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# Axiom directed Focusing

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**Abstract.** Superdeduction is a focusing method especially designed to ease the use of first-order theories in predicate logic. Superdeduction modulo, which results of the combination of deduction modulo and superdeduction, enables then the user to make a distinct use of computational and reasoning axioms. Although soundness and completeness of superdeduction modulo are squarely ensured, it is not the case of cut-free superdeduction modulo since both deduction modulo and superdeduction may break cut-elimination. In this paper we prove the soundness and completeness of *cut-free* superdeduction modulo: we show that by using permutations of inference rules, any cut-free proof in deduction modulo can be transformed into a cut-free proof in superdeduction modulo and conversely. As a corollary we obtain that cut-elimination for deduction modulo (which is well-studied) is equivalent to cut-elimination for superdeduction modulo provided that some hypotheses on the synchrony of reasoning axioms are verified. Finally we propose a tableau method based on superdeduction modulo which is sound and complete provided that cut-elimination holds.

**Key words:** proof theory, focusing, sequent calculus, rewriting, tableaux

## 1 Introduction

Computer science and mathematics are both concerned with the construction of formal proofs. These proofs are used as certificates for assertions, theorems or programs since once they have been constructed, they can be easily communicated, replayed and understood. Therefore formal proofs are the highest criterion for trusting a piece of software. The construction of proofs usually relies on the definition of some proof system which contains both a discipline which the user has to follow, and deductive as well as some computing abilities which empower the user. The proof system may then be used to backup automated theorem provers such as Simplify, Harvey, Zenon or implemented in interactive proof assistants like Coq, Isabelle, PVS or HOL that work as a framework for the proof engineering process. In this context, a proof is usually constructed with respect to a theory (a set of axioms) which is plugged into the deduction process. Indeed

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the deductive power of a deduction system comes from its association with a theory. In frameworks such as first-order natural deduction or sequent calculus, the use of some theory is always uniform since the proofs only express atomic steps which correspond to decompositions of logical connectors. Higher-level notions such as sets, set inclusion, natural numbers or addition have to be encoded in the first-order language and handled through these atomic inference steps, leading often to long, hardly-readable and redundant “assembly like” proofs.

Several paradigms propose *ad hoc* systems for specific theories. For example Definitional Reflection [1], extended with induction in [2] is especially related to logic programming. Another point of view is Huang’s Assertion level [3] mainly motivated by the presentation of machine found proofs in natural language. The approach proposed by Negri and von Plato in [4] is closely related to the handling of axioms which we will use here. They express first-order axioms through so-called non-logical rules. However these inference rules only act on the left-hand side of sequents. Consequently they can only poorly interact with cuts during an elimination of cuts.

The approach we are about to study in this paper is quite related to Focusing introduced by Andreoli in [5] which was meant to remove irrelevant choices in backward reasoning for sequent calculus: Indeed syntactically different proofs can still be identical up to some permutations or simplifications of the applications of the inference rules. This approach was also more recently adapted to forward reasoning in [6]. Our point of view is that usually when one wishes to reason with an axiom (such as one used in the definition of natural numbers), in sequent calculus for instance, it is necessary to destruct the logical connectors of the axiom. These steps are typically always identical, at least *up to permutations or simplifications of the applications of the inference rules*. An interesting way to deal with this is superdeduction, introduced in [7], which allows to use the deductive part of some theory to enrich the deductive system, obtaining thus a custom-made system devoted to the theory. Superdeduction is founded on the originative work of Benjamin Wack in [8] on supernatural deduction which was introduced in order to provide a logical interpretation to the rewriting calculus [9,10]. Based on Urban’s work on the classical sequent calculus [11], a proof-term language for superdeduction is proposed in [7] along with a cut-elimination procedure whose normalisation is proved in [12] under appropriate hypotheses.

In this paper superdeduction is combined with a dual approach which is also designed to structure the use of some first-order theory: Deduction modulo, which was introduced ten years ago by Dowek, Hardin and Kirchner [13]. It is a paradigm which allows one to transform any formula at any point of some deduction process. Since the transformation is not explicit in the constructed proof, it has to be redone during proofchecking and therefore cannot be *more than* computation. For instance it is commonly defined as a congruence on formulæ which is constructed from a convergent rewrite system [14]. Since the rewrite system is convergent, verifying some formula transformation is decidable. Hence deduction modulo allows to make use of the computational part of a theory *Th* for true computations modulo which deduction is performed. Deduction modulo

also leads to interesting automated theorem proving procedures like ENAR [13] or TaMeD [15]. Another line of work is to formalize theories of interest, such as Heyting arithmetic [16,17], Zermelo’s set theory [18] or higher-order logics [19,20,21], using rewrite systems so that they can be used in deduction modulo. Finally, in addition to simplicity, deduction modulo admits unbounded proof size speed-up [22].

In this work, these two approaches are combined in the following way, as it was already put forward in [7]. Instead of constructing a proof of  $\mathcal{Th} \vdash \Delta$  in natural deduction or sequent calculus using the theory  $\mathcal{Th}$  uniformly, we propose to split  $\mathcal{Th}$  in three parts  $\Gamma \cup \mathcal{Th}_1 \cup \mathcal{Th}_2$ . While  $\Gamma$  will still be used as a context to our deductions,  $\mathcal{Th}_1$  will represent computational axioms and will be handled by the deduction modulo paradigm, and  $\mathcal{Th}_2$  will represent reasoning axioms and therefore will be handled by the superdeduction paradigm. We obtain then a custom deduction system in which we prove a sequent  $\Gamma \vdash_{\equiv_{\mathcal{Th}_1}}^{+\mathcal{Th}_2} \Delta$ .

As already explained in [7], superdeduction modulo may be used as an innovative foundation for new proof assistants. First it allows to naturally encode custom reasoning and computational schemes, such as induction over natural numbers and addition. Furthermore it suggests a convenient representation of proofs in contrast with the proofs that are typically constructed with existing proof-assistants such as Coq or Isabelle. These usually consist in tactics or proof-terms which are bound to convince the user of the correctness of the proof (through a typechecking process) but not actually explain it. Indeed the main steps are flooded with a multitude of usually not explicit and often straightforward logical arguments caused by both the underlying calculus and the presence of purely computational parts. While deduction modulo addresses the issue of computational parts, superdeduction eliminates trivial logical arguments in a proof. Moreover the expressiveness of superdeduction modulo is promising. For instance an encoding of Pure Type Systems in (a weak version of) superdeduction modulo is shown in [23].

Deduction modulo and superdeduction can both break cut-elimination (admissibility of the cut-rule) and strong normalisation (of a cut-elimination procedure), even when the underlying theory is consistent. However several criteria have been proved to ensure cut-elimination or strong normalisation of deduction modulo: Criteria have been developed for strong normalisation, using reducibility candidates [24] leading to the notion of truth values algebras and superconsistency [25,26], and for cut-elimination using semantic methods [27,28] or abstract completion [29]. Our aim in this paper is to relate superdeduction modulo and sheer deduction modulo by proving that cut-elimination in one system is equivalent to cut-elimination in the other. Indeed we prove that any *cut-free* proof of  $\Gamma \vdash_{\equiv_{\mathcal{Th}_1 \cup \mathcal{Th}_2}}^{+\emptyset} \Delta$  can be turned into a *cut-free* proof of  $\Gamma \vdash_{\equiv_{\mathcal{Th}_1}}^{+\mathcal{Th}_2} \Delta$  and conversely. As a consequence, the numerous criteria for cut-elimination of deduction modulo also hold for superdeduction modulo. Finally this cut-elimination result allows us to write a sound and complete tableaux method for superdeduction modulo.

This paper is organized as following: In Section 2 we recall the definition of superdeduction modulo and prove its soundness and completeness with respect

to predicate logic. In Section 3, we prove the equivalence between the superdeduction approach and the usual use of first-order axioms *in the cut-free setting*: we show that provided that some synchrony hypothesis on  $\mathcal{Th}_2$  is verified, a cut-free proof in deduction modulo ( $\vdash_{\equiv_{\mathcal{Th}_1 \cup \mathcal{Th}_2}}$ ) can be transformed into a cut-free proof in superdeduction modulo ( $\vdash_{\equiv_{\mathcal{Th}_1}}^{+\mathcal{Th}_2}$ ). The converse being obvious, we obtain that cut-elimination for deduction modulo is equivalent to cut-elimination for superdeduction modulo. Therefore we get as corollaries that all the cut-elimination theorems that have already been shown for deduction modulo also hold for superdeduction modulo. Finally in Section 4 we propose a sound and complete tableau method for superdeduction modulo, provided that the corresponding deduction system enjoys cut-elimination. Detailed proofs of lemmas and theorems are written in Appendix A.

## 2 Superdeduction Modulo

In this section we formally define superdeduction modulo which is the combination of two different extensions of classical sequent calculus for predicate logic: deduction modulo on one hand and superdeduction on the other hand. Let us recall the classical sequent calculus LK which contains the *deduction core* of our superdeduction modulo systems:

$$\begin{array}{c}
\text{Ax} \frac{}{\Gamma, \varphi \vdash \varphi, \Delta} \quad \text{Cut} \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta} \quad \perp_L \frac{}{\Gamma, \perp \vdash \Delta} \quad \top_R \frac{}{\Gamma \vdash \top, \Delta} \\
\wedge_L \frac{\Gamma, \varphi_1, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \vdash \Delta} \quad \wedge_R \frac{\Gamma \vdash \varphi_1, \Delta \quad \Gamma \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \wedge \varphi_2, \Delta} \quad \vee_L \frac{\Gamma, \varphi_1 \vdash \Delta \quad \Gamma, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \vdash \Delta} \\
\Rightarrow_R \frac{\Gamma, \varphi_1 \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \Rightarrow \varphi_2, \Delta} \quad \Rightarrow_L \frac{\Gamma \vdash \varphi_1, \Delta \quad \Gamma, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \Rightarrow \varphi_2 \vdash \Delta} \quad \vee_R \frac{\Gamma \vdash \varphi_1, \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \vee \varphi_2, \Delta} \\
\forall_R \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x. \varphi, \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta) \quad \forall_L \frac{\Gamma, \varphi[t/x] \vdash \Delta}{\Gamma, \forall x. \varphi \vdash \Delta} \quad \exists_R \frac{\Gamma \vdash \varphi[t/x], \Delta}{\Gamma \vdash \exists x. \varphi, \Delta} \\
\text{CONTR}_R \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta} \quad \text{CONTR}_L \frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \quad \exists_L \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \exists x. \varphi \vdash \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta)
\end{array}$$

A *term rewrite rule* is a rewrite rule that rewrites first-order terms into first-order terms and a *proposition rewrite rule* is a rewrite rule that rewrites an atomic proposition into an arbitrary formula. For instance  $\text{plus}(\text{zero}, x) \rightarrow x$  is a term rewrite rule while  $a \subseteq b \rightarrow \forall x. x \in a \Rightarrow x \in b$  is a proposition rewrite rule. We will consider two sets of rewrite rules:  $\mathcal{Th}_1$  is a set containing both term and proposition rewrite rules ;  $\mathcal{Th}_2$  is a set of proposition rewrite rules. As  $\mathcal{Th}_1$  will be used to extend the deduction system through deduction modulo, our intuition is that it contains the computational axioms. Dually as  $\mathcal{Th}_2$  will be used to enrich the deduction system through superdeduction, our intuition is that it contains the deductive axioms. We suppose that each rewrite rule  $P \rightarrow \varphi$  of  $\mathcal{Th}_2$  is associated with some name R then denoted  $\mathbf{R} : P \rightarrow \varphi$ .

