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About a Brooks-type theorem for improper colouring

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About a Brooks-type theorem for improper colouring

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Abstract: A graph is k -improperly ℓ -colourable if its vertices can be partitioned into ℓ parts such that each part induces a subgraph of maximum degree at most k . A result of Lovász states that for any graph G , such a partition exists if $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. When $k = 0$, this bound can be reduced by Brooks' Theorem, unless G is complete or an odd cycle. We study the following question, which can be seen as a generalisation of the celebrated Brooks' Theorem to improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k -improper chromatic number at most $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$? We show that the answer is no, unless $\mathcal{P} = \mathcal{NP}$, when $\Delta = \ell(k+1)$, $k \geq 1$ and $\ell + \sqrt{\ell} \leq 2k + 3$. We also show that, if G is planar, $k = 1$ or $k = 2$, $\Delta = 2k + 2$, and $\ell = 2$, then the answer is still no, unless $\mathcal{P} = \mathcal{NP}$. These results answer some questions of Cowen et al. [Journal of Graph Theory 24(3):205-219, 1997].

Key-words: Brooks, improper colouring, defective colouring, complexity

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Sur un analogue au théorème de Brooks pour la coloration impropre

Résumé : Un graphe est k -improprement ℓ -colorable si ses sommets peuvent être partitionnés en ℓ parts de sorte que chaque part induise un sous-graphe de degré maximum au plus k . Un résultat de Lovász stipule que pour tout graphe G , une telle partition existe si $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. Lorsque k est nul, cette borne peut-être réduite en vertu du théorème de Brooks, sauf si G est un graphe complet ou un cycle impair. Nous étudions le problème suivant, qui peut être vu comme une généralisation du théorème de Brooks à la coloration impropre : existe-t-il un algorithme polynomial pour décider si le nombre chromatique k -impropre d'un graphe G de degré maximum Δ est au plus $\lceil \frac{\Delta+1}{k+1} \rceil - 1$? Nous apportons une réponse négative, à moins que $\mathcal{P} = \mathcal{NP}$, lorsque $\Delta = \ell(k+1)$, $k \geq 1$ et $\ell + \sqrt{\ell} \leq 2k + 3$. Nous prouvons également que dans le cas où G est planaire, et $k = 1$ or $k = 2$, $\Delta = 2k + 2$ et $\ell = 2$, la réponse est également négative (sauf si $\mathcal{P} = \mathcal{NP}$). Ces résultats apportent des réponses à des questions de Cowen et al. [Journal of Graph Theory 24(3):205-219, 1997].

Mots-clés : Brooks, coloration impropre, complexité

Introduction

An ℓ -colouring of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, 2, \dots, \ell\}$. For any vertex $v \in V$, the *impropriety of v under c* is

$$\text{im}_c(v) := |\{u \in V : uv \in E \text{ and } c(u) = c(v)\}|.$$

A colouring is *k -improper* provided that the impropriety of every vertex is at most k . A 0-improper colouring is *proper*. A graph is *k -improperly ℓ -colourable* if it admits a k -improper ℓ -colouring. The *k -improper chromatic number* is

$$c_k(G) := \min\{\ell : G \text{ is } k\text{-improperly } \ell\text{-colourable}\}.$$

In particular, $c_0(G)$ is the chromatic number $\chi(G)$ of the graph G . Since the early nineties, a lot of work has been devoted to various aspects of improper colourings, both from a purely theoretical point of view [5, 6, 12, 18, 19] and in relation with frequency assignment issues [10, 11, 14].

For all integers k and ℓ , let k -IMP ℓ -COL be the following problem:

INSTANCE: a graph G .

QUESTION: is G k -improperly ℓ -colourable?

Cowen et al. [6] showed that the problem k -IMP ℓ -COL is \mathcal{NP} -complete for all pairs (k, ℓ) of integers with $k \geq 1$ and $\ell \geq 2$. When $\ell \geq 3$, this is not very surprising since it is already hard to determine whether a given graph is properly 3-colourable. On the contrary, determining if a graph is 2-colourable, i.e. bipartite, can be done in polynomial-time, whereas it is \mathcal{NP} -complete to know if it is k -improper 2-colourable as soon as $k > 0$.

