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Comparison of the Discriminatory Processor Sharing Policies

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Abstract: Discriminatory Processor Sharing policy introduced by Kleinrock is of a great interest in many application areas, including telecommunications, web applications and TCP flow modelling. Under the DPS policy the job priority is controlled by the vector of weights. Verifying the vector of weights it is possible to modify the service rates of the jobs and optimize system characteristics. In the present paper we present the results concerning the comparison of two DPS policies with different weight vectors. We show the monotonicity of the expected sojourn time of the system depending on the weight vector under certain condition on the system. Namely, the system has to consist of classes with means which are quite different from each other. The classes with similar means can be organized together and considered as one class, so the given restriction can be overcome.

Key-words: Discriminatory Processor Sharing, exponential service times, optimization.

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Comparaison des politiques DPS

Résumé : L'ordre de service DPS (Discriminatory Processor Sharing) qui était introduit par Kleinrock est un problème très intéressant et peut être appliqué dans beaucoup de domaines comme les télécommunications, les applications web et la modélisation de flux TCP. Avec le DPS, les jobs qui viennent dans le système sont contrôlés par un vecteur de poids. En modifiant le vecteur de poids, il est possible de contrôler les taux de service des jobs, donner la priorité à certaines classes de jobs et optimiser certaines caractéristiques du système. Le problème du choix des poids est donc très important et très difficile en raison de la complexité du système. Dans le présent papier, nous comparons deux politiques DPS avec les vecteurs de poids différents et nous présentons des résultats sur la monotonie du temps moyen de service du système en fonction du vecteur de poids, sous certaines conditions sur le système. Le système devrait consister en plusieurs classes avec des moyennes très différentes. En considérant que toutes les classes qui ont une moyenne très proche peuvent être considérées comme une seule classe, cette restriction peut être enlevée.

Mots-clés : Discriminatory Processor Sharing, le temps de service exponentielle, optimisation.

1 Introduction

The Discriminatory processor Sharing (DPS) was first introduced by Kleinrock ([11]). Under the DPS policy jobs are organized in classes, which share a single server. The capacity which each class obtains depends on the number of jobs currently presented in all classes. All jobs present in the system are served simultaneously at rates controlled by the vector of weights $\{g_k > 0, k = 1, \dots, M\}$, where M is the number of classes. If there are N_j jobs in the class j , then the jobs of this class are served with the rate $g_j / \sum_{k=1}^M g_k N_k$. When all weights are equal, DPS system is equivalent to the standard PS policy.

The DPS application is very wide. In the Internet many services provide the possibility of payment for the quality of obtained service, so the priority of users depends on their payment or in other words, weight vectors. DPS could be applied to model flow level sharing of TCP flows with different flow characteristics such as different RTT. DPS could model the weighted round-robin discipline, which is used in operating systems for task scheduling.

Varying DPS weights it is possible to give priority to different classes at the expense of others, control their instantaneous service rates and optimize different system characteristics as mean sojourn time and so on. So, the proper weight selection is an important task, which is not easy to solve because of the model complexity.

The previously obtained results on DPS model are not very numerous. After the work of Kleinrock [11] the paper of Fayolle et al. [1] provided results for the DPS model. The authors obtained the expression of the expected sojourn time as a solution of a system of linear equations. Also they provide the analysis of the system in case of the exponentially distributed required service times and give analytical solution for the case when there are two classes in the system. The authors show that independent of the weights the slowdown for the expected conditional response time under the DPS policy tends to the constant slowdown of the PS policy as the service requirements increase to infinity.

For exponential service time distributions Rege and Sengupta [5] obtained higher moments of the queue length distribution as the solutions of linear equations system and also provide the theorem for the heavy-traffic regime. These results were extended by Van Kessel et al [9], [2]. For general distributions of the required service times the approximation analysis was carried out by Guo and Matta [7]. Altman et al [8] study the behavior of the DPS in overload. Most of the results obtained for the DPS queue were collected together in the paper of Altman et al [3].

Kim and Kim in [2] compare the PS and DPS policies for the exponential required service time distributions. They show that DPS policy decreases expected sojourn time of the system in comparison with PS policy when the weights are selected in such a way that they decrease with the means of the classes. Also they give the conjecture about the monotonicity of the expected sojourn time of the DPS policies, but do not give mathematical proof for it.

In the present paper we prove the monotonicity of the DPS policy on the weight selection, which was formulated by Kim and Kim in [2], under certain conditions on the system. Namely, the system has to contain of classes with very different means.

The paper is organized as follows. In Section 2 we give general definitions of the DPS policy and formulate the main theorem. In Section 3 we give the additional Lemmas which prove the Theorem statement. In Section 4 we give the experimental results. The proofs could be found in the appendix.

2 Definitions and previous results

We consider the Discriminatory Processor Sharing (DPS) model. All jobs are organized in classes and share the single server. There are M classes in the system and in every class there are N_k jobs. Jobs of class $k = 1, \dots, M$ arrive with a Poisson process with rate λ_k and have required service-time distribution $F_k(x) = 1 - e^{-\mu_k x}$ with mean $1/\mu_k$. The load of the system is $\rho = \sum_{i=1}^M \rho_i$ and $\rho_k = \lambda_k / \mu_k, k = 1, \dots, M$. We consider that the system is stable, $\rho < 1$. Let us denote $\lambda = \sum \lambda_k$.

The state of the system is controlled by the vector of weights (g_1, \dots, g_M) , which denote the priority for the job classes. So, each job of class k is served with the rate equal to $g_j / \sum_{k=1}^M g_k N_k$, which depends on the current system state, or on the number of jobs in each class.

Let \bar{T}^{DPS} be the expected sojourn time of the DPS system. Then, we have

$$\bar{T}^{DPS} = \sum_{k=1}^M \frac{\lambda_k}{\lambda} \bar{T}_k,$$

where \bar{T}_k are expected sojourn times for class k .