For  $i \in \{1, 2\}$ , the one-step rewrite reduction associated with  $\mathcal{Th}_i$  is denoted  $\rightarrow_i$ . The reflexive and transitive closure of  $\rightarrow_i$  is denoted  $\rightarrow_i^*$ . The symmetric, reflexive and transitive closure of  $\rightarrow_i$  is denoted  $\equiv_i$ . The notations  $\rightarrow_{1,2}$ ,  $\rightarrow_{1,2}^*$ , and  $\equiv_{1,2}$  are used for  $\mathcal{Th}_1 \cup \mathcal{Th}_2$ .

The first-order axiom associated with a proposition rewrite rule  $P \rightarrow \varphi$  is  $\forall \bar{x}.(P \Leftrightarrow \varphi)$ , where  $\bar{x}$  represents the free variables of  $P$  and  $\varphi$ , denoted  $\mathcal{FV}(P, \varphi)$ . The first-order axiom associated with a term rewrite rule  $l \rightarrow r$  is  $\forall \bar{x}.(l = r)$ , where  $\bar{x}$  represents the free variables of  $l$  and  $r$ . This way of representing a term rewrite rule as a first-order axiom supposes that the logic contains an equality symbol. If it does not, one may add this predicate and the corresponding axioms in a conservative way as detailed in [30]. An other way to proceed with the term rewrite rules of  $\mathcal{Th}_1$  is to use axioms of the form  $\forall \bar{x}. P \Leftrightarrow Q$  for all  $P \equiv_1 Q$ .

When writing  $\vdash_{\equiv_{\mathcal{Th}_1}^+}$ ,  $\mathcal{Th}_1$  and  $\mathcal{Th}_2$  will represent the rewrite rules, and when writing  $\mathcal{Th}_1 \vdash$  or  $\mathcal{Th}_2 \vdash$ , they will represent the first-order axioms.

For some deduction system  $\vdash^*$ , we will denote  $\Gamma \vdash^* \Delta$  the sentence *there is a proof of  $\Gamma \vdash^* \Delta$* . We will denote  $\Gamma \stackrel{ef}{\vdash} \Delta$  the sequents in the corresponding cut-free deduction system.

Now let us define superdeduction modulo. Superdeduction stands for the addition to classical sequent calculus of new superdeduction inference rules that are computed from  $\mathcal{Th}_2$  in the following way.

**Definition 1 (Superdeduction rules computation [7])** *Let Calc be a set of rules composed by the subset of the sequent calculus deduction rules formed of  $\text{Ax}$ ,  $\perp_L$ ,  $\top_R$ ,  $\vee_L$ ,  $\vee_R$ ,  $\wedge_L$ ,  $\wedge_R$ ,  $\Rightarrow_L$ ,  $\Rightarrow_R$ ,  $\forall_L$ ,  $\forall_R$ ,  $\exists_L$  and  $\exists_R$ , as well as of the rules  $\top_L \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}$  and  $\perp_R \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}$ .*

Let  $R : P \rightarrow \varphi$  be a proposition rewrite rule of  $\mathcal{Th}_2$ .

1. To get the right rule associated with  $R$ , initialise the procedure with the sequent  $\Gamma \vdash \varphi, \Delta$ . Next, apply the rules of *Calc* until no more open leaves remain on which they can be applied. Then, collect the premises, the side conditions and the conclusion and replace  $\varphi$  by  $P$  to obtain the rule  $R_R$ .
2. To get the left rule  $R_L$  associated with  $R$ , initialise the procedure with the sequent  $\Gamma, \varphi \vdash \Delta$ . Apply the rules of *Calc* and get the new left rule the same way as for the right one.

For instance, the rules associated with  $\subseteq_{\text{def}} : a \subseteq b \rightarrow \forall x.x \in a \Rightarrow x \in b$  are

$$\subseteq_{\text{def}R} \frac{\Gamma, x \in a \vdash x \in b, \Delta}{\Gamma \vdash a \subseteq b, \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta) \quad \subseteq_{\text{def}L} \frac{\Gamma, t \in b \vdash \Delta \quad \Gamma \vdash t \in a, \Delta}{\Gamma, a \subseteq b \vdash \Delta}$$

Since the propositional rules of *Calc* commute with any other rules, they may be applied in any order to reach axioms. However the application order of rules concerning quantifiers can be significant. Even if it will not affect the resulting premises, the side condition may differ: Decomposing  $P \rightarrow (\exists x.A(x)) \Rightarrow (\exists x.B(x))$  on the right can lead to

$$\frac{\Gamma, A(x) \vdash B(t), \Delta}{\Gamma \vdash P, \Delta} x \notin \mathcal{FV}(\Gamma, \Delta) \quad \text{or} \quad \frac{\Gamma, A(x) \vdash B(t), \Delta}{\Gamma \vdash P, \Delta} x \notin \mathcal{FV}(\Gamma, \Delta, t)$$

depending on which existential quantifier is decomposed first. Although both rules are sound, only the first is complete. Therefore when computing superdeduction rules, we give  $\exists_L$  and  $\forall_R$  a higher priority than the other rules, and we consequently ensure that we will always obtain the weakest (but still sound) side condition.

Completeness of new superdeduction rules has been proved in [7] provided that  $\mathcal{Th}_2$  contains no permutability problem:

**Definition 2 (Polarity of a subformula)** *The polarity  $pol_\varphi(\psi)$  of  $\psi$  in  $\varphi$  where  $\psi$  is an occurrence of a subformula of  $\varphi$  is a boolean defined as follows:*

- if  $\varphi = \psi$ , then  $pol_\varphi(\psi) = \text{true}$ ;
- if  $\varphi = \varphi_1 \wedge \varphi_2$  or  $\varphi_1 \vee \varphi_2$ , then  $pol_\varphi(\psi) = pol_{\varphi_1}(\psi)$  if  $\psi$  is a subformula of  $\varphi_1$ ,  $pol_{\varphi_2}(\psi)$  otherwise;
- if  $\varphi = \forall x.\varphi_1$  or  $\exists x.\varphi_1$ , then  $pol_\varphi(\psi) = pol_{\varphi_1}(\psi)$ ;
- if  $\varphi = \varphi_1 \Rightarrow \varphi_2$ , then  $pol_\varphi(\psi) = \neg pol_{\varphi_1}(\psi)$  if  $\psi$  is a subformula of  $\varphi_1$ ,  $pol_{\varphi_2}(\psi)$  otherwise.

**Definition 3 (Set of permutability problems)** *A formula  $\psi$  is in the set  $PP(\varphi)$  of permutability problems of  $\varphi$  if there exists  $\varphi'$  a subformula of  $\varphi$  such that  $\psi$  is an occurrence of a subformula of  $\varphi'$  and one of these propositions holds:*

- $\varphi' = \forall x.\varphi'_1$ ,  $\psi = \forall x.\psi'_1$  and  $pol_{\varphi'}(\psi) = \text{false}$
- $\varphi' = \exists x.\varphi'_1$ ,  $\psi = \exists x.\psi'_1$  and  $pol_{\varphi'}(\psi) = \text{false}$
- $\varphi' = \forall x.\varphi'_1$ ,  $\psi = \exists x.\psi'_1$  and  $pol_{\varphi'}(\psi) = \text{true}$
- $\varphi' = \exists x.\varphi'_1$ ,  $\psi = \forall x.\psi'_1$  and  $pol_{\varphi'}(\psi) = \text{true}$

To summarize, a permutability problem is a positive  $\forall$  or a negative  $\exists$  inside a  $\exists$  quantifier, or a negative  $\forall$  or a positive  $\exists$  inside a  $\forall$  quantifier. Consequently we make the following hypothesis.

**Hypothesis 1** *If  $P \rightarrow \varphi \in \mathcal{Th}_2$ , then  $PP(\varphi) = \emptyset$ .*

It has been proved in [7] that it implies the completeness of the corresponding superdeduction system. Finally a procedure has been proposed in [7] which transforms any set of proposition rewrite rules into an equivalent one verifying this property, namely replacing every subformula  $\psi$  of  $\varphi$  leading to a permutability problem by a fresh predicate symbol  $Q$  parametrised by the free variables of the subformula and then adding the rule  $Q \rightarrow \psi$ .

Let us formally define superdeduction modulo.

**Definition 4 (Superdeduction modulo)** *The superdeduction modulo system associated with  $(\mathcal{Th}_1, \mathcal{Th}_2)$  is formed of the rules of classical sequent calculus and the rules built upon  $\mathcal{Th}_2$ , where each first-order term or proposition is considered modulo rewriting through  $\mathcal{Th}_1$ . Sequents in this system are denoted  $\Gamma \vdash_{\equiv_{\mathcal{Th}_1}}^{+\mathcal{Th}_2} \Delta$ , or simply  $\Gamma \vdash_{\equiv_1}^{+\mathcal{Th}_2} \Delta$ .*

If  $\mathcal{T}h_2$  is empty, then we are in raw deduction modulo and sequents are denoted  $\Gamma \vdash_{\equiv_1} \Delta$ . If  $\mathcal{T}h_1$  is empty, then we are in raw superdeduction and sequents are denoted  $\Gamma \vdash^{+2} \Delta$ .

There are two equivalent ways to present the use of inference rules where propositions and first-order terms are considered modulo rewriting through  $\mathcal{T}h_1$ : One can use new inference rules such as

$$\equiv_{1R} \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi, \Delta} \psi \equiv_1 \varphi \qquad \equiv_{1L} \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \psi \equiv_1 \varphi$$

or one can replace the inference rules of the system by rules such as

$$\begin{array}{c} \text{Ax} \frac{}{\Gamma, \varphi \vdash \psi, \Delta} \varphi \equiv_1 \psi \qquad \Rightarrow_R \frac{\Gamma, \varphi_1 \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi, \Delta} \varphi \equiv_1 \varphi_1 \Rightarrow \varphi_2 \\ \\ \text{Cut} \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma \vdash \Delta} \varphi \equiv_1 \psi \qquad \dots \end{array}$$

For example in this latter presentation, the rules associated with  $\subseteq_{\text{def}}$  become

$$\subseteq_{\text{def}R} \frac{\Gamma, x \in a \vdash x \in b, \Delta}{\Gamma \vdash \varphi, \Delta} \left\{ \begin{array}{l} x \notin \mathcal{FV}(\Gamma, \Delta) \\ \varphi \equiv_1 a \subseteq b \end{array} \right. \subseteq_{\text{def}L} \frac{\Gamma, t \in b \vdash \Delta \quad \Gamma \vdash t \in a, \Delta}{\Gamma, \varphi \vdash \Delta} \varphi \equiv_1 a \subseteq b$$

Let us recall that we allow  $\mathcal{T}h_1$  to contain proposition rewrite rules (rewriting on formulæ). This can lead to odd situations: For instance a conjunction  $\varphi_1 \wedge \varphi_2$  may be congruent to a disjunction  $\psi_1 \vee \psi_2$ , and then one can perform a cut between a decomposition of the first on the left and a decomposition of the second on the right.

$$\text{Cut} \frac{\vee_R \frac{\Gamma \vdash_{\equiv_1}^{+2} \varphi_1, \varphi_2, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} \varphi_1 \vee \varphi_2, \Delta} \quad \wedge_L \frac{\Gamma, \psi_1, \psi_2 \vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \psi_1 \wedge \psi_2 \vdash_{\equiv_1}^{cf+2} \Delta}}{\Gamma \vdash_{\equiv_1}^{+2} \Delta}$$

Obviously there is no way to eliminate this cut *a priori*, especially when  $\varphi_i \not\equiv_1 \psi_j$  for all  $i, j \in \{1, 2\}$ . In order to avoid this kind of awkward situation, deduction modulo is usually only considered for confluent rewrite systems: Since  $\mathcal{T}h_1$  only rewrites first-order terms and atomic predicates, if  $\varphi_1 \wedge \varphi_2 \equiv_1 \psi_1 \vee \psi_2$  and if  $\mathcal{T}h_1$  is confluent, then the common reduct of  $\varphi_1 \wedge \varphi_2$  and  $\psi_1 \vee \psi_2$  is both a conjunction and a disjunction, which is contradictory. Consequently we suppose the following hypothesis.