Of even more interest is the question of complexity of k -IMP ℓ -COL when restricted to graphs with maximum degree $(k+1)\ell$. Indeed, Lovász [15] proved that, for any graph G , it holds that $c_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$, where $\Delta(G)$ is the maximum degree of G . When $k = 0$, this is the usual bound $\chi(G) \leq \Delta(G) + 1$. Brooks' Theorem [4] states that this upper bound can be decreased by one, provided that G is neither complete nor an odd cycle, which can be checked in polynomial-time.

Extensions of Brooks' Theorem have also been considered. A well-known conjecture of Borodin and Kostochka [3] states that every graph of maximum degree $\Delta \geq 9$ and chromatic number at least Δ has a Δ -clique. Reed [17] proved that this is true when Δ is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [2]. Further information about this problem can be found in the monograph of Jensen and Toft [13, Problem 4.8]. Generalisation of this problem has also been studied by Farzad, Molloy and Reed [9] and Molloy and Reed [16]. In particular, it is proved [16] that determining whether a graph with large constant maximum degree Δ is $(\Delta - k)$ -colourable can be done in linear time if $(k+1)(k+2) \leq \Delta$. This threshold is optimal by a result of Emden-Weinert, Hougardy and Kreuter [8], since they proved that for any two constants Δ and $k \leq \Delta - 3$ such that $(k+1)(k+2) > \Delta$, determining whether a graph of maximum degree Δ is $(\Delta - k)$ -colourable is \mathcal{NP} -complete.

It is natural to ask whether analogous results can be found for improper colouring. This first problem to grapple with is the existence, or not, of a Brooks-like theorem for improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k -improper chromatic number at most $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$? Proving that k -IMP ℓ -COL is \mathcal{NP} -complete when restricted to graphs with maximum degree $(k+1)\ell$ would provide a negative answer to this question unless $\mathcal{P} = \mathcal{NP}$. Cowen et al. [6] proved that k -IMP 2-COL is \mathcal{NP} -complete for the class of graphs with maximum degree $2(k+1)$, and asked what happens when $\ell \geq 3$. In this paper, we prove that k -IMP ℓ -COL restricted to graphs with maximum degree $(k+1)\ell$ is \mathcal{NP} -complete for all integers $k \geq 1$ and $\ell \in \{3, \dots, s\}$, where s is the biggest integer such that $s + \sqrt{s} \leq 2k+3$ (Theorem 2). An intriguing question that remains unanswered is the complexity of this problem for larger values of ℓ .

Problem 1. What is the complexity of k -IMP ℓ -COL restricted to graphs with maximum degree $(k+1)\ell$ when $\ell + \sqrt{\ell} > 2k+3$?

We conjecture that is always \mathcal{NP} -complete. As an evidence, we prove the \mathcal{NP} -completeness when $k = 1$ and $\ell = 4$ (Theorem 6).

In view of these negative results, one may ask what happens for planar graphs. It is known that every planar graph is 4-colourable [1], 2-improperly 3-colourable [7, 19], and Cowen et al. [6] proved that is is \mathcal{NP} -complete to know whether a planar graph is 1-improperly 3-colourable, but without any restriction on the maximum degree.

Cowen et al. [6] also proved that k -IMP 2-COL is \mathcal{NP} -complete for planar graphs, again without any restriction on the degree. In particular, they asked if 1-IMP 2-COL is still \mathcal{NP} -complete for planar graphs with maximum degree four — they could prove it only for maximum degree five. In more general terms, we consider the following problem.

Problem 2. What is the complexity of k -IMP 2-COL restricted to planar graphs of maximum degree $2k+2$?

We show in Section 2 that it is \mathcal{NP} -complete when $k \in \{1, 2\}$. Note that for $k = 1$, it settles Cowen et al. [6] question. However, we conjecture that if k is sufficiently large then k -IMP 2-COL can be polynomially decided, as the answer is always affirmative.

Conjecture 1. *There exists an integer $k_0 \geq 3$ such that for any $k \geq k_0$, any planar graph with maximum degree at most $2k+2$ is k -improperly 2-colourable.*

We end the introduction with two definitions. Given an undirected graph G , an *orientation* of G is any directed graph obtained from G by assigning a unique direction to each edge. Then, a vertex u is *dominated* by a vertex v if there is an edge between v and u directed from v to u .