We can find \bar{T}_k using the fact that $\bar{T}_k = \int_0^\infty T_k(t) \mu_k e^{-\mu_k t} dt$, where $T_k(t)$ are the expected conditional response time of a class k and is equal to

$$T_k(t) = \frac{t}{1-\rho} + \sum_{j=1}^M \left(\frac{g_k c_j \alpha_j + d_j}{\alpha_j^2} \right) (1 - e^{-\alpha_j t / g_k}), \quad k = 1, \dots, M,$$

where the coefficients α_j, c_j, d_j are some constant values which are quite complex. Also expressions for them are not analytical and have to be found as a solution of a system of linear equations.

Also the expressions for the expected sojourn time \bar{T}_k can be found as a solution of the system of linear equations, see [1].

$$\bar{T}_k \left(1 - \sum_{j=1}^M \frac{\lambda_j g_j}{\mu_j g_j + \mu_k g_k} \right) - \sum_{j=1}^M \frac{\lambda_j g_j \bar{T}_j}{\mu_j g_j + \mu_k g_k} = \frac{1}{\mu_k}, \quad k = 1, \dots, M. \quad (1)$$

Let us notice that for the standard Processor Sharing system

$$\bar{T}^{PS} = \frac{m}{1-\rho}.$$

One of the problems when studying DPS is to minimize the expected sojourn time \bar{T}^{DPS} with some weight selection. Or to find the optimal strategy to minimize expected sojourn time of the system. Namely, find g^* such as

$$\bar{T}^{DPS}(g^*) = \min_g \bar{T}^{DPS}(g).$$

This is a general problem and to simplify it the following subcase is considered. Find such G that

$$\forall g^* \in G, \quad \bar{T}^{DPS}(g^*) < \bar{T}^{PS}. \quad (2)$$

For the case when job size distributions are exponential and the means of classes are decreasing the solution of 2 is given by Kim and Kim in [2] and is as follows. If the means of the classes are such as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_M$, then G consists of all such vectors which satisfy

$$G = \{g \mid g_1 \geq g_2 \geq \dots \geq g_M\}.$$

Using the approach of [2] we solve more general problem about DPS monotonicity which we formulate in the following section as Theorem 1.

3 Expected sojourn time monotonicity

3.1 Problem formulation

Let us prove the following theorem.

Theorem 1. *Let the job size distributions for every class be exponential with means μ_i and we can remunerate them in the following way*

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_M. \quad (3)$$

Let us consider two different weight policies for the DPS system, which we denote as α and β . Let $\alpha, \beta \in G$, or

$$\begin{aligned} \alpha_1 &\geq \alpha_2 \geq \dots \geq \alpha_M, \\ \beta_1 &\geq \beta_2 \geq \dots \geq \beta_M \end{aligned}$$

and let the following property is true for α and β :

$$\frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\beta_{i+1}}{\beta_i}, \quad i = 1, \dots, M-1. \quad (4)$$

Then

$$\overline{T}^{DPS}(\alpha) < \overline{T}^{DPS}(\beta), \quad (5)$$

when the following condition is true : for every j

$$\frac{\mu_{j+1}}{\mu_j} < 1 - \rho, \quad \mu_{j+1} \neq \mu_j, \quad \text{or} \quad \mu_{j+1} = \mu_j. \quad (6)$$

Condition (6) gives us the limit on the quantity of systems to which the result is applied. It means that the result is true for the system with very different means of classes. On the other hand, all the classes with means such as $\frac{\mu_{j+1}}{\mu_j} > 1 - \rho$ can be organized together, so we consider that this restriction can be overcome.

To prove Theorem 1, let us first give some notations and prove additional Lemmas.

3.2 Notations and Lemmas

Let us rewrite linear system (1) in the matrix form. Let $\overline{T}^g = [\overline{T}_1, \dots, \overline{T}_M]^T$ be the vector of \overline{T}_k , $k = 1, \dots, M$. Here by $[\]^T$ we mean transpose sign, so $[\]^T$ is a vector. By $[\]^g$ we note that this vector depends on the weight selection $g \in G$. Let define matrixes A and D in the following way.

$$A^{(g)} = \begin{pmatrix} \frac{\lambda_1 g_1}{\mu_1 g_1 + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_1 g_1 + \mu_2 g_2} & \dots & \frac{\lambda_M g_M}{\mu_1 g_1 + \mu_M g_M} \\ \frac{\lambda_1 g_1}{\mu_2 g_2 + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_2 g_2 + \mu_2 g_2} & \dots & \frac{\lambda_M g_M}{\mu_2 g_2 + \mu_M g_M} \\ \dots & \dots & \dots & \dots \\ \frac{\lambda_1 g_1}{\mu_M g_M + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_M g_M + \mu_2 g_2} & \dots & \frac{\lambda_M g_M}{\mu_M g_M + \mu_M g_M} \end{pmatrix} \quad (7)$$

$$D^{(g)} = \begin{pmatrix} \sum_i \frac{\lambda_i g_i}{\mu_1 g_1 + \mu_i g_i} & 0 & \dots & 0 \\ 0 & \sum_i \frac{\lambda_i g_i}{\mu_2 g_2 + \mu_i g_i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sum_i \frac{\lambda_i g_i}{\mu_M g_M + \mu_i g_i} \end{pmatrix} \quad (8)$$

Then (1) becomes

$$(E - D^{(g)} - A^{(g)})\overline{T}^g = \left[\frac{1}{\mu_1}, \dots, \frac{1}{\mu_M} \right]^T. \quad (9)$$

We need to find the expected sojourn time of the DPS system \overline{T}^{DPS} . According to the definition of \overline{T}^{DPS} and equation (9) we have

$$\overline{T}^{DPS} = \frac{1}{\lambda} [\lambda_1, \dots, \lambda_M] \overline{T}^g = \frac{1}{\lambda} [\lambda_1, \dots, \lambda_M] (E - D^{(g)} - A^{(g)})^{-1} \left[\frac{1}{\mu_1}, \dots, \frac{1}{\mu_M} \right]^T. \quad (10)$$

Let us consider the case when $\lambda_i = 1$ for $i = 1, \dots, M$. This results can be extended for the case when λ_i are different, we prove it following the approach of [2] in Subsection .