**Hypothesis 2**  $\mathcal{T}h_1$  is confluent.

It is interesting to notice that our superdeduction modulo systems are combination of superdeduction and deduction modulo in the following order: first the superdeduction inference rules are computed, then they are used modulo rewriting. Therefore deduction modulo does not interfere with the computation of the superdeduction rules, but only with their use. However we could have chosen to compute these new inference rules by already using deduction modulo. This computation is deterministic if and only if  $\mathcal{T}h_1$  is confluent: Indeed in this case, rewriting through  $\mathcal{T}h_1$  cannot erase the logical connectors of some formula

(since it only rewrites *atomic predicates*) and cannot spawn distinct logical connectors at the same position of the formula (since it is confluent). Besides the computation is terminating if and only if  $\mathcal{Th}_1$  is terminating.

The first properties of superdeduction modulo systems that we will prove are its soundness and completeness *w.r.t.* classical predicate logic. They are straightforward consequences of soundness and completeness of deduction modulo (Theorem 1) and superdeduction (Theorem 2):

**Theorem 1 ([13])**  $\Gamma \vdash_{\equiv_1} \Delta$  if and only if  $\mathcal{Th}_1, \Gamma \vdash \Delta$ .

**Theorem 2 ([7])**  $\Gamma \vdash^{+2} \Delta$  if and only if  $\mathcal{Th}_2, \Gamma \vdash \Delta$ .

The soundness of superdeduction modulo is then a direct consequence of the following lemma.

**Lemma 1 (From  $\vdash_{\equiv_1}^{+2}$  to  $\vdash_{\equiv_{1,2}}$ )**

If  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ , then  $\Gamma \vdash_{\equiv_{1,2}} \Delta$ . If  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$ , then  $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$ .

*Proof.* By definition of the superdeduction rules for  $P \rightarrow \varphi \in \mathcal{Th}_2$ , they can be replaced by several rules of Calc followed by a conversion step  $\varphi \equiv_2 P$ . The proof is detailed in Appendix A.

**Theorem 3 (Soundness of superdeduction modulo)**

If  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ , then  $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$ .

*Proof.* Using Lemma 1, the proof of  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$  is first translated into a proof of  $\Gamma \vdash_{\equiv_{1,2}} \Delta$ , then by soundness of deduction modulo, we obtain a proof of  $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$ .

**Theorem 4 (Completeness of superdeduction modulo)**

If  $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$ , then  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ .

*Proof.* From the completeness of superdeduction, for all  $\varphi \in \mathcal{Th}_2$ , there exists a proof of  $\vdash^{+2} \varphi$ . From the completeness of deduction modulo, for all  $\varphi \in \mathcal{Th}_1$ , there exists a proof of  $\vdash_{\equiv_1} \varphi$ . Starting from a proof of  $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$ , using cuts with the proofs of  $\vdash_{\equiv_1}^{+2} \varphi$  for all  $\varphi \in \mathcal{Th}_1 \cup \mathcal{Th}_2$ , we get a proof of  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ .

The use of superdeduction modulo through an example is demonstrated in Appendix B.

### 3 Cut-free setting

Deduction modulo and superdeduction are known to jeopardize cut-elimination (completeness of the cut-free deduction system) and normalisation (termination of a cut-elimination procedure): For example if  $A \equiv_{1,2} A \Rightarrow A$ , one can easily build a proof of  $\vdash_{\equiv_1}^{+2} A$ <sup>1</sup>, but there exists no cut-free proof of it. However cut-elimination and normalisation are well-studied for deduction modulo. Several proofs and criteria have been presented for cut-elimination and normalisation. The following theorems hold for classical sequent calculus modulo:

<sup>1</sup> The proof may be written  $(\lambda x. x x) (\lambda x. x x)$  in a  $\lambda$ -calculus style.

**Theorem 5 ([24])** *If the light double negation<sup>2</sup> of a rewrite system  $\mathcal{R}$  has a premodel<sup>2</sup>, cut-elimination holds for the classical sequent calculus modulo  $\mathcal{R}$ .*

**Theorem 6 ([27])** *If  $\mathcal{R}$  is a rewrite system compatible with a well-founded order, if  $\mathcal{R}^+$  is a positive rewrite system<sup>3</sup> whose right-hand sides are  $\mathcal{R}$ -normal forms and if  $\mathcal{R} \cup \mathcal{R}^+$  is confluent, then cut-elimination holds for the classical sequent calculus modulo  $\mathcal{R} \cup \mathcal{R}^+$ .*

**Theorem 7 ([29])** *A sequent has a proof in classical sequent calculus modulo some theory  $\mathcal{R}$  if and only if it has a cut-free proof in the saturated theory corresponding to  $\mathcal{R}$ .*

Our aim is now to relate superdeduction modulo to deduction modulo in order to prove that all these criteria are extendible to superdeduction modulo. We will only consider cut-free deduction systems and show that they allow to prove exactly the same sequents: a sequent  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$  is provable if and only if  $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$  is. The translation of a proof in superdeduction modulo into sheer deduction modulo is already shown in Lemma 1. Now let us prove the converse: if  $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$ , then  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$ . To achieve this goal, we will first consider some proposition rewrite rule  $R : P \rightarrow \varphi$  of  $\mathcal{Th}_2$  and prove that if  $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$ , then  $\Gamma \Vdash_{\equiv_1}^{cf+2} P, \Delta$ ; if  $\Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta$ , then  $\Gamma, P \Vdash_{\equiv_1}^{cf+2} \Delta$ . This property then implies the admissibility of rules  $\equiv_{2R} \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \psi, \Delta} \varphi \equiv_2 \psi$  and  $\equiv_{2L} \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \psi \vdash \Delta} \varphi \equiv_2 \psi$  in *cut-free*

superdeduction modulo, which in turn implies that if  $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$ , then  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$ . The proof of the property deals with permutability problems in classical sequent calculus: Indeed the intuition is to unite the steps of the proof that decompose  $\varphi$ . There are four cases for which two steps of LK cannot be permuted:

$$\begin{array}{l} \exists_R \frac{\Gamma \Vdash_{\equiv_1}^{cf+2} P(x), Q(x), \Delta}{\Gamma \Vdash_{\equiv_1}^{cf+2} P(x), \psi, \Delta} \psi \equiv_1 \exists x.Q(x) \\ \forall_R \frac{\Gamma \Vdash_{\equiv_1}^{cf+2} P(x), \psi, \Delta}{\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \psi, \Delta} \varphi \equiv_1 \forall x.P(x) \end{array}$$

and similarly where  $\forall_R$  is replaced by  $\exists_L$  and/or  $\exists_R$  is replaced by  $\forall_L$ . Consequently we can easily build a proof of some sequent  $\Gamma \vdash_{\equiv_1}^{\pm 2} \varphi, \Delta$  where the steps decomposing  $\varphi$  cannot be united using sheer permutations:

$$\begin{array}{l} \dots \\ \forall_L \frac{\dots}{P(y_0) \Rightarrow Q(x_0), P(y_0) \vdash Q(x_0)} \\ \exists_L \frac{\forall_L \frac{\dots}{P(y_0) \Rightarrow Q(x_0), (\forall x.P(x)) \vdash Q(x_0)}}{P(y_0) \Rightarrow Q(x_0), (\forall x.P(x)) \vdash Q(x_0)} \\ \forall_L \frac{\exists_L \frac{\dots}{\exists y.(P(y) \Rightarrow Q(x_0)), (\forall x.P(x)) \vdash Q(x_0)}}{\exists y.(P(y) \Rightarrow Q(x_0)), (\forall x.P(x)) \vdash Q(x_0)} \\ \forall_R \frac{\forall_L \frac{\dots}{\forall x.\exists y.(P(y) \Rightarrow Q(x)), (\forall x.P(x)) \vdash Q(x_0)}}{\forall x.\exists y.(P(y) \Rightarrow Q(x)), (\forall x.P(x)) \vdash (\forall x.Q(x))} \\ \Rightarrow_R \frac{\forall_R \frac{\dots}{\forall x.\exists y.(P(y) \Rightarrow Q(x)), (\forall x.P(x)) \vdash (\forall x.Q(x))}}{\forall x.\exists y.(P(y) \Rightarrow Q(x)) \vdash (\forall x.P(x)) \Rightarrow (\forall x.Q(x))} \end{array}$$

where  $\varphi$  is  $(\forall x.P(x)) \Rightarrow (\forall x.Q(x))$ . Let us notice that  $PP(\varphi)$  is empty. A solution is to combine the permutations with contractions as follows.

<sup>2</sup> Both definitions of light double negation and pre-model can be found in [24].

<sup>3</sup> The definition of a *positive* rewrite system can be found in [27].

$$\begin{array}{c}
\dots \\
\frac{}{\vdash P(y_0) \Rightarrow Q(x_0), P(x_0), P(y_0) \vdash Q(x_1), Q(x_0)} \\
\frac{}{\vdash P(y_0) \Rightarrow Q(x_0), P(x_0), P(y_0) \vdash \forall x.Q(x), Q(x_0)} \\
\frac{}{\vdash P(y_0) \Rightarrow Q(x_0), P(x_0), \forall x.P(x) \vdash \forall x.Q(x), Q(x_0)} \\
\Rightarrow_R \frac{}{\vdash P(y_0) \Rightarrow Q(x_0), P(x_0) \vdash \varphi, Q(x_0)} \\
\frac{}{\vdash \exists y.P(y) \Rightarrow Q(x_0), P(x_0) \vdash \varphi, Q(x_0)} \\
\frac{}{\vdash \forall x.\exists y.P(y) \Rightarrow Q(x), P(x_0) \vdash \varphi, Q(x_0)} \\
\frac{}{\vdash \forall x.\exists y.P(y) \Rightarrow Q(x), \forall x.P(x) \vdash \varphi, Q(x_0)} \\
\frac{}{\vdash \forall x.\exists y.P(y) \Rightarrow Q(x), \forall x.P(x) \vdash \varphi, \forall x.Q(x)} \\
\Rightarrow_R \frac{}{\vdash \forall x.\exists y.P(y) \Rightarrow Q(x) \vdash \varphi, (\forall x.P(x)) \Rightarrow (\forall x.Q(x))} \\
\text{CONTR}_R \frac{}{\vdash \forall x.\exists y.(P(y) \Rightarrow Q(x)) \vdash (\forall x.P(x)) \Rightarrow (\forall x.Q(x))}
\end{array}$$

In this proof, all the steps decomposing  $(\forall x.P(x)) \Rightarrow (\forall x.Q(x))$  have been duplicated and then united. It seems that this manipulation allows to transform any proof using several LK inference rules to decompose a formula  $\varphi$  into a proof using several occurrences of a single superdeduction inference rule decomposing the corresponding predicate  $P$  ( $P \rightarrow \varphi \in \mathcal{Th}_2$ ). However proving that this manipulation always solves the problem remains an open question to our knowledge. We choose to avoid this kind of manipulation by making the following hypothesis.

**Hypothesis 3** *If  $P \rightarrow \varphi \in \mathcal{Th}_2$ , then  $\varphi$  is in prenex normal form.*

We do not know however if the final result holds without Hypothesis 2, 1 or 3, since we did not find any counterexample. Let us notice that together with Hypothesis 1, Hypothesis 3 implies the following lemma.

