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► **To cite this version:**

Didier Bresch, Thierry Colin, Emmanuel Grenier, Benjamin Ribba, Olivier Saut. A viscoelastic model for avascular tumor growth. [Research Report] 2008. inria-00267292v1

**HAL Id: inria-00267292**

**<https://inria.hal.science/inria-00267292v1>**

Submitted on 27 Mar 2008 (v1), last revised 10 Jul 2009 (v4)

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# A viscoelastic model for avascular tumor growth

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## Abstract

In this article, we present a continuous model for tumor growth . This model describes the evolution of three cellular densities: the density of sane tissue, cancer cells and extracellular medium. In order to render correctly the cellular division, this model use a discrete description of the cell cycle (the set of steps a cell has to undergo in order to divide) presented in [7]. To render cellular adhesion and the mechanics which may influence the growth, we use a viscoelastic approach. This model extends the one presented in [7] with a more realistic description of the movement.

## 1 Introduction

A tumor arises after several mutations of cells that have made them less sensible to anti-growth factors or lack of nutrients for instance. This may lead to uncontrolled division of these cells.

In order to divide, a cell needs nutrients (such as oxygen for instance), which is obtained from its close environment in the avascular phase. As the tumor grows, some cells stop getting enough nutrients and turn to a quiescent state where they no longer divide waiting for the environment to become favorable again. Therefore, for a realistic description of cancer growth, one has to describe the evolution of the concentration of nutrients. Tumor cells have also the ability to produce their energy from glucose, whose production lowers the pH of the medium. This toxicity may harm cancer or healthy cells and favor tumoral invasion. Finally, it is known that cellular adhesion has an influence on tumor growth [3]. This influence was not possible to study with model [7].

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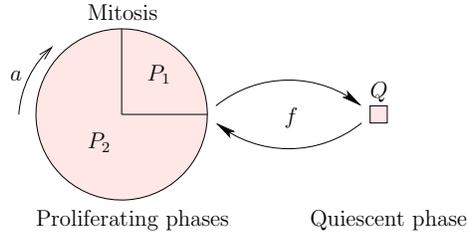


Figure 1: Scheme of the simplified cell-cycle used in this paper and presented in [7].

There are many mathematical models describing solid tumor growth. Roughly they can be divided in two classes. In discrete models, one describe the evolution of cells individually. Among these models, we find for instance models based on cellular automata or agent-based models. If these models are very efficient to describe the microscopic aspects of tumor growth, they are computationally expensive, which makes them difficult to use for a large number of cells. Furthermore, it is also difficult to account for the mechanical effects influencing tumor growth.

Continuous models compute the evolution of cellular densities or of the boundary of the tumor [4]. They are much less expensive to discretize but as they deal with averages over a large number of cells, they can not always render correctly the microscopic or genetic aspects of tumor growth. Most of these models are based on partial differential equations, such as reaction-diffusion equations or advection equations [2]. In this paper, we present such a model using advection equations on cell densities, in which the velocity is obtained from a viscoelastic approach. The outline of this paper is the following one. In Sec. 2, we present the mathematical model and in Sec. 3, we show a numerical experiment performed with this model.

## 2 Description of the model

We describe the cellular species by their densities. The density of sane tissue will be denoted by  $S(t, \mathbf{x})$ , the density of extra-cellular medium or interstitial liquid by  $L(t, \mathbf{x})$ . Cancer cells are described by their densities in each of the two proliferation phases  $P_1(t, a, \mathbf{x})$ ,  $P_2(t, a, \mathbf{x})$  and quiescent phase  $Q(t, \mathbf{x})$ . The proliferation phases are age-structured as shown in Fig. 1. We refer the reader to [7] for a detailed presentation.

We assume that the total number of cells is constant per unit volume, *i.e.*

$$S + L + \int_a (P_1 + P_2) da + Q = N_0. \quad (1)$$

In the sequel, we will take  $N_0 = 1$ .

We denote by  $C(t, \mathbf{x})$  the concentration of nutrients and  $H(t, \mathbf{x})$  the acidity. We also consider that the cellular division and the corresponding growth of volume generate a movement described by a velocity denoted by  $\mathbf{v}(t, \mathbf{x})$ .

We make the following biological assumptions:

- Healthy cells may die if the concentration of nutrient is too low or if the total cell density is too high or if the acidity is important. We neglect division of these cells.
- Cancer cells may die for the same reasons than healthy cells but there are more resistant to harsh conditions. (Therefore the corresponding survival thresholds will be less restrictive.)
- Proliferating cells are dividing. If the environment is not favorable enough (in term of hypoxia, overpopulation and acidity) they become quiescent and stop dividing.
- Dead cells are metabolized and are accounted for in the extracellular phase.

