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Axiom directed Focusing

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Abstract

This paper presents a highly configurable deduction system for predicate logic called superdeduction modulo, resulting of the combination of deduction modulo and superdeduction which are two orthogonal presentations of predicate logic. It enables the user to make a distinct use of computational and reasoning axioms in a sound and complete manner. We show that by using permutations of inference rules, any cut-free proof in deduction modulo can be transformed into a cut-free proof in superdeduction modulo and conversely. As a corollary we obtain that cut-elimination for deduction modulo (which is well-studied) is equivalent to cut-elimination for superdeduction modulo provided that some hypotheses on the synchrony of reasoning axioms are verified. Finally we propose a tableau method based on superdeduction modulo which is sound and complete provided that cut-elimination holds.

1. Introduction

Computer science and mathematics are both concerned with the construction of formal proofs. These proofs are used as certificates for assertions, theorems or programs since once they have been constructed, they can be easily communicated, replayed and understood. Therefore formal proofs are the highest criterion for trusting a piece of software. The construction of proofs usually relies on the definition of some proof system which contains both a discipline which the user has to follow, and deductive as well as some computing abilities which empower the user. The proof system may then be used to backup automated theorem provers such as Simplify, Harvey, Zenon or implemented in interac-

tive proof assistants like Coq, Isabelle, PVS or HOL that work as a framework for the proof engineering process.

In this context, a proof is usually constructed with respect to a theory (a set of axioms) which is plugged into the deduction process. Indeed the deductive power of a deduction system comes from its association with a theory. In frameworks such as first-order natural deduction or sequent calculus, the use of some theory is always uniform since the proofs only express atomic steps which correspond to decompositions of logical connectors. Higher-level notions such as sets, set inclusion, natural numbers or addition have to be encoded in the first-order language and handled through these atomic inference steps, consequently leading often to long, hardly-readable and redundant “assembly like” proofs.

Several paradigms propose *ad hoc* systems for specific theories. For example Definitional Reflection [25], extended with induction in [30] is especially related to logic programming. Another point of view is Huang’s Assertion level [29] mainly motivated by the presentation of machine found proofs in natural language.

The approach we will use in this paper is related to Focusing introduced by Andreoli in [2] which was meant to remove irrelevant choices in backward reasoning for sequent calculus: indeed syntactically different proofs can still be identical up to some permutations or simplifications of the applications of the inference rules. This approach was also more recently adapted to forward reasoning in [12]. Our point of view is that usually when one wishes to reason with an axiom (such as one used in the definition of natural numbers), in sequent calculus for instance, it is necessary to destruct the logical connectors of the axiom. These steps are typically always identical, at least *up to permutations or simplifications of the applications of the inference rules*. An interesting way to deal with this is superdeduction, introduced in [7], which allows to use the deductive part of some theory to enrich the deductive system, obtaining

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thus a custom-made system devoted to the theory. Typically one may define the object $\{a, b, c\}$ by stating that $x \in \{a, b, c\}$ if and only if $x = a$ or $x = b$ or $x = c$. It is translated in a first-order language by the axiom

$$\forall a. \forall b. \forall c. \quad x \in \{a, b, c\} \Leftrightarrow (x = a \vee x = b \vee x = c)$$

Instead of using this axiom directly in natural deduction or in sequent calculus, superdeduction proposes to construct new deduction rules representing the deductive power that it brings to the system. In classical sequent calculus, these are

$$\text{TRIO}_R \frac{\Gamma \vdash t = a, t = b, t = c, \Delta}{\Gamma \vdash t \in \{a, b, c\}, \Delta}$$

$$\text{TRIO}_L \frac{\Gamma, t = a \vdash \Delta \quad \Gamma, t = b \vdash \Delta \quad \Gamma, t = c \vdash \Delta}{\Gamma, t \in \{a, b, c\} \vdash \Delta}$$

These inference rules roughly correspond to the decomposition of the logical connectors of $t = a \vee t = b \vee t = c$, which is crushed into a single focusing step. Besides and similarly to Prawitz's folding/unfolding [32], the inference rules directly construct proofs of $t \in \{a, b, c\}$. They translate exactly the reasonings that one can make about trios:

- to prove that some object t is in $\{a, b, c\}$, one must prove that it is equal to a , to b or to c ;
- to prove something (Δ) from $t \in \{a, b, c\}$, one must prove it for $t = a$, for $t = b$ and for $t = c$.

This approach which is equivalent to the corresponding first-order axiom reveals the reasoning that is included inside the definition of pairs. Hence superdeduction allows to use the reasoning part of the theory to customize the deduction system. Superdeduction is based on the seminal work of Benjamin Wack in [34] on supernatural deduction which was introduced in order to provide a logical interpretation to the rewriting calculus [13, 14]. Based on Urban's work on the classical sequent calculus [33], a proof-term language for superdeduction is proposed in [7] along with a cut-elimination procedure whose normalisation is proved in [8] under appropriate hypotheses. The expressivity of superdeduction is promising. For instance an encoding of Pure Type Systems in superdeduction modulo is shown in [10].