1 Complexity of k -IMP ℓ -COL for graphs with maximum degree $(k + 1)\ell$

In this section, we study the complexity of k -IMP ℓ -COL restricted to graphs with maximum degree $(k + 1)\ell$. The main result is the following theorem establishing the NP-completeness of the problem when $k \geq 1$ and $\ell + \sqrt{\ell} \leq 2k + 3$.

Theorem 2. *Fix a positive integer k , and an integer $\ell \geq 3$ such that $\ell + \sqrt{\ell} \leq 2k + 3$. The following problem is NP-complete:*

INSTANCE: *a graph G with maximum degree at most $(k + 1)\ell$.*

QUESTION: *is G k -improperly ℓ -colourable?*

Remark 3. The maximum value of ℓ is approximately $2k + 4 - \sqrt{2k + 2}$.

To prove Theorem 2, we need some preliminaries. Let k and ℓ be two positive integers, and let $H(k, \ell)$ be the graph with vertex set $X \cup Y \cup \{z\}$ where $|X| = (k + 1)(\ell - 1)$ and $|Y| = (k + 1)$ such that xy is an edge unless $x = z$ and $y \in Y$ (see Figure 1).

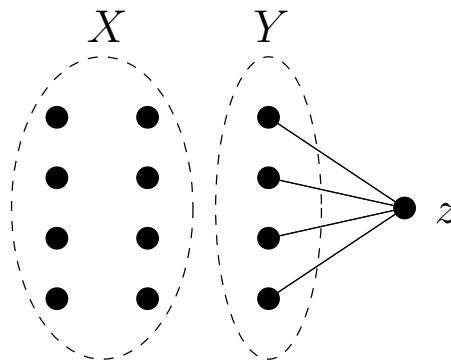


Figure 1: The complement of the graph $H(3, 3)$.

Proposition 4. *The graph $H(k, \ell)$ is k -improperly ℓ -colourable, and in any k -improper ℓ -colouring of $H(k, \ell)$ the vertices of $Y \cup \{z\}$ are coloured the same.*

Proof. Since $H(k, \ell)$ has $(k + 1)\ell + 1$ vertices, at least one colour class must contain at least $k + 2$ vertices. Observe that a vertex of X must be in a colour class containing at most $k + 1$ vertices since it is connected to every other vertex. Hence, the colour class with $k + 2$ vertices is $Y \cup \{z\}$. □

Lemma 5. *Let G be a graph with maximum degree at most $2k + 2$. Then G has an orientation D such that every vertex has indegree and outdegree at most $k + 1$.*

Proof. Since every graph with maximum degree at most $2k + 2$ is a subgraph of a $(2k + 2)$ -regular graph, it suffices to prove the assertion for $(2k + 2)$ -regular graphs. Let G' be such a graph, then it admits an Eulerian cycle C . Let D be the orientation of G' such that (u, v) is an arc if and only if u precedes v in C . Then D has indegree and outdegree at most $k + 1$. \square

Proof of Theorem 2. Reduction to the following problem:

INSTANCE: a graph G with maximum degree at most $2k + 2 \geq \ell + 1 \geq 4$.

QUESTION: is G ℓ -colourable?

Thanks to the choice of ℓ , this problem is \mathcal{NP} -complete by the result of Emden-Weinert et al. [8] cited in the introduction.

Let $G = (V, E)$ be a graph of maximum degree at most $2k + 2$, and let D be an orientation of G with in- and outdegree at most $k + 1$. Such an orientation exists by Lemma 5.

Let G' be the graph constructed as follows: replace each vertex v of G by a copy $H(v)$ of $H(k, \ell)$; if v dominates u in D then connect $z(v)$ to an element of $Y(u)$, in such a way that every vertex of $Y(u)$ is connected to a single vertex not in $H(u)$. The maximum degree of G' is $(k + 1)\ell$.

Let us now prove that G is ℓ -colourable if and only if G' is k -improperly ℓ -colourable. If G admits an ℓ -colouring c , then for any vertex v we assign the colour $c(v)$ to the vertices of $Y(v) \cup \{z(v)\}$ and the $\ell - 1$ other colours to $\ell - 1$ disjoint sets of $k + 1$ vertices of $X(v)$. This yields a k -improper ℓ -colouring of G' .