## 2.1 Equation for the populations of cells

In order to obtain the equations giving the evolution of cellular densities, we use the mass-balance conservation principle. The equation for the healthy cells is

$$\partial_t S + \nabla \cdot (\mathbf{v}S) = -\alpha_S f_{AS} S, \quad (2)$$

where  $f_{AS}$  is the function giving the rate of apoptosis for healthy cells. It has the form

$$f_{AS} = \mathbf{1}_{\{C < \tau_{0,S} \text{ or } \Sigma_p > K_S \text{ or } H > H_{0,S}\}}, \quad (3)$$

where  $\Sigma_p$  is the indicator for overpopulation and is computed by  $\Sigma_p = S + \int_0^{a_{max,P_1}} P_1(a) da + \int_0^{a_{max,P_2}} P_2(a) da + Q$ . We have denoted by  $\tau_{0,S}$ ,  $K_S$  and  $H_{0,S}$  the thresholds for respectively hypoxia, overpopulation and toxicity.

For the tumor cells in phase  $P_1$ , the equation is:

$$\partial_t P_1 + \partial_a P_1 + \nabla \cdot (\mathbf{v}P_1) = -\alpha_P f_{AP} P_1, \quad (4)$$

with  $a$  ranging from 0 to  $a_{max,P_1}$ . The function  $f_{AP}$  describing apoptosis has the same expression as Eq. (3) with different (less restrictive) parameters ( $\tau_{0,P} < \tau_{0,S}$ ,  $K_P > K_S$  and  $H_{0,P} < H_{0,S}$ ). The equation for cells in phase  $P_2$  is similar:

$$\partial_t P_2 + \partial_a P_2 + \nabla \cdot (\mathbf{v}P_2) = -\alpha_P f_{AP} P_2, \quad (5)$$

for  $a = 0$  to  $a_{max,P_2}$ . The boundary conditions accounting for the transitions between phases in Fig. 1 are:

$$P_1(a = 0) = 2P_2(a_{max,P_2}), \quad (6)$$

which describes the mitosis and

$$P_2(a=0) = f_Q P_1(a_{max,P_1}) + \left[ \frac{d}{dt} f_Q \right]^+ Q(t^-), \quad (7)$$

where  $f_Q$  is the boolean function checking the environmental factors for a cell to go or come back from the quiescent state:

$$f_Q = 1 - \mathbf{1}_{\{C < \tau_Q \text{ or } \Sigma_p > K_Q \text{ or } H < H_{0,Q}\}}, \quad (8)$$

where again the thresholds are different of those of apoptosis:  $\tau_{0,P} < \tau_{0,S} < \tau_Q$ ,  $K_P > K_S > K_Q$  and  $H_Q < H_{0,P} < H_{0,S}$ . That means that if the environmental conditions are not favorable, then the proliferating cells go into quiescent state and if they become even worse, cells go to apoptosis.

The equation for the quiescent cells is

$$\partial_t Q + \nabla \cdot (\mathbf{v}Q) = (1-f)P_1(a = a_{max,P_1}) - \left[ \frac{d}{dt} f_Q \right]^+ Q(t^-) - \alpha_P f_{AP} Q \quad (9)$$

with the same notation than above.

We still have to deal with the extra-cellular phase  $L$ :

$$\begin{aligned} \partial_t L + \nabla \cdot (\mathbf{v}L) = & \alpha_S f_{AS} S + \int_0^{a_{max,P_1}} \alpha_P f_{AP} P_1 \\ & + \int_0^{a_{max,P_2}} \alpha_P f_{AP} P_2 + \alpha_p f_{AQ} Q \end{aligned} \quad (10)$$

where we have added every source terms coming from the different equations (2), (4), (5) and (9) in order to ensure conservation (1). This means from the biological point of view that every waste produced by the death of any kind of cells enter the extracellular medium.

Collecting the equations leads to

$$\nabla \cdot \mathbf{v} = P_2(a = a_{max,P_2}). \quad (11)$$

## 2.2 Equation on the velocity

Eq. (11) is not sufficient to determine the velocity  $\mathbf{v}$ . The medium is described as being a viscoelastic material: the relation between the stress, pressure and velocity is:

$$\nabla \cdot \sigma - \nabla p = -\nu \Delta \mathbf{v}, \quad (12)$$

where  $\nu$  denotes the viscosity of the surrounding liquid like for emulsions. The constitutive law for the stress is

$$\partial_t \sigma + \mathbf{v} \cdot \nabla \sigma - \nabla \mathbf{v}^t \sigma - \sigma \nabla \mathbf{v} + \frac{1}{\tau} \sigma = \frac{\beta(\tau)}{\tau} D(\mathbf{v}), \quad (13)$$

where  $D(\mathbf{v}) = \frac{\nabla \mathbf{v} + \nabla \mathbf{v}^t}{2}$  and  $\beta(\tau)$  is a function describing the rheological properties of the tissues. In this paper, we neglect the terms  $\mathbf{v} \cdot \nabla \sigma - \nabla \mathbf{v}^t \sigma - \sigma \nabla \mathbf{v}$ . The parameter  $\tau$  is given by