In this paper superdeduction is combined with a dual approach which is also designed to structure the use of some first-order theory: deduction modulo, which was introduced ten years ago by Dowek, Hardin and Kirchner [20]. It is a paradigm which allows one to transform any formula at any point of some deduction process. Since the transformation is not explicit in the constructed proof, it has to be redone during proofchecking

and therefore cannot be *more than* computation. For instance it is commonly defined as a congruence on formulae which is constructed from a convergent rewrite system [17]. Since the rewrite system is convergent, verifying some formula transformation is decidable. Hence deduction modulo allows to make use of the computational part of a theory Th for true computations modulo which deduction is performed. Deduction modulo also leads to interesting automated theorem proving procedures like ENAR [20] or TaMeD [5]. Let us notice that in general neither strong normalisation nor cut-elimination are ensured for deduction modulo. However criteria have been developed for strong normalisation, using reducibility candidates [23] leading to the notion of truth values algebras and superconsistency [18, 21], and for cut-elimination using semantic methods [27, 28] or abstract completion [11]. Another line of work is to formalize theories of interest, such as Heyting algebra [24, 1], Zermelo's set theory [22] or higher-order logics [19, 16, 15], using rewrite systems so that they can be used in deduction modulo. Finally, in addition to simplicity, deduction modulo admits unbounded proof size speed-up [9].

In this work, we propose to combine these two approaches, as it was already put forward in [7], in the following way. Instead of constructing a proof of $Th \vdash \Delta$ in natural deduction or sequent calculus using the theory Th uniformly, we propose to split Th in three parts $\Gamma \cup Th_1 \cup Th_2$. While Γ will still be used as a context to our deductions, Th_1 will represent computational axioms and will be handled by the deduction modulo paradigm, and Th_2 will represent reasoning axioms and therefore will be handled by the superdeduction paradigm. We obtain then a custom deduction system in which we prove a sequent $\Gamma \vdash_{\equiv_{Th_2}}^{+Th_1} \Delta$.

As already explained in [7], superdeduction modulo may be used as an innovative foundation for new proof assistants. First it allows to naturally encode custom reasoning and computational schemes, such as induction over natural numbers and addition. Furthermore it suggests a convenient representation of proofs in contrast with the proofs that are typically constructed with existing proof-assistants such as Coq or Isabelle. These usually consist in tactics or proof-terms which are bound to convince the user of the correctness of the proof (through a typechecking process) but not actually explain it. Indeed the main steps are flooded with a multitude of usually not explicited and often straightforward logical arguments caused by both the underlying calculus and the presence of purely computational parts. While deduction modulo addresses the issue of compu-

tational parts, superdeduction eliminates trivial logical arguments in a proof.

After recalling the definitions of superdeduction and deduction modulo, we formally define superdeduction modulo in Section 2 and prove its soundness and completeness with respect to predicate logic. In Section 3, we prove the equivalence between the superdeduction approach and the usual use of first-order axioms *in the cut-free setting*: we show that provided that some synchrony hypothesis on \mathcal{Th}_2 is verified, a cut-free proof in deduction modulo ($\vdash_{\equiv \mathcal{Th}_1 \cup \mathcal{Th}_2}$) can be transformed into a cut-free proof in superdeduction modulo ($\vdash_{\equiv \mathcal{Th}_2}^{+\mathcal{Th}_1}$). The converse being obvious, we obtain that cut-elimination for deduction modulo is equivalent to cut-elimination for superdeduction modulo, therefore we get as corollaries that all the cut-elimination theorems that have already been shown for deduction modulo also hold for superdeduction modulo. Finally in Section 4 we propose a sound and complete tableau method for superdeduction modulo, provided that the corresponding deduction system enjoys cut-elimination.

2. Superdeduction Modulo

In this section we formally define superdeduction modulo which is the combination of two different extensions of classical sequent calculus for predicate logic: deduction modulo on one hand and superdeduction on the other hand. Let us recall the classical sequent calculus LK which contains the *deduction core* of our superdeduction modulo systems:

$$\begin{array}{c}
\text{AX} \frac{}{\Gamma, \varphi \vdash \varphi, \Delta} \quad \text{CUT} \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta} \\
\perp_L \frac{}{\Gamma, \perp \vdash \Delta} \quad \top_R \frac{}{\Gamma \vdash \top, \Delta} \\
\wedge_L \frac{\Gamma, \varphi_1, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \vdash \Delta} \quad \wedge_R \frac{\Gamma \vdash \varphi_1, \Delta \quad \Gamma \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \wedge \varphi_2, \Delta} \\
\vee_L \frac{\Gamma, \varphi_1 \vdash \Delta \quad \Gamma, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \vdash \Delta} \quad \vee_R \frac{\Gamma \vdash \varphi_1, \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \vee \varphi_2, \Delta} \\
\Rightarrow_R \frac{\Gamma, \varphi_1 \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi_1 \Rightarrow \varphi_2, \Delta} \quad \Rightarrow_L \frac{\Gamma \vdash \varphi_1, \Delta \quad \Gamma, \varphi_2 \vdash \Delta}{\Gamma, \varphi_1 \Rightarrow \varphi_2 \vdash \Delta} \\
\forall_R \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x. \varphi, \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta) \quad \forall_L \frac{\Gamma, \varphi[t/x] \vdash \Delta}{\Gamma, \forall x. \varphi \vdash \Delta} \\
\exists_R \frac{\Gamma \vdash \varphi[t/x], \Delta}{\Gamma \vdash \exists x. \varphi, \Delta} \quad \exists_L \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \exists x. \varphi \vdash \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta) \\
\text{CONTR}_R \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta} \quad \text{CONTR}_L \frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}
\end{array}$$

