



HAL
open science

About a Brooks-type theorem for improper colouring

Ricardo Correa, Frédéric Havet, Jean-Sébastien Sereni

► **To cite this version:**

Ricardo Correa, Frédéric Havet, Jean-Sébastien Sereni. About a Brooks-type theorem for improper colouring. [Research Report] RR-6432, 2008, pp.13. inria-00223009v2

HAL Id: inria-00223009

<https://inria.hal.science/inria-00223009v2>

Submitted on 31 Jan 2008 (v2), last revised 8 Jul 2010 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

About a Brooks-type theorem for improper colouring

Ricardo Corrêa — Frédéric Havet — Jean-Sébastien Sereni

N° 1

Janvier 2008

Thème COM



*Rapport
de recherche*

About a Brooks-type theorem for improper colouring

Ricardo Corrêa* , Frédéric Havet[†] , Jean-Sébastien Sereni[‡]

Thème COM — Systèmes communicants
Projets Mascotte

Rapport de recherche n° 1 — Janvier 2008 — 13 pages

Abstract: A graph is k -improperly ℓ -colourable if its vertices can be partitioned into ℓ parts such that each part induces a subgraph of maximum degree at most k . A result of Lovász states that for any graph G , such a partition exists if $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. When $k = 0$, this bound can be reduced by Brooks' Theorem, unless G is complete or an odd cycle. We study the following question, which can be seen as a generalisation of the celebrated Brooks' Theorem to improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k -improper chromatic number at most $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$? We show that the answer is no, unless $\mathcal{P} = \mathcal{NP}$, when $\Delta = \ell(k+1)$, $k \geq 1$ and $\ell + \sqrt{\ell} \leq 2k + 3$. We also show that, if G is planar, $k = 1$ or $k = 2$, $\Delta = 2k + 2$, and $\ell = 2$, then the answer is still no, unless $\mathcal{P} = \mathcal{NP}$. These results answer some questions of Cowen et al. [Journal of Graph Theory 24(3):205-219, 1997].

Key-words: Brooks, improper colouring, defective colouring, complexity

This work was partially supported by the Égide ECO-NET project 16305SB.

* Federal University of Ceará, Campus do Pici, Bloco 910, 60455-760 Fortaleza, CE, Brazil. Partially supported by the brazilian agencies CNPq and CAPES. E-mail: correa@lia.ufc.br.

[†] MASCOTTE, I3S (CNRS-UNSA) - INRIA, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex, France. Partially supported by European Project IST FET AEOLUS. E-mail: fhavet@sophia.inria.fr.

[‡] Institute for Theoretical Computer Science (ITI) and Department of Applied Mathematics (KAM), Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic. The research in this paper was carried while this author was a doctoral research student in MASCOTTE. This author is now supported by the European project IST FET AEOLUS. E-mail: sereni@kam.mff.cuni.cz.

Sur un analogue au théorème de Brooks pour la coloration impropre

Résumé : Un graphe est k -improprement ℓ -colorable si ses sommets peuvent être partitionnés en ℓ parts de sorte que chaque part induise un sous-graphe de degré maximum au plus k . Un résultat de Lovász stipule que pour tout graphe G , une telle partition existe si $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. Lorsque k est nul, cette borne peut-être réduite en vertu du théorème de Brooks, sauf si G est un graphe complet ou un cycle impair. Nous étudions le problème suivant, qui peut être vu comme une généralisation du théorème de Brooks à la coloration impropre : existe-t-il un algorithme polynomial pour décider si le nombre chromatique k -impropre d'un graphe G de degré maximum Δ est au plus $\lceil \frac{\Delta+1}{k+1} \rceil - 1$? Nous apportons une réponse négative, à moins que $\mathcal{P} = \mathcal{NP}$, lorsque $\Delta = \ell(k+1)$, $k \geq 1$ et $\ell + \sqrt{\ell} \leq 2k + 3$. Nous prouvons également que dans le cas où G est planaire, et $k = 1$ or $k = 2$, $\Delta = 2k + 2$ et $\ell = 2$, la réponse est également négative (sauf si $\mathcal{P} = \mathcal{NP}$). Ces résultats apportent des réponses à des questions de Cowen et al. [Journal of Graph Theory 24(3):205-219, 1997].