$$\tau = (1-L) + \frac{1-L}{1-S}, \quad (14)$$

and the function  $\beta$  by

$$\beta(\tau) = \beta_0 + \tau\beta_\infty. \quad (15)$$

Then,  $\tau \rightarrow 0$  leads to  $\sigma = \beta(0)D(\mathbf{v})$ , which describes the behavior of a Newtonian liquid (*i.e.* the extracellular medium  $L$  is liquid). When  $\tau \rightarrow \infty$ , we obtain  $\partial_t \sigma + \mathbf{v} \cdot \nabla \sigma = \beta_\infty D(\mathbf{v})$ , which is the law for linear elasticity *i.e.* healthy tissue is considered as elastic. Cell-to-cell adhesion is weaker for cancer cells than for healthy ones and therefore we consider their behavior as being viscoelastic.

### 2.3 Nutrients and acidity

The oxygen (or nutrient) is consumed by cancer and healthy cells. As proliferating cells consume much more oxygen, their consumption rate is much higher than the ones of quiescent or healthy cells. This leads to the equation

$$\begin{cases} -\nabla \cdot (D \nabla C) = -\alpha_C \left( \int_0^{a_{max,P_1}} P_1(a) da + \int_0^{a_{max,P_2}} P_2(a) da \right) C \\ \quad \quad \quad -\alpha_S S - \alpha_Q Q - \Gamma_C C \text{ on } \Omega/O, \\ C = C_0 \text{ on } \partial\Omega, \\ C = C_{\max} \text{ on } O, \end{cases} \quad (16)$$

where we have made the stationary assumption because the diffusion time-scale of oxygen is much lower than the time-scale of cellular division.

The acidity is produced by the cancer cells and is evacuated by the blood vessels:

$$\begin{cases} -\nabla \cdot (D \nabla H) = \alpha_H^P \left( \int_0^{a_{max,P_1}} P_1(a) da + \int_0^{a_{max,P_2}} P_2(a) da \right) + \alpha_H^Q Q - \alpha_H^B \mathbf{1}_O H, \\ H = H_0 \text{ on } \partial\Omega. \end{cases} \quad (17)$$

## 3 Numerical result

In this section, we present a numerical result of our domain. In this experiment, we study the effect of mechanical stress on the shape of the tumor.

The initial tumor is a spheroid, the oxygen distribution is isotropic. Hence, with a constant stress, the spheroid should keep its spherical form through the computations. In our case, the mechanical stress is lowered in the horizontal direction. This is achieved by changing the density of sane tissue and liquid according to

$$S = 0.99(1 - N) \times \begin{cases} \frac{1}{2} & \text{if } |x - \frac{L}{2}| < 5\delta x \\ 1 & \text{otherwise,} \end{cases}$$

$N$  being the total density of cancer cells and  $L = 1 - N - S$ .

The oxygen source is represented by the domain  $O = \{|\mathbf{x}| > \frac{L}{2} - 4\delta x\}$ , on this domain, we set  $C = 0.1$ .

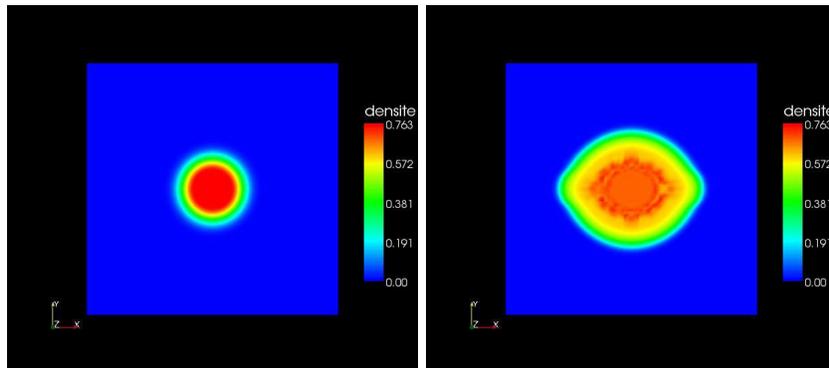


Figure 2: Evolution of a spheroid when the mechanical stress is not isotropic.

With a isotropic stress (or a simpler model as [7]), the tumor would have stayed disc-shaped. The kind of shape obtained in this run is observed in *in-vitro* experiments.

## 4 Conclusion

In this paper, we have presented a mathematical model for avascular tumor growth. It uses advection equations on cellular densities and accounts in the same time with microscopic (cell-cycle) and macroscopic (cellular-adhesion, mechanical effects) aspects of the tumoral growth.

This model was initially described to study avascular growth. Yet, through the domain  $O$  appearing in Eq. (16) and (17), a coupling could be made with a model describing the evolution of the density of blood vessels during the process of angiogenesis.

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