{th}}$  jet remains in  $\Omega$  to smooth maps  $I \rightarrow J^r(M')$ ; it is indeed the differential operator  $\mathcal{D}_{\Phi^{[r]}}^{k+r}$ , of order  $k+r$ , with  $\Phi^{[r]}$  the unique smooth map  $\pi_{k+r,k}^{-1}(\Omega) \rightarrow J^r(M')$  such that

$$j^r \circ \Phi \circ j^k = \Phi^{[r]} \circ j^{k+r} . \quad (2.5)$$

We call  $\Phi^{[r]}$  the  $r^{\text{th}}$  prolongation of  $\Phi$ . One has  $\pi_{r,0} \circ \Phi^{[r]} = \Phi \circ \pi_{k+r,k}$  and more generally, for  $s < r$ ,

$$\pi_{r,s} \circ \Phi^{[r]} = \Phi^{[s]} \circ \pi_{k+r,k+s} . \quad (2.6)$$

### 3. Systems and equivalence.

**3.1. Systems.** DEFINITION 3.1. A  $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$  regular system with  $m$  controls on a smooth manifold  $M$  is a  $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$  sub-bundle  $\Sigma$  of the tangent bundle  $\text{TM}$

$$\begin{array}{ccc} \Sigma & \xhookrightarrow{i} & \text{TM} \\ \pi \searrow & & \downarrow \\ & & M \end{array} \quad (3.1)$$

with fiber  $\Upsilon$ , a  $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$  manifold of dimension  $m$  (e.g. an open subset of  $\mathbb{R}^m$ ). The velocity set at a point  $x \in M$  is the fiber  $\Sigma_x = \pi^{-1}(\{x\})$ , a submanifold of  $\text{T}_x M$  diffeomorphic to  $\Upsilon$ .

DEFINITION 3.2 (Solutions of a system). A *solution of system  $\Sigma$  on the real interval  $I$*  is a smooth ( $\mathbf{C}^\infty$ )  $x(\cdot) : I \rightarrow M$  such that  $j^1(x(\cdot))(t) \in \Sigma$  for all  $t \in I$ .

Although a general solution of a system need not be smooth, we only consider *smooth* solutions. They form a rich enough class in the sense that these systems are fully characterized by their set of smooth solutions.

Locally, one may write “explicit” equations of  $\Sigma$  in the following form. Of course there are many choices of coordinates and the map  $f$  depends on this choice.

PROPOSITION 3.3. For each  $\xi \in \Sigma$ , with  $\Sigma \hookrightarrow \text{TM}$  a regular system (3.1), there is

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<sup>1</sup> “Of order no larger than  $k$ ” would be more accurate: if  $\Phi$  does not depend on  $k^{\text{th}}$  derivatives, the order in the usual sense would be smaller than  $k$ . See for instance  $\Psi$  in example (3.19).

- an open neighborhood  $\mathcal{U}$  of  $\xi$  in  $\text{TM}$ ,  $\mathcal{U}_0$  its projection on  $M$ ,
  - a system of local coordinates  $(x_I, x_{\text{II}})$  on  $\mathcal{U}_0$ , with  $x_I$  a block of dimension  $n - m$  and  $x_{\text{II}}$  of dimension  $m$ ,
  - an open subset  $U$  of  $\mathbb{R}^{n+m}$  and a smooth ( $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$ ) map  $f : U \rightarrow \mathbb{R}^{n-m}$ ,
- such that the equation of  $\Sigma \cap \mathcal{U}$  in these coordinates is

$$\dot{x}_I = f(x_I, x_{\text{II}}, \dot{x}_{\text{II}}), \quad (x_I, x_{\text{II}}, \dot{x}_{\text{II}}) \in U. \quad (3.2)$$

*Proof.* Consequence of the implicit function theorem.  $\square$

**Prolongations of  $\Sigma$ .** For integers  $k \geq 1$ , we denote by  $\Sigma_k$  the prolongation of the system  $\Sigma$  to  $k^{\text{th}}$  order; it is the subbundle  $\Sigma_k \hookrightarrow J^k(M)$  with the following property: for any smooth map  $x(\cdot) : I \rightarrow M$ ,

$$j^1(x(\cdot))(t) \in \Sigma, \quad t \in I \quad \Leftrightarrow \quad j^k(x(\cdot))(t) \in \Sigma_k, \quad t \in I. \quad (3.3)$$

Recall that  $j^k(x(\cdot)) : I \rightarrow J^k(M)$  is the  $k^{\text{th}}$  jet of  $x(\cdot)$ ; the left-hand side means that  $x(\cdot)$  is a *solution* of  $\Sigma$  according to Definition 3.2. Obviously,  $\Sigma_1 = \Sigma$ . We may describe  $\Sigma_k$  in coordinates.

**PROPOSITION 3.4.** *Let  $K$  be a positive integer. There is a unique sub-bundle  $\Sigma_K \hookrightarrow J^K(M)$  such that:*

$$\begin{aligned} & \text{a smooth map } x(\cdot) : I \rightarrow M \text{ is a solution of system } \Sigma \text{ on the real interval } I \\ & \text{if and only if } j^K(x(\cdot))(t) \in \Sigma_K \text{ for all } t \in I. \end{aligned} \quad (3.4)$$

For all  $\xi \in \Sigma_K$ , its projection  $\xi_1 = \pi_{K,1}(\xi)$  is in  $\Sigma$  and, with  $\mathcal{U}$  the neighborhood of  $\xi_1$ ,  $(x_I, x_{\text{II}})$  the coordinates on  $\mathcal{U}_0$ ,  $U$  the open subset  $\mathbb{R}^{n+m}$  and  $f : U \rightarrow \mathbb{R}^m$  the map given by Proposition 3.3, the equations of  $\Sigma_K \cap \mathcal{U}_K$  in  $J^K(M)$  are, in the coordinates  $(x_I, x_{\text{II}}, \dot{x}_I, \dot{x}_{\text{II}}, \dots, x_I^{(K)}, x_{\text{II}}^{(K)})$  induced on  $\mathcal{U}$  by  $(x_I, x_{\text{II}})$ ,

$$\begin{aligned} x_I^{(i)} &= f^{(i-1)}(x_I, x_{\text{II}}, \dot{x}_{\text{II}}, \dots, x_{\text{II}}^{(i)}), \quad 1 \leq i \leq K, \\ (x_I, x_{\text{II}}, \dot{x}_{\text{II}}, \dots, x_{\text{II}}^{(K)}) &\in U \times \mathbb{R}^{(K-1)m}, \end{aligned} \quad (3.5)$$

where, for a smooth map  $f : U \rightarrow \mathbb{R}^{n-m}$ , and  $\ell \geq 0$ ,  $f^{(\ell)}$  is the smooth map  $U \times \mathbb{R}^{Km} \rightarrow \mathbb{R}^{n-m}$  defined by  $f^{(0)} = f$  and, for  $i \geq 1$ ,

$$f^{(i)}(x_I, x_{\text{II}}, \dot{x}_{\text{II}}, \dots, x_{\text{II}}^{(i+1)}) = \frac{\partial f^{(i-1)}}{\partial x_I} f(x_I, x_{\text{II}}, \dot{x}_{\text{II}}) + \sum_{i=0}^i \frac{\partial f^{(i-1)}}{\partial x_{\text{II}}^{(i)}} x_{\text{II}}^{(i+1)}. \quad (3.6)$$

*Proof.* This is classical, and obvious in coordinates.  $\square$

**REMARK 3.5.** Each  $\Sigma_{K+1}$  ( $K \geq 1$ ) is an affine bundle over  $\Sigma_K$ , and may be viewed as an affine sub-bundle of  $\text{T}\Sigma_K$ , i.e. it is a system in the sense of Section 3.1 on the manifold  $\Sigma_K$  instead of  $M$ .

In particular  $\Sigma_2 \hookrightarrow \text{T}\Sigma$  is the system obtained by “adding an integrator in each control” of the system  $\Sigma \hookrightarrow \text{TM}$ . It is an affine system (i.e. affine sub-bundle) even when  $\Sigma$  is a “general” system.

**3.2. Control systems and implicit systems of ODEs.** This section, devoted to the possibly more familiar notions of control systems or implicit systems of ODEs, will not be needed in the sequel, where “systems” are as defined in section 3.1.