Mots-clés : Brooks, coloration impropre, complexité

Introduction

An ℓ -colouring of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, 2, \dots, \ell\}$. For any vertex $v \in V$, the *impropriety of v under c* is

$$\text{im}_c(v) := |\{u \in V : uv \in E \text{ and } c(u) = c(v)\}|.$$

A colouring is *k -improper* provided that the impropriety of every vertex is at most k . A 0-improper colouring is *proper*. A graph is *k -improperly ℓ -colourable* if it admits a k -improper ℓ -colouring. The *k -improper chromatic number* is

$$c_k(G) := \min\{\ell : G \text{ is } k\text{-improperly } \ell\text{-colourable}\}.$$

In particular, $c_0(G)$ is the chromatic number $\chi(G)$ of the graph G . Since the early nineties, a lot of work has been devoted to various aspects of improper colourings, both from a purely theoretical point of view [5, 6, 12, 18, 19] and in relation with frequency assignment issues [10, 11, 14].

For all integers k and ℓ , let k -IMP ℓ -COL be the following problem:

INSTANCE: a graph G .

QUESTION: is G k -improperly ℓ -colourable?

Cowen et al. [6] showed that the problem k -IMP ℓ -COL is \mathcal{NP} -complete for all pairs (k, ℓ) of integers with $k \geq 1$ and $\ell \geq 2$. When $\ell \geq 3$, this is not very surprising since it is already hard to determine whether a given graph is properly 3-colourable. On the contrary, determining if a graph is 2-colourable, i.e. bipartite, can be done in polynomial-time, whereas it is \mathcal{NP} -complete to know if it is k -improper 2-colourable as soon as $k > 0$.

Of even more interest is the question of complexity of k -IMP ℓ -COL when restricted to graphs with maximum degree $(k+1)\ell$. Indeed, Lovász [15] proved that, for any graph G , it holds that $c_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$, where $\Delta(G)$ is the maximum degree of G . When $k = 0$, this is the usual bound $\chi(G) \leq \Delta(G) + 1$. Brooks' Theorem [4] states that this upper bound can be decreased by one, provided that G is neither complete nor an odd cycle, which can be checked in polynomial-time.

Extensions of Brooks' Theorem have also been considered. A well-known conjecture of Borodin and Kostochka [3] states that every graph of maximum degree $\Delta \geq 9$ and chromatic number at least Δ has a Δ -clique. Reed [17] proved that this is true when Δ is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [2]. Further information about this problem can be found in the monograph of Jensen and Toft [13, Problem 4.8]. Generalisation of this problem has also been studied by Farzad, Molloy and Reed [9] and Molloy and Reed [16]. In particular, it is proved [16] that determining whether a graph with large constant maximum degree Δ is $(\Delta - k)$ -colourable can be done in linear time if $(k+1)(k+2) \leq \Delta$. This threshold is optimal by a result of Emden-Weinert, Hougardy and Kreuter [8], since they proved that for any two constants Δ and $k \leq \Delta - 3$ such that $(k+1)(k+2) > \Delta$, determining whether a graph of maximum degree Δ is $(\Delta - k)$ -colourable is \mathcal{NP} -complete.

It is natural to ask whether analogous results can be found for improper colouring. This first problem to grapple with is the existence, or not, of a Brooks-like theorem for improper colouring: does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k -improper chromatic number at most $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$? Proving that k -IMP ℓ -COL is \mathcal{NP} -complete when restricted to graphs with maximum degree $(k+1)\ell$ would provide a negative answer to this question unless $\mathcal{P} = \mathcal{NP}$. Cowen et al. [6] proved that k -IMP 2-COL is \mathcal{NP} -complete for the class of graphs with maximum degree $2(k+1)$, and asked what happens when $\ell \geq 3$. In this paper, we prove that k -IMP ℓ -COL restricted to graphs with maximum degree $(k+1)\ell$ is \mathcal{NP} -complete for all integers $k \geq 1$ and $\ell \in \{3, \dots, s\}$, where s is the biggest integer such that $s + \sqrt{s} \leq 2k+3$ (Theorem 2). An intriguing question that remains unanswered is the complexity of this problem for larger values of ℓ .

Problem 1. What is the complexity of k -IMP ℓ -COL restricted to graphs with maximum degree $(k+1)\ell$ when $\ell + \sqrt{\ell} > 2k+3$?