Then (10) becomes

$$\overline{T}^{DPS}(g) = \underline{1}'(E - D^{(g)} - A^{(g)})^{-1}[\rho_1, \dots, \rho_M]^T \lambda^{-1}. \quad (11)$$

Let us give the following notations.

$$\sigma_{ij}^{(g)} = \frac{g_j}{\mu_i g_i + \mu_j g_j}, \quad g \in G. \quad (12)$$

Then $\sigma_{ij}^{(g)}$ have the following properties.

Proposition 2. For $\forall g \in G$

$$\begin{aligned} \sigma_{ij}^{(g)} g_i &= \sigma_{ji}^{(g)} g_j, \\ \frac{\sigma_{ij}^{(g)}}{\mu_i} + \frac{\sigma_{ji}^{(g)}}{\mu_j} &= \frac{1}{\mu_i \mu_j}. \end{aligned} \quad (13)$$

Proof. Follows from the definition of $\sigma_{i,j}^{(g)}$. □

Then matrixes $A^{(g)}$ and $D^{(g)}$ given by (7) and (8) can be rewritten in the terms of $\sigma_{ij}^{(g)}$.

$$\begin{aligned} A_{i,j}^{(g)} &= \sigma_{ij}^{(g)}, \quad i, j = 1, \dots, M, \\ D_{i,i}^{(g)} &= \sum_j \sigma_{ij}^{(g)}, \quad i = 1, \dots, M, \\ D_{i,j}^{(g)} &= 0, \quad i, j = 1, \dots, M, \quad i \neq j. \end{aligned}$$

Let us prove the following property of the weights α and β

Lemma 3. If α and β are such that (4) is true then

$$\frac{\alpha_j}{\alpha_i} \leq \frac{\beta_j}{\beta_i}, \quad i = 1, \dots, M-1, \quad \forall j > i. \quad (14)$$

Proof. Let us notice that if $a < b$ and $c < d$, then $ac < bd$ when a, b, c, d are positive. Also if $j > i$ then there exist such $l > 0$ that $j = i + l$. Then

$$\frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\beta_{i+1}}{\beta_i}, \quad \frac{\alpha_{i+2}}{\alpha_{i+1}} \leq \frac{\beta_{i+2}}{\beta_{i+1}}, \quad \dots \quad \frac{\alpha_{i+l}}{\alpha_{i+l-1}} \leq \frac{\beta_{i+l}}{\beta_{i+l-1}}, \quad i = 1, \dots, M-2.$$

Multiplying left and right parts of the previous inequalities we get the following:

$$\frac{\alpha_{i+l}}{\alpha_i} \leq \frac{\beta_{i+l}}{\beta_i}, \quad i = 1, \dots, M-2,$$

which proves Lemma. □

Lemma 4. If α and β are such that (14) is true, then

$$\sigma_{ij}^{(\alpha)} \leq \sigma_{ij}^{(\beta)}, \quad i \leq j, \quad (15)$$

$$\sigma_{ij}^{(\alpha)} \geq \sigma_{ij}^{(\beta)}, \quad i \geq j, \quad (16)$$

Proof. As (14) then

$$\begin{aligned}\frac{\alpha_j}{\alpha_i} &\leq \frac{\beta_j}{\beta_i}, \quad i \leq j, \\ \alpha_j \mu_i \beta_i &\leq \beta_j \mu_i \alpha_i, \quad i \leq j, \\ \alpha_j (\mu_i \beta_i + \mu_j \beta_j) &\leq \beta_j (\mu_i \alpha_i + \mu_j \alpha_j), \quad i \leq j, \\ \frac{\alpha_j}{\mu_i \alpha_i + \mu_j \alpha_j} &\leq \frac{\beta_j}{\mu_i \beta_i + \mu_j \beta_j}, \quad i \leq j, \\ \sigma_{ij}^{(\alpha)} &\leq \sigma_{ij}^{(\beta)}, \quad i \leq j.\end{aligned}$$

Property (16) is not evident as $\sigma_{i,j}^{g'} \neq \sigma_{j,i}^g$, $g \in G$. So, as (14) then

$$\begin{aligned}\frac{\alpha_i}{\alpha_j} &\leq \frac{\beta_i}{\beta_j}, \quad i \geq j, \\ \frac{\alpha_j}{\alpha_i} &\geq \frac{\beta_j}{\beta_i}, \quad i \geq j, \\ \sigma_{ij}^{(\alpha)} &\geq \sigma_{ij}^{(\beta)}, \quad i \geq j.\end{aligned}$$

The the statement of the Lemma is true. □

Lemma 5. *If α, β are such as (4) then*

$$\overline{T}^{DPS}(\alpha) < \overline{T}^{DPS}(\beta)$$

with the same conditions as the elements of vector $y = \underline{1}'(E - B^{(\alpha)})^{-1}M$ decrease on index.