A smooth *control system* on  $M$  with  $m$  controls is given by

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in \mathcal{B}, \quad (3.7)$$

where  $\mathcal{B}$  is a manifold of dimension  $m$  (e.g. an open subset of  $\mathbb{R}^m$ ) and  $f$  in  $\mathbf{C}^\infty(M \times \mathcal{B}, \text{TM})$  or  $\mathbf{C}^\omega(M \times \mathcal{B}, \text{TM})$ ; we call it *regular* if the map  $u \mapsto f(x, u)$  is, for all  $x$ , an immersion  $\mathcal{B} \rightarrow \text{T}_x M$ , i.e. if, for all  $(x, u)$ ,

$$\text{rank} \frac{\partial f}{\partial u}(x, u) = m . \quad (3.8)$$

It defines a system in the sense of section 3.1, where  $\Sigma$  is the set of  $(x, f(x, u)) \in \text{TM}$  where  $x$  varies in  $M$  and  $u$  in  $\mathcal{U}$ ; this is indeed a sub-bundle from assumption (3.8), and the fiber, or velocity set over  $x$  is

$$\Sigma_x = \pi^{-1}(\{x\}) = \{ f(x, u), u \in \mathcal{B} \} . \quad (3.9)$$

A smooth *system of  $n - m$  ODEs* on  $M$ :

$$F(x, \dot{x}) = 0 , \quad x \in M , \quad (3.10)$$

where  $F \in \mathbf{C}^\infty(\text{TM}, \mathbb{R}^{n-m})$ , is called *regular* if, for all  $(x, \dot{x})$  such that  $F(x, \dot{x}) = 0$ ,

$$\text{rank} \frac{\partial F}{\partial \dot{x}}(x, \dot{x}) = n - m . \quad (3.11)$$

It defines a system in the sense of section 3.1, where  $F(x, \dot{x}) = 0$  is the equation of  $\Sigma$  as a submanifold of  $\text{TM}$ ; this is indeed a sub-bundle from assumption (3.11), and the fiber, or velocity set over  $x$  is

$$\Sigma_x = \pi^{-1}(\{x\}) = \{ \dot{x} \in \text{T}_x M, F(x, \dot{x}) = 0 \} . \quad (3.12)$$

A system in the sense of Definition 3.1 may always be described, at least locally, either by an explicit parametrization (3.7) of the fibers, or by their equation (3.10). Note however that, by assuming (3.11), (3.8) or  $\Sigma$  being a smooth sub-bundle, we have carefully avoided singularities, where these descriptions no longer coincide, and where Theorem 4.2 might fail.

**3.3. Ruled systems.** Recall that a smooth submanifold of an affine space is ruled if and only if it is a union of straight lines, i.e. if through each point of the submanifold passes a straight line contained in the submanifold. Such a manifold must be unbounded; since we want to consider the intersection of a submanifold with an arbitrary open set and allow this patch to be “ruled”, we use the same slightly abusive notion as [15]: a submanifold  $N$  is ruled if and only if, through each point of it, passes a straight line which is contained in  $N$  “until it reaches the boundary of  $N$ ”. Here, the boundary of the submanifold  $N$  is  $\partial N = \overline{N} \setminus N$ .

A system will be called ruled if and only if  $\Sigma_x$  is, for all  $x$ , a ruled submanifold of  $\text{T}_x M$ . This is formalized below in a self-contained manner.

**DEFINITION 3.6.** Let  $\mathcal{O}$  be an open subset of  $\text{TM}$ . System  $\Sigma$  (see (3.1)) is ruled in  $\mathcal{O}$  if and only if, for all  $(x, \dot{x}) \in (\mathcal{O} \cap \Sigma)$ , there is a nonzero vector  $w \in \text{T}_x M \setminus \{0\}$  and two possibly infinite numbers  $\lambda^- \in [-\infty, 0)$  and  $\lambda^+ \in (0, +\infty]$  such that  $(x, \dot{x} + \lambda w) \in \mathcal{O} \cap \Sigma$  for all  $\lambda$ ,  $\lambda^- < \lambda < \lambda^+$  and

$$\begin{aligned} \lambda^- > -\infty &\Rightarrow (x, \dot{x} + \lambda^- w) \in \partial(\mathcal{O} \cap \Sigma) , \\ \lambda^+ < +\infty &\Rightarrow (x, \dot{x} + \lambda^+ w) \in \partial(\mathcal{O} \cap \Sigma) . \end{aligned} \quad (3.13)$$

Recall that, by definition,  $\partial(\mathcal{O} \cap \Sigma) = \overline{\mathcal{O} \cap \Sigma} \setminus (\mathcal{O} \cap \Sigma)$ .

We shall need the following characterisation.

**PROPOSITION 3.7** ([15]). *Let  $\mathcal{O}$  be an open subset of  $TM$ .  $\Sigma$  is ruled in  $\mathcal{O}$  if and only if, for all  $\xi = (x, \dot{x})$  in  $\Sigma \cap \mathcal{O}$ , there is a straight line in  $T_x M$  passing through  $\dot{x}$  that has contact of infinite order with  $\Sigma_x$  at  $\dot{x}$ .*

*Proof.* From [15, Theorem 1], a “patch of” submanifold of dimension  $m$  in a manifold of dimension  $n$  is ruled if and only if there is, through each point, a straight line that has contact of order  $n + 1$ . This is of course implied by infinite order.  $\square$

**3.4. Dynamic equivalence.** The following notion is usually called dynamic equivalence, or equivalence by (endogenous) dynamic feedback transformations in control theory, see [16, 7, 12, 17]. It is in fact also the notion of Lie-Bäcklund transformation, limited to ordinary differential equation, as noted in [7] or [17].

**DEFINITION 3.8.** *Let  $\Sigma \hookrightarrow TM$  and  $\Sigma' \hookrightarrow TM'$  be  $\mathbf{C}^\infty$  (resp.  $\mathbf{C}^\omega$ ) regular systems (see (3.1)) on two manifolds  $M$  and  $M'$  of dimension  $n$  and  $n'$ ,  $K, K'$  two integers,  $\Omega \subset J^K(M)$  and  $\Omega' \subset J^{K'}(M')$  two open subsets.*

*Systems  $\Sigma$  and  $\Sigma'$  are dynamic equivalent over  $\Omega$  and  $\Omega'$  if and only if there exists two mappings of class  $\mathbf{C}^\infty$  (resp.  $\mathbf{C}^\omega$ )*

$$\Phi : \Omega \rightarrow M', \quad \Psi : \Omega' \rightarrow M \quad (3.14)$$

*inducing differential operators  $\mathcal{D}_\Phi^K$  and  $\mathcal{D}_\Psi^{K'}$  –see (2.4)– such that, for any interval  $I$ ,*

- *for any solution  $x(\cdot) : I \rightarrow M$  of  $\Sigma$  whose  $K^{\text{th}}$  jet remains inside  $\Omega$ ,  $\mathcal{D}_\Phi^K(x(\cdot))$  is a solution of  $\Sigma'$  whose  $K'^{\text{th}}$  jet remains inside  $\Omega'$  and  $\mathcal{D}_\Psi^{K'}(\mathcal{D}_\Phi^K(x(\cdot))) = x(\cdot)$ ,*
- *for any solution  $z(\cdot) : I \rightarrow M'$  of  $\Sigma'$  whose  $K'^{\text{th}}$  jet remains inside  $\Omega'$ ,  $\mathcal{D}_\Psi^{K'}(z(\cdot))$  is a solution of  $\Sigma$  whose  $K^{\text{th}}$  jet remains inside  $\Omega$  and  $\mathcal{D}_\Phi^K(\mathcal{D}_\Psi^{K'}(z(\cdot))) = z(\cdot)$ .*

**REMARK 3.9.** In the theorems, we shall require that  $\Omega$  and  $\Omega'$  satisfy

$$\Omega_1 \cap \Sigma \subset (\Omega \cap \Sigma_K)_1 \quad \text{and} \quad \Omega'_1 \cap \Sigma' \subset (\Omega' \cap \Sigma'_{K'})_1, \quad (3.15)$$

*i.e.* any (jet of) solution whose first jet is in  $\Omega_1$  lifts to at least one (jet of) solution whose  $K^{\text{th}}$  jet is in  $\Omega$ . Note the following facts about this requirement.