We conjecture that is always \mathcal{NP} -complete. As an evidence, we prove the \mathcal{NP} -completeness when $k = 1$ and $\ell = 4$ (Theorem 6).

In view of these negative results, one may ask what happens for planar graphs. It is known that every planar graph is 4-colourable [1], 2-improperly 3-colourable [7, 19], and Cowen et al. [6] proved that is is \mathcal{NP} -complete to know whether a planar graph is 1-improperly 3-colourable, but without any restriction on the maximum degree.

Cowen et al. [6] also proved that k -IMP 2-COL is \mathcal{NP} -complete for planar graphs, again without any restriction on the degree. In particular, they asked if 1-IMP 2-COL is still \mathcal{NP} -complete for planar graphs with maximum degree four — they could prove it only for maximum degree five. In more general terms, we consider the following problem.

Problem 2. What is the complexity of k -IMP 2-COL restricted to planar graphs of maximum degree $2k+2$?

We show in Section 2 that it is \mathcal{NP} -complete when $k \in \{1, 2\}$. Note that for $k = 1$, it settles Cowen et al. [6] question. However, we conjecture that if k is sufficiently large then k -IMP 2-COL can be polynomially decided, as the answer is always affirmative.

Conjecture 1. *There exists an integer $k_0 \geq 3$ such that for any $k \geq k_0$, any planar graph with maximum degree at most $2k+2$ is k -improperly 2-colourable.*

We end the introduction with two definitions. Given an undirected graph G , an *orientation* of G is any directed graph obtained from G by assigning a unique direction to each edge. Then, a vertex u is *dominated* by a vertex v if there is an edge between v and u directed from v to u .

1 Complexity of k -IMP ℓ -COL for graphs with maximum degree $(k + 1)\ell$

In this section, we study the complexity of k -IMP ℓ -COL restricted to graphs with maximum degree $(k + 1)\ell$. The main result is the following theorem establishing the NP-completeness of the problem when $k \geq 1$ and $\ell + \sqrt{\ell} \leq 2k + 3$.

Theorem 2. *Fix a positive integer k , and an integer $\ell \geq 3$ such that $\ell + \sqrt{\ell} \leq 2k + 3$. The following problem is NP-complete:*

INSTANCE: *a graph G with maximum degree at most $(k + 1)\ell$.*

QUESTION: *is G k -improperly ℓ -colourable?*

Remark 3. The maximum value of ℓ is approximately $2k + 4 - \sqrt{2k + 2}$.

To prove Theorem 2, we need some preliminaries. Let k and ℓ be two positive integers, and let $H(k, \ell)$ be the graph with vertex set $X \cup Y \cup \{z\}$ where $|X| = (k + 1)(\ell - 1)$ and $|Y| = (k + 1)$ such that xy is an edge unless $x = z$ and $y \in Y$ (see Figure 1).

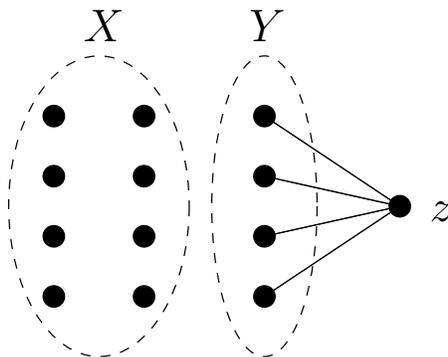


Figure 1: The complement of the graph $H(3, 3)$.

Proposition 4. *The graph $H(k, \ell)$ is k -improperly ℓ -colourable, and in any k -improper ℓ -colouring of $H(k, \ell)$ the vertices of $Y \cup \{z\}$ are coloured the same.*

Proof. Since $H(k, \ell)$ has $(k + 1)\ell + 1$ vertices, at least one colour class must contain at least $k + 2$ vertices. Observe that a vertex of X must be in a colour class containing at most $k + 1$ vertices since it is connected to every other vertex. Hence, the colour class with $k + 2$ vertices is $Y \cup \{z\}$. \square

Lemma 5. *Let G be a graph with maximum degree at most $2k + 2$. Then G has an orientation D such that every vertex has indegree and outdegree at most $k + 1$.*

Proof. Since every graph with maximum degree at most $2k + 2$ is a subgraph of a $(2k + 2)$ -regular graph, it suffices to prove the assertion for $(2k + 2)$ -regular graphs. Let G' be such a graph, then it admits an Eulerian cycle C . Let D be the orientation of G' such that (u, v) is an arc if and only if u precedes v in C . Then D has indegree and outdegree at most $k + 1$. \square

Proof of Theorem 2. Reduction to the following problem:

INSTANCE: a graph G with maximum degree at most $2k + 2 \geq \ell + 1 \geq 4$.