Conversely, suppose that G' admits a k -improper ℓ -colouring c' . Let c be defined by $c(v) := c'(z(v))$. We prove now that c is a proper ℓ -colouring of G : let u and v be two neighbours. Without loss of generality, v is the predecessor of u in D . Thus, the vertex $z(v)$ is connected to an element $y(u)$ of $Y(u)$. Note that $c'(z(v)) \neq c'(z(u))$, otherwise by Proposition 4, all the vertices of $Y(u) \cup Y(v) \cup \{z(u), z(v)\}$ are coloured the same. Then $y(u)$ would have degree $k + 1$ in this set which is impossible. This concludes the proof. \square

We now extend this result to the case when $k = 1$ and $\ell = 4$.

Theorem 6. *The following problem is \mathcal{NP} -complete:*

INSTANCE: a graph G with maximum degree at most 8.

QUESTION: is G 1-improperly 4-colourable?

Let B be the graph with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and xy is an edge except if there exists $i \in \{1, 2, 3\}$ such that $\{x, y\} = \{a_i, b_i\}$.

Let A be the graph with vertex set $\{x_1, x_2, y_1, y_2\}$ and with the single edge x_1x_2 . Let A' the graph obtained from A by adding the edge y_1y_2 . For all $i \in \{1, 2, 3, 4\}$, let J_i be the union of a copy A_i of A and a copy B_i of B , to which we add every edge xy such that $(x, y) \in A_i \times B_i$. Let J'_1 be a copy of J_1 , with A'_1 and B'_1 defined analogously.

Let $H := J'_1 \cup \bigcup_{i=1}^4 J_i$, to which we add the following edges (see Figure 2):

$$\begin{array}{ccc} y_1^{J_1} y_1^{J_2}, & y_1^{J_1} y_1^{J_3}, & y_2^{J_1} y_1^{J_4} \\ y_1^{J'_1} y_2^{J_2}, & y_1^{J'_1} y_2^{J_3}, & y_2^{J'_1} y_2^{J_4} \\ x_1^{J_2} x_2^{J_4}, & x_2^{J_2} x_1^{J_3}, & x_2^{J_3} x_1^{J_4} \end{array}$$

Proposition 7. *The graph H is 1-improperly 4-colourable, and for any 1-improper 4-colouring of H , the sets A_i , for $i \in \{1, 2, 3, 4\}$, and $A_1 \cup A'_1$ are monochromatic.*

Proof. Consider a 1-improper 4-colouring of H . For every $i \in \{1, 2, 3, 4\}$, the colour of each vertex not belonging to A_i is assigned at most twice. Therefore, all the vertices of A_i must be coloured the same. The same holds also for A'_i . Moreover, for every $j \in \{2, 3, 4\}$, the colour of the vertices of A_j must be different from the colour of the vertices of A_i for every $i \in \{1, 2, 3, 4\} \setminus \{j\}$. Hence, A_1 and A'_1 are coloured the same. \square

Proof of Theorem 6. Reduction to the following problem, which is \mathcal{NP} -complete [8]:

INSTANCE: a graph G with degree at most 6.

QUESTION: is G 4-colourable?

Let $G = (V, E)$ be a graph of maximum degree 6. By Lemma 5, let D be an orientation of G with in- and outdegree at most $k + 1$.

Let G' be the graph obtained by replacing each vertex v of G by a copy $H(v)$ of H ; we set $X(v) := \{x_1^{J_1}(v), x_2^{J_2}(v), x_1^{J'_1}(v)\}$ and $Y(v) := \{y_2^{J_1}(v), y_2^{J'_1}(v), x_2^{J'_1}(v)\}$; if v dominates u in D , then we connect an element of $Y(v)$ to an element of $X(u)$ in such a way that every vertex of $X(v) \cup Y(v)$ is connected to a single vertex not in $H(v)$, see Figure 2.