Proof. Let us denote $B^{(g)} = A^{(g)} + D^{(g)}$, $g = \alpha, \beta$. Then as (11)

$$\overline{T}^{(g)} = \lambda^{-1} \underline{1}'(E - B^{(g)})^{-1} [\rho_1, \dots, \rho_M]^T, \quad g = \alpha, \beta.$$

Following the method described in [2] we get the following.

$$\begin{aligned}\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \lambda^{-1} \underline{1}'(E - B^{(\alpha)})^{-1} [\rho_1, \dots, \rho_M]^T - \lambda^{-1} \underline{1}'(E - B^{(\beta)})^{-1} [\rho_1, \dots, \rho_M]^T = \\ &= \lambda^{-1} \underline{1}'((E - B^{(\alpha)})^{-1} - (E - B^{(\beta)})^{-1}) [\rho_1, \dots, \rho_M]^T = \\ &= \lambda^{-1} \underline{1}'((E - B^{(\alpha)})^{-1}(B^{(\alpha)} - B^{(\beta)})(E - B^{(\beta)})^{-1}) [\rho_1, \dots, \rho_M]^T.\end{aligned}$$

Let us denote M as a diagonal matrix $M = \text{diag}(\mu_1, \dots, \mu_M)$ and

$$y = \underline{1}'(E - B^{(\alpha)})^{-1}M.$$

Then

$$\begin{aligned}\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \underline{1}'(1 - B^{(\alpha)})^{-1}MM^{-1}(B^{(\alpha)} - B^{(\beta)})\overline{T}^{(\beta)} = \\ &= yM^{-1}(B^{(\alpha)} - B^{(\beta)})\overline{T}^{(\beta)} = \\ &= \sum_{i,j} \left(\frac{y_j}{\mu_j} \sigma_{ji}^{(\alpha)} + \frac{y_i}{\mu_i} \sigma_{ij}^{(\alpha)} - \left(\frac{y_j}{\mu_j} \sigma_{ji}^{(\beta)} + \frac{y_i}{\mu_i} \sigma_{ij}^{(\beta)} \right) \right) T_j^{(\beta)} = \\ &= \sum_{i,j} \left(y_j \left(\frac{\sigma_{ji}^{(\alpha)}}{\mu_j} - \frac{\sigma_{ji}^{(\beta)}}{\mu_j} \right) + \frac{y_i}{\mu_i} (\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \right) T_j^{(\beta)}.\end{aligned}$$

As (13):

$$\frac{\sigma_{ji}^{(g)}}{\mu_j} = \frac{1}{\mu_i \mu_j} - \frac{\sigma_{ij}^{(g)}}{\mu_i}, \quad g = \alpha, \beta,$$

then

$$\begin{aligned}
\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \sum_{i,j} \left(-y_j \left(\frac{\sigma_{ij}^{(\alpha)}}{\mu_i} - \frac{\sigma_{ij}^{(\beta)}}{\mu_i} \right) + \frac{y_i}{\mu_i} (\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \right) T_j^{(\beta)} = \\
&= \sum_{i,j} \left(-\frac{y_j}{\mu_i} (\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) + \frac{y_i}{\mu_i} (\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \right) T_j^{(\beta)} = \\
&= \sum_{i,j} \left((\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) (y_i - y_j) \frac{1}{\mu_i} \right) T_j^{(\beta)}.
\end{aligned}$$

Using Lemma 4 we get $(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) (y_i - y_j)$ is negative for $i, j = 1, \dots, M$ with the same conditions as the elements of the vector y decrease on index. This proves the statement of Lemma. \square

Lemma 6. *Elements of vector y decrease on i or*

$$y_1 \geq y_2 \geq \dots \geq y_M,$$

when for every j

$$\frac{\mu_{j+1}}{\mu_j} < 1 - \rho, \quad \mu_{j+1} \neq \mu_j \quad \text{or} \quad \mu_{j+1} = \mu_j.$$

Proof. The proof could be found in the appendix. \square

Combining the results of Lemmas 3, 4, 5 and 7 we prove the statement of the Theorem 1.

3.3 Extension on the case unequal λ_i .

Let us show that the proved Theorem1 is extended on the case when $\lambda_i \neq 1$.

If λ_i are rational, then they could be written in $\lambda_i = \frac{p_i}{q}$, where p_i and q are positive integers. So each class can be presented as p classes with equal means $1/\mu_i$ and intensity $1/q$. So, the DPS system can be considered as a DPS system with $p_1 + \dots + p_K$ classes with the same arrival rates $1/q$. The result of Theorem 1 is extended on this case: as q is the same for all jobs, and as the results of the Theorem 1 and particularly Lemma 6 are correct when there are several classes of the same mean and weight in the system.

If λ_i are positive and real we apply the previous case and use continuity.

4 Experimental results for different weight vectors

Let us consider the case of three classes, each exponential and let us consider three vectors of weights.

$$\begin{aligned}
g_1^{(1)} &= \frac{x + c_1}{3x + c_1 + c_2}, & g_2^{(1)} &= \frac{x + c_2}{3x + c_1 + c_2}, & g_3^{(1)} &= \frac{x}{3x + c_1 + c_2} \\
g_1^{(2)} &= \frac{x + a_1}{3x + a_1 + a_2}, & g_2^{(2)} &= \frac{x + a_2}{3x + a_1 + a_2}, & g_3^{(2)} &= \frac{x}{3x + a_1 + a_2} \\
g_1^{(3)} &= \frac{x + b_1}{3x + b_1 + b_2}, & g_2^{(3)} &= \frac{x + b_2}{3x + b_1 + b_2}, & g_3^{(3)} &= \frac{x}{3x + b_1 + b_2}.
\end{aligned}$$

$$c_1 = 150, \quad c_2 = 25, \quad a_1 = 25, \quad a_2 = 6, \quad b_1 = 4, \quad b_2 = 2.$$

For this vector selection the following is true:

$$\frac{g_{i+1}^{(1)}}{g_i^{(1)}} \leq \frac{g_{i+1}^{(2)}}{g_i^{(2)}} \leq \frac{g_{i+1}^{(3)}}{g_i^{(3)}}, \quad i = 1, 2, \quad \forall x.$$