A *term rewrite rule* is a rewrite rule that rewrites first-order terms into first-order terms and a *proposition rewrite rule* is a rewrite rule that rewrites an atomic proposition into an arbitrary formula. For instance $\text{plus}(\text{zero}, x) \rightarrow x$ is a term rewrite rule while $a \subseteq b \rightarrow \forall x. x \in a \Rightarrow x \in b$ is a proposition rewrite rule. We will consider two sets of rewrite rules: \mathcal{Th}_1 is a set containing both term and proposition rewrite rules ; \mathcal{Th}_2 is a set of proposition rewrite rules. As \mathcal{Th}_1 will be used to extend the deduction system through deduction modulo, our intuition is that it contains the computational axioms. Dually as \mathcal{Th}_2 will be used to enrich the deduction system through superdeduction, our intuition is that it contains the deductive axioms. We suppose that each rewrite rule of \mathcal{Th}_2 is associated with a name. If $P \rightarrow \varphi$ is associated with R, it is denoted $R : P \rightarrow \varphi$.

For $i \in \{1, 2\}$, the one-step rewrite reduction associated with \mathcal{Th}_i is denoted \rightarrow_i . The transitive closure of \rightarrow_i is denoted \rightarrow_i^+ . The reflexive and transitive closure of \rightarrow_i is denoted \rightarrow_i^* . The symmetric, reflexive and transitive closure of \rightarrow_i is denoted \equiv_i . The notations $\rightarrow_{1,2}$, $\rightarrow_{1,2}^+$, $\rightarrow_{1,2}^*$, and $\equiv_{1,2}$ are used for $\mathcal{Th}_1 \cup \mathcal{Th}_2$.

The first-order axiom associated with a proposition rewrite rule $P \rightarrow \varphi$ is $\forall \bar{x}. (P \Leftrightarrow \varphi)$, where \bar{x} represents the free variables of P and φ , denoted $\mathcal{FV}(P, \varphi)$. The first-order axiom associated with a term rewrite rule $l \rightarrow r$ is $\forall \bar{x}. (l = r)$, where \bar{x} represents the free variables of l and r . When writing $\vdash_{\equiv \mathcal{Th}_1}^{+\mathcal{Th}_2}$, \mathcal{Th}_1 and \mathcal{Th}_2 will represent the rewrite rules, and when writing $\mathcal{Th}_1 \vdash$ or $\mathcal{Th}_2 \vdash$, they will represent the first-order axioms.

For some deduction system \vdash_{\equiv}^* , we will denote $\Gamma \vdash_{\equiv}^* \Delta$ the sentence *there is a proof of $\Gamma \vdash_{\equiv}^* \Delta$* . We will denote $\Gamma \not\vdash_{\equiv}^* \Delta$ the sentence *there is no proof of $\Gamma \vdash_{\equiv}^* \Delta$* . We will denote $\Gamma \vdash_{\equiv}^{cf*} \Delta$ the sequents in the corresponding cut-free deduction system.

First let us recall the definition of deduction modulo.

Definition 1 (Deduction modulo [20]). *The sequent calculus modulo associated with \mathcal{Th}_1 is the classical sequent calculus where each first-order term and each proposition is considered modulo rewriting through \mathcal{Th}_1 . The sequents in such a system are denoted $\Gamma \vdash_{\equiv \mathcal{Th}_1} \Delta$ or simply $\Gamma \vdash_{\equiv_1} \Delta$.*

There are two main presentations for deduction modulo. One can replace the inference rules of LK by rules such as depicted in Figure 1 or one can use them together with the new rules

$$\begin{array}{c}
\equiv_{1R} \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi, \Delta} \quad \psi \equiv_1 \varphi \quad \equiv_{1L} \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \quad \psi \equiv_1 \varphi
\end{array}$$

We will use both alternatively. Deduction modulo has

$$\begin{array}{c}
\text{AX} \frac{}{\Gamma, \varphi \vdash \psi, \Delta} \varphi \equiv_1 \psi \\
\Rightarrow_R \frac{\Gamma, \varphi_1 \vdash \varphi_2, \Delta}{\Gamma \vdash \varphi, \Delta} \varphi \equiv_1 \varphi_1 \Rightarrow \varphi_2 \\
\text{CUT} \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma \vdash \Delta} \varphi \equiv_1 \psi \quad \dots
\end{array}$$

Figure 1. Rules of deduction modulo.

been proved to be sound and complete *w.r.t.* classical predicate logic.

Theorem 1 ([20]). $\Gamma \vdash_{\equiv_1} \Delta$ if and only if $\text{Th}_1, \Gamma \vdash \Delta$.

Now let us recall the definition of superdeduction.