- These inclusions are equalities for the reverse inclusions always hold.
- Replacing the original  $\Omega$  with  $\Omega \setminus \overline{((\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1)}_K$  and  $\Omega'$  accordingly forces (3.15); alternatively, keeping arbitrary open sets, Theorem 4.2 and Theorem 4.1 would hold with  $\Omega_1$  replaced with  $\Omega_1 \setminus \overline{((\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1)}$ .
- When  $\Sigma' = TM'$  is the trivial system (see section 3.6), any open  $\Omega'$  satisfies (3.15).

**3.5. Static equivalence.** **DEFINITION 3.10.** *Let  $\mathcal{O} \subset TM$  and  $\mathcal{O}' \subset TM'$  be open subsets. Systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{O}$  and  $\mathcal{O}'$  if and only if there is a smooth diffeomorphism  $\Phi : \mathcal{O}_0 \rightarrow \mathcal{O}'_0$  such that the following holds:*

$$\left. \begin{array}{l} \text{a smooth map } t \mapsto x(t) \text{ is a solution of } \Sigma \text{ whose first jet remains in } \mathcal{O} \\ \text{if and only if } t \mapsto \Phi(x(t)) \text{ is a solution of } \Sigma' \text{ whose first jet remains in } \mathcal{O}'. \end{array} \right\} \quad (3.16)$$

**DEFINITION 3.11** (Local static equivalence). *Let  $\mathcal{O} \subset TM$  and  $\mathcal{O}' \subset TM'$  be open subsets. Systems  $\Sigma$  and  $\Sigma'$  are locally static equivalent over  $\mathcal{O}$  and  $\mathcal{O}'$  if and only if there are coverings of  $\mathcal{O} \cap \Sigma$  and  $\mathcal{O}' \cap \Sigma'$  :*

$$\Sigma \cap \mathcal{O} \subset \Sigma \cap \bigcup_{\alpha \in A} \mathcal{O}^\alpha, \quad \Sigma' \cap \mathcal{O}' \subset \Sigma' \cap \bigcup_{\alpha \in A} \mathcal{O}'^\alpha$$

where  $A$  is a set of indices,  $\mathcal{O}^\alpha$  and  $\mathcal{O}'^\alpha$  are open subsets of  $\mathcal{O}$  and  $\mathcal{O}'$ , such that, for all  $\alpha$ , systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{O}^\alpha$  and  $\mathcal{O}'^\alpha$ .

These definitions, stated in terms of solutions, can be translated in a property that only relies on geometry of  $\Sigma$  and  $\Sigma'$  as submanifolds. Point (b) is used for instance in [14, 23] where invariant “centro-affine” geometry of each  $\Sigma_x$  are studied.

**PROPOSITION 3.12.** (a) *Systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{O} \subset TM$  and  $\mathcal{O}' \subset TM'$  if and only there is a smooth diffeomorphism  $\Phi : \mathcal{O}_0 \rightarrow \mathcal{O}'_0$  such that  $\Phi_*$  maps  $\mathcal{O} \cap \Sigma$  to  $\mathcal{O}' \cap \Sigma'$ .*

(b) *If systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{O} \subset TM$  and  $\mathcal{O}' \subset TM'$ , there is, for each  $x \in \mathcal{O}_0$  a linear isomorphism  $T_x M \rightarrow T_{\Phi(x)} M'$  that maps  $\Sigma_x$  to  $\Sigma'_{\Phi(x)}$ .*

*Proof.* (b) is an easy consequence of (a), which in turn is clear by differentiating solutions in Definition 3.2.  $\square$

**3.6. Examples.** (i) On a smooth manifold  $M$ , we call *trivial system on  $M$*  the tangent bundle itself  $TM$ . Any smooth  $x(\cdot) : I \rightarrow M$  is a solution of this system; it corresponds to “no equation” or to the control system  $\dot{x} = u$  (see section 3.2).

A trivial system corresponds to the “affine diffieties” in [7]. Following [6, 7], we call differentially flat a system  $\sigma \hookrightarrow TM$  which is dynamic equivalent to the trivial system  $TM'$  for some manifold  $M'$  (possibly in restriction to some subsets  $\Omega$  and  $\Omega'$ ).

(ii) The easiest (and somehow trivial) example of two dynamic equivalent systems is a system  $\Sigma \hookrightarrow M$  and the one obtained by “adding integrators”. It was described in Remark 3.5 as an affine sub-bundle  $\Sigma_2 \hookrightarrow T\Sigma$ ;  $\Sigma$  and  $\Sigma_2$  are equivalent in the sense of Definition 3.8 with  $M' = \Sigma$ ,  $K = 1$ ,  $K' = 0$ ,  $\Omega$  an open neighborhood of  $\Sigma$  in  $J^1(M) = TM$  such that there is a  $\Phi : \Omega \rightarrow \Sigma$  that coincides with identity on  $\Sigma$ ,  $\Omega' = M' = \Sigma$  and  $\Psi = \pi$  (see (3.1)).

This may be easier to follow in the coordinates of Proposition 3.3. The prolongation of (3.2) has state  $(y_I, y_{II}) \in U$ , with  $y_I$  a block of dimension  $n$  and  $y_{II}$  of dimension  $m$ , and equation  $\dot{y}_I = (f(y_I, y_{II}), y_{II})$ . In coordinates, the transformations  $\Phi : J^1(U_0) \rightarrow U$  and  $\Psi : U \rightarrow U_0$  are given by  $(y_I, y_{II}) = \Phi(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}) = (x, \dot{x}_{II})$  and  $x = \Psi(y) = y_I$ .

Static equivalence between these systems of different dimension does not hold.

(iii) Let us now give –mostly to illustrate the role of the integers  $K, K'$  and the open sets  $\Omega$  and  $\Omega'$ – two more specific examples of systems  $\Sigma \hookrightarrow T\mathbb{R}^3$  and  $\Sigma' \hookrightarrow T\mathbb{R}^3$  with the following equations in  $T\mathbb{R}^3$ , with coordinates  $(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)$  or  $(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)$ , clearly defining sub-bundles with fiber diffeomorphic to  $\mathbb{R}^2$ :

$$\Sigma : \dot{x}_1 = x_2, \quad \Sigma' : \dot{y}_1 = y_2 + (\dot{y}_2 - y_1 \dot{y}_3) \dot{y}_3. \quad (3.17)$$

These equations are even globally in the “explicit” form given by Proposition 3.3.

First of all,  $\Sigma$  is dynamic equivalent to the trivial system  $\Sigma'' = T\mathbb{R}^2$ , with  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $\Phi(x_1, x_2, x_3) = (x_1, x_3)$  and  $\Psi : J^1(\mathbb{R}^2) \rightarrow \mathbb{R}^3$  given by  $\Psi(z_1, z_2, \dot{z}_1, \dot{z}_2) = (z_1, \dot{z}_1, z_2)$ . Here  $K = 0, K' = 1, \Omega = \mathbb{R}^2, \Omega' = J^1(\mathbb{R}^2)$ .

Also, with  $K = 1$  and  $K' = 2$ , systems  $\Sigma$  and  $\Sigma'$  are dynamic equivalent over  $\Omega \subset J^1(\mathbb{R}^3)$  and  $\Omega' \subset J^2(\mathbb{R}^3)$  defined by

$$\begin{aligned} \Omega &= \{(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3), 1 - \dot{x}_2 - x_2^3 \neq 0\}, \\ \Omega' &= \{(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3, \ddot{y}_1, \ddot{y}_2, \ddot{y}_3), 1 - \ddot{y}_3 - \dot{y}_3^3 \neq 0\}. \end{aligned}$$

The maps  $\Phi : \Omega \rightarrow \mathbb{R}^3$  and  $\Psi : \Omega' \rightarrow \mathbb{R}^3$  are given by

$$\Phi(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) = \left( \frac{(1 - \dot{x}_2)x_3 + x_2 \dot{x}_3}{1 - \dot{x}_2 - x_2^3}, \frac{x_2^2 x_3 + \dot{x}_3}{1 - \dot{x}_2 - x_2^3}, x_1 \right), \quad (3.18)$$

$$\Psi(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3, \ddot{y}_1, \ddot{y}_2, \ddot{y}_3) = (y_3, \dot{y}_3, y_1 - \dot{y}_3 y_2). \quad (3.19)$$

REMARK 3.13. Since  $\Psi$  does not depend on second derivatives,  $K' = 2$  is not the order of the differential operator  $\mathcal{D}_\Psi^{K'}$  in the usual sense; this illustrates the footnote after (2.4); it is however necessary to go to second jets to describe the domain  $\Omega'$  where the restriction to solutions of  $\Sigma'$  of this first order operator can be inverted.