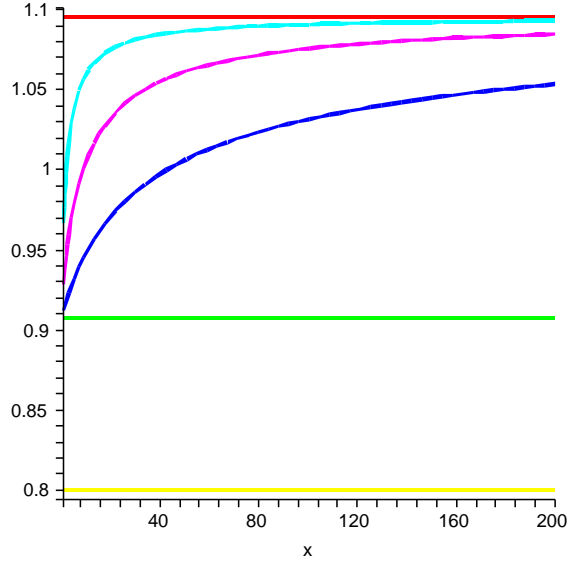


Figure 1:

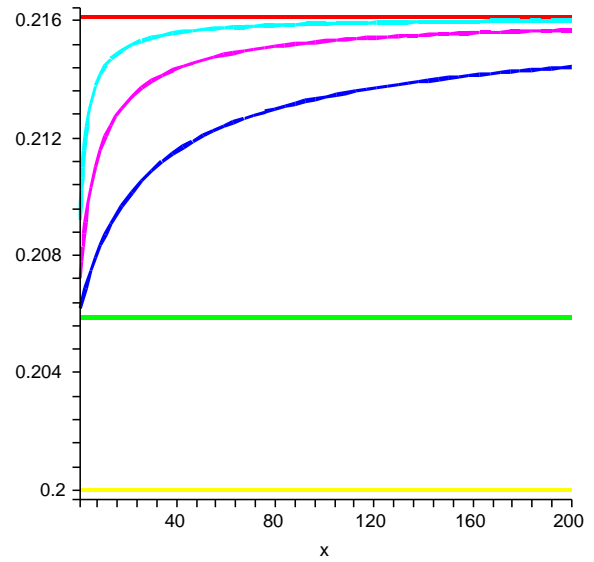


Figure 2:

Then we plot $\bar{T}(g^{(1,2,3)})$ as a function of x , see Figure 1. Also on Figure 1 we plot $\bar{T}^{DPS} = \bar{T}^{DPS}(1/3, 1/3, 1/3)$, which has the largest value and which is depicted by the dash line. Also we plot $\bar{T}(\frac{10^4 x}{10^4 x + 10^2 x + x}, \frac{10^4 x}{10^2 x + 10^2 x + x}, \frac{x}{10^4 x + 10^2 x + x})$, which is depicted by the straight dot line and approximates the strict priority policy. From the plot one can see that when

$$\bar{T}^{DPS}(g^{(1)}) \leq \bar{T}^{DPS}(g^{(2)}) \leq \bar{T}^{DPS}(g^{(3)}).$$

On the Figure 1 we plot the result when the means μ_i satisfy the restriction $\frac{\mu_{i+1}}{\mu_i} < 1 - \rho$. We take $\mu_1 = 100$, $\mu_2 = 20$, $\mu_3 = 3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. Here $\rho = 59/140$. For the Figure 2 we plot the case when the means do not satisfy this condition. We take $\mu_1 = 10$, $\mu_2 = 6$, $\mu_3 = 2$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.

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5 Appendix

In the following proof we do not use in the notation the dependency of the parameters on g to simplify the notations.

Lemma 7. *Elements of vector $y = \mathbf{1}(E - B)^{-1}M$ decrease on i or*

$$y_1 \geq y_2 \geq \dots \geq y_M,$$

when for every j

$$\begin{aligned} \mu_{j+1} &= \mu_j, \\ \frac{\mu_{j+1}}{\mu_j} &< 1 - \rho, \quad \mu_{j+1} \neq \mu_j. \end{aligned}$$

Proof. Using the results of the following Lemmas we prove the statement of the Lemma. \square

Let us give the following notations

$$\tilde{\mu} = \mu^T(E - D)^{-1}, \quad (17)$$

$$\tilde{A} = M^{-1}AM(E - D)^{-1}. \quad (18)$$

Let us notice the following

$$(E - D)_j^{-1} = \frac{1}{1 - \sum_{k=1}^M \frac{g_k}{\mu_j g_j + \mu_k g_k}} = \frac{1}{1 - \rho + \sum_{k=1}^M \frac{\mu_j g_j}{\mu_k(\mu_j g_j + \mu_k g_k)}} > 0, \quad j = 1, \dots, M,$$

$$\tilde{A}_{ij} = \frac{\frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)}}{1 - \sum_{k=1}^M \frac{g_k}{\mu_j g_j + \mu_k g_k}} = \frac{\frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)}}{1 - \rho + \sum_{k=1}^M \frac{\mu_j g_j}{\mu_k(\mu_j g_j + \mu_k g_k)}} > 0, \quad i, j = 1, \dots, M$$

Let us give the following notation

$$f(x) = \sum_{k=1}^M \frac{x}{\mu_k(x + \mu_k g_k)}.$$

Then

$$(E - D)_j^{-1} = \frac{1}{1 - \rho + f(\mu_j g_j)}, \quad j = 1, \dots, M,$$

$$\tilde{A}_{ij} = \frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)(1 - \rho + f(\mu_j g_j))}, \quad i, j = 1, \dots, M.$$

Lemma 8. *Matrix*

$$\tilde{A} = M^{-1}AM(E - D)^{-1}$$

is a positive contraction.