The maximum degree of G' is 8. Moreover G' is 1-improper 4-colourable if and only if G is 4-colourable. Indeed, consider any proper 4-colouring of G . For every vertex $v \in V$, assign to every vertex of $A_1(v) \cup A'_1(v)$ the colour of v . This partial colouring of G' can be extended to a 1-improper 4-colouring of G' . Conversely, if C' is a 1-improper 4-colouring of G' , then by Proposition 7, for each $v \in V$ the vertices of $A_1(v)$ are monochromatic. For all $v \in V$, we define $C(v)$ to be the colour assigned to every vertex of $A_1(v)$. Then C is a proper 4-colouring of G : consider an edge uv of G , and say it is oriented from u to v in D . By Proposition 7, the vertex of $X(u)$ to which the corresponding edge of G' is linked has impropriety one in $H(u)$. Thus, the vertex of $Y(v)$ to which it is linked is coloured differently. Consequently, by the definition of C , we deduce that $C(u) \neq C(v)$, as desired. \square

2 Planar graphs

In this section we show that 1-IMP 2-COL and 2-IMP 2-COL are \mathcal{NP} -complete when restricted to planar graphs of bounded maximum degree.

Theorem 8. *The following problem is \mathcal{NP} -complete:*

INSTANCE: a planar graph G with maximum degree 4.

QUESTION: is there a 1-improper 2-colouring of G ?

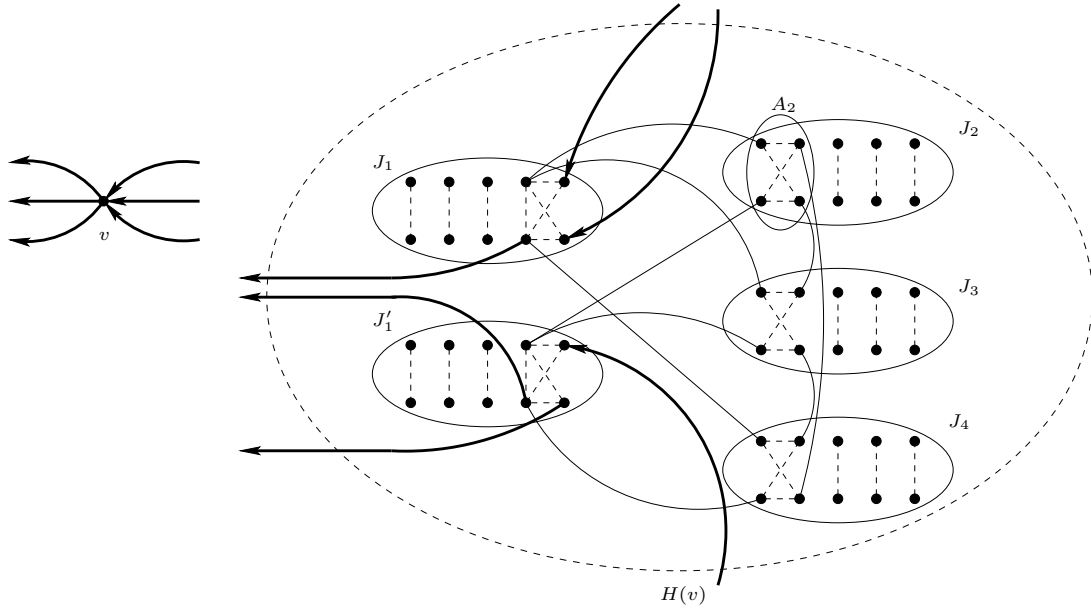


Figure 2: Replacing a vertex v of G by a copy $H(v)$ of H . For each subgraph J_i or J'_1 , dotted lines indicate missing edges.

Proof. This result is proved by slightly modifying the proof of Cowen *et al.* [6]. The only change is in the crossing gadget. The crossing gadget of Cowen *et al.* [6] has maximum degree five, and they asked whether one with maximum degree four exists. We shall exhibit such a crossing gadget. We make the whole proof here for completeness.

The reduction is from 3-SAT. Let Φ be a 3-CNF. We shall construct, in polynomial time, a planar graph G_Φ of maximum degree 4 such that Φ is satisfiable if and only if G_Φ is 1-improperly 2-colourable. We use several gadgets with useful properties.

An *xy-regulator* is depicted in Figure 3. There is a unique 1-improper 2-colouring of this graph, in which x , u_1 , u_2 and y form a colour-class and all have impropriety zero. Note also that both x and y have degree one within an *xy-regulator*. An *xy-inversor*, depicted in Figure 4, is a $K_{2,3}$, with x and y being any two vertices not in the same part of the bipartition. There is a unique 1-improper 2-colouring of this graph, in which x and y receive different colours, and both have impropriety zero. Note that one out of x and y has degree two in the inversor while the other has degree three. The variable-gadget that represents the literals of a particular variable is constructed from the two aforementioned graphs as shown in Figure 5. We put one variable-gadget for each variable, with as many literals as needed. Each of the literals will be linked to exactly one clause.