Finally, note that systems  $\Sigma$  and  $\Sigma'$  are not static equivalent because, from Proposition 3.12-(b), this would imply that each  $\Sigma_x$  is sent to some  $\Sigma'_y$  by a linear isomorphism  $T_x M \rightarrow T_y M'$ , which is not possible because each  $\Sigma_x$  is an affine subspace of  $T_x M$  and  $\Sigma'_y$  a non degenerate quadric of  $T_y M'$ .

(iv) Consider two more systems,  $\Sigma \hookrightarrow T\mathbb{R}^3$  and  $\Sigma' \hookrightarrow T\mathbb{R}^3$  described as in (3.17):

$$\Sigma : \dot{x}_1 = x_2 + (\dot{x}_2 - x_1 \dot{x}_3)^2 \dot{x}_3, \quad \Sigma' : \dot{y}_1 = y_2 + (\dot{y}_2 - y_1 \dot{y}_3)^2 \dot{y}_3. \quad (3.20)$$

These equations also define sub-bundles with fiber diffeomorphic to  $\mathbb{R}^2$  and are also globally in the “explicit” form given by Proposition 3.3.

Systems  $\Sigma$  and  $\Sigma'$  are not static equivalent because each  $\Sigma_x$  is the union of lines  $\dot{x}_2 - x_1 \dot{x}_3 = c$ ,  $\dot{x}_1 = x_2 + c^2 \dot{x}_3$  for  $c$  in  $\mathbb{R}$ , hence a ruled submanifold of  $T_x M$ , while the submanifold  $\Sigma'_y$  is not ruled in  $T_y M'$  for any  $y$  (use Proposition 3.12-(b) as in Example 3.6-(iii)). This example is further discussed with Theorems 4.1 and 4.2.

**4. Main Result.** It has been known since [18, 21] that a system which is dynamic equivalent to a *trivial system* –see the beginning of section 3.6; such a system is called differentially flat– must be ruled; of course, at least in the smooth case, this is true only on the domain where equivalence is assumed.

THEOREM 4.1 ([18, 21]). *If  $\Sigma$  is dynamic equivalent to the trivial system  $\Sigma' = TM'$  over  $\Omega \subset J^K(M)$  and  $\Omega' \subset J^{K'}(M')$  satisfying (3.15), then  $\Sigma$  is ruled in  $\Omega_1$ .*

Consider systems  $\Sigma$  and  $\Sigma'$  in (3.20). System  $\Sigma'$  is not ruled, hence, according to this theorem, it is not flat, i.e. not dynamic equivalent to the trivial system  $T\mathbb{R}^2$ , or to the first system in (3.17). Since  $\Sigma$  is ruled, the theorem does not apply to it; in fact, one conjectures that this system is not ruled, but no proof is available; see [3].

The contribution of this paper is the following strong necessary condition for dynamic equivalence between two general systems.  $\Omega_1$  and  $\Omega'_1$  are defined by (2.3).

THEOREM 4.2. *Let  $\Sigma$  and  $\Sigma'$  be systems on manifolds of dimension  $n$  and  $n'$ ,  $K, K'$  two integers and  $\Omega \subset J^K(M)$ ,  $\Omega' \subset J^{K'}(M')$  two open subsets satisfying (3.15).*

*If  $\Sigma$  and  $\Sigma'$  are dynamic equivalent over  $\Omega$  and  $\Omega'$ , then*

**if  $n > n'$ ,** *system  $\Sigma$  is ruled in  $\Omega_1$ ,*

**if  $n < n'$ ,** *system  $\Sigma'$  is ruled in  $\Omega'_1$ ,*

**if  $n = n'$ ,** *then (see Definition 3.11 for “locally static equivalent”)*

- *in the real analytic case, and if  $\Omega \cap \Sigma$  and  $\Omega' \cap \Sigma'$  are connected,*  
*either systems  $\Sigma$  and  $\Sigma'$  are ruled in  $\Omega_1$  and  $\Omega'_1$  respectively,*  
*or they are locally static equivalent over  $\Omega_1$  and  $\Omega'_1$ ,*

- *in the smooth ( $C^\infty$ ) case, there are open subsets  $\mathcal{R}, \mathcal{S}$  of  $\Omega_1$  and  $\mathcal{R}', \mathcal{S}'$  of  $\Omega'_1$  such that  $\Omega_1$  and  $\Omega'_1$  are covered as*

$$\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}, \quad \Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'} \quad (4.1)$$

*and the systems have the following properties on these sets:*

1.  $\Sigma$  and  $\Sigma'$  are ruled in  $\mathcal{R}$  and  $\mathcal{R}'$  respectively,
2.  $\Sigma$  and  $\Sigma'$  are locally static equivalent over  $\mathcal{S}$  and  $\mathcal{S}'$ .

The proof of this result is to be found in section 5.2; let us make a brief discussion.

It is obviously a generalization of Theorem 4.1. Here, static equivalence has to appear explicitly because two general systems can be static equivalent without being ruled, whereas in Theorem 4.1 static equivalence implied the ruled property because one of the two systems is assumed linear, hence ruled and static equivalence preserves (see Proposition 3.12) the property of being ruled.

The case  $n = n'$  states that dynamic equivalence, except when it reduces to static equivalence, forces both systems to be ruled (in the real analytic case, the added rigidity prevent the two situations to occur simultaneously). Theorem 4.2 can be used to prove that some systems are *not* dynamic equivalent:

**COROLLARY 4.3.**  *$\Sigma$  and  $\Sigma'$  in (3.20) are not dynamic equivalent over any domains.*

*Proof.* On the one hand  $\Sigma'$  is not ruled anywhere while on the other hand they are not static equivalent, as noted just below equation (3.20).  $\square$

Since being ruled is non-generic, we have the following general consequence, stated in terms of germs of systems because the conclusion in the theorem is only local.

**COROLLARY 4.4.** *Generic static equivalence classes for germs of systems of the same dimension at a point are also dynamic equivalence classes.* Note that this is in the mathematical sense of “generic”: this does not prevent many interesting systems from being dynamic equivalent without being static equivalent... it might even be that “most” interesting systems fall in this case !

The part “ $n > n'$  or  $n < n'$ ” can be rephrased as follows: if a system is not ruled, it cannot be dynamic equivalent to any system of smaller dimension. No necessary condition is given on the system of lower dimension; indeed *any* system is dynamic equivalent to at least its first prolongation, see Example 3.6-(ii).

**5. Proofs.** Recall that subscripts always refer to the order of the jet space. The notation (2.3) is constantly used.

### 5.1. Preliminaries: a re-formulation of dynamic and static equivalence.

The maps  $\Phi$  and  $\Psi$  are always applied to jets of solutions, and, according to (3.3), the  $K^{\text{th}}$  jets of solutions of  $\Sigma$  remain in  $\Sigma_K$ ; hence the only information to retain about  $\Phi$  and  $\Psi$  is their restriction to, respectively,

$$\tilde{\Omega} = \Omega \cap \Sigma_K \quad \text{and} \quad \tilde{\Omega}' = \Omega' \cap \Sigma'_{K'}. \quad (5.1)$$

We need one more piece of notation: according to Section 2.3, the  $\ell^{\text{th}}$  prolongation of a smooth map  $\tilde{\Phi} : \tilde{\Omega} \rightarrow M'$ , is a map  $\pi_{K+\ell, \ell}^{-1}(\tilde{\Omega}) \rightarrow J^\ell M'$ ; again, only its restriction to  $\Omega_{K+\ell}$  will matter; for this reason, the notations  $\tilde{\Phi}^{[\ell]}$  and  $\tilde{\Psi}^{[\ell]}$  will not stand for the prolongations as defined earlier, but rather these restrictions:

$$\tilde{\Phi}^{[\ell]} : \tilde{\Omega}_{K+\ell} \rightarrow J^\ell(M'), \quad \tilde{\Psi}^{[\ell]} : \tilde{\Omega}'_{K'+\ell} \rightarrow J^\ell(M), \quad (5.2)$$

$$\text{with } \tilde{\Omega}_{K+\ell} = \Omega_{K+\ell} \cap \Sigma_{K+\ell}, \quad \tilde{\Omega}'_{K'+\ell} = \Omega'_{K'+\ell} \cap \Sigma'_{K'+\ell}. \quad (5.3)$$

We may now state the following proposition. Note that smooth ( $\mathbf{C}^\infty$  or  $\mathbf{C}^\omega$ ) maps on  $\tilde{\Omega}_{K+\ell}$  or  $\tilde{\Omega}'_{K'+\ell}$  can be defined in a standard way because these are smooth embedded submanifolds, see Proposition 3.3 for instance.