Definition 2 (Superdeduction rules computation [7]). Let *Calc* be a set of rules composed by the subset of the sequent calculus deduction rules formed of AX, \perp_L , \top_R , \forall_L , \forall_R , \wedge_L , \wedge_R , \Rightarrow_L , \Rightarrow_R , \forall_L , \forall_R , \exists_L and \exists_R , as well as of the two following rules \top_L and \perp_R :

$$\top_L \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta} \quad \perp_R \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}$$

Let $R : P \rightarrow \varphi$ be a proposition rewrite rule of Th_2 .

1. To get the right rule associated with R , initialise the procedure with the sequent $\Gamma \vdash \varphi, \Delta$. Next, apply the rules of *Calc* until no more open leaves remain on which they can be applied. Then, collect the premises, the side conditions and the conclusion and replace φ by P to obtain the rule R_R .
2. To get the left rule R_L associated with R , initialise the procedure with the sequent $\Gamma, \varphi \vdash \Delta$. Apply the rules of *Calc* and get the new left rule the same way as for the right one.

For instance, the rules associated with

$\subseteq_{\text{def}} : a \subseteq b \rightarrow \forall x. x \in a \Rightarrow x \in b$ are

$$\begin{array}{c}
\subseteq_{\text{def}R} \frac{\Gamma, x \in a \vdash x \in b, \Delta}{\Gamma \vdash a \subseteq b, \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta) \\
\subseteq_{\text{def}L} \frac{\Gamma, t \in b \vdash \Delta \quad \Gamma \vdash t \in a, \Delta}{\Gamma, a \subseteq b \vdash \Delta}
\end{array}$$

Definition 3 (Superdeduction system [7]). The superdeduction system associated with Th_2 is formed of the rules of classical sequent calculus and the rules built upon Th_2 . The sequents in such a system are written $\Gamma \vdash^{+\text{Th}_2} \Delta$ or simply $\Gamma \vdash^{+2} \Delta$.

Let us notice that in order to obtain a superdeduction system which is sound and complete *w.r.t.* classical predicate logic, Th_2 must contain no permutability problem as defined in [7].

Definition 4 (Polarity of a subformula). The polarity $\text{pol}_\varphi(\psi)$ of ψ in φ where ψ is an occurrence of a subformula of φ is a boolean defined as follows:

- if $\varphi = \psi$, then $\text{pol}_\varphi(\psi) = \text{true}$;
- if $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$, then $\text{pol}_\varphi(\psi) = \text{pol}_{\varphi_1}(\psi)$ if ψ is a subformula of φ_1 , $\text{pol}_{\varphi_2}(\psi)$ otherwise;
- if $\varphi = \forall x. \varphi_1$ or $\exists x. \varphi_1$, then $\text{pol}_\varphi(\psi) = \text{pol}_{\varphi_1}(\psi)$;
- if $\varphi = \varphi_1 \Rightarrow \varphi_2$, then $\text{pol}_\varphi(\psi) = \neg \text{pol}_{\varphi_1}(\psi)$ if ψ is a subformula of φ_1 , $\text{pol}_{\varphi_2}(\psi)$ otherwise.

Definition 5 (Set of permutability problems). A formula ψ is in the set $PP(\varphi)$ of φ permutability problems if there exists φ' a subformula of φ such that ψ is an occurrence of a subformula of φ' and one of these propositions holds:

- $\varphi' = \forall x. \varphi'_1$, $\psi = \forall x. \psi'_1$ and $\text{pol}_{\varphi'}(\psi) = \text{false}$
- $\varphi' = \exists x. \varphi'_1$, $\psi = \exists x. \psi'_1$ and $\text{pol}_{\varphi'}(\psi) = \text{false}$
- $\varphi' = \forall x. \varphi'_1$, $\psi = \exists x. \psi'_1$ and $\text{pol}_{\varphi'}(\psi) = \text{true}$
- $\varphi' = \exists x. \varphi'_1$, $\psi = \forall x. \psi'_1$ and $\text{pol}_{\varphi'}(\psi) = \text{true}$

Completeness of superdeduction only holds if whenever φ occurs in a rewrite rule $R : P \rightarrow \varphi \in \text{Th}_2$, $PP(\varphi)$ is empty. Besides a procedure has been proposed in [7] which transforms any set of proposition rewrite rules into an equivalent one verifying this property. Consequently we suppose that this property holds for Th_2 . Therefore the corresponding superdeduction system is sound and complete *w.r.t.* classical predicate logic.

Theorem 2 ([7]). $\Gamma \vdash^{+2} \Delta$ if and only if $\text{Th}_2, \Gamma \vdash \Delta$.

Now let us define formally superdeduction modulo.

Definition 6 (Superdeduction modulo). The superdeduction modulo system associated with $(\text{Th}_1, \text{Th}_2)$ is the superdeduction system associated with Th_2 where each first-order term or proposition is considered modulo rewriting through Th_1 . Sequents in this system are denoted $\Gamma \vdash_{\equiv_{\text{Th}_1}}^{+\text{Th}_2} \Delta$, or simply $\Gamma \vdash_{\equiv_1}^{+2} \Delta$.