Let C_1, C_2, \dots, C_m be the clauses of Φ . For each clause C_i , we put a copy G'_i of the graph G' . It is left to the reader to check out that the graph G' , depicted in Figure 6, has

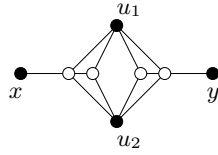


Figure 3: A regulator and its unique 1-improper 2-colouring.

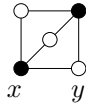


Figure 4: An inverter and its unique 1-improper 2-colouring.

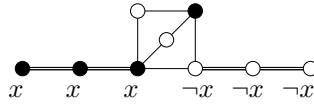


Figure 5: The vertex gadget. A double edge stands for a regulator.

the following property: in any 1-improper 2-colouring, z is coloured 1 if and only if at least one of p, q, r is. Then we add the edges $z_i z_{i+1}$, for $i \in \{1, 2, \dots, m - 1\}$.

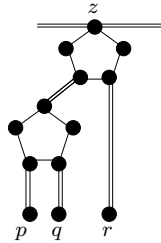


Figure 6: The clause gadget. A double edge stands for a regulator.

The obtained graph has maximum degree four, and the vertex-gadgets and the clauses can be arranged so that the only edges that can cross are the ones joining vertex-gadgets to clauses. We uncross two crossing edges by using the crossing gadget CG depicted in Figure 7. The graph CG has maximum degree four, and fulfils the following properties:

- (i) the graph CG is 1-improperly 2-colourable but not 1-improperly 1-colourable;

- (ii) in any 1-improper 2-colouring of CG , for $i \in \{1, 2\}$, the vertices a_i and b_i are coloured the same; and
- (iii) there exist two 1-improper 2-colourings c_1 and c_2 of CG such that $c_1(a_1) = c_1(b_1) = c_1(a_2) = c_1(b_2)$ and $c_2(a_1) = c_2(b_1) \neq c_2(a_2) = c_2(b_2)$.

Properties (i) and (iii) can be directly checked. For property (ii), suppose that c is a 1-improper 2-colouring of CG . First, observe that, necessarily, three vertices among o_1, o_2, o_3 , and o_4 are coloured the same, because the vertex o_0 is linked to all of them. Thus, the vertices o_2 and o_3 must be coloured differently. If $c(a_1) \neq c(b_1)$, then $c(o_4) \neq c(o_1)$ and by the preceding observation, the vertex o_0 has two neighbours of each colour, a contradiction. Now, suppose that $c(a_2) \neq c(b_2)$ and so $c(o_3) \neq c(o_7)$. By the construction, $c(o_7) = c(o_6) \neq c(o_5)$, because o_6 and o_7 are joined by a regulator and o_5 and o_6 are joined by an inverter. Hence $c(o_3) = c(o_5)$, and so $c(o_4) \neq c(o_3)$. Therefore, $c(o_4) = c(o_2) = c(o_1)$. This is not possible since $c(o_4) = c(o_7)$, so the vertex o_1 would have impropriety two.

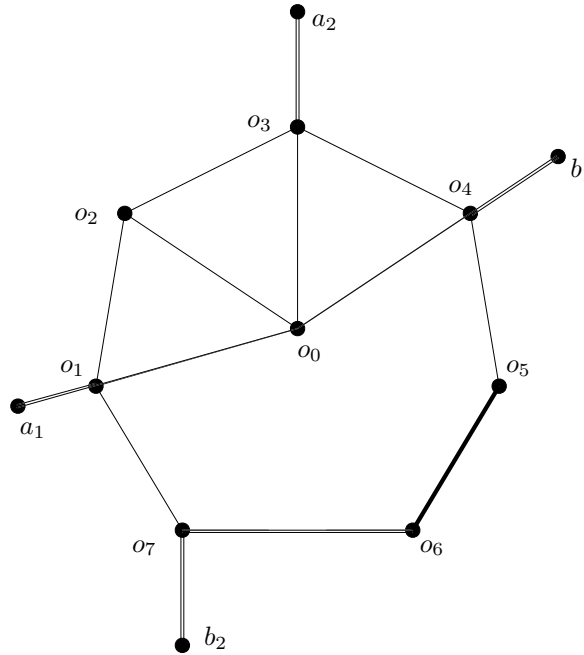


Figure 7: The crossing gadget CG . A double edge stands for a regulator, and the bold edge stands for an inverter.