**PROPOSITION 5.1 (Dynamic Equivalence).** *Let  $K, K'$  be integers,  $\Omega \subset J^K(M)$  and  $\Omega' \subset J^{K'}(M')$  two open subsets. Systems  $\Sigma$  and  $\Sigma'$  are dynamic equivalent*

over  $\Omega$  and  $\Omega'$  if and only if, with  $\tilde{\Omega}, \tilde{\Omega}'$  defined in (5.1), there exist two smooth (real analytic, in the real analytic case) mappings

$$\tilde{\Phi} : \tilde{\Omega} \rightarrow M' \quad \text{and} \quad \tilde{\Psi} : \tilde{\Omega}' \rightarrow M,$$

such that

$$\tilde{\Phi}^{[1]}(\tilde{\Omega}_{K+1}) \subset \Sigma', \quad \tilde{\Psi}^{[1]}(\tilde{\Omega}'_{K'+1}) \subset \Sigma, \quad (5.4)$$

and, with  $\tilde{\Phi}^{[K]}$  and  $\tilde{\Psi}^{[K]}$  defined by (5.2),

$$\tilde{\Phi}^{[K']}(\tilde{\Omega}_{K+K'}) \subset \Omega', \quad \tilde{\Psi}^{[K]}(\tilde{\Omega}'_{K'+K'}) \subset \Omega, \quad (5.5)$$

$$\tilde{\Psi} \circ \tilde{\Phi}^{[K']} = \pi_{K+K',0} \Big|_{\tilde{\Omega}_{K+K'}}, \quad \tilde{\Phi} \circ \tilde{\Psi}^{[K]} = \pi_{K+K',0} \Big|_{\tilde{\Omega}'_{K+K'}}. \quad (5.6)$$

*Proof.* If the above conditions on  $\Phi$  and  $\Psi$  are satisfied, and  $x(\cdot) : I \rightarrow M$  is a solution of  $\Sigma$  whose  $K^{\text{th}}$  jet remains inside  $\Omega$ , then the first part of (5.4) implies that  $\mathcal{D}_{\Phi}^K(x(\cdot))$  is a solution of  $\Sigma'$ , the first part of (5.5) implies that its  $K^{\text{th}}$  jet remains inside  $\Omega'$ , and the first part of (5.6) implies that  $\mathcal{D}_{\Psi}^{K'}(\mathcal{D}_{\Phi}^K(x(\cdot))) = x(\cdot)$ . This proves the first item of Definition 3.8; the second item follows in the same way from the second part of (5.4), (5.5) and (5.6).

Conversely, if  $\Phi$  and  $\Psi$  satisfy the properties of Definition 3.8, their restrictions  $\tilde{\Phi}$  and  $\tilde{\Psi}$  to  $\tilde{\Omega}$  and  $\tilde{\Omega}'$  respectively satisfy the above relations because through each point in  $\tilde{\Omega}_{K+1}$ ,  $\tilde{\Omega}'_{K'+1}$ ,  $\tilde{\Omega}_{K+K'}$  or  $\tilde{\Omega}'_{K'+K'}$  passes a jet of order  $K+1$ ,  $K'+1$  or  $K+K'$  of a solution of  $\Sigma$  or  $\Sigma'$ ; differentiating yields the required relations.  $\square$

**PROPOSITION 5.2 (Static Equivalence).** *With  $\Omega_1 \subset J^1(M) = TM$  and  $\Omega'_1 \subset J^1(M') = TM$  two open subsets, systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\Omega_1$  and  $\Omega'_1$  if and only if, with  $\tilde{\Omega}_1, \tilde{\Omega}'_1$  defined in (5.1), there exist a smooth diffeomorphism  $\Phi_0 : \tilde{\Omega}_0 \rightarrow \tilde{\Omega}'_0$ , and its inverse  $\Psi_0$  such that  $\tilde{\Phi}^{[1]}(\tilde{\Omega}_1) = \tilde{\Omega}'_1$  (and  $\tilde{\Psi}^{[1]}(\tilde{\Omega}'_1) = \tilde{\Omega}_1$ ).*

*Proof.* This is a re-phrasing of Proposition 3.12-(a).  $\square$

**5.2. Proof of Theorem 4.2.** Assume that  $\Sigma$  and  $\Sigma'$  are dynamic equivalent over the open sets  $\Omega \subset J^K(M)$  and  $\Omega' \subset J^{K'}(M')$ ; let  $\tilde{\Phi} : \tilde{\Omega} \rightarrow M'$  and  $\tilde{\Psi} : \tilde{\Omega}' \rightarrow M$  be the smooth maps given by Proposition 5.1 (recall that  $\tilde{\Omega}$  and  $\tilde{\Omega}'$  are open subsets of  $\Sigma_K$  and  $\Sigma'_{K'}$ ). We define open subsets  $\tilde{\Omega}^S \subset \tilde{\Omega}$  and  $\tilde{\Omega}'^S \subset \tilde{\Omega}'$  and state four lemmas concerning these :

$$\begin{aligned} \xi \in \tilde{\Omega}^S &\Leftrightarrow \text{There is a neighborhood } V \text{ of } \xi \text{ in } \tilde{\Omega} \text{ and a smooth map} \\ &\tilde{\Phi}_0 : V_0 \rightarrow M' \text{ such that } \tilde{\Phi} \Big|_V = \tilde{\Phi}_0 \circ \pi_{K,0}, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \xi' \in \tilde{\Omega}'^S &\Leftrightarrow \text{There is a neighborhood } V' \text{ of } \xi' \text{ in } \tilde{\Omega}' \text{ and a smooth map} \\ &\tilde{\Psi}_0 : V'_0 \rightarrow M \text{ such that } \tilde{\Psi} \Big|_{V'} = \tilde{\Psi}_0 \circ \pi_{K,0}. \end{aligned} \quad (5.8)$$

**LEMMA 5.3.** *In the analytic case, and if  $\tilde{\Omega} = \Omega \cap \Sigma$  and  $\tilde{\Omega}' = \Omega' \cap \Sigma'$  are connected, one has either  $\tilde{\Omega}^S = \tilde{\Omega}$  or  $\tilde{\Omega}^S = \emptyset$ , and either  $\tilde{\Omega}'^S = \tilde{\Omega}'$  or  $\tilde{\Omega}'^S = \emptyset$ .*

**LEMMA 5.4.** *One has the following identities, where the two first ones hold for any subsets  $S \subset \tilde{\Omega}$ ,  $S' \subset \tilde{\Omega}'$  and any integer  $\ell$ ,  $0 \leq \ell \leq K+K'$ ,*

$$\begin{aligned} \pi_{K+K',\ell} \left( \tilde{\Phi}^{[K']^{-1}}(S') \right) &= \tilde{\Psi}^{[\ell]}(S'_{K'+\ell}), \quad \pi_{K+K',\ell} \left( \tilde{\Psi}^{[K]^{-1}}(S) \right) = \tilde{\Phi}^{[\ell]}(S_{K+\ell}), \quad (5.9) \\ \tilde{\Phi}^{[1]}(\tilde{\Omega}_{K+1}) &= \tilde{\Omega}'_1, \quad \tilde{\Psi}^{[1]}(\tilde{\Omega}'_{K'+1}) = \tilde{\Omega}_1. \end{aligned} \quad (5.10)$$

LEMMA 5.5. *If  $n < n'$ , then  $\tilde{\Omega}^S = \emptyset$ . If  $n > n'$ , then  $\tilde{\Omega}'^S = \emptyset$ .*