**Lemma 2** *If  $P \rightarrow \varphi \in \mathcal{Th}_2$ , then  $\varphi$  is either  $\forall x_1 \dots x_n.\psi$ , either  $\exists x_1 \dots x_n.\psi$  for some propositional formula  $\psi$ . We will respectively say that  $\varphi$  is right-handed and left-handed.*

Hypotheses 3 and 1 can be replaced by the following single hypothesis (which we will call Hypothesis 3'): The only quantifiers of some  $\varphi$  in  $P \rightarrow \varphi \in \mathcal{Th}_2$  are either positive  $\forall$  and negative  $\exists$ , either negative  $\forall$  and positive  $\exists$ . Indeed in this case, one can always find a prenex normal form  $\psi$  classically equivalent to  $\varphi$  which satisfies Hypotheses 3 and 1 and such as the rules associated with  $P \rightarrow \varphi$  are the same as the rules associated with  $P \rightarrow \psi$ . The deduction system is consequently unchanged and then our final result also holds if we just replace Hypotheses 3 and 1 by Hypothesis 3'. Furthermore the procedure proposed in [7] which turns any set of proposition rewrite rules into an equivalent one verifying Hypothesis 1 can easily be strengthened into a procedure which turns any set of proposition rewrite rules into an equivalent one verifying this new single hypothesis.

Let us illustrate this with an example: The proposition rewrite rule  $R : P \rightarrow (\forall x.A(x)) \Rightarrow (\exists y.B(y))$  verifies Hypothesis 3' (which in turn implies Hypothesis 1), but does not verify Hypothesis 3. However  $(\forall x.A(x)) \Rightarrow (\exists y.B(y))$  is classically equivalent to  $\exists x.\exists y.(A(x) \Rightarrow B(y))$ . Furthermore we obtain the same

superdeduction inference rules if we replace the proposition rewrite rule R by  $R' : P \rightarrow \exists x. \exists y. (A(x) \Rightarrow B(y))$ , namely

$$\frac{\Gamma, A(t_1) \vdash B(t_2), \Delta}{\Gamma \vdash P, \Delta} \quad \frac{\Gamma, B(y) \vdash \Delta \quad \Gamma \vdash A(x), \Delta}{\Gamma, P \vdash \Delta} \quad y \notin \mathcal{FV}(\Gamma, \Delta)$$

Finally  $R'$  satisfies both Hypotheses 1 and 3.

Now let us prove that  $\vdash_{\equiv_1}^{cf+2}$  is equivalent to  $\vdash_{\equiv_{1,2}}^{cf}$ . Dealing with right-handed formulæ on the right and left-handed formulæ on the left of a sequent needs the following lemma, adapted from a central lemma in Hermant's semantic proofs of cut-elimination for deduction modulo [27].

**Lemma 3 (Kleene's lemma adapted to  $\vdash_{\equiv_1}^{+2}$ )**

Let  $A_1 \equiv_1 A_2 \equiv_1 \dots A_n \equiv_1 \varphi$  be some propositions. Let  $\Theta = A_1, A_2 \dots A_n$ .

- If  $\varphi = \neg A$  and  $\Gamma, \Theta \vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma \vdash_{\equiv_1}^{cf+2} A, \Delta$ .
- If  $\varphi = A \wedge B$  and  $\Gamma, \Theta \vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, A, B \vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \vee B$  and  $\Gamma, \Theta \vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, A \vdash_{\equiv_1}^{cf+2} \Delta$  and  $\Gamma, B \vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \Rightarrow B$  and  $\Gamma, \Theta \vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, B \vdash_{\equiv_1}^{cf+2} \Delta$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} A, \Delta$ .
- If  $\varphi = \exists x. Q$  and  $\Gamma, \Theta \vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, Q[c/x] \vdash_{\equiv_1}^{cf+2} \Delta$  for some fresh variable  $c$ .
- If  $\varphi = \neg A$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma, A \vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \wedge B$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \vdash_{\equiv_1}^{cf+2} A, \Delta$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} B, \Delta$ .
- If  $\varphi = A \vee B$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \vdash_{\equiv_1}^{cf+2} A, B, \Delta$ .
- If  $\varphi = A \Rightarrow B$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} \Theta, \Delta$ , then  $\Gamma, A \vdash_{\equiv_1}^{cf+2} B, \Delta$ .
- If  $\varphi = \forall x. Q$  and  $\Gamma \vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \vdash_{\equiv_1}^{cf+2} Q[c/x], \Delta$  for some fresh variable  $c$ .

*Proof.* For each assertion, The proof is done by induction on the derivation and detailed in Appendix A.

It implies the following lemma.

**Lemma 4** Let us consider some  $R : P \rightarrow \varphi \in \mathcal{Th}_2$ .

- If  $\varphi$  is right-handed and  $\Gamma \vdash_{\equiv_1}^{cf+2} \varphi, \Delta$ , then there exists (cut-free) proofs of each premise of  $R_R$ . Therefore  $\Gamma \vdash_{\equiv_1}^{cf+2} P, \Delta$ .
- If  $\varphi$  is left-handed and  $\Gamma, \varphi \vdash_{\equiv_1}^{cf+2} \Delta$ , then there exists (cut-free) proofs of each premise of  $R_L$ . Therefore  $\Gamma, P \vdash_{\equiv_1}^{cf+2} \Delta$ .

*Proof.* By iteration of Lemma 3. The proof is detailed in Appendix A. The idea is to unite the decomposition of  $\varphi$  at the root of the proof: All these steps can move downward since  $\varphi$  is right-handed and decomposed on the right. The proof is detailed in Appendix A.

Lemma 5 deals with right-handed formulæ on the left and left-handed formulæ on the right.

**Lemma 5** Let us consider  $R : P \rightarrow \varphi \in \mathcal{Th}_2$ .

- If  $\varphi$  is right-handed and  $\Gamma, \varphi \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ , then  $\Gamma, P \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .
- If  $\varphi$  is left-handed and  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \varphi, \Delta$ , then  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} P, \Delta$ .

*Proof.* The proof is detailed in Appendix A. The idea is to fully decompose  $\varphi$ , partially eliminate contractions, and then to pull up the decomposition of  $\varphi$ .

Lemmas 4 and 5 are concentrated in the following theorem, which proves that rewriting through  $\mathcal{Th}_1 \cup \mathcal{Th}_2$  preserves provability in superdeduction modulo.

**Theorem 8** *If  $\varphi \equiv_{1,2} \psi$ , then* 
$$\begin{cases} \Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \varphi, \Delta & \text{if and only if} & \Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \psi, \Delta \\ \Gamma, \varphi \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta & \text{if and only if} & \Gamma, \psi \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta \end{cases}$$

*Proof.* First by Lemmas 4 and 5, provability is preserved by one-step head reduction through  $\mathcal{Th}_2$  (i.e. if  $\varphi = P\sigma$  for some substitution  $\sigma$  and  $\psi = \phi\sigma$  and  $P \rightarrow \phi \in \mathcal{Th}_2$ ). By induction on  $\varphi$ , we prove that it extends to any one-step reduction ( $\varphi \rightarrow_2 \psi$ ). Then by induction on  $\varphi \equiv_{1,2} \psi$ , we obtain the final result.

**Lemma 6 (From  $\Vdash_{\equiv_{1,2}}^{\mathcal{C}f}$  to  $\Vdash_{\equiv_1}^{\mathcal{C}f+2}$ )** *If  $\Gamma \Vdash_{\equiv_{1,2}}^{\mathcal{C}f} \Delta$ , then  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .*

*Proof.* By induction on the proof of  $\Gamma \Vdash_{\equiv_{\mathcal{Th}_1 \cup \mathcal{Th}_2}}^{\mathcal{C}f} \Delta$ : The raw deductive steps and the steps modulo  $\mathcal{Th}_1$  are unchanged. A step  $\frac{\Gamma \Vdash_{\equiv_{1,2}}^{\mathcal{C}f} \psi, \Delta}{\Gamma \Vdash_{\equiv_{1,2}}^{\mathcal{C}f} \varphi, \Delta} \psi \equiv_2 \varphi$  is translated (by induction hypothesis) into a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \psi, \Delta$  which is translated using Theorem 8 into a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \varphi, \Delta$ .

Let us notice that the transformation from  $\Vdash_{\equiv_{1,2}}^{\mathcal{C}f}$  to  $\Vdash_{\equiv_1}^{\mathcal{C}f+2}$  includes a *structuring* of the proof. Hence the final result may only use superdeduction inferences, as in the proof of the example of Appendix B, therefore forcing a structured use of (superdeduction) axioms. Finally we get the equivalence of deduction modulo and superdeduction modulo:

**Theorem 9**  $\Gamma \Vdash_{\equiv_{1,2}}^{\mathcal{C}f} \Delta$  *if and only if*  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .

*Proof.* By Lemmas 1 and 6.

As a corollary we directly obtain that cut-elimination holds for deduction modulo  $\mathcal{Th}_1 \cup \mathcal{Th}_2$  if and only if it holds for the superdeduction modulo system associated with  $(\mathcal{Th}_1, \mathcal{Th}_2)$ . In particular it holds if  $(\mathcal{Th}_1, \mathcal{Th}_2)$  verifies the criterion of Theorem 5, 6 or 7.

## 4 A tableau method for superdeduction

An interesting use of the cut-elimination property for superdeduction modulo is to express the system through a sound and complete tableau method. In this section we choose to extend TaMeD —Tableau Method for Deduction modulo [31,15]— whose rules are depicted in Appendix C. We use the presentation of

TaMeD of [31], where each formula  $\varphi$  is annotated with a set of first-order variables  $l$ , denoted  $\varphi^l$ . This set at least contains the free variables of  $\varphi$ . Besides each formula  $\psi$  of the form  $\forall x.\varphi$  or  $\neg(\exists x.\varphi)$  is annotated with an integer  $n$ , denoted  $\psi_n$ . This integer is the maximal number of decomposition of  $\psi$  that are allowed, in order to keep the method terminating.

The extension of TaMeD is done in the following manner. Each superdeduction rule on the left

$$\frac{(I, \Gamma_i \vdash \Delta_i, \Delta)_{1 \leq i \leq n}}{\Gamma, P \vdash \Delta} \quad X \cap \mathcal{FV}(\Gamma, \Delta) = \emptyset$$

is translated into  $\mathcal{B}_1, P_{n_P}^l \mid \mathcal{T} \xrightarrow{tab} \mathcal{B}_1, P_{n_P-1}^l, \Gamma_1^{l \cup Y} \sigma, \neg \Delta_1^{l \cup Y} \sigma \mid \dots \mid \mathcal{B}_1, P_{n_P-1}^l, \Gamma_n^{l \cup Y} \sigma, \neg \Delta_n^{l \cup Y} \sigma \mid \mathcal{T}$  if  $n_P > 0$  where  $\Gamma_i^l$  and  $\Delta_i^l$  stand respectively for  $\Gamma_i$  and  $\Delta_i$  where each meta-term (such as  $t$  in the  $\exists_R$ -rule) is replaced by a fresh variable,  $Y$  stands for the set of these fresh variables, and  $\sigma$  stands for a substitution replacing each variable  $x_i$  of  $X = x_1, x_2, \dots$  by a fresh skolem function  $f_i(l)$ . The construction of the tableau rule corresponding to superdeduction rules on the right is done in a similar way.

Together with the rules of Appendix C, we obtain a sound and complete tableau method for superdeduction modulo, provided that the superdeduction modulo system enjoys cut-elimination: Each use of a superdeduction tableau transformation can be translated into a use of the corresponding superdeduction inference rule and conversely. Since it also holds for the other rules of TaMeD (soundness and completeness of TaMeD [31,15]), we consequently obtain the equivalence between the superdeduction modulo tableau method and the corresponding cut-free superdeduction modulo system.