Proof. Matrix \tilde{A} is a positive operator as elements of matrices M and A are positive and elements of matrix $(E - D)^{-1}$ are positive. Let $\Omega = \{X | x_i \geq 0, i = 1, \dots, M\}$. If $X \in \Omega$, then $\tilde{A}X \in \Omega$. Also $\tilde{\mu} \in \Omega$. Then as $y^{(0)} \in \Omega$, then $y^{(n)} \in \Omega, \forall n$. Then to prove that matrix \tilde{A} is a contraction it is enough to show that

$$\exists q, \quad 0 < q < 1, \quad \|\tilde{A}X\| \leq q\|X\|, \quad \forall X \in \Omega. \quad (19)$$

As $X \in \Omega$, then we can take $\|X\| = \mathbf{1}'X = \sum_i x_i$. Then

$$\begin{aligned} \mathbf{1}'\tilde{A}X &= \sum_{j=1}^M x_j \sum_{i=1}^M \tilde{A}_{ij} = \sum_{j=1}^M x_j \frac{\sum_{i=1}^M \frac{\mu_j g_j}{\mu_i(\mu_j g_j + \mu_i g_i)}}{(1 - \rho + f(\mu_j g_j))} = \\ &= \sum_j x_j \frac{f(\mu_j g_j)}{1 - \rho + f(\mu_j g_j)} = \sum_j x_j \left(1 - \frac{1 - \rho}{1 - \rho + f(\mu_j g_j)}\right) = \sum_j x_j - (1 - \rho) \sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}. \end{aligned}$$

Let us find the value of q , which satisfies condition (19).

$$\sum_j x_j - (1 - \rho) \sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)} \leq q \sum_j x_j$$

$$q \geq 1 - (1 - \rho) \frac{\sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}}{\sum_j x_j}.$$

As $f(\mu_j g_j) > 0$ then

$$1 - (1 - \rho) \frac{\sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}}{\sum_j x_j} > 1 - (1 - \rho) \frac{\sum_j \frac{x_j}{1 - \rho}}{\sum_j x_j} = 1 - (1 - \rho) \frac{1}{1 - \rho} = 0,$$

$$1 - (1 - \rho) \frac{\sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}}{\sum_j x_j} < 1.$$

Let us define δ in the following way:

$$\frac{\sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}}{\sum_j x_j} > \frac{1}{1 - \rho + \max_j f(\mu_j g_j)} = \delta,$$

$$1 - (1 - \rho) \frac{\sum_j \frac{x_j}{1 - \rho + f(\mu_j g_j)}}{\sum_j x_j} < 1 - (1 - \rho)\delta,$$

Let us notice that $\max_j f(\mu_j g_j)$ always exists as the values of $\mu_j g_j$, $j = 1, \dots, M$ are finite. Then we can select

$$q = 1 - (1 - \rho)\delta, \quad 0 < q < 1.$$

Which completes the proof. \square

Lemma 9. *If*

$$y_1^{(0)} = [0, \dots, 0], \quad (20)$$

$$y^{(n)} = \tilde{\mu} + y^{(n-1)}\tilde{A}, \quad n = 1, 2, \dots, \quad (21)$$

then $y^{(n)} \rightarrow y$, when $n \rightarrow \infty$.

Proof. Let us present y in the following way. As $B = E - A - D$, then

$$\begin{aligned} y &= \underline{1}(E - B)^{-1}M, \\ yM^{-1}(E - D - A) &= \underline{1}, \\ yM^{-1}(E - D) &= -yM^{-1}A + \underline{1}, \\ y(E - D)^{-1}M &= -yM^{-1}A(E - D)^{-1}M + \underline{1}(E - D)^{-1}M. \end{aligned}$$

As matrixes D and M are diagonal, the $MD = DM$ and then

$$y = \mu^T(E - D)^{-1} + yM^{-1}AM(E - D)^{-1},$$

where $\mu = [\mu_1, \dots, \mu_M]$. According to (17) and (18) we have the following

$$y = \tilde{\mu} + y\tilde{A}.$$

Let us denote $y^{(n)} = [y_1^{(n)}, \dots, y_1^{(n)}]$, $n = 0, 1, 2, \dots$ and let define the $y_1^{(0)} = [0, \dots, 0]$ and y , and $y^{(n)} = \tilde{\mu} + y^{(n-1)}\tilde{A}$, $n = 1, 2, \dots$ as (20) (21). According to Lemma 8 reflection \tilde{A} is a positive reflection and is a contraction. Also $\tilde{\mu}_i$ are positive. Then $y^{(n)} \rightarrow y$, when $n \rightarrow \infty$ and we prove the statement of Lemma. \square

Lemma 10. *Let $y^{(n)}$ is defined by (21) with $y^{(0)}$ given by (20), then*

$$y_1^{(n)} \geq y_2^{(n)} \geq \dots \geq y_M^{(n)}, \quad n = 1, 2, \dots \quad (22)$$

Proof. We prove the statement (22) by induction. For $y^{(0)}$ the statement (22) is true. Let us assume that (22) is true for the $(n-1)$ step, $y_1^{(n-1)} \geq y_2^{(n-1)} \geq \dots \geq y_M^{(n-1)}$. To prove the induction statement we have to show that $y_1^{(n)} \geq y_2^{(n)} \geq \dots \geq y_M^{(n)}$, which is equal to that $y_j^{(n)} \geq y_p^{(n)}$, if $j \leq p$. As

$$y_j^{(n)} = \tilde{\mu}_j + \sum_{i=1}^M y_i^{(n-1)} \tilde{A}_{ij},$$

then

$$\begin{aligned} y_j^{(n)} - y_p^{(n)} &= \tilde{\mu}_j + \sum_{i=1}^M y_i^{(n-1)} \tilde{A}_{ij} - \left(\tilde{\mu}_p + \sum_{i=1}^M y_i^{(n-1)} \tilde{A}_{ip} \right) = \\ &= \tilde{\mu}_j - \tilde{\mu}_p + \sum_{i=1}^M y_i^{(n-1)} (\tilde{A}_{ij} - \tilde{A}_{ip}). \end{aligned}$$