*If  $n = n'$ , there is, for all  $\xi_K \in \tilde{\Omega}^S$ , a neighborhood  $\mathcal{V}_1$  of  $\xi_1 = \pi_{K,1}(\xi_K)$  in  $\Omega_1$  and an open subset  $\mathcal{V}'_1$  of  $\Omega'_1$  such that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{V}_1$  and  $\mathcal{V}'_1$ . There is also, for all  $\xi'_{K'} \in \tilde{\Omega}'^S$ , a neighborhood  $\mathcal{W}'$  of  $\xi'_1 = \pi_{K',1}(\xi'_{K'})$  in  $\Omega'_1$  and an open subset  $\mathcal{W}_1$  of  $\Omega_1$  such that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{W}_1$  and  $\mathcal{W}'_1$ . Finally,*

$$\pi_{K+K',K'} \left( \tilde{\Psi}^{[K]}{}^{-1} \left( \tilde{\Omega}^S \right) \right) = \tilde{\Phi}^{[K']} \left( \tilde{\Omega}_{K+K'}^S \right) = \tilde{\Omega}'^S, \quad (5.11)$$

$$\pi_{K+K',K} \left( \tilde{\Phi}^{[K']}{}^{-1} \left( \tilde{\Omega}'^S \right) \right) = \tilde{\Psi}^{[K]} \left( \tilde{\Omega}_{K'+K}^S \right) = \tilde{\Omega}^S. \quad (5.12)$$

LEMMA 5.6. *For all  $\xi_{K+1} \in \tilde{\Omega}_{K+1}$  such that  $\xi_K = \pi_{K+1,K}(\xi_{K+1}) \in \tilde{\Omega} \setminus \tilde{\Omega}^S$ , there is a straight line in  $T_{\tilde{\Phi}(\xi_K)} M'$  that has contact of infinite order with  $\Sigma'$  at  $\tilde{\Phi}^{[1]}(\xi_{K+1})$ .*

These lemmas will be proved later. Let us finish the proof of the Theorem.

If  $n < n'$ , (5.10) implies existence, for each  $\xi' \in \tilde{\Omega}'_1 = \Omega_1 \cap \Sigma'$ , of some  $\xi_{K+1} \in \tilde{\Omega}_{K+1}$  such that  $\tilde{\Phi}^{[1]}(\xi_{K+1}) = \xi'$  and finally, since  $\tilde{\Omega}^S$  is empty according to Lemma 5.5, Lemma 5.6 yields a straight line in  $T_{\xi'_0} M'$  that has contact of infinite order with  $\Sigma'$  at  $\xi'$ ; from Proposition 3.7, this implies that system  $\Sigma'$  is ruled over  $\Omega_1$ . If  $n > n'$ , one concludes in the same way.

Now assume  $n = n'$ . For all  $\xi'$  in  $\tilde{\Phi}^{[1]} \left( (\tilde{\Omega} \setminus \tilde{\Omega}^S)_{K+1} \right)$ , there is, according to Lemma 5.6, a straight line in  $T_{\xi'_0} M'$  that has contact of infinite order with  $\Sigma'$  at  $\xi'$ . By continuity, this is also true for all  $\xi'$  in the topological closure

$$\tilde{R}' = \overline{\tilde{\Phi}^{[1]} \left( (\tilde{\Omega} \setminus \tilde{\Omega}^S)_{K+1} \right)} = \overline{\pi_{K+K',1} \left( \tilde{\Psi}^{[K]}{}^{-1} \left( \tilde{\Omega} \setminus \tilde{\Omega}^S \right) \right)}, \quad (5.13)$$

where the second equality come from (5.9). Let  $i(\tilde{R}')$  be the interior of  $\tilde{R}'$  for the induced topology on  $\Sigma'$ ; since  $\tilde{R}' = \overline{i(\tilde{R}')}$ , there is an open subset  $\mathcal{R}'$  of  $\Omega'_1 \subset TM'$ , enjoying the property that it is the interior of its topological closure, and such that  $\mathcal{R}' \cap \Sigma' = i(\tilde{R}')$  and  $\overline{\mathcal{R}'} \cap \Sigma' = \tilde{R}'$ . From Proposition 3.7,  $\Sigma'$  is ruled over  $\mathcal{R}'$ . Setting  $\mathcal{S}' = \Omega'_1 \setminus \overline{\mathcal{R}'}$ , one has  $\Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'}$ . Along the same lines,  $\Sigma$  is ruled over  $\mathcal{R}$ , open subset of  $\Omega_1 \subset TM$  such that  $\mathcal{R} \cap \Sigma$  is the relative interior of

$$\tilde{R} = \overline{\tilde{\Psi}^{[1]} \left( (\tilde{\Omega}' \setminus \tilde{\Omega}'^S)_{K'+1} \right)} = \overline{\pi_{K+K',1} \left( \tilde{\Phi}^{[K']}{}^{-1} \left( \tilde{\Omega}' \setminus \tilde{\Omega}'^S \right) \right)}, \quad (5.14)$$

and such that  $\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}$  with  $\mathcal{S} = \Omega_1 \setminus \mathcal{R}$ .

We have proved (4.1) and point 1; let us prove point 2. Obviously,

$$\mathcal{S} \cap \Sigma \subset \pi_{K+K',1} \left( \tilde{\Phi}^{[K']}{}^{-1} \left( \tilde{\Omega}'^S \right) \right) \quad \text{and} \quad \mathcal{S}' \cap \Sigma' \subset \pi_{K+K',1} \left( \tilde{\Psi}^{[K]}{}^{-1} \left( \tilde{\Omega}^S \right) \right).$$

Using identities (5.11) and (5.12), this implies

$$\mathcal{S} \cap \Sigma \subset \pi_{K,1}(\tilde{\Omega}^S) \quad \text{and} \quad \mathcal{S}' \cap \Sigma' \subset \pi_{K',1}(\tilde{\Omega}'^S). \quad (5.15)$$

For all  $\xi$  in  $\mathcal{S} \cap \Sigma$ , there is one  $\xi_K \in \tilde{\Omega}^S$  such that  $\xi = \pi_{K,1}(\xi_K)$  and, from Lemma 5.5, a neighborhood  $\mathcal{V}_1^\xi$  of  $\xi$  in  $\Omega_1$  and an open subset  $\mathcal{V}'_1{}^\xi$  of  $\Omega'_1$  such that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{V}_1^\xi$  and  $\mathcal{V}'_1{}^\xi$ . For all  $\xi'$  in  $\mathcal{S}' \cap \Sigma'$ , there is

one  $\xi'_{K'} \in \tilde{\Omega}^S$  such that  $\xi' = \pi_{K',1}(\xi'_{K'})$  and, from Lemma 5.5, a neighborhood  $\mathcal{W}'^{\xi'}$  of  $\xi'_1 = \pi_{K',1}(\xi'_{K'})$  in  $\Omega'_1$  and an open subset  $\mathcal{W}_1^{\xi'}$  of  $\Omega_1$  such that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{W}_1^{\xi'}$  and  $\mathcal{W}'^{\xi'}$ .

Now,  $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S} \cap \Sigma}$  is an open covering of  $\mathcal{S} \cap \Sigma$  and  $(\mathcal{W}_1^{\xi'})_{\xi' \in \mathcal{S}' \cap \Sigma'}$  is an open covering of  $\mathcal{S}' \cap \Sigma'$ . Take for  $(\tilde{\mathcal{S}}^\alpha)_{\alpha \in A}$  the union of  $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S} \cap \Sigma}$  and  $(\mathcal{W}_1^{\xi'})_{\xi' \in \mathcal{S}' \cap \Sigma'}$ ; take for  $(\tilde{\mathcal{S}}'^\alpha)_{\alpha \in A}$  the union of  $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S} \cap \Sigma}$  and  $(\mathcal{W}'^{\xi'})_{\xi' \in \mathcal{S}' \cap \Sigma'}$ .

This proves the smooth case, and obviously implies the real analytic one from Lemma 5.3.  $\square$

Let us now prove the four lemmas used in the above proof.

*Proof of Lemma 5.3.* If  $\tilde{\Omega}^S \neq \emptyset$ , then there is at least an open set in  $\tilde{\Omega}$  derivatives of  $\tilde{\Phi}$  along any vertical vector field (preserving fibers of  $\Sigma_K \rightarrow M$ ) are identically zero; since these are real analytic they must be zero all over  $\tilde{\Omega}$ , assumed connected, hence  $\tilde{\Omega}^S = \tilde{\Omega}$ . The proof is similar in  $\tilde{\Omega}'$ .  $\square$

*Proof of Lemma 5.4.* The first relation in (5.9) is a consequence of the two identities

$$\pi_{K'+\ell, K'} \circ \tilde{\Phi}^{[K'+\ell]} = \tilde{\Phi}^{[K']} \circ \pi_{K+K'+\ell, K+K'} \quad \text{and} \quad \tilde{\Psi}^{[\ell]} \circ \tilde{\Phi}^{[K'+\ell]} = \pi_{K+K'+\ell, \ell}, \quad (5.16)$$

respectively (2.6) with  $(r, s) = (K' + \ell, K')$  and the  $\ell^{\text{th}}$  prolongation of (5.6). The second relation follows from interchanging  $K, \Phi, S$  with  $K', \Psi, S'$ .