It remains to show that Φ is satisfiable if and only if the obtained graph G_Φ , which is planar and of maximum degree four, is 1-improperly 2-colourable. Consider a 1-improper

2-colouring of G_Φ . Without loss of generality, say that each clause — vertices z of the clause gadgets — is coloured 1. Then, at least one of the input vertices of each clause is coloured 1. Therefore, associating 1 with TRUE and 2 with FALSE yields a truth assignment for Φ . Conversely, starting from a truth assignment of Φ , one can derive a 1-improper 2-colouring of G_Φ as follows. Vertices corresponding to literals are coloured 1 if the corresponding literal is TRUE, and 2 otherwise. Thanks to the properties of the gadgets, such a partial colouring can be extended to a 1-improper 2-colouring of G_Φ . \square

Theorem 9. *The following problem is NP-complete:*

INSTANCE: *a planar graph G of maximum degree 6.*

QUESTION: *is there a 2-improper 2-colouring of G ?*

Proof. We shall reduce the problem to 1-improper 2-colouring of planar graphs with maximum degree four. Let G be a planar graph of maximum degree four: we construct, in polynomial time, a planar graph \hat{G} of maximum degree 6 that is 2-improperly 2-colourable if and only if G is 1-improperly 2-colourable. The graph H showed in Figure 8 fulfils the following properties:

- (i) it is planar and has maximum degree 6;
- (ii) in any 2-improper 2-colouring of H , the vertex v must have impropriety at least one; and
- (iii) there exists a 2-improper 2-colouring of H in which the vertex v has impropriety exactly one.

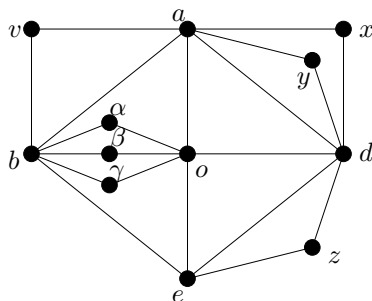


Figure 8: The graph H .

Property (i) can be directly checked. To prove (ii), it is sufficient to show that no 2-improper 2-colouring of H such that a and b are coloured the same exists. So, suppose that $c(a) = c(b) = 1$.

If $c(o) = 1$, then d, y and x must be coloured 2, so e and z both receive colour 1. Therefore e has impropriety three, because of b, o and z , a contradiction.

If $c(o) = 2$, then the three vertices α, β, γ cannot be coloured the same, otherwise b or o would have impropriety at least three. Moreover, since $c(b) = c(a) = 1$, exactly two vertices among α, β, γ are coloured with colour 2. Hence, the vertices b and o both have impropriety two in the subgraph of G induced by the vertices $a, b, o, \alpha, \beta, \gamma$. But the vertex e does not belong to this subgraph, and is linked to both b and o , a contradiction. This proves **(ii)**.

Assigning 1 to $\{v, a, d, e, \alpha, \beta\}$ and 2 to $\{b, o, x, y, z, \gamma\}$, we obtain the colouring of **(iii)**.

To construct the graph \hat{G} , put a copy $H(x)$ of H for each vertex $x \in V(G)$. Then, for each edge $xy \in E(G)$, we put an edge between the vertex v of $H(x)$ and the vertex v of $H(y)$. Note that H has maximum degree 6 and v has degree two in H , so, as G has maximum degree four, the graph \hat{G} has maximum degree 6. Furthermore, the graph \hat{G} is planar.

Now let c be a 1-improper 2-colouring of G . For any $x \in V(G)$, assign the colour $c(x)$ to the vertex v of $H(x)$, and then extend the colouring to each copy of H by property **(iii)**.

If c is a 2-improper 2-colouring of \hat{G} , then for each $x \in V(G)$, assign to x the colour of the vertex v of $H(x)$. The obtained 2-colouring of G is 1-improper because of property **(ii)** of H . \square

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