QUESTION: is G ℓ -colourable?

Thanks to the choice of ℓ , this problem is \mathcal{NP} -complete by the result of Emden-Weinert et al. [8] cited in the introduction.

Let $G = (V, E)$ be a graph of maximum degree at most $2k + 2$, and let D be an orientation of G with in- and outdegree at most $k + 1$. Such an orientation exists by Lemma 5.

Let G' be the graph constructed as follows: replace each vertex v of G by a copy $H(v)$ of $H(k, \ell)$; if v dominates u in D then connect $z(v)$ to an element of $Y(u)$, in such a way that every vertex of $Y(u)$ is connected to a single vertex not in $H(u)$. The maximum degree of G' is $(k + 1)\ell$.

Let us now prove that G is ℓ -colourable if and only if G' is k -improperly ℓ -colourable. If G admits an ℓ -colouring c , then for any vertex v we assign the colour $c(v)$ to the vertices of $Y(v) \cup \{z(v)\}$ and the $\ell - 1$ other colours to $\ell - 1$ disjoint sets of $k + 1$ vertices of $X(v)$. This yields a k -improper ℓ -colouring of G' .

Conversely, suppose that G' admits a k -improper ℓ -colouring c' . Let c be defined by $c(v) := c'(z(v))$. We prove now that c is a proper ℓ -colouring of G : let u and v be two neighbours. Without loss of generality, v is the predecessor of u in D . Thus, the vertex $z(v)$ is connected to an element $y(u)$ of $Y(u)$. Note that $c'(z(v)) \neq c'(z(u))$, otherwise by Proposition 4, all the vertices of $Y(u) \cup Y(v) \cup \{z(u), z(v)\}$ are coloured the same. Then $y(u)$ would have degree $k + 1$ in this set which is impossible. This concludes the proof. \square

We now extend this result to the case when $k = 1$ and $\ell = 4$.

Theorem 6. *The following problem is \mathcal{NP} -complete:*

INSTANCE: a graph G with maximum degree at most 8.

QUESTION: is G 1-improperly 4-colourable?

Let B be the graph with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and xy is an edge except if there exists $i \in \{1, 2, 3\}$ such that $\{x, y\} = \{a_i, b_i\}$.

Let A be the graph with vertex set $\{x_1, x_2, y_1, y_2\}$ and with the single edge x_1x_2 . Let A' the graph obtained from A by adding the edge y_1y_2 . For all $i \in \{1, 2, 3, 4\}$, let J_i be the union of a copy A_i of A and a copy B_i of B , to which we add every edge xy such that $(x, y) \in A_i \times B_i$. Let J'_1 be a copy of J_1 , with A'_1 and B'_1 defined analogously.

Let $H := J'_1 \cup \bigcup_{i=1}^4 J_i$, to which we add the following edges (see Figure 2):

$$\begin{array}{ccc} y_1^{J_1} y_1^{J_2}, & y_1^{J_1} y_1^{J_3}, & y_2^{J_1} y_1^{J_4} \\ y_1^{J'_1} y_2^{J_2}, & y_1^{J'_1} y_2^{J_3}, & y_2^{J'_1} y_2^{J_4} \\ x_1^{J_2} x_2^{J_4}, & x_2^{J_2} x_1^{J_3}, & x_2^{J_3} x_1^{J_4} \end{array}$$

Proposition 7. *The graph H is 1-improperly 4-colourable, and for any 1-improper 4-colouring of H , the sets A_i , for $i \in \{1, 2, 3, 4\}$, and $A_1 \cup A'_1$ are monochromatic.*

Proof. Consider a 1-improper 4-colouring of H . For every $i \in \{1, 2, 3, 4\}$, the colour of each vertex not belonging to A_i is assigned at most twice. Therefore, all the vertices of A_i must be coloured the same. The same holds also for A'_i . Moreover, for every $j \in \{2, 3, 4\}$, the colour of the vertices of A_j must be different from the colour of the vertices of A_i for every $i \in \{1, 2, 3, 4\} \setminus \{j\}$. Hence, A_1 and A'_1 are coloured the same. \square

Proof of Theorem 6. Reduction to the following problem, which is \mathcal{NP} -complete [8]:

INSTANCE: a graph G with degree at most 6.