From equations (5.4) and (5.5), one has, for any positive integer  $\ell$ ,

$$\tilde{\Phi}^{[\ell]}(\tilde{\Omega}_{K+\ell}) \subset \tilde{\Omega}'_\ell \quad \text{and} \quad \tilde{\Psi}^{[\ell]}(\tilde{\Omega}'_{K'+\ell}) \subset \tilde{\Omega}_\ell \quad (5.17)$$

(for instance, (5.4) implies  $\tilde{\Phi}^{[\ell]}(\tilde{\Omega}_{K+\ell}) \subset \Sigma'_\ell$ , (5.5) implies  $\tilde{\Phi}^{[\ell]}(\tilde{\Omega}_{K+\ell}) \subset \Omega'_\ell$ , hence the first relation above because  $\tilde{\Omega}'_\ell = \Omega'_\ell \cap \Sigma'_\ell$ ). We only need to prove the reverse inclusions for  $\ell = 1$ . Let us do it for the second one. The second relation in (5.16) for  $\ell = 1$  implies  $\tilde{\Omega}_1 = \tilde{\Psi}^{[1]}(\tilde{\Phi}^{[K'+1]}(\tilde{\Omega}_{K+K'+1}))$ , and finally  $\tilde{\Omega}_1 \subset \tilde{\Psi}^{[1]}(\tilde{\Omega}'_{K'+1})$  from the first relation in (5.16) with  $\ell = K' + 1$ .  $\square$

*Proof of Lemma 5.5.* Assume for instance that  $\tilde{\Omega}^S$  is non-empty; then it contains an open subset  $V$  and there is a smooth  $\tilde{\Phi}_0 : V_0 \rightarrow M'$  such that, in restriction to  $V$ ,  $\tilde{\Phi} = \tilde{\Phi}_0 \circ \pi_{K,0}$ . Hence (5.6) implies, on the open subset  $V' = (\tilde{\Psi}^{[K]})^{-1}(V)$  of  $\Sigma'_{K+K'}$ ,

$$\tilde{\Phi}_0 \circ \pi_{K,0} \circ \tilde{\Psi}^{[K]} = \pi_{K+K',0}|_{V'}. \quad (5.18)$$

The rank of the map on the left-hand side is  $n'$  while the rank of the right-hand side is no larger than  $n$  (rank of  $\pi_{K,0}$ ), hence  $\tilde{\Omega}^S \neq \emptyset$  implies  $n' \leq n$ . By interchanging the two systems, this proves the first sentence of the Lemma.

Let us now turn to the case where  $n = n'$ . Consider  $\xi_K$  in  $\tilde{\Omega}^S$ . By definition of  $\tilde{\Omega}^S$ , there is a neighborhood  $V$  and a smooth (real analytic in the real analytic case) map  $\tilde{\Phi}_0 : V_0 \rightarrow M'$  such that  $\tilde{\Phi} = \tilde{\Phi}_0 \circ \pi_{K,0}$  on  $V$ . Let  $V'$  be defined from  $V$  as

$$V' = \pi_{K+K',K'} \left( (\tilde{\Psi}^{[K]} )^{-1}(V) \right) = \tilde{\Phi}^{[K']} (V_{K+K'}), \quad (5.19)$$

where the second equality comes from (5.9). Applying  $\tilde{\Psi}$  and  $\tilde{\Psi}^{[1]}$  to both sides of the first equality in (2.6) and using (5.19) with  $(r, s) = (K, 0)$  and  $(r, s) = (K, 1)$  yields

$$\tilde{\Psi}(V') = V_0, \quad \tilde{\Psi}^{[1]}(V'_{K'+1}) = V_1. \quad (5.20)$$

Substituting  $\tilde{\Phi} = \tilde{\Phi}_0 \circ \pi_{K,0}$  in (5.6), one has  $\tilde{\Phi}_0 \circ \tilde{\Psi} \circ \pi_{K+K',K'} = \pi_{K+K',0}$  on  $\tilde{\Psi}^{[K]}{}^{-1}(V)$ , and finally

$$\tilde{\Phi}_0 \circ \tilde{\Psi} = \pi_{K',0} \text{ on } V'; \quad (5.21)$$

in a similar way, substituting  $\tilde{\Phi}^{[1]} = \tilde{\Phi}_0^{[1]} \circ \pi_{K+1,1}$  in the first prolongation of (5.6),

$$\tilde{\Phi}_0^{[1]} \circ \tilde{\Psi}^{[1]} = \pi_{K'+1,1} \text{ on } V'_{K'+1}. \quad (5.22)$$

Applying  $\tilde{\Phi}_0$  to both sides of the first relation and  $\tilde{\Phi}_0^{[1]}$  to both sides of the second relation in (5.20), one has, using (5.21) and (5.22),

$$\tilde{\Phi}_0(V_0) = V'_0, \quad \tilde{\Phi}_0^{[1]}(V_1) = V'_1. \quad (5.23)$$

Since the rank of  $\pi_{K',0}$  in the right-hand side of (5.21) is  $n' = n$  at all points of  $V'$ ,  $\tilde{\Phi}_0$  must be a local diffeomorphism at all point of  $\tilde{\Psi}(V') = V_0$  and in particular at  $\xi_0$ : by the inverse function theorem, there is a neighborhood  $O$  of  $\xi_0 = \pi_{K,0}(\xi)$  in  $V_0$  and a neighborhood  $O'$  of  $\Phi_0(\xi_0)$  in  $M'$  such that  $\tilde{\Phi}_0$  defines a diffeomorphism  $O \rightarrow O'$ .

Let us now replace  $V$  with  $V \cap \pi_{K,0}^{-1}(O)$ , a smaller neighborhood of  $\xi_K$ ;  $V'$  is still defined by (5.19) from this smaller  $V$ , one has  $V_0 = O$ , the former  $\tilde{\Phi}_0$  is replaced by its restriction to this smaller  $V_0$ , and the above relations still hold. In particular,  $O' = \tilde{\Phi}_0(O)$  must be all  $V'_0$  according to (5.23), i.e.  $\tilde{\Phi}_0$  defines a diffeomorphism  $V_0 \rightarrow V'_0$ ; let  $\tilde{\Psi}_0$  be its inverse. Composing each side of (5.21) with  $\tilde{\Psi}_0$ , one gets  $\tilde{\Psi} = \tilde{\Psi}_0 \circ \pi_{K',0}$  on  $V'$ ; hence, by (5.8), one has  $V' \subset \tilde{\Omega}'^S$  and, since this is true for all  $\xi_K$  in  $\tilde{\Omega}^S$ , one has

$$\pi_{K+K',K'} \left( \tilde{\Psi}^{[K]}{}^{-1}(\tilde{\Omega}^S) \right) = \tilde{\Phi}^{[K']}(\tilde{\Omega}_{K+K'}^S) \subset \tilde{\Omega}'^S. \quad (5.24)$$

Let  $\mathcal{V}_1$  and  $\mathcal{V}'_1$  and be open subsets of  $\Omega_1$  and  $\Omega'_1$  such that

$$V_1 = \Sigma \cap \mathcal{V}_1, \quad V'_1 = \Sigma \cap \mathcal{V}'_1. \quad (5.25)$$

From Proposition 5.2, the second relation in (5.23) implies that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{V}_1$  and  $\mathcal{V}'_1$ . Interchanging the two systems, one proves that

$$\pi_{K+K',K} \left( \tilde{\Phi}^{[K']}{}^{-1}(\tilde{\Omega}'^S) \right) = \tilde{\Psi}^{[K]}(\tilde{\Omega}_{K+K'}^S) \subset \tilde{\Omega}^S. \quad (5.26)$$

and that, for all  $\xi'_{K'} \in \tilde{\Omega}'^S$ , there are a neighborhood  $\mathcal{W}'$  of  $\xi'_1 = \pi_{K',1}(\xi'_{K'})$  in  $\Omega'_1$  and an open subset  $\mathcal{W}_1$  of  $\Omega_1$  such that systems  $\Sigma$  and  $\Sigma'$  are static equivalent over  $\mathcal{W}_1$  and  $\mathcal{W}'_1$ .