Let us show how it operates on the example of Section 2 The tableau rule which is associated with the  $\mathbb{N}_L$  inference rule is

$$\frac{\mathcal{B}, \mathbb{N}(n)_n^l \mid \mathcal{T} \xrightarrow{tab} \mathcal{B}, \mathbb{N}(n)_{n-1}^l, \neg(0 \in Y)^{l \cup \{Y\}} \mid \mathcal{B}, \mathbb{N}(n)_{n-1}^l, \neg(\mathfrak{H}(Y))^{l \cup \{Y\}} \mid \mathcal{B}, \mathbb{N}(n)_{n-1}^l, (n \in Y)^{l \cup \{Y\}} \mid \mathcal{T}}{\mathcal{B}, \mathbb{N}(n)_n^l \mid \mathcal{T}}$$

It corresponds to the first step of the proof in the example of Section 2, except that a first-order variable  $Y$  replaces the first-order constant  $\tilde{P}$  which relates to  $\text{sum}(x) = x * x$ . However  $Y$  may be instantiated to  $\tilde{P}$  through the extended narrowing rule (see [31] or Appendix C)

$$\frac{\mathcal{B}, U \mid \mathcal{T} \quad [C]}{\mathcal{T}' \mid \mathcal{T} \quad [C \cup \{U_{|w} \stackrel{?}{=} l\}]} \begin{cases} l \rightarrow_1 r \\ U_{|w} \text{ is an atomic proposition} \\ \mathcal{T}' = \text{Tab}(\mathcal{B}, U[r]_w) \end{cases}$$

where  $U_{|w}$  denotes the subformula of  $U$  at some position  $w$  and  $U[r]_w$  denotes  $U$  where this subformula at this position is replaced by  $r$ . We can use this TaMeD rule to instantiate  $Y$  to  $\tilde{P}$  in any  $t \in Y$  which may appear, using the fact that  $t \in \tilde{P} \rightarrow_1 P(t)$ . The generated constraint  $(U_{|w} \stackrel{?}{=} l)$  is then  $t \in Y \stackrel{?}{=} t \in \tilde{P}$  which will always be equivalent to  $Y \stackrel{?}{=} \tilde{P}$ .

## 5 Conclusion

Superdeduction modulo is the combination of superdeduction and deduction modulo with both inference rules systematically derived from an axiomatic theory and the ability to conduct deduction modulo computation. First we proved

that superdeduction modulo is sound and complete with respect to predicate logic. As a main result we proved that cut-free deduction modulo can be translated into cut-free superdeduction modulo using permutations of the applications of the inference rules, provided that some hypothesis on the synchrony of the superdeduction axioms are verified. We showed how this translation corresponds to a structuring of the proof. The inverse translation being trivial, we acquired as a corollary that cut-elimination for superdeduction modulo is equivalent to cut-elimination for deduction modulo, consequently obtaining that the numerous criteria for cut-elimination in deduction modulo also hold for superdeduction modulo. Finally we proposed a tableau method for superdeduction modulo which is sound and complete provided that the corresponding deduction system enjoys cut-elimination.

Superdeduction modulo is a promising framework for proof engineering. Our tableau method may be in particular the foundation of an automated theorem prover based on superdeduction modulo. Besides let us notice that superdeduction modulo is already the core of a small proof assistant named Lemuridæ which can be downloaded at <http://rho.loria.fr/lemuridae.html>. Superdeduction modulo is also used in [23] in a restricted manner since modulo is only used on first-order terms (and *not* first-order propositions). Its expressiveness is nevertheless demonstrated by an encoding of functional PTS.

Filling the gap between superdeduction and deduction modulo has also been done in a related approach by Brauner and Dowek [32]. Whereas we proved the equivalence of cut-elimination for superdeduction modulo and deduction modulo, they proved the equivalence of normalisation for deduction modulo and supernatural deduction. Superdeduction and supernatural deduction stand for the same paradigm applied to classical sequent calculus for the first and to intuitionistic natural deduction for the second. However in this latter system, permutability problems forbid the paradigm to handle disjunctions or existential quantifications. Therefore an interesting extension would be to apply the approach of [32] to (classical sequent calculus) superdeduction.

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## A Proofs of theorems and lemmas

**Lemma 1** *If  $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ , then  $\Gamma \vdash_{\equiv_{1,2}} \Delta$ . If  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$ , then  $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$ .*

*Proof.* For some  $R : P \rightarrow_2 \varphi \in \mathcal{Th}_2$  the superdeduction step  $\mathbb{R}_R \frac{H_1 \ H_2 \ \dots \ H_n}{\Gamma \vdash_{\equiv_1}^{+2} P, \Delta}$  can be translated by definition into

$$\frac{\frac{H_1 \ H_2 \ \dots \ H_n}{[\text{Calc}]}}{\frac{\Gamma \vdash_{\equiv_{1,2}} \varphi, \Delta}{\Gamma \vdash_{\equiv_{1,2}} P, \Delta}} P \equiv_2 \varphi$$

and similarly for superdeduction steps on the left.

In the rest of this appendix, we use the second presentation of deduction modulo (*i.e.* not the  $\equiv_{1R}/\equiv_{1L}$  one).

**Definition 5** *A proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$  only rewrites atomic predicates if when using rules such as*

$$\wedge_R \frac{\Gamma \vdash \psi_1, \Delta \quad \Gamma \vdash \psi_2, \Delta}{\Gamma \vdash \varphi, \Delta} \varphi \equiv_1 \psi_1 \wedge \psi_2 \quad \wedge_L \frac{\Gamma, \psi_1, \psi_2 \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \varphi \equiv_1 \psi_1 \wedge \psi_2$$

*$\varphi$  is either an atomic predicate or syntactically equal to  $\psi_1 \wedge \psi_2$ , and similarly for all the rules of the cut-free deduction system, except axioms.*

**Lemma 7** *Any cut-free proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$  can be turned into a cut-free proof of the same sequent which only rewrites atomic predicates.*

*Proof.* By induction on the proof:

**axiom** the proof already only rewrites atomic predicates.

**conjunction on the right** the proof is

$$\wedge_R \frac{\Gamma \Vdash_{\equiv_1}^{cf+2} \psi_1, \Delta \quad \Gamma \Vdash_{\equiv_1}^{cf+2} \psi_2, \Delta}{\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta} \varphi \equiv_1 \psi_1 \wedge \psi_2$$

Since  $\varphi \equiv_1 \psi_1 \wedge \psi_2$ , since  $\mathcal{Th}_1$  is confluent and since  $\mathcal{Th}_1$  only rewrites atomic predicates, either  $\varphi = \varphi_1 \wedge \varphi_2$  with  $\varphi_i \equiv_1 \psi_i$ , either  $\varphi$  is atomic.

- If  $\varphi$  is atomic, then by induction hypothesis, we obtain proofs of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \psi_1, \Delta$  and of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \psi_2, \Delta$  which only rewrites atomic predicates and through  $\wedge_R$  we get a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$  which only rewrites atomic predicates.
- If  $\varphi = \varphi_1 \wedge \varphi_2$  with  $\varphi_i \equiv_1 \psi_i$ , then a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \psi_i, \Delta$  is also a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi_i, \Delta$ . Then by induction hypothesis we obtain proofs of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi_i, \Delta$  which only rewrite atomic predicates and we get through  $\wedge_R$  a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$  which only rewrites atomic predicates.

**other cases** are handled the same way.

**Definition 6**  $\Gamma', \Delta'$  is fully decomposed in some proof of  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$  if this proof only rewrites atomic predicates and when applying AX-rules in this proof, no logical connector of  $\Gamma', \Delta'$  remain (they have been fully decomposed).

Let us notice that if a proof only rewrites atomic predicates, the only way a logical connector of  $\Gamma', \Delta'$  can disappear is if it is decomposed by some rule of LK.

Let us first prove the following auxiliary lemma.

**Lemma 8** A proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta$  can be transformed into a proof of the same sequent where  $\Gamma, \Delta$  is fully decomposed.

*Proof.* First by Lemma 7, we can suppose that the proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta$  only rewrites atomic predicates. Then we proceed by induction on the multiset containing the number of logical connectors of  $\Gamma, \Delta$  that remains in an AX-rule, for each AX-rule of the proof.

- If no logical connector of  $\Gamma, \Delta$  remain, the proof is fully decomposed.
- If a logical connector of  $\Gamma, \Delta$  appears in a principal formula of an AX-rule, for instance a conjunction on the right:

$$\text{AX} \frac{}{\Gamma', \varphi \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1 \wedge \psi_2, \Delta'} \varphi \equiv_1 \psi_1 \wedge \psi_2$$

Since  $\mathcal{Th}_1$  only rewrites atomic predicates and is confluent, either  $\varphi$  is atomic, either  $\varphi = \varphi_1 \wedge \varphi_2$  with  $\varphi_i \equiv_1 \psi_i$ .

**In the first case** the corresponding subproof can be turned into

$$\wedge_R \frac{\text{AX} \frac{}{\Gamma', \psi_1, \psi_2, \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1, \Delta'} \quad \text{AX} \frac{}{\Gamma', \psi_1, \psi_2, \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_2, \Delta'}}{\Gamma', \psi_1, \psi_2 \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1 \wedge \psi_2, \Delta'} \left\{ \begin{array}{l} \varphi \equiv_1 \psi_1 \wedge \psi_2 \\ \varphi \text{ atomic} \end{array} \right.$$

Since  $\varphi$  is atomic, the modified proof still only rewrites atomic predicates. Besides the logical connectors of  $\Gamma, \Delta$  which remains in  $\Gamma', \psi_1, \psi_2 \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_i, \Delta'$  for some  $i \in \{1, 2\}$  are those of  $\Gamma', \Delta'$  and  $\psi_i$ . (The occurrence of  $\psi_1, \psi_2$  on the left-hand side must not be taken into account since it comes from the decomposition of  $\varphi$  which is atomic.) These logical connectors consequently already appeared in the AX-rule which we are replacing, and which also contains an additionnal logical connector of  $\Gamma, \Delta$  (the  $\wedge$  we are decomposing). Therefore the induction hypothesis may be applied on the global proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta$  obtaining consequently a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta$  which fully decomposes  $\Gamma, \Delta$ .

**In the second case** ( $\varphi = \varphi_1 \wedge \varphi_2$ ), the corresponding subproof can be turned into

$$\wedge_R \frac{\text{AX} \frac{}{\Gamma', \varphi_1, \varphi_2 \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1, \Delta'} \varphi_1 \equiv_1 \psi_1 \quad \text{AX} \frac{}{\Gamma', \varphi_1, \varphi_2 \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_2, \Delta'} \varphi_2 \equiv_1 \psi_2}{\wedge_L \frac{\Gamma', \varphi_1, \varphi_2 \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1 \wedge \psi_2, \Delta'}{\Gamma', \varphi \Vdash_{\equiv_1}^{\mathcal{E}f+2} \psi_1 \wedge \psi_2, \Delta'}}$$

The modified proof still only rewrites atomic predicates. Besides there are fewer logical connectors of  $\Gamma, \Delta$  in a sequent  $\Gamma', \varphi_1, \varphi_2 \vdash_{\equiv_1}^{cf+2} \psi_i$  for some  $i \in \{1, 2\}$  than in  $\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi_1 \wedge \psi_2, \Delta'$ . Therefore the induction hypothesis may be applied on the global proof of  $\Gamma \vdash_{\equiv_1}^{cf+2} \Delta$  obtaining consequently a proof of  $\Gamma \vdash_{\equiv_1}^{cf+2} \Delta$  which fully decomposes  $\Gamma, \Delta$ .

- If a logical connector appears in a non-principal formula of an AX-rule, for instance a disjunction on the right:

$$\text{Ax} \frac{}{\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_1 \vee \phi_2, \Delta'} \varphi \equiv_1 \psi$$

Then the corresponding subproof can be turned into

$$\text{Ax} \frac{}{\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_1, \Delta'} \varphi \equiv_1 \psi \quad \text{Ax} \frac{}{\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_2, \Delta'} \varphi \equiv_1 \psi \\ \text{v}_R \frac{}{\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_1 \vee \phi_2, \Delta'}$$

The modified proof still only rewrites atomic predicates. Besides there are fewer logical connectors of  $\Gamma, \Delta$  in a sequent  $\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_i, \Delta'$  for some  $i \in \{1, 2\}$  than in  $\Gamma', \varphi \vdash_{\equiv_1}^{cf+2} \psi, \phi_1 \vee \phi_2, \Delta'$ . Therefore the induction hypothesis may be applied on the global proof of  $\Gamma \vdash_{\equiv_1}^{cf+2} \Delta$  obtaining consequently a proof of  $\Gamma \vdash_{\equiv_1}^{cf+2} \Delta$  which fully decomposes  $\Gamma, \Delta$ .