QUESTION: is G 4-colourable?

Let $G = (V, E)$ be a graph of maximum degree 6. By Lemma 5, let D be an orientation of G with in- and outdegree at most $k + 1$.

Let G' be the graph obtained by replacing each vertex v of G by a copy $H(v)$ of H ; we set $X(v) := \{x_1^{J_1}(v), x_2^{J_2}(v), x_1^{J'_1}(v)\}$ and $Y(v) := \{y_2^{J_1}(v), y_2^{J'_1}(v), x_2^{J'_1}(v)\}$; if v dominates u in D , then we connect an element of $Y(v)$ to an element of $X(u)$ in such a way that every vertex of $X(v) \cup Y(v)$ is connected to a single vertex not in $H(v)$, see Figure 2.

The maximum degree of G' is 8. Moreover G' is 1-improper 4-colourable if and only if G is 4-colourable. Indeed, consider any proper 4-colouring of G . For every vertex $v \in V$, assign to every vertex of $A_1(v) \cup A'_1(v)$ the colour of v . This partial colouring of G' can be extended to a 1-improper 4-colouring of G' . Conversely, if C' is a 1-improper 4-colouring of G' , then by Proposition 7, for each $v \in V$ the vertices of $A_1(v)$ are monochromatic. For all $v \in V$, we define $C(v)$ to be the colour assigned to every vertex of $A_1(v)$. Then C is a proper 4-colouring of G : consider an edge uv of G , and say it is oriented from u to v in D . By Proposition 7, the vertex of $X(u)$ to which the corresponding edge of G' is linked has impropriety one in $H(u)$. Thus, the vertex of $Y(v)$ to which it is linked is coloured differently. Consequently, by the definition of C , we deduce that $C(u) \neq C(v)$, as desired. \square

2 Planar graphs

In this section we show that 1-IMP 2-COL and 2-IMP 2-COL are \mathcal{NP} -complete when restricted to planar graphs of bounded maximum degree.

Theorem 8. *The following problem is \mathcal{NP} -complete:*

INSTANCE: a planar graph G with maximum degree 4.

QUESTION: is there a 1-improper 2-colouring of G ?

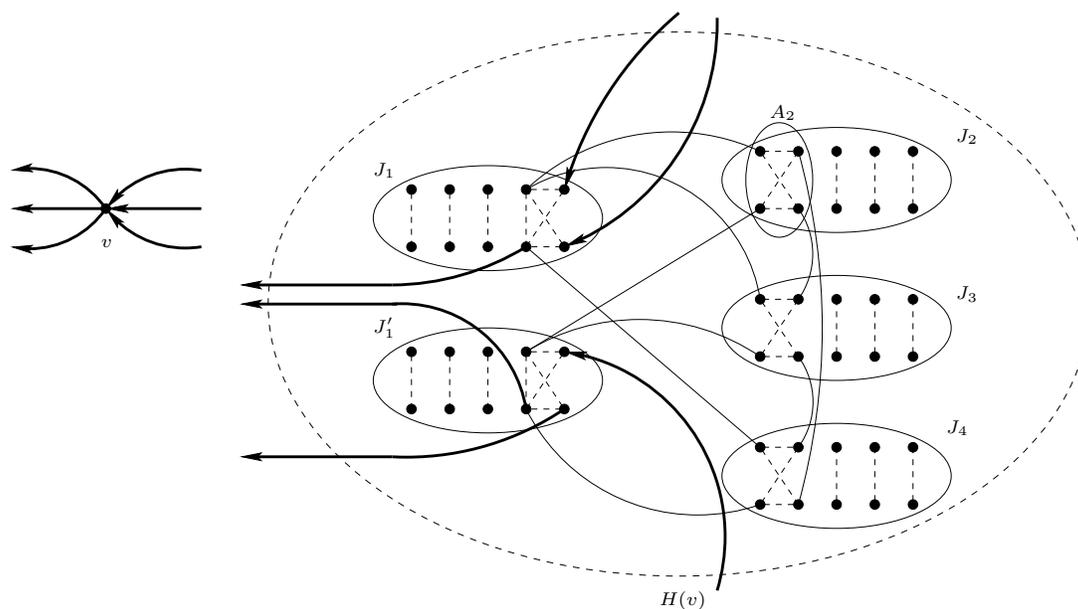


Figure 2: Replacing a vertex v of G by a copy $H(v)$ of H . For each subgraph J_i or J'_1 , dotted lines indicate missing edges.