For instance, using the presentation of deduction modulo depicted in Figure 1, the rules associated with \subseteq_{def} become

$$\begin{array}{c}
\subseteq_{\text{def}R} \frac{\Gamma, x \in a \vdash x \in b, \Delta}{\Gamma \vdash \varphi, \Delta} \left\{ \begin{array}{l} x \notin \mathcal{FV}(\Gamma, \Delta) \\ \varphi \equiv_1 a \subseteq b \end{array} \right. \\
\subseteq_{\text{def}L} \frac{\Gamma, t \in b \vdash \Delta \quad \Gamma \vdash t \in a, \Delta}{\Gamma, \varphi \vdash \Delta} \quad \varphi \equiv_1 a \subseteq b
\end{array}$$

It is interesting to notice that the combination of superdeduction and deduction modulo is done in the following order: first the superdeduction inference rules are computed, then they are used modulo a theory. Therefore deduction modulo does not interfere with the computation of the superdeduction rules, but only with their use. However we could have chosen to compute these new inference rules by already using deduction modulo. This computation is deterministic if and only if \mathcal{Th}_1 is confluent: indeed in this case, rewriting through \mathcal{Th}_1 cannot erase the logical connectors of some formula (since it only rewrites *atomic predicates*) and cannot spawn distinct logical connectors at the same position of the formula (since it is confluent). Besides the computation is terminating if and only if \mathcal{Th}_1 is terminating.

The first properties of superdeduction modulo systems that we will prove are its soundness and completeness *w.r.t.* classical predicate logic. Soundness is a direct consequence of the following lemma.

Lemma 1 (From $\vdash_{\equiv_1}^{+2}$ to $\vdash_{\equiv_{1,2}}$).
If $\Gamma \vdash_{\equiv_1}^{+2} \Delta$, then $\Gamma \vdash_{\equiv_{1,2}} \Delta$.
If $\Gamma \vdash_{\equiv_1}^{cf+2} \Delta$, then $\Gamma \vdash_{\equiv_{1,2}}^{cf} \Delta$.

Proof. the superdeduction step

$$R_R \frac{H_1 \quad H_2 \quad \dots \quad H_n}{\Gamma \vdash_{\equiv_1}^{+2} P, \Delta} R : P \rightarrow_2 \varphi$$

can be translated by definition into

$$\frac{\frac{H_1 \quad H_2 \quad \dots \quad H_n}{[LK]} \quad \frac{\Gamma \vdash_{\equiv_{1,2}} \varphi, \Delta}{\Gamma \vdash_{\equiv_{1,2}} P, \Delta}}{P \equiv_2 \varphi}$$

and similarly for superdeduction steps on the left. \square

Theorem 3 (Soundness of superdeduction modulo).

If $\Gamma \vdash_{\equiv_1}^{+2} \Delta$, then $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$.

Proof. Using Lemma 1, the proof of $\Gamma \vdash_{\equiv_1}^{+2} \Delta$ is first translated into a proof of $\Gamma \vdash_{\equiv_{1,2}} \Delta$, then by soundness of deduction modulo, we obtain a proof of $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$. \square

Theorem 4 (Completeness of superdeduction modulo).

If $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$, then $\Gamma \vdash_{\equiv_1}^{+2} \Delta$.

Proof. From the completeness of superdeduction, for all $\varphi \in \mathcal{Th}_2$, there exists a proof of $\vdash_{\equiv_1}^{+2} \varphi$. From the completeness of deduction modulo, for all $\varphi \in \mathcal{Th}_1$, there exists a proof of $\vdash_{\equiv_1} \varphi$. Starting from a proof of $\mathcal{Th}_1, \mathcal{Th}_2, \Gamma \vdash \Delta$, using cuts with the proofs of $\vdash_{\equiv_1}^{+2} \varphi$ for all $\varphi \in \mathcal{Th}_1 \cup \mathcal{Th}_2$, we get a proof of $\Gamma \vdash_{\equiv_1}^{+2} \Delta$. \square

Let us now demonstrate the use of superdeduction modulo through an example. On one hand, deduction modulo is convenient for representing the computational part of some theory. For instance one can define the addition and the multiplication using the following rules.

$$\begin{cases} 0 + y \rightarrow y \\ S(x) + y \rightarrow S(x + y) \\ 0 * x \rightarrow 0 \\ S(x) * y \rightarrow y + (y * x) \end{cases} \quad (1)$$

Let us also define $\text{sum}(x) = \sum_{k=0}^{x-1} (2 * k + 1)$ with the rules

$$\begin{cases} \text{sum}(0) \rightarrow 0 \\ \text{sum}(S(x)) \rightarrow S(x + (x + \text{sum}(x))) \end{cases} \quad (2)$$

We obtain a convergent rewrite system, which is suitable for representing computation in deduction modulo.