- Definition 7** – A partial decomposition of a right-handed formula  $\forall x_1 \dots x_n. \varphi$  is a formula  $\forall x_k \dots x_n. \varphi \sigma$  for some substitution  $\sigma$  on the variables  $x_1 \dots x_{k-1}$ .  
 – A partial decomposition of a left-handed formula  $\exists x_1 \dots x_n. \varphi$  is a formula  $\exists x_k \dots x_n. \varphi \sigma$  for some substitution  $\sigma$  on the variables  $x_1 \dots x_{k-1}$ .

**Definition 8** A right-handed formula  $\varphi = \forall x_1 \dots x_n. \phi$  is non-contracted in a proof of  $\Gamma, \varphi \dots \varphi \vdash_{\equiv_1}^{cf+2} \Delta$  if no contraction is done on some formula  $\forall x_k \dots x_n. \phi \sigma$  coming from the partial decomposition of  $\varphi$ . A left-handed formula  $\varphi = \exists x_1 \dots x_n. \phi$  is non-contracted in a proof of  $\Gamma \vdash_{\equiv_1}^{cf+2} \varphi \dots \varphi, \Delta$  if no contraction is done on some formula  $\exists x_k \dots x_n. \phi \sigma$  coming from the partial decomposition of  $\varphi$ .

**Lemma 9** Any proof of  $\Gamma, \varphi_1 \dots \varphi_n \vdash_{\equiv_1}^{cf+2} \Delta$  where  $\varphi_i$  is right-handed and fully decomposed can be transformed into a proof of  $\Gamma, \varphi_1 \dots \varphi_1, \varphi_2 \dots \varphi_2 \dots \varphi_n \vdash_{\equiv_1}^{cf+2} \Delta$  where each  $\varphi_i$  is fully decomposed and non-contracted.

The same holds for left-handed formulæ on the right.

*Proof.* The proof is conducted by induction on the proof of  $\Gamma, \varphi_1 \dots \varphi_n \vdash_{\equiv_1}^{cf+2} \Delta$ .

**axiom** on  $\varphi_i$ :

$$\text{Ax} \frac{}{\Gamma, \varphi_1 \dots \varphi_n \vdash_{\equiv_1}^{cf+2} \psi, \Delta'} \varphi_i \equiv_1 \psi$$

The  $\varphi_k$  are already non-contracted in the proof.

**axiom** on  $\Gamma, \Delta$ :

$$\text{Ax} \frac{}{\Gamma', \psi_1, \varphi_1 \dots \varphi_n \vdash_{\equiv_1}^{cf+2} \psi_2, \Delta} \psi_1 \equiv_1 \psi_2$$

The  $\varphi_k$  is already non-contracted in the proof.

**contraction** on some formula of  $\Gamma$  or  $\Delta$ : We just apply the induction hypothesis.

**contraction** on some  $\varphi_i$  ( $i = 1$  without loss of generality):

$$\frac{\Gamma, \varphi_1, \varphi_1 \dots \varphi_n, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \varphi_1 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta}$$

By induction hypothesis applied on the proof of  $\Gamma, \varphi_1, \varphi_1 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta$ , we obtain the proof where each  $\varphi_i$  is fully decomposed and non-contracted.

**decomposition** of some formula of  $\Gamma, \Delta$ : we just apply the induction hypothesis on the premisses.

**decomposition** of some  $\varphi_i = \forall x_1 \dots x_n. \phi$  ( $i = 1$  without loss of generality):

$$\frac{\Gamma, \forall x_2 \dots x_n. \phi[t/x_1], \varphi_2 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \varphi_1, \varphi_2 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta} \forall_L$$

Then by induction hypothesis, we obtain a proof of  $\Gamma, \forall x_2 \dots x_n. \phi[t/x_1] \dots \forall x_2 \dots x_n. \phi[t/x_1], \varphi_2 \dots \varphi_2 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta$  where  $\forall x_2 \dots x_n. \phi[t/x]$  and each  $\varphi_i$  ( $i > 1$ ) is fully decomposed and non-contracted. Then using several  $\forall_L$  we obtain a proof of  $\Gamma, \varphi_1 \dots \varphi_1, \varphi_2 \dots \varphi_2 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta$ .

**decomposition** of some propositional  $\varphi_i$  ( $i = 1$  without loss of generality): it is sufficient to apply the induction on each premiss, obtaining proofs where each  $\varphi_i$  ( $i > 1$ ) is non-contracted and fully decomposed. Then using the same inference rule we obtain a proof of  $\Gamma, \varphi_1 \dots \varphi_1, \varphi_2 \dots \varphi_2 \dots \varphi_n \Vdash_{\equiv_1}^{cf+2} \Delta$  where each  $\varphi_i$  is non-contracted and fully decomposed.

Let us prove Lemma 3.

### Lemma 3

Let  $A_1 \equiv_1 A_2 \equiv_1 \dots A_n \equiv_1 \varphi$  be some propositions. Let  $\Theta = A_1, A_2 \dots A_n$ .

- If  $\varphi = \neg A$  and  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$ .
- If  $\varphi = A \wedge B$  and  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, A, B \Vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \vee B$  and  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$  and  $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \Rightarrow B$  and  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$ .
- If  $\varphi = \exists x. Q$  and  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$  then  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$  for some fresh variable  $c$ .
- If  $\varphi = \neg A$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$ .
- If  $\varphi = A \wedge B$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} B, \Delta$ .
- If  $\varphi = A \vee B$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \Vdash_{\equiv_1}^{cf+2} A, B, \Delta$ .
- If  $\varphi = A \Rightarrow B$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Theta, \Delta$ , then  $\Gamma, A \Vdash_{\equiv_1}^{cf+2} B, \Delta$ .
- If  $\varphi = \forall x. Q$  and  $\Gamma \Vdash_{\equiv_1}^{cf+2} \Theta, \Delta$  then  $\Gamma \Vdash_{\equiv_1}^{cf+2} Q[c/x], \Delta$  for some fresh variable  $c$ .

*Proof.* Let us prove the first proposition by induction on the proof of  $\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta$ .

If the last step of the proof of  $\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta$  decomposes  $A_i \equiv_1 \neg A$  (we suppose without loss of generality that  $i = 1$ ), then since  $\mathcal{Th}_1$  is confluent (Hypothesis 2) and only rewrites atomic propositions and first-order terms, the decomposition of  $A_1$  is consequently  $\neg_L$ , and we have a proof of  $\Gamma, A_2 \dots A_n \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A', \Delta$  with  $A \equiv_1 A'$ . Therefore by induction hypothesis, we obtain a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, A', \Delta$  and using a contraction, we finally get a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \Delta$  (since  $A' \equiv_1 A$ ).

Otherwise let us consider the inference rule applied at the root of the proof.

**contraction** the contraction may be applied on a formula of  $\Theta$ ,  $\Gamma$  or  $\Delta$ , but in all cases it is sufficient to apply the induction hypothesis on the premiss of the contraction.

**axiom** if the proof is

$$\text{Ax} \frac{}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}$$

then there exists some  $C \in \Gamma$  and  $D \in \Delta$  such that  $C \equiv_1 D$  or some  $E \equiv_1 A_i$  appears in  $\Delta$ . In the first case we can write the proof

$$\text{Ax} \frac{}{\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \Delta}$$

In the second case the proof

$$\text{Ax} \frac{}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} E, \Delta'}$$

where  $\Delta = E, \Delta'$  can be transformed into

$$\frac{\text{Ax} \frac{}{\Gamma, A \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \Delta'}}{\neg_R \frac{}{\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, E, \Delta'}} E \equiv_1 \neg A$$

**implication-right** if the proof is

$$\Rightarrow_R \frac{\dots \frac{}{\Gamma, \Theta, B \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} C, \Delta'}}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \psi, \Delta'}}{\psi \equiv_1 B \Rightarrow C}$$

then by induction hypothesis we obtain a proof of  $\Gamma, B \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, C, \Delta'$  and through  $\Rightarrow_R$  we get a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \psi, \Delta'$ .

**implication-left** if the proof is

$$\Rightarrow_L \frac{\dots \frac{}{\Gamma, \Theta, C \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta} \quad \dots \frac{}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} B, \Delta}}{\Gamma, \Theta, \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}}{\psi \equiv_1 B \Rightarrow C}$$

then by induction hypothesis we obtain proofs of  $\Gamma, C \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \Delta'$  and  $\Gamma \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, B, \Delta'$ . Finally through  $\Rightarrow_L$  we get a proof of  $\Gamma, \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} A, \Delta$ .

**exists-right** if the proof is

$$\exists_R \frac{\overline{\dots} \Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} Q[t/x], \Delta}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \psi, \Delta} \psi \equiv_1 \exists x.Q$$

then by induction hypothesis we obtain a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, Q[t/x], \Delta$  and through  $\exists_R$  we get a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, \psi, \Delta$ .

**exists-left** if the proof is

$$\exists_L \frac{\overline{\dots} \Gamma, \Theta, Q \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta}{\Gamma, \Theta, \psi \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta} \left\{ \begin{array}{l} x \notin \mathcal{FV}(\Gamma, \Theta, \Delta) \\ \psi \equiv_1 \exists x.Q \end{array} \right.$$

This proof holds for any  $x \notin \mathcal{FV}(\Gamma, \Theta, \Delta)$ , especially one such that  $x \notin \mathcal{FV}(A)$ . Then by induction hypothesis we obtain a proof of  $\Gamma, Q \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, \Delta$  and through  $\exists_L$  we get a proof of  $\Gamma, \psi \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, \Delta$ .

**superdeduction-right** if the proof is

$$\text{R}_R \frac{\overline{\dots} \Gamma, \Gamma_1, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta_1, \Delta \quad \overline{\dots} \Gamma, \Gamma_2, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta_2, \Delta \quad \dots \quad \overline{\dots} \Gamma, \Gamma_n, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta_n, \Delta}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} P, \Delta} \mathcal{C}$$

with  $\text{R} : P \rightarrow \varphi \in \mathcal{Th}_2$ . Let us remark that since the variables concerned with the side condition  $\mathcal{C}$  do not appear in  $\mathcal{FV}(\Gamma, \Theta, \Delta)$  (because  $\mathcal{C}$  holds), they can even be chosen such that they do not appear in  $\mathcal{FV}(A)$ . Then by induction hypothesis we obtain proofs of the  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, \Delta_i, \Delta)_i$  for  $1 \leq i \leq n$  and therefore through  $\text{R}_R$  we obtain a proof of  $\Gamma \Vdash_{\equiv_1}^{\mathcal{C}f+2} A, P, \Delta$  (in particular the side condition of this instance of  $\text{R}_R$  is verified).

**other cases** are handled the same way.

Let us prove the fifth proposition by induction on the proof of  $\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .

If the last step of the proof of  $\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$  decomposes  $A_i \equiv_1 \exists x.Q$  (we suppose without loss of generality that  $i = 1$ ), then since  $\mathcal{Th}_1$  is confluent (Hypothesis 2) and only rewrites atomic propositions and first-order terms, the decomposition of  $A_1$  is  $\exists_L$ , and we have a proof of  $\Gamma, Q'[c/x], A_2 \dots A_n \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$  for  $Q \equiv_1 Q'$  and for some fresh variable  $c$ . Therefore by induction hypothesis, we obtain a proof of  $\Gamma, Q'[c/x], Q[d/x] \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .  $c$  and  $d$  being two distinct fresh variables, this latter proof also holds if we replace them by another fresh variable  $e$ . Therefore we obtain a proof of  $\Gamma, Q'[e/x], Q[e/x] \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ . Using a contraction (since  $Q'[e/x] \equiv_1 Q[e/x]$ ), we get a proof of  $\Gamma, Q[e/x] \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .

Otherwise let us consider the inference rule applied at the root of the proof of  $\Gamma, \exists x.Q \Vdash_{\equiv_1}^{\mathcal{C}f+2} \Delta$ .

**contraction** the contraction may be applied on a formula of  $\Theta$ ,  $\Gamma$  or  $\Delta$ , but in all cases it is sufficient to apply the induction hypothesis on the premiss of the contraction.

**axiom** if the proof is

$$\text{Ax} \frac{}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}$$

then there exists  $B \in \Gamma \cup \Theta$  and  $C \in \Delta$  such that  $B \equiv_1 C$ . If  $B \in \Gamma$  then we have a trivial proof of  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta$ . If  $B \in \Theta$  then the proof is

$$\text{Ax} \frac{}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} C, \Delta'}$$

where  $\Delta = C, \Delta'$  and it can be transformed into

$$\exists_R \frac{\text{Ax} \frac{}{\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} Q[c/x], \Delta'}}{\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} C, \Delta'}}{C \equiv_1 \exists x.Q}$$

**implication-right** if the proof is

$$\Rightarrow_R \frac{\dots}{\Gamma, \Theta, B \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} C, \Delta'}}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \psi, \Delta'}} \psi \equiv_1 B \Rightarrow C$$

then by induction hypothesis we obtain a proof of  $\Gamma, Q[c/x], B \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} C, \Delta'$  and through  $\Rightarrow_R$  we get a proof of  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \psi, \Delta'$ .

**implication-left** if the proof is

$$\Rightarrow_L \frac{\dots}{\Gamma, \Theta, C \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta} \quad \frac{\dots}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} B, \Delta'}}{\Gamma, \Theta, \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}} \psi \equiv_1 B \Rightarrow C$$

then by induction hypothesis we obtain proofs of  $\Gamma, Q[c/x], C \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta'$  and  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} B, \Delta'$ . Finally through  $\Rightarrow_L$  we get a proof of  $\Gamma, Q[c/x], \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta$ .

**exists-right** if the proof is

$$\exists_R \frac{\dots}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} P[t/y], \Delta}}{\Gamma, \Theta \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \psi, \Delta}} \psi \equiv_1 \exists y.P$$

then by induction hypothesis we obtain a proof of  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} P[t/y], \Delta$  and through  $\exists_R$  we get a proof of  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \psi, \Delta$ .

**exists-left** if the proof is

$$\exists_L \frac{\dots}{\Gamma, \Theta, P \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}}{\Gamma, \Theta, \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta}} \left\{ \begin{array}{l} y \notin \mathcal{FV}(\Gamma, \Theta, \Delta) \\ \psi \equiv_1 \exists y.P \end{array} \right.$$

This proof holds for any  $y \notin \mathcal{FV}(\Gamma, \Theta, \Delta)$ , especially one such that  $y \notin \mathcal{FV}(Q)$ . Then by induction hypothesis we obtain a proof of  $\Gamma, Q[c/x], P \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta$  and through  $\exists_L$  we get a proof of  $\Gamma, Q[c/x], \psi \Vdash_{\equiv_1}^{\mathcal{L}^{f+2}} \Delta$  ( $c$  being a fresh variable, we may suppose it is not  $y$  and therefore  $y \notin \mathcal{FV}(\Gamma, Q[c/x], \Delta)$ ).

**superdeduction-right** if the proof is

$$\mathbf{R}_R \frac{\overline{\Gamma, \Gamma_1, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta_1, \Delta} \quad \overline{\Gamma, \Gamma_2, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta_2, \Delta} \quad \dots \quad \overline{\Gamma, \Gamma_n, \Theta \Vdash_{\equiv_1}^{cf+2} \Delta_n, \Delta}}{\Gamma, \Theta \Vdash_{\equiv_1}^{cf+2} P, \Delta} \mathcal{C}$$

with  $\mathbf{R} : P \rightarrow \varphi \in \mathcal{Th}_2$ . Let us remark that since the variables concerned with the side condition  $\mathcal{C}$  do not appear in  $\mathcal{FV}(\Gamma, \Theta, \Delta)$  (because  $\mathcal{C}$  holds), they can even be chosen such that they do not appear in  $\mathcal{FV}(Q)$ . Then by induction hypothesis we obtain proofs of the  $(\Gamma, \Gamma_i, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$  for  $1 \leq i \leq n$  and for some fresh variable  $c$  and therefore through  $\mathbf{R}_R$  we obtain a proof of  $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} P, \Delta$  (in particular the side condition of this instance of  $\mathbf{R}_R$  is verified).

**other cases** are handled the same way.

The other propositions are proved the same way.

Then we prove the following lemma.

- Lemma 10** – *For some right-handed  $\phi$ , if there is a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \phi, \Delta$  and  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$  is a list of sequents representing a step of the decomposition using rule of **Calc** of  $\phi$  in  $\Gamma \Vdash_{\equiv_1}^{cf+2} \phi, \Delta$ , then for all  $i$ , there is a proof of  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$ .*
- *For some left-handed  $\phi$ , if there is a proof of  $\Gamma, \phi \Vdash_{\equiv_1}^{cf+2} \Delta$  and  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$  is a list of sequents representing a step of the decomposition using rule of **Calc** of  $\phi$  in  $\Gamma, \phi \Vdash_{\equiv_1}^{cf+2} \Delta$ , then for all  $i$ , there is a proof of  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$ .*

*Proof.* Let us prove the first proposition. By induction on the number of steps from  $\Gamma \Vdash_{\equiv_1}^{cf+2} \phi, \Delta$  to  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$  using rules of **Calc**.

- If the list of sequents contains only  $\Gamma \Vdash_{\equiv_1}^{cf+2} \phi, \Delta$ , by hypothesis there is a proof of  $\Gamma \Vdash_{\equiv_1}^{cf+2} \phi, \Delta$ .
- If the step  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$  is obtained from previous step  $(\Gamma, \Gamma'_k \Vdash_{\equiv_1}^{cf+2} \Delta'_k, \Delta)_k$  using the rule

**axiom** then all the sequents  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$  are already in the list  $(\Gamma, \Gamma'_k \Vdash_{\equiv_1}^{cf+2} \Delta'_k, \Delta)_k$ . Therefore by induction hypothesis, for all  $i$  there exists a proof of  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$ .

**implication left** then all the sequents  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$  are already in the list  $(\Gamma, \Gamma'_k \Vdash_{\equiv_1}^{cf+2} \Delta'_k, \Delta)_k$  except some sequents  $\Gamma, \Gamma', \psi \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$  and  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta', \Delta$  that comes from decomposing  $\varphi \Rightarrow \psi$  in  $\Gamma, \Gamma', \varphi \Rightarrow \psi \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$ . From induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of  $\Gamma, \Gamma', \varphi \Rightarrow \psi \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$ . By Lemma 3, we obtain proofs of  $\Gamma, \Gamma', \psi \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$  and  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta', \Delta$  and consequently we have proofs of all sequents  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)$ .

**implication right** then all the sequents  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{\text{cf}+2} \Delta_i, \Delta$  are already in the list  $(\Gamma, \Gamma'_k \Vdash_{\equiv_1}^{\text{cf}+2} \Delta'_k, \Delta)_k$  except some sequent  $\Gamma, \Gamma', \varphi \Vdash_{\equiv_1}^{\text{cf}+2} \psi, \Delta', \Delta$  that comes from decomposing  $\varphi \Rightarrow \psi$  in  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \varphi \Rightarrow \psi, \Delta', \Delta$ . From induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \varphi \Rightarrow \psi, \Delta', \Delta$ . By Lemma 3, we obtain a proof of  $\Gamma, \Gamma', \varphi \Vdash_{\equiv_1}^{\text{cf}+2} \psi, \Delta', \Delta$  and consequently we have proofs of all sequents  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{\text{cf}+2} \Delta_i, \Delta)$ .

**forall right** then all the sequents  $\Gamma, \Gamma_i \Vdash_{\equiv_1}^{\text{cf}+2} \Delta_i, \Delta$  are already in the list  $(\Gamma, \Gamma'_k \Vdash_{\equiv_1}^{\text{cf}+2} \Delta'_k, \Delta)_k$  except some sequent  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \psi[c/x], \Delta', \Delta$  that comes from decomposing  $\forall x.\psi$  in  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \forall x.\psi, \Delta', \Delta$  where  $c$  is a fresh variable. By induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \forall x.\psi, \Delta', \Delta$ . By Lemma 3, we obtain a proof of  $\Gamma, \Gamma' \Vdash_{\equiv_1}^{\text{cf}+2} \psi[c/x], \Delta', \Delta$  and hence we have proofs of all sequents  $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{\text{cf}+2} \Delta_i, \Delta)$ .

**exists right** This case is contradictory since  $\phi$  is right-handed.

**forall left** This case is contradictory since  $\phi$  is right-handed.

**exists left** This case is contradictory since  $\phi$  is right-handed.

**other cases** are handled the same way.

The second proposition is symmetrical.

Let us prove now Lemma 4.

**Lemma 4** *Let us consider some  $R : P \rightarrow \varphi \in \mathcal{Th}_2$ .*

- *If  $\varphi$  is right-handed and  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} \varphi, \Delta$ , then there exists (cut-free) proofs of each premise of the introduction of  $P$  on the right. Therefore  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} P, \Delta$ .*
- *If  $\varphi$  is left-handed and  $\Gamma, \varphi \Vdash_{\equiv_1}^{\text{cf}+2} \Delta$ , then there exists (cut-free) proofs of each premise of the introduction of  $P$  on the left. Therefore  $\Gamma, P \Vdash_{\equiv_1}^{\text{cf}+2} \Delta$ .*

*Proof.* Let us prove the first proposition: Since there exists a proof of  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} \varphi, \Delta$ , by Lemma 10 and since the premises of the  $R_R$  rule represent a step (the last one) of the decomposition of  $\varphi$  in  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} \varphi, \Delta$ , by Lemma 10 there exists proofs of each premise (where the variables appearing in the side condition of  $R_R$  are replaced by fresh variables: consequently this side condition is verified). Therefore through  $R_R$  we obtain a proof of  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} P, \Delta$ .

The second proposition is proved in a symmetrical manner.

Before proving Lemma 5, we prove this auxiliary lemma:

**Lemma 11** *Let us consider  $R : P \rightarrow \varphi$ .*

- *If  $\varphi$  is right-handed, if  $\varphi'$  is some partial decomposition of  $\varphi$  and if  $\Gamma, \varphi' \Vdash_{\equiv_1}^{\text{cf}+2} \Delta$ , then there is a proof of  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} P, \Delta$ .*
- *If  $\varphi$  is left-handed, if  $\varphi'$  is some partial decomposition of  $\varphi$  and if  $\Gamma \Vdash_{\equiv_1}^{\text{cf}+2} \varphi', \Delta$ , then there is a proof of  $\Gamma, P \Vdash_{\equiv_1}^{\text{cf}+2} \Delta$ .*

*Proof.* Let us prove the first proposition. First by Lemmas 7, 8, 9, we can suppose that  $\varphi'$  is fully decomposed and non-contracted in the proof. Then we proceed by induction on this proof.