Proof. This result is proved by slightly modifying the proof of Cowen *et al.* [6]. The only change is in the crossing gadget. The crossing gadget of Cowen *et al.* [6] has maximum degree five, and they asked whether one with maximum degree four exists. We shall exhibit such a crossing gadget. We make the whole proof here for completeness.

The reduction is from 3-SAT. Let Φ be a 3-CNF. We shall construct, in polynomial time, a planar graph G_Φ of maximum degree 4 such that Φ is satisfiable if and only if G_Φ is 1-improperly 2-colourable. We use several gadgets with useful properties.

An xy -regulator is depicted in Figure 3. There is a unique 1-improper 2-colouring of this graph, in which x , u_1 , u_2 and y form a colour-class and all have impropriety zero. Note also that both x and y have degree one within an xy -regulator. An xy -inversor, depicted in Figure 4, is a $K_{2,3}$, with x and y being any two vertices not in the same part of the bipartition. There is a unique 1-improper 2-colouring of this graph, in which x and y receive different colours, and both have impropriety zero. Note that one out of x and y has degree two in the inversor while the other has degree three. The variable-gadget that represents the literals of a particular variable is constructed from the two aforementioned graphs as shown in Figure 5. We put one variable-gadget for each variable, with as many literals as needed. Each of the literals will be linked to exactly one clause.

Let C_1, C_2, \dots, C_m be the clauses of Φ . For each clause C_i , we put a copy G'_i of the graph G' . It is left to the reader to check out that the graph G' , depicted in Figure 6, has

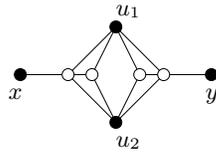


Figure 3: A regulator and its unique 1-improper 2-colouring.

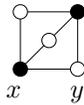


Figure 4: An inverter and its unique 1-improper 2-colouring.

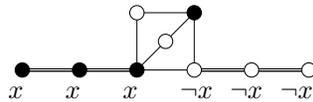


Figure 5: The vertex gadget. A double edge stands for a regulator.

the following property: in any 1-improper 2-colouring, z is coloured 1 if and only if at least one of p, q, r is. Then we add the edges $z_i z_{i+1}$, for $i \in \{1, 2, \dots, m - 1\}$.

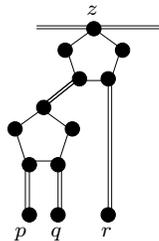


Figure 6: The clause gadget. A double edge stands for a regulator.

The obtained graph has maximum degree four, and the vertex-gadgets and the clauses can be arranged so that the only edges that can cross are the ones joining vertex-gadgets to clauses. We uncross two crossing edges by using the crossing gadget CG depicted in Figure 7. The graph CG has maximum degree four, and fulfils the following properties:

- (i) the graph CG is 1-improperly 2-colourable but not 1-improperly 1-colourable;

- (ii) in any 1-improper 2-colouring of CG , for $i \in \{1, 2\}$, the vertices a_i and b_i are coloured the same; and
- (iii) there exist two 1-improper 2-colourings c_1 and c_2 of CG such that $c_1(a_1) = c_1(b_1) = c_1(a_2) = c_1(b_2)$ and $c_2(a_1) = c_2(b_1) \neq c_2(a_2) = c_2(b_2)$.

Properties (i) and (iii) can be directly checked. For property (ii), suppose that c is a 1-improper 2-colouring of CG . First, observe that, necessarily, three vertices among o_1, o_2, o_3 , and o_4 are coloured the same, because the vertex o_0 is linked to all of them. Thus, the vertices o_2 and o_3 must be coloured differently. If $c(a_1) \neq c(b_1)$, then $c(o_4) \neq c(o_1)$ and by the preceding observation, the vertex o_0 has two neighbours of each colour, a contradiction. Now, suppose that $c(a_2) \neq c(b_2)$ and so $c(o_3) \neq c(o_7)$. By the construction, $c(o_7) = c(o_6) \neq c(o_5)$, because o_6 and o_7 are joined by a regulator and o_5 and o_6 are joined by an inverter. Hence $c(o_3) = c(o_5)$, and so $c(o_4) \neq c(o_3)$. Therefore, $c(o_4) = c(o_2) = c(o_1)$. This is not possible since $c(o_4) = c(o_7)$, so the vertex o_1 would have impropriety two.