On the other hand, superdeduction is convenient for representing the deductive part of some theory. Yet when Poincaré stated the difference between computation and deduction in [31], he took induction on natural numbers as the typical example of a deduction. Let us define natural numbers (and induction) using the rules

$$\begin{cases} \mathbb{N}(n) \rightarrow \forall P. 0 \in P \Rightarrow \mathfrak{H}(P) \Rightarrow n \in P \\ \mathfrak{H}(P) \rightarrow \forall k. k \in P \Rightarrow S(k) \in P \end{cases} \quad (3)$$

Let us also define Leibniz's equality with the rule

$$x = y \rightarrow \forall P. x \in P \Rightarrow y \in P \quad (4)$$

In the three latter rules, we wrote $x \in P$ instead of $P(x)$, for some formula P . This is allowed if we use (in \mathcal{Th}_1) the following axioms coming from the theory of classes

$$x \in \tilde{P} \rightarrow P(x) \quad (5)$$

where \tilde{P} denotes fresh constants associated with each formula P . The new inference rules (associated with the rules (3) and (4)) are

$$\begin{aligned} & \mathbb{N}_R \frac{0 \in P, H(P) \vdash_{\equiv_1}^{+2} n \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} \mathbb{N}(n), \Delta} P \notin \mathcal{FV}(\Gamma, \Delta) \\ & \Gamma \vdash_{\equiv_1}^{+2} 0 \in P, \Delta \\ & \vdots \quad \Gamma \vdash_{\equiv_1}^{+2} \mathfrak{H}(P), \Delta \quad \Gamma, n \in P \vdash_{\equiv_1}^{+2} \Delta \\ & \mathbb{N}_L \frac{\Gamma, \mathbb{N}(n) \vdash_{\equiv_1}^{+2} \Delta}{\Gamma, m \in P \vdash_{\equiv_1}^{+2} S(m) \in P, \Delta} \\ & \mathfrak{H}_R \frac{\Gamma, m \in P \vdash_{\equiv_1}^{+2} S(m) \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} \mathfrak{H}(P), \Delta} \\ & \mathfrak{H}_L \frac{\Gamma \vdash_{\equiv_1}^{+2} m \in P, \Delta \quad \Gamma, S(m) \in P \vdash_{\equiv_1}^{+2} \Delta}{\Gamma, \mathfrak{H}(P) \vdash_{\equiv_1}^{+2} \Delta} \\ & =_R \frac{\Gamma, x \in P \vdash_{\equiv_1}^{+2} y \in P, \Delta}{\Gamma \vdash_{\equiv_1}^{+2} x = y, \Delta} P \notin \mathcal{FV}(\Gamma, \Delta) \end{aligned}$$

$$=L \frac{\Gamma, y \in P \vdash_{\equiv_1}^{+2} \Delta \quad \Gamma \vdash_{\equiv_1}^{+2} x \in P, \Delta}{\Gamma, x = y \vdash_{\equiv_1}^{+2} \Delta}$$

Besides, the system is considered modulo the rules (1), (2) and (5). Then one can easily prove in the system that

$$\mathbb{N}(n) \Rightarrow \sum_{k=0}^{n-1} (2 * k + 1) = n^2$$

The proof is depicted in Figure 2. Let us notice that the only inference rules we use are superdeduction rules and Ax rules. The same proof where computational steps are detailed is written in Appendix B.

3. Cut-free setting

Both cut-elimination (completeness of the cut-free deduction system) and normalisation (normalisation of a cut-elimination procedure) are well-studied for deduction modulo. Several proofs and criteria have been presented for cut-elimination and normalisation. Let us cite first the seminal work of Dowek and Werner [23], where the following theorem for classical sequent calculus modulo is proved.

Theorem 5 ([23]). *If the light double negation of the rewrite system \mathcal{R} has a pre-model then the cut rule is redundant in the classical sequent calculus modulo \mathcal{R} .*

Both definitions of light double negation and pre-model can be found in [23]. Dowek also proposed in [18] an extension of Heyting algebras called *truth values algebras* which allows to distinguish provable equivalence from computational equivalence. A theory is said to be *superconsistent* if it has a model in all truth values algebras. Superconsistency of some theory has been proved in [18] and in [21] to imply strong normalisation in the intuitionistic natural deduction modulo the theory. However this criterion does not apply here since we consider classical sequent calculus.

Hermant deeply studied cut-elimination for the intuitionistic and classical sequent calculus particularly using semantic methods in [28, 26, 27]. Among others, he proved the following cut-elimination theorem for classical sequent calculus modulo.

Theorem 6 ([27]). *If \mathcal{R} is a rewrite system compatible with a well-founded order, if \mathcal{R}^+ is a positive rewrite system whose right-hand sides are \mathcal{R} -normal forms and if $\mathcal{R} \cup \mathcal{R}^+$ is confluent, then cut-elimination holds for the classical sequent calculus modulo $\mathcal{R} \cup \mathcal{R}^+$.*

The definition of a *positive* rewrite system can be found in [27]. Bonichon and Hermant also developed

a constructive proof of cut-elimination for the intuitionistic sequent calculus modulo in [4].

Finally Burel and Kirchner proposed another approach in [11] where they use abstract canonical systems and abstract completion in order to mechanically transform a classical sequent calculus system into an equivalent one having the cut-elimination property.