To show that $y_j^{(n)} - y_p^{(n)} \geq 0$ we need to show that $\tilde{\mu}_j - \tilde{\mu}_p \geq 0$ and $\sum_{i=1}^M y_i^{(n-1)} (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, when $j \leq p$. To show that $\sum_{i=1}^M y_i^{(n-1)} (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$ it is enough to show that $\sum_{i=1}^r (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$, $r = 1, \dots, M$. Let us show this. If we regroup this sum we can get the following

$$\begin{aligned} &\sum_{i=1}^M y_i^{(n-1)} (\tilde{A}_{ij} - \tilde{A}_{ip}) = \\ &\sum_{i=1}^M (y_i^{(n-1)} - y_{i+1}^{(n-1)} + y_{i+1}^{(n-1)} - \dots - y_M^{(n-1)} + y_M^{(n-1)}) (\tilde{A}_{ij} - \tilde{A}_{ip}) = \\ &= \sum_{i=1}^{M-1} (y_i^{(n-1)} - y_{i+1}^{(n-1)}) \left[(\tilde{A}_{1j} - \tilde{A}_{1p}) + (\tilde{A}_{2j} - \tilde{A}_{2p}) + \dots + (\tilde{A}_{ij} - \tilde{A}_{ip}) \right] + \\ &\quad + y_M^{(n-1)} ((\tilde{A}_{1j} - \tilde{A}_{1p}) + \dots + (\tilde{A}_{(M-1)j} - \tilde{A}_{(M-1)p}) + (\tilde{A}_{Mj} - \tilde{A}_{Mp})) = \\ &= \sum_{i=1}^{M-1} (y_i^{(n-1)} - y_{i+1}^{(n-1)}) \sum_{k=1}^r (\tilde{A}_{kj} - \tilde{A}_{kp}) + y_M^{(n-1)} \sum_{k=1}^M (\tilde{A}_{kj} - \tilde{A}_{kp}). \end{aligned}$$

As $y_i^{(n-1)} \geq y_{i+1}^{(n-1)}$, $i = 1, \dots, M$, according to the induction step, then to show that $\sum_{i=1}^M y_i^{(n-1)} (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$ it is enough to show that $\sum_{i=1}^r (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$, $r = 1, \dots, M$. We show this in Lemma 12. In Lemma 11 we show that $\tilde{\mu}_j \geq \tilde{\mu}_p$, $j \leq p$ when condition (...) is true. Then we prove the induction statement and so prove the statement of Lemma. \square

Lemma 11. *If for every j*

$$\begin{aligned} \mu_{j+1} &= \mu_j, \\ \frac{\mu_{j+1}}{\mu_j} &< 1 - \rho, \quad \mu_{j+1} \neq \mu_j. \end{aligned}$$

then

$$\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \dots \geq \tilde{\mu}_M.$$

Proof. Let us notice that $\tilde{\mu}_j = \tilde{\mu}_p$ if $\mu_j = \mu_p$ and $g_j = g_p$. Let us denote

$$f_2(x) = \sum_{i=k}^M \frac{g_k}{x + \mu_k g_k},$$

which has the following properties

$$\begin{aligned} 0 < f_2(x) < \rho, \\ 0 < \frac{\rho}{1 + \frac{x}{\mu_N g_N}} < f_2(x) < \frac{\rho}{1 + \frac{x}{\mu_1 g_1}} < \rho. \end{aligned} \quad (23)$$

From (...)

$$\tilde{\mu}_i = \frac{\mu_i}{1 - \sum_j \frac{g_j}{\mu_i g_i + \mu_j g_j}} = \frac{\mu_i}{1 - f_2(\mu_i g_i)}.$$

We need to prove that $\tilde{\mu}_j \geq \tilde{\mu}_p$, $j \leq p$. Let us find

$$\tilde{\mu}_j - \tilde{\mu}_p = \frac{\mu_j}{1 - f_2(\mu_j g_j)} - \frac{\mu_p}{1 - f_2(\mu_p g_p)} = \frac{\mu_j - \mu_p - (\mu_j f_2(\mu_p g_p) - \mu_p f_2(\mu_j g_j))}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_p))}.$$

As (23) then

$$\mu_j f_2(\mu_p g_p) - \mu_p f_2(\mu_j g_j) < \mu_j \rho.$$

then

$$\tilde{\mu}_j - \tilde{\mu}_p > \frac{(\mu_j - \mu_p)}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_p))} \left(1 - \rho \left(\frac{\mu_j}{\mu_j - \mu_p}\right)\right) = \frac{(\mu_j - \mu_p)}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_p))} \left(1 - \rho \left(\frac{1}{1 - \frac{\mu_p}{\mu_j}}\right)\right)$$

Then

$$\tilde{\mu}_j - \tilde{\mu}_{j+1} > \frac{(\mu_j - \mu_{j+1})}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_{j+1} g_{j+1}))} \left(1 - \rho \left(\frac{1}{1 - \frac{\mu_{j+1}}{\mu_j}}\right)\right) \geq 0,$$

when

$$\begin{aligned} \mu_{j+1} &= \mu_j, \\ \frac{\mu_{j+1}}{\mu_j} &< 1 - \rho, \quad \mu_{j+1} \neq \mu_j, \end{aligned}$$

which proves Lemma. □

Lemma 12.

$$\sum_{i=1}^r \tilde{A}_{i1} \geq \sum_{i=1}^r \tilde{A}_{i2} \geq \dots \geq \sum_{i=1}^r \tilde{A}_{iM}, \quad r = 1, \dots, M.$$

Proof. Let us remember $\tilde{A} = M^{-1} A M (E - D)^{-1}$. Then as $\rho = \sum_{k=1}^M \frac{1}{\mu_k}$, then

$$\sum_{i=1}^r \tilde{A}_{ij} = \frac{\sum_{i=1}^r \frac{\mu_j g_j}{\mu_i (\mu_j g_j + \mu_i g_i)}}{1 - \sum_{k=1}^M \frac{g_k}{\mu_j g_j + \mu_k g_k}} = \frac{\sum_{i=1}^r \frac{\mu_j g_j}{\mu_i (\mu_j g_j + \mu_i g_i)}}{1 - \rho + \sum_{k=1}^M \frac{\mu_j g_j}{\mu_k (\mu_j g_j + \mu_k g_k)}}$$