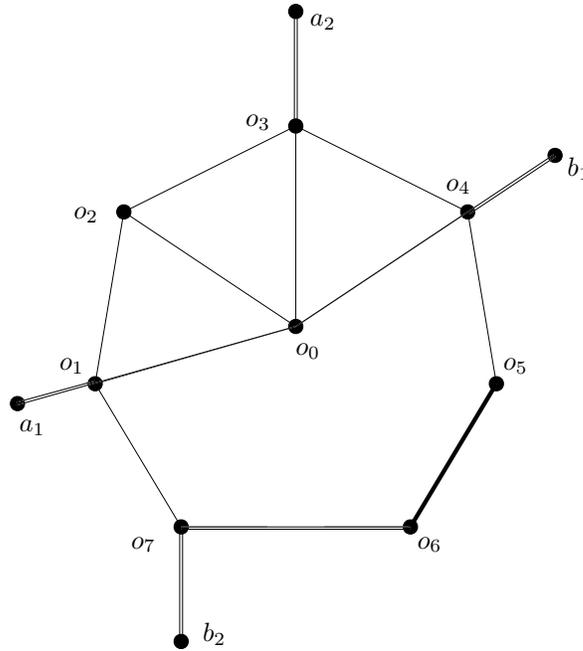


Figure 7: The crossing gadget CG . A double edge stands for a regulator, and the bold edge stands for an inverter.

It remains to show that Φ is satisfiable if and only if the obtained graph G_Φ , which is planar and of maximum degree four, is 1-improperly 2-colourable. Consider a 1-improper

2-colouring of G_Φ . Without loss of generality, say that each clause — vertices z of the clause gadgets — is coloured 1. Then, at least one of the input vertices of each clause is coloured 1. Therefore, associating 1 with TRUE and 2 with FALSE yields a truth assignment for Φ . Conversely, starting from a truth assignment of Φ , one can derive a 1-improper 2-colouring of G_Φ as follows. Vertices corresponding to literals are coloured 1 if the corresponding literal is TRUE, and 2 otherwise. Thanks to the properties of the gadgets, such a partial colouring can be extended to a 1-improper 2-colouring of G_Φ . \square

Theorem 9. *The following problem is NP-complete:*

INSTANCE: *a planar graph G of maximum degree 6.*

QUESTION: *is there a 2-improper 2-colouring of G ?*

Proof. We shall reduce the problem to 1-improper 2-colouring of planar graphs with maximum degree four. Let G be a planar graph of maximum degree four: we construct, in polynomial time, a planar graph \hat{G} of maximum degree 6 that is 2-improperly 2-colourable if and only if G is 1-improperly 2-colourable. The graph H showed in Figure 8 fulfils the following properties:

- (i) it is planar and has maximum degree 6;
- (ii) in any 2-improper 2-colouring of H , the vertex v must have impropriety at least one; and
- (iii) there exists a 2-improper 2-colouring of H in which the vertex v has impropriety exactly one.

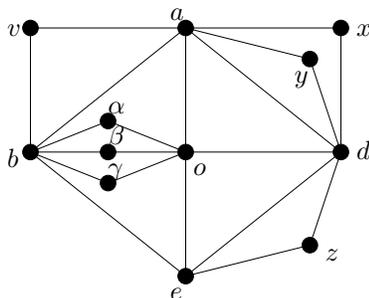


Figure 8: The graph H .

Property (i) can be directly checked. To prove (ii), it is sufficient to show that no 2-improper 2-colouring of H such that a and b are coloured the same exists. So, suppose that $c(a) = c(b) = 1$.

If $c(o) = 1$, then d, y and x must be coloured 2, so e and z both receive colour 1. Therefore e has impropriety three, because of b, o and z , a contradiction.