Theorem 7 ([11]). *A sequent has a proof in classical sequent calculus modulo some theory \mathcal{R} if and only if it has a cut-free proof in the saturated theory corresponding to \mathcal{R} .*

Our aim is now to relate superdeduction modulo to deduction modulo in order to prove that all these criteria are extendible to superdeduction modulo. We will only consider cut-free deduction systems and show that they allow to prove exactly the same sequents: a sequent $\Gamma \vdash_{\equiv_{\mathcal{T}h_1}}^{cf+Th_2} \Delta$ is provable if and only if $\Gamma \vdash_{\equiv_{\mathcal{T}h_1 \cup \mathcal{T}h_2}}^{cf} \Delta$ is. The translation of a proof in superdeduction modulo into sheer deduction modulo is already shown in Lemma 1. Now let us prove the converse: if $\Gamma \vdash_{\equiv_{\mathcal{T}h_1 \cup \mathcal{T}h_2}}^{cf} \Delta$, then $\Gamma \vdash_{\equiv_{\mathcal{T}h_1}}^{cf+Th_2} \Delta$. To achieve this goal, we will first consider some proposition rewrite rule $R : P \rightarrow \varphi$ of $\mathcal{T}h_2$ and prove that

- if $\Gamma \vdash_{\equiv_1}^{cf+2} \varphi, \Delta$, then $\Gamma \vdash_{\equiv_1}^{cf+2} P, \Delta$;
- if $\Gamma, \varphi \vdash_{\equiv_1}^{cf+2} \Delta$, then $\Gamma, P \vdash_{\equiv_1}^{cf+2} \Delta$.

Then from a proof

$$\frac{\Gamma \vdash_{\equiv_{1,2}}^{cf} \varphi, \Delta}{\Gamma \vdash_{\equiv_{1,2}}^{cf} P, \Delta} \varphi \equiv_2 P$$

we will construct (by induction hypothesis) a proof of $\Gamma \vdash_{\equiv_1}^{cf+2} \varphi, \Delta$ and using the above property we will obtain a proof of $\Gamma \vdash_{\equiv_1}^{cf+2} P, \Delta$ (and similarly for proving P on the left). The proof of the property deals with permutability problems in classical sequent calculus: indeed the intuition is to unite the steps of the proof that decompose φ . There are four cases for which two steps in superdeduction modulo cannot be permuted:

$$\begin{aligned} \exists_R & \frac{\Gamma \vdash_{\equiv_1}^{cf+2} P(x), Q(x), \Delta}{\Gamma \vdash_{\equiv_1}^{cf+2} P(x), \psi, \Delta} \psi \equiv_1 \exists x.Q(x) \\ \forall_R & \frac{\Gamma \vdash_{\equiv_1}^{cf+2} P(x), \psi, \Delta}{\Gamma \vdash_{\equiv_1}^{cf+2} \varphi, \psi, \Delta} \varphi \equiv_1 \forall x.P(x) \end{aligned}$$

and similarly where \forall_R is replaced by \exists_L and/or \exists_R is replaced by \forall_L . Consequently we can easily build a proof of some sequent $\Gamma \vdash_{\equiv_1}^{+2} \varphi, \Delta$ where the steps de-

$$\begin{array}{c}
\text{AX} \frac{S(m + (m + \text{sum}(m))) = S(m + (m + (m * m))) \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + (m * m)))}{\vdots} \\
\text{AX} \frac{S(m + (m + \text{sum}(m))) \in A \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) \in A}{\vdots} \\
\text{=}_R \frac{\vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))}{\vdots} \\
\text{=}_L \frac{\text{sum}(m) = m * m \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + (m * m)))}{\text{AX} \frac{\text{sum}(n) = n * n \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n}{\text{=}_R \frac{0 \in A \vdash_{\equiv_1}^{+2} 0 \in A}{\text{NL} \frac{\vdash_{\equiv_1}^{+2} 0 = 0}{\text{NL}}}}} \\
\text{NL} \frac{\vdash_{\equiv_1}^{+2} 0 = 0}{\text{NL}} \quad \text{AX} \frac{\text{sum}(n) = n * n \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n}{\text{=}_R} \quad \text{NL} \frac{\vdash_{\equiv_1}^{+2} 0 = 0}{\text{NL}} \\
\text{NL}(n) \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n
\end{array}$$

where \tilde{P} is the first-order constant associated with the formula $\text{sum}(m) = m * m$.

Figure 2. A proof of $\mathbb{N}(n) \implies \sum_{k=0}^{n-1} (2 * k - 1) = n^2$.

composing φ cannot be united using sheer permutations:

$$\begin{array}{c}
\vdots \\
\forall_L \frac{P(y_0) \Rightarrow Q(x_0), P(y_0) \vdash Q(x_0)}{\exists_L \frac{P(y_0) \Rightarrow Q(x_0), (\forall x.P(x)) \vdash Q(x_0)}{\forall_L \frac{\exists y.(P(y) \Rightarrow Q(x_0)), (\forall x.P(x)) \vdash Q(x_0)}{\forall_L \frac{\forall x.\exists y.(P(y) \Rightarrow Q(x)), (\forall x.P(x)) \vdash Q(x_0)}{\forall_R \frac{\forall x.\exists y.(P(y) \Rightarrow Q(x)), (\forall x.P(x)) \vdash (\forall x.Q(x))}{\Rightarrow_R \frac{\forall x.\exists y.(P(y) \Rightarrow Q(x)) \vdash (\forall x.P(x)) \Rightarrow (\forall x.Q(x))}}}}}
\end{array}$$

where φ is $(\forall x.P(x)) \Rightarrow (\forall x.Q(x))$. A solution is to combine the permutations with contractions as follows.