Let us define

$$f_3(x) = \frac{\sum_{i=1}^r \frac{x}{\mu_i (x + \mu_i g_i)}}{1 - \rho + \sum_{k=1}^M \frac{x}{\mu_k (x + \mu_k g_k)}} = \frac{h_1(x)}{1 - \rho + h_1(x) + h_2(x)},$$

where

$$\begin{aligned} h_1(x) &= \sum_{i=1}^r \frac{x}{\mu_i (x + \mu_i g_i)} > 0, \\ h_2(x) &= \sum_{j=r+1}^M \frac{x}{\mu_j (x + \mu_j g_j)} > 0. \end{aligned}$$

Let us show that $f_3(x)$ is increasing on x . For that it enough to show that $\frac{df_3(x)}{dx} \geq 0$. Let us consider

$$\frac{df_3(x)}{dx} = \frac{h'_1(x)(1-\rho) + h'_1(x)h_2(x) - h_1(x)h'_2(x)}{(1-\rho + h_1(x) + h_2(x))^2}$$

Since $h'_1(x) > 0$ and $1-\rho > 0$:

$$\frac{df_3(x)}{dx} \geq 0 \quad \text{if} \quad h'_1(x)h_2(x) - h_1(x)h'_2(x) \geq 0.$$

Let us consider

$$\begin{aligned} h'_1(x)h_2(x) - h_1(x)h'_2(x) &= \sum_{i=1}^r \frac{g_i}{(x + \mu_i g_i)^2} \sum_{k=r+1}^M \frac{x}{\mu_k(x + \mu_k g_k)} - \sum_{i=1}^r \frac{x}{\mu_i(x + \mu_i g_i)} \sum_{k=r+1}^M \frac{g_k}{(x + \mu_k g_k)^2} = \\ &= \sum_{i=1}^r \sum_{k=r+1}^M \left(\frac{g_i x}{(x + \mu_i g_i)^2 (x + \mu_k g_k) \mu_k} - \frac{g_k x}{\mu_i (x + \mu_i g_i) (x + \mu_k g_k)^2} \right) = \\ &= \sum_{i=1}^r \sum_{k=r+1}^M \frac{x}{(x + \mu_i g_i)(x + \mu_k g_k)} \left(\frac{g_i}{\mu_k (x + \mu_i g_i)} - \frac{g_k}{\mu_i (x + \mu_k g_k)} \right) = \\ &= \sum_{i=1}^r \sum_{k=r+1}^M \frac{x}{(x + \mu_i g_i)(x + \mu_k g_k)} \left(\frac{\mu_i g_i (x + g_k \mu_k) - \mu_k g_k (x + \mu_i g_i)}{\mu_i \mu_k (x + \mu_k g_k)(x + \mu_i g_i)} \right) = \\ &= \sum_{i=1}^r \sum_{k=r+1}^M \frac{x^2 (\mu_i g_i - \mu_k g_k)}{(x + \mu_i g_i)^2 (x + \mu_k g_k)^2 \mu_k \mu_i} \geq 0, \end{aligned}$$

Then $\frac{df_3(x)}{dx} \geq 0$ and $f_3(x)$ is an increasing function of x . As $\mu_j g_j \geq \mu_p g_p$, $j < p$, then we prove the statement of Lemma. \square

References

- [1] G. Fayolle, I. Mitrani, R. Iasnogorodski, Sharing a Processor Among Many Job Classes, Journal of the ACM (JACM), Volume 27, Issue 3 (July 1980), pp. 519 - 532, 1980, ISSN:0004-5411
- [2] B. Kim and J. Kim, Comparison of DPS and PS Systems According to DPS Weights
- [3] E. Altman, K. Avrachenkov, U. Ayesta, A survey on discriminatory processor sharing, Queueing Systems, Volume 53, Numbers 1-2, June 2006, pp. 53-63(11).
- [4] K. Avrachenkov, U. Ayesta, P. Brown, R. Nunez-Queija, Discriminatory Processor Sharing Revisited, url = "citeseer.ist.psu.edu/720573.html"
- [5] Kiran M. Rege, Bhaskar Sengupta, Queue-Length Distribution for the Discriminatory Processor-Sharing Queue, Operations Research, Vol. 44, No. 4 (Jul. - Aug., 1996), pp. 653-657.
- [6] Kiran M. Rege, Bhaskar Sengupta, A decomposition theorem and related results for the discriminatory processor sharing queue, Queueing Systems, Springer Netherlands ISSN 0257-0130, 1572-9443, Issue Volume 18, Numbers 3-4 / September, 1994, pp. 333-351, 2005.
- [7] Liang Guo and Ibrahim Matta, Scheduling flows with unknown sizes: approximate analysis, in Proceedings of the 2002 ACM SIGMETRICS international conference on Measurement and modeling of computer systems, 2002, pp. 276-277.
- [8] Eitan Altman, Tanja Jimenez, and Daniel Kofman, "DPS queues with stationary ergodic service times and the performance of TCP in overload," in Proceedings of IEEE Infocom, Hong-Kong, March 2004.

- [9] G van Kessel, R Nunez-Queija, S Borst, Differentiated bandwidth sharing with disparate flow sizes, INFOCOM 2005. 24th Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings IEEE, Vol. 4 (2005), pp. 2425-2435 vol. 4.
- [10] G van Kessel, R Nunez-Queija, S Borst, Asymptotic regimes and approximations for discriminatory processor sharing, SIGMETRICS Perform. Eval. Rev., vol. 32, pp. 44-46, 2004.
- [11] L. Kleinrock, Time-shared systems: A theoretical treatment, J.ACM, vol.14, no. 2, pp. 242-261, 1967.

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