If $c(o) = 2$, then the three vertices α, β, γ cannot be coloured the same, otherwise b or o would have impropriety at least three. Moreover, since $c(b) = c(a) = 1$, exactly two vertices among α, β, γ are coloured with colour 2. Hence, the vertices b and o both have impropriety two in the subgraph of G induced by the vertices $a, b, o, \alpha, \beta, \gamma$. But the vertex e does not belong to this subgraph, and is linked to both b and o , a contradiction. This proves **(ii)**.

Assigning 1 to $\{v, a, d, e, \alpha, \beta\}$ and 2 to $\{b, o, x, y, z, \gamma\}$, we obtain the colouring of **(iii)**.

To construct the graph \hat{G} , put a copy $H(x)$ of H for each vertex $x \in V(G)$. Then, for each edge $xy \in E(G)$, we put an edge between the vertex v of $H(x)$ and the vertex v of $H(y)$. Note that H has maximum degree 6 and v has degree two in H , so, as G has maximum degree four, the graph \hat{G} has maximum degree 6. Furthermore, the graph \hat{G} is planar.

Now let c be a 1-improper 2-colouring of G . For any $x \in V(G)$, assign the colour $c(x)$ to the vertex v of $H(x)$, and then extend the colouring to each copy of H by property **(iii)**.

If c is a 2-improper 2-colouring of \hat{G} , then for each $x \in V(G)$, assign to x the colour of the vertex v of $H(x)$. The obtained 2-colouring of G is 1-improper because of property **(ii)** of H . \square

References

- [1] K. Appel and W. Haken. *Every planar map is four colorable*, volume 98 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989. With the collaboration of J. Koch.
- [2] A. Beutelspacher and P.-R. Hering. Minimal graphs for which the chromatic number equals the maximal degree. *Ars Combin.*, 18:201–216, 1984.
- [3] O. V. Borodin and A. V. Kostochka. On an upper bound of a graph’s chromatic number, depending on the graph’s degree and density. *J. Combinatorial Theory Ser. B*, 23(2-3):247–250, 1977.
- [4] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.
- [5] L. Cowen, R. H. Cowen, and D. R. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency. *J. Graph Theory*, 10(2):187–195, 1986.
- [6] L. Cowen, W. Goddard, and C. E. Jesurum. Defective coloring revisited. *J. Graph Theory*, 24(3):205–219, 1997.
- [7] N. Eaton and T. Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.
- [8] T. Emden-Weinert, S. Hougardy, and B. Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combin. Probab. Comput.*, 7(4):375–386, 1998.

-
- [9] B. Farzad, M. Molloy, and B. Reed. $(\Delta - k)$ -critical graphs. *J. Combin. Theory Ser. B*, 93(2):173–185, 2005.
- [10] F. Havet, R. J. Kang, and J.-S. Sereni. Improper colouring of unit disk graphs. Submitted. INRIA Research Report 6206, 2007.
- [11] F. Havet and J.-S. Sereni. Channel assignment and improper choosability of graphs. In *Proceedings of the 31st Workshop on Graph-Theoretic Concepts in Computer Science (WG'05)*, volume 3787 of *Lecture Notes in Computer Science*, pages 81–90. Springer Verlag, June 2005.
- [12] F. Havet and J.-S. Sereni. Improper choosability of graphs and maximum average degree. *Journal of Graph Theory*, 52(3):181–199, 2006.
- [13] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New-York, 1995.
- [14] R. J. Kang, T. Müller, and J.-S. Sereni. Improper colouring of (random) unit disk graphs. *Discrete Mathematics*, 2008, to appear in the Special Issue devoted to Euro-Comb 2005.
- [15] L. Lovász. On decompositions of graphs. *Studia Sci. Math. Hungar.*, 1:237–238, 1966.
- [16] M. Molloy and B. Reed. Colouring graphs when the number of colours is nearly the maximum degree. In *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing*, pages 462–470 (electronic), New York, 2001. ACM.
- [17] B. Reed. A strengthening of Brooks' theorem. *J. Combin. Theory Ser. B*, 76(2):136–149, 1999.
- [18] R. Škrekovski. A Grötzsch-type theorem for list colourings with impropriety one. *Combin. Probab. Comput.*, 8(5):493–507, 1999.
- [19] R. Škrekovski. List improper colourings of planar graphs. *Combin. Probab. Comput.*, 8(3):293–299, 1999.



Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)

Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399