**axiom** on  $\varphi'$ : The proof is

$$\text{Ax} \frac{}{\Gamma, \varphi' \vdash_{\equiv_1}^{\text{cf}+2} \psi, \Delta'} \varphi' \equiv_1 \psi$$

Since  $\varphi'$  is fully decomposed in the proof, it is then an atomic predicate ( $= \phi\sigma$  for some  $\sigma$ ). Then the  $\text{R}_L$  rule is

$$\text{R}_L \frac{\Gamma, \phi\rho \vdash \Delta}{\Gamma, P \vdash \Delta}$$

(for any substitution  $\rho$ ) and then the following proof holds.

$$\begin{array}{c} \text{Ax} \frac{}{\Gamma, \phi\sigma \vdash_{\equiv_1}^{\text{cf}+2} \psi, \Delta'} \phi\sigma \equiv_1 \psi \\ \text{R}_L \frac{}{\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \psi, \Delta} \end{array}$$

**axiom** on  $\Gamma, \Delta$ . This case is obvious.

**contraction** on  $\Gamma, \Delta$ . This case is obvious.

**contraction** on  $\varphi'$ . Since  $\varphi'$  is non-contracted in the proof, it can only be a propositional formula  $\phi\sigma$ . Then by Lemma 4, there are proofs of each premise  $\Gamma, \Gamma_i\sigma \vdash_{\equiv_1}^{\text{cf}+2} \Delta_i\sigma, \Delta$  of a decomposition of  $\phi\sigma$  on the left in  $\Gamma, \varphi' \vdash_{\equiv_1}^{\text{cf}+2} \Delta$ . Besides the  $\text{R}_L$  inference rule is

$$\text{R}_L \frac{(\Gamma, \Gamma_i\rho \vdash \Delta_i\rho, \Delta)_i}{\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \Delta}$$

(for any substitution  $\rho$ ). Then through  $\text{R}_L$  we get a proof of  $\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \Delta$ .

**decomposition** of a formula of  $\Gamma, \Delta$ . We simply apply the induction hypothesis on each premise.

**decomposition** of  $\varphi' = \phi\sigma$ : This case is handled the same way as a contraction on  $\varphi' = \phi\sigma$ .

**decomposition** of  $\varphi' = \forall x_k \dots x_n. \phi\sigma$ :

$$\exists_L \frac{\Gamma, \forall x_{k+1} \dots x_n. \phi\sigma' \vdash_{\equiv_1}^{\text{cf}+2} \Delta}{\Gamma, \forall x_k \dots x_n. \phi\sigma \vdash_{\equiv_1}^{\text{cf}+2} \Delta}$$

Then by induction hypothesis, there exists a proof of  $\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \Delta$ .

Since the second proposition is symmetrical, it is proved in a symmetrical manner.

Let us prove now Lemma 5.

**Lemma 5** *Let us consider  $\text{R} : P \rightarrow \varphi \in \text{Th}_2$ .*

- If  $\varphi$  is right-handed and  $\Gamma, \varphi \vdash_{\equiv_1}^{\text{cf}+2} \Delta$ , then  $\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \Delta$ .
- If  $\varphi$  is left-handed and  $\Gamma \vdash_{\equiv_1}^{\text{cf}+2} \varphi, \Delta$ , then  $\Gamma \vdash_{\equiv_1}^{\text{cf}+2} P, \Delta$ .

*Proof.* If  $\varphi$  is right-handed (resp. left-handed), then  $\varphi$  is a partial decomposition of itself and consequently by Lemma 11, if there is a proof of  $\Gamma, \varphi \vdash_{\equiv_1}^{\text{cf}+2} \Delta$  (resp.  $\Gamma \vdash_{\equiv_1}^{\text{cf}+2} \varphi, \Delta$ ), then there is a proof of  $\Gamma, P \vdash_{\equiv_1}^{\text{cf}+2} \Delta$  (resp.  $\Gamma \vdash_{\equiv_1}^{\text{cf}+2} P, \Delta$ ).

## B A simple example

Let us now demonstrate the use of superdeduction modulo through an example. On one hand, deduction modulo is convenient for representing the computational part of some theory. For instance one can define the addition and the multiplication using the following rules.

$$\left\{ \begin{array}{l} 0 + y \rightarrow y \\ S(x) + y \rightarrow S(x + y) \end{array} \right. \quad \left\{ \begin{array}{l} 0 * x \rightarrow 0 \\ S(x) * y \rightarrow y + (y * x) \end{array} \right. \quad (1)$$

Let us also define  $\text{sum}(x) = \sum_{k=0}^{x-1} (2 * k + 1)$  with the rules

$$\text{sum}(0) \rightarrow 0 \quad \text{sum}(S(x)) \rightarrow S(x + (x + \text{sum}(x))) \quad (2)$$

We obtain a convergent rewrite system, which is suitable for representing computation in deduction modulo.

On the other hand, superdeduction is convenient for representing the deductive part of some theory. Let us define natural numbers (and induction) using the rules

$$\left\{ \begin{array}{l} \mathbb{N}(n) \rightarrow \forall P. 0 \in P \Rightarrow \mathfrak{H}(P) \Rightarrow n \in P \\ \mathfrak{H}(P) \rightarrow \forall k. k \in P \Rightarrow S(k) \in P \end{array} \right. \quad (3)$$

Let us also define Leibniz's equality with the rule

$$x = y \rightarrow \forall P. x \in P \Rightarrow y \in P \quad (4)$$

In the three latter rules, we wrote  $x \in P$  instead of  $P(x)$ , for some formula  $P$ . This is allowed if we use (in  $\mathcal{Th}_1$ ) the following axioms coming from the theory of classes

$$x \in \tilde{P} \rightarrow P(x) \quad (5)$$

where  $\tilde{P}$  denotes fresh constants associated with each formula  $P$ . The new inference rules (associated with the rules (3) and (4)) are

$$\begin{array}{c} \mathbb{N}_R \frac{\Gamma, 0 \in P, \mathfrak{H}(P) \vdash_{\equiv_1}^{+2} n \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} \mathbb{N}(n), \Delta} P \notin \mathcal{FV}(\Gamma, \Delta) \\ \mathbb{N}_L \frac{\Gamma \vdash_{\equiv_1}^{+2} 0 \in P, \Delta \quad \Gamma \vdash_{\equiv_1}^{+2} \mathfrak{H}(P), \Delta \quad \Gamma, n \in P \vdash_{\equiv_1}^{+2} \Delta}{\Gamma, \mathbb{N}(n) \vdash_{\equiv_1}^{+2} \Delta} \\ \mathfrak{H}_R \frac{\Gamma, m \in P \vdash_{\equiv_1}^{+2} S(m) \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} \mathfrak{H}(P), \Delta} \quad \mathfrak{H}_L \frac{\Gamma \vdash_{\equiv_1}^{+2} m \in P, \Delta \quad \Gamma, S(m) \in P \vdash_{\equiv_1}^{+2} \Delta}{\Gamma, \mathfrak{H}(P) \vdash_{\equiv_1}^{+2} \Delta} \\ =_R \frac{\Gamma, x \in P \vdash_{\equiv_1}^{+2} y \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} x = y, \Delta} P \notin \mathcal{FV}(\Gamma, \Delta) \quad =_L \frac{\Gamma, y \in P \vdash_{\equiv_1}^{+2} \Delta \quad \Gamma \vdash_{\equiv_1}^{+2} x \in P, \Delta}{\Gamma, x = y \vdash_{\equiv_1}^{+2} \Delta} \end{array}$$

Besides, the system is considered modulo the rules (1), (2) and (5). Then one can easily prove in the system that  $\mathbb{N}(n) \Rightarrow \sum_{k=0}^{n-1} (2 * k + 1) = n^2$ . The proof is

$$\begin{array}{c}
 \text{Ax}(S(m + (m + \text{sum}(m)))) = S(m + (m + (m * m))) \\
 \vdots \\
 \text{Ax}(S(m + (m + \text{sum}(m))) \in A) \\
 \vdots \\
 \text{=}R \frac{\vdash_{\equiv 1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))}{\vdash_{\equiv 1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))} \\
 \text{=}L \frac{\text{sum}(m) = m * m \vdash_{\equiv 1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))}{\text{sum}(m) = m * m \vdash_{\equiv 1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))} \\
 \mathfrak{H}_R \frac{\text{Ax}(0 \in A) \quad \vdash_{\equiv 1}^{+2} \mathfrak{H}(\tilde{P})}{\text{Ax}(0 \in A) \quad \vdash_{\equiv 1}^{+2} \mathfrak{H}(\tilde{P})} \\
 \text{=}R \frac{\vdash_{\equiv 1}^{+2} 0 = 0 \quad \vdots \quad \text{Ax}(\text{sum}(n) = n * n)}{\vdash_{\equiv 1}^{+2} 0 = 0 \quad \vdots \quad \text{Ax}(\text{sum}(n) = n * n)} \\
 \mathbb{N}_L \frac{\text{Ax}(0 \in A) \quad \vdash_{\equiv 1}^{+2} \mathfrak{H}(\tilde{P}) \quad \text{Ax}(\text{sum}(n) = n * n)}{\mathbb{N}(n) \vdash_{\equiv 1}^{+2} \text{sum}(n) = n * n}
 \end{array}$$

where  $\tilde{P}$  is the first-order constant associated with the formula  $\text{sum}(m) = m * m$  and  $\text{Ax}(\varphi)$  stands for the proof  $\frac{\text{Ax}}{\varphi \vdash_{\equiv 1}^{+2} \varphi}$ . Let us notice that the only inference rules we use are superdeduction rules and Ax rules.

## C TaMeD

*Tableau form transformation*  $\text{Tab}(T)$

$$\begin{array}{ll}
 \mathcal{B}, (\varphi_1 \vee \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l \mid \mathcal{B}, \varphi_2^l \mid \mathcal{T} & \mathcal{B}, \neg \perp^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B} \mid \mathcal{T} \\
 \mathcal{B}, \neg(\varphi_1 \wedge \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \neg \varphi_1^l \mid \mathcal{B}, \neg \varphi_2^l \mid \mathcal{T} & \mathcal{B}, \top^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B} \mid \mathcal{T} \\
 \mathcal{B}, (\varphi_1 \wedge \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l, \varphi_2^l \mid \mathcal{T} & \mathcal{B}, \neg \neg \varphi^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \varphi^l \mid \mathcal{T} \\
 \mathcal{B}, \neg(\varphi_1 \vee \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \neg \varphi_1^l, \neg \varphi_2^l \mid \mathcal{T} & \mathcal{B}, \perp^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{T} \\
 \mathcal{B}, (\varphi_1 \Rightarrow \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \neg \varphi_1^l \mid \mathcal{B}, \varphi_2^l \mid \mathcal{T} & \mathcal{B}, \neg \top^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{T} \\
 \mathcal{B}, \neg(\varphi_1 \Rightarrow \varphi_2)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l, \neg \varphi_2^l \mid \mathcal{T} & \\
 \mathcal{B}, (\forall x.P)^l_{n_x} \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, (\forall x.P)^l_{n_x-1}, P^{l,x} \mid \mathcal{T} & (x \text{ fresh and } n_x > 1) \\
 \mathcal{B}, \neg(\exists x.P)^l_{n_x} \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, \neg(\exists x.P)^l_{n_x-1}, \neg P^{l,x} \mid \mathcal{T} & (x \text{ fresh and } n_x > 1) \\
 \mathcal{B}, (\exists x.P)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, (P[f(l)/x])^l \mid \mathcal{T} & (f \text{ fresh skolem symbol}) \\
 \mathcal{B}, \neg(\forall x.P)^l \mid \mathcal{T} \xrightarrow{\text{tab}} \mathcal{B}, (\neg P[f(l)/x])^l \mid \mathcal{T} & (f \text{ fresh skolem symbol})
 \end{array}$$

$$\text{TaMeD rules} \quad \frac{\mathcal{B}, \varphi_1, \neg \varphi_2 \mid \mathcal{T} \quad [C]}{\mathcal{T} \quad [C \cup \{\varphi_1 \stackrel{?}{=} \varphi_2\}]} \quad \frac{\mathcal{B}, U \mid \mathcal{T} \quad [C]}{\mathcal{T}' \mid \mathcal{T} \quad [C \cup \{U_{|w} \stackrel{?}{=} l\}]}$$

if  $l \rightarrow_1 r$ ,  $U_{|w}$  is an atomic proposition and  $\mathcal{T}' = \text{Tab}(\mathcal{B}, U[r]_w)$