Now,  $\tilde{\Phi}^{[K']}(\tilde{\Omega}_{K+K'}^S) \subset \tilde{\Omega}'^S$  in (5.24) implies  $\tilde{\Omega}_{K+K'}^S \subset \tilde{\Phi}^{[K']}{}^{-1}(\tilde{\Omega}'^S)$ , and hence  $\tilde{\Omega}^S \subset \pi_{K+K',K} \left( \tilde{\Phi}^{[K']}{}^{-1}(\tilde{\Omega}'^S) \right)$ . Hence (5.24) implies the converse inclusion in (5.26); in a similar way (5.26) implies the converse inclusion in (5.24). This proves (5.11) and (5.12), and ends the proof of Lemma 5.5.  $\square$

*Proof of Lemma 5.6.* Denote by  $\bar{\xi}_{K+1}$  the point  $\xi_{K+1}$  in the lemma statement and set  $\bar{\xi}_K = \pi_{K+1,K}(\bar{\xi}_{K+1}) \in \tilde{\Omega} \setminus \tilde{\Omega}^S$ ,  $\bar{\xi}_0 = \pi_{K,0}(\bar{\xi}_{K+1})$ ,  $\bar{\xi}_1 = \pi_{K,1}(\bar{\xi}_{K+1})$ . From Proposition 3.4, and after possibly shrinking  $\mathcal{U}_K$  so that it is contained in  $\Omega$ , there exist a neighborhood  $\mathcal{U}_K \subset \Omega$  of  $\bar{\xi}_K$  in  $J^K(M)$ , coordinates  $(x_I, x_{II})$  on  $\mathcal{U}_0 = \pi_{K,0}(\mathcal{U}_K)$  inducing coordinates  $(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_I^{(K)}, x_{II}^{(K)})$  on  $\mathcal{U}_K$ , and an open subset  $U_K \subset \mathbb{R}^{n+K^m}$  such that the equations of  $\tilde{\mathcal{U}}_K = \mathcal{U}_K \cap \Sigma_K$  in  $J^K(M)$  in these coordinates are

$$\begin{aligned} x_I^{(i)} &= f^{(i-1)}(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(i)}), \quad 1 \leq i \leq K, \\ (x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(K)}) &\in U_K. \end{aligned} \quad (5.27)$$

By substitution, there is a unique smooth map  $\phi_K : U_K \rightarrow M'$  such that  $\tilde{\Phi}(\xi) = \phi_K(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(K)})$  for all  $\xi$  in  $\tilde{\mathcal{U}}_K$  with coordinate vector  $(x_I, x_{II}, \dots, x_I^{(K)}, x_{II}^{(K)})$ .

Let  $\bar{X}_i = (\bar{x}_I, \bar{x}_{II}, \bar{\dot{x}}_I, \bar{\dot{x}}_{II}, \dots, \bar{x}_I^{(i)}, \bar{x}_{II}^{(i)})$  be the coordinate vector of  $\bar{\xi}_i$  for  $i \leq K+1$  and  $\bar{\rho}$  the smallest integer such that  $\phi_K$  does not depend on  $x_{II}^{(\bar{\rho}+1)}, \dots, x_{II}^{(K)}$  on at least one neighborhood of  $\bar{X}_K$ . Shrinking  $U_K$  to this neighborhood, and  $\tilde{\mathcal{U}}_K$  accordingly, we may define  $\phi : U_{\bar{\rho}} \rightarrow M'$ , with  $U_{\bar{\rho}}$  the projection of  $U_K$  on  $\mathbb{R}^{n+\bar{\rho}^m}$ , such that  $\tilde{\Phi}(\xi) = \phi_K(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(K)}) = \phi(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(\bar{\rho})})$ . If  $\bar{\rho}$  was zero, one would have  $\tilde{\Phi}(\xi) = \phi(x_I, x_{II})$ , hence the right-hand side of (5.7) would be satisfied for  $\xi = \bar{\xi}_K$  with  $V = \tilde{\mathcal{U}}_K$ ; this is impossible because we assumed  $\bar{\xi}_K \in \tilde{\Omega} \setminus \tilde{\Omega}^S$ . Hence  $\bar{\rho} \geq 1$ .

For all  $\xi_{K+1}$  in  $\tilde{\mathcal{U}}_{K+1}$  with coordinate vector  $(x_I, x_{II}, \dots, x_{II}^{(K+1)})$ , one has

$$\tilde{\Phi}^{[1]}(\xi_{K+1}) = \chi(x_I, x_{II}, \dot{x}_I, \dot{x}_{II}, \dots, x_{II}^{(\bar{\rho})}, x_{II}^{(\bar{\rho}+1)}) \quad (5.28)$$

with  $\chi : U_{\bar{\rho}+1} \rightarrow TM'$  the map defined by

$$\chi(x_I \dots x_{II}^{(\bar{\rho}+1)}) = \left( \phi(x_I \dots x_{II}^{(\bar{\rho})}), a(x_I \dots x_{II}^{(\bar{\rho})}) + \frac{\partial \phi}{\partial x_{II}^{(\bar{\rho})}}(x_I \dots x_{II}^{(\bar{\rho})}) x_{II}^{(\bar{\rho}+1)} \right) \quad (5.29)$$

with  $a = \frac{\partial \phi}{\partial x_I} f + \sum_{i=0}^{\bar{\rho}-1} \frac{\partial \phi}{\partial x_{II}^{(i)}} x_{II}^{(i+1)}$ . According to (5.5), (5.27) and (5.28),  $\Sigma'$  contains  $\chi(U_{\bar{\rho}+1})$ . Now, for any  $(x_I, \dots, x_{II}^{(\bar{\rho}+1)}) \in U_{\bar{\rho}+1}$  such that the linear map

$$\frac{\partial \phi}{\partial x_{II}^{(\bar{\rho})}}(x_I, \dots, x_{II}^{(\bar{\rho})}) : \mathbb{R}^m \rightarrow T_{\phi(x_I, \dots, x_{II}^{(\bar{\rho})})} M'$$

is nonzero, picking  $\underline{w} \neq 0$  in its range, (5.29) implies that the straight line  $\Delta$  in  $T_{\phi(x_I, \dots, x_{II}^{(\bar{\rho})})} M'$  passing through  $\chi(x_I \dots x_{II}^{(\bar{\rho}+1)})$  with direction  $\underline{w}$  has a segment around  $\chi(x_I \dots x_{II}^{(\bar{\rho}+1)})$  contained in  $\Sigma'$ , hence in particular  $\Delta$  has contact of infinite order with  $\Sigma'$  at point  $\chi(x_I, \dots, x_{II}^{(\bar{\rho}+1)})$ . To sum up, we have proved so far that, for all  $\xi_{K+1}$  in  $\tilde{\mathcal{U}}_{K+1}$  with coordinate vector  $(x_I, x_{II}, \dots, x_{II}^{(K+1)})$  such that  $\frac{\partial \phi}{\partial x_{II}^{(\bar{\rho})}}(x_I \dots x_{II}^{(\bar{\rho})})$  is nonzero, there is a straight line  $\Delta_{\xi_{K+1}}$  in  $T_{\tilde{\Phi}(\xi_K)} M'$  passing through  $\tilde{\Phi}^{[1]}(\xi_{K+1})$  that has contact of infinite order with  $\Sigma'$  at  $\tilde{\Phi}^{[1]}(\xi_{K+1})$ . The set of such points  $\xi_{K+1}$  may not contain  $\bar{\xi}_{K+1}$  but its topological closure does, by minimality of  $\bar{\rho}$ ; taking a sequence of points  $\xi_{K+1}$  that converges to  $\bar{\xi}_{K+1}$ , any accumulation point of the compact sequence  $(\Delta_{\xi_{K+1}})$  is a straight line in  $T_{\tilde{\Phi}(\bar{\xi}_K)} M'$  passing through  $\tilde{\Phi}^{[1]}(\bar{\xi}_{K+1})$  that has contact of infinite order with  $\Sigma'$  at  $\tilde{\Phi}^{[1]}(\bar{\xi}_{K+1})$ .  $\square$

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