$$\begin{array}{c}
\vdots \\
\forall_R \frac{P(y_0) \Rightarrow Q(x_0), P(x_0), P(y_0) \vdash Q(x_1), Q(x_0)}{\forall_L \frac{P(y_0) \Rightarrow Q(x_0), P(x_0), P(y_0) \vdash \forall x.Q(x), Q(x_0)}{\Rightarrow_R \frac{P(y_0) \Rightarrow Q(x_0), P(x_0), \forall x.P(x) \vdash \forall x.Q(x), Q(x_0)}{\exists_L \frac{P(y_0) \Rightarrow Q(x_0), P(x_0) \vdash \varphi, Q(x_0)}{\forall_L \frac{\exists y.P(y) \Rightarrow Q(x_0), P(x_0) \vdash \varphi, Q(x_0)}{\forall_L \frac{\forall x.\exists y.P(y) \Rightarrow Q(x), P(x_0) \vdash \varphi, Q(x_0)}{\forall_R \frac{\forall x.\exists y.P(y) \Rightarrow Q(x), \forall x.P(x) \vdash \varphi, Q(x_0)}{\Rightarrow_R \frac{\forall x.\exists y.P(y) \Rightarrow Q(x) \vdash \varphi, (\forall x.P(x)) \Rightarrow (\forall x.Q(x))}{\text{CONTR}_R \frac{\forall x.\exists y.(P(y) \Rightarrow Q(x)) \vdash (\forall x.P(x)) \Rightarrow (\forall x.Q(x))}}}}}
\end{array}$$

In this proof, all the steps decomposing $(\forall x.P(x)) \Rightarrow (\forall x.Q(x))$ have been duplicated and then united. It seems that this manipulation always allows to transform a proof using several LK inference rules to decompose a formula φ into a proof using a single superdeduction inference rule decomposing the corresponding predicate

P . However proving it remains an open problem to our knowledge. We choose to avoid this kind of manipulation by making the following hypothesis.

Definition 7. A formula which does not contain existential formulæ in positive position or universal formulæ in negative position is said to be right-handed. A formula which does not contain existential formulæ in negative position or universal formulæ in positive position is said to be left-handed. These two definitions are extended to contexts in the obvious way.

For example $\forall x.P(x)$ and $(\exists x.P(x)) \Rightarrow (\forall y.Q(y))$ are right-handed, $\exists x.P(x)$ and $(\forall x.P(x)) \Rightarrow (\exists y.Q(y))$ are left-handed and $(\forall x.P(x)) \Rightarrow (\forall y.Q(y))$ and $(\exists x.P(x)) \Rightarrow (\exists y.Q(y))$ are neither right-handed nor left-handed

Hypothesis 1. If $P \rightarrow \varphi \in Th_2$, then φ is either left-handed or right-handed.

We do not know however if the final result holds without it. Besides, we will only consider the case where φ is right-handed, since the other case is symmetrical. Let us also remark that a set of proposition rewrite rules can always be transformed into an equivalent one which satisfies Hypothesis 1 in the same manner in which permutability problems are avoided in [7]. We also suppose that Th_1 is confluent.

Hypothesis 2. Th_1 is confluent.

Dealing with right-handed formulæ on the right needs the following lemma, adapted from a central lemma in Hermant's semantic proofs of cut-elimination for deduction modulo [27].

Lemma 2 (Kleene's lemma adapted to $\vdash_{\equiv_1}^{+2}$).

- If $\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$.
- If $\Gamma, A \wedge B \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma, A, B \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma, A \vee B \Vdash_{\equiv_1}^{cf+2} \Delta$ then
 $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$ and $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma, A \Rightarrow B \Vdash_{\equiv_1}^{cf+2} \Delta$ then
 $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$.
- If $\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \Delta$ then
 $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$ for some fresh variable c .
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} \neg A, \Delta$ then $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \wedge B, \Delta$ then
 $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \vee B, \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} A, B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \Rightarrow B, \Delta$, then $\Gamma, A \Vdash_{\equiv_1}^{cf+2} B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} \forall x.Q, \Delta$ then
 $\Gamma \Vdash_{\equiv_1}^{cf+2} Q[c/x], \Delta$ for some fresh variable c .

Proof. For each assertion, The proof is done by induction on the derivation and detailed in Appendix A. \square

It implies the following lemma.

Lemma 3. *If $R : P \rightarrow \varphi \in \mathcal{Th}_2$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$, then there exists (cut-free) proofs of each premise of the introduction of P on the right. Therefore $\Gamma \Vdash_{\equiv_1}^{cf+2} P, \Delta$.*

Proof. By iteration of Lemma 2. The proof is detailed in Appendix A. The idea is to unite the decomposition of φ at the root of the proof: all these steps can move downward since φ is right-handed and decomposed on the right. The proof is detailed in Appendix A. \square

Now let us deal with right-handed formulæ on the left. We need the following lemma, which is in some sense dual to Lemma 2.

Lemma 4. *Let us suppose that Γ' is right-handed and that Δ' is left-handed. Then a proof of $\Gamma, \Gamma' \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$ can be transformed into a proof of the same sequent where either no formula of Γ, Δ is decomposed, either Γ' and Δ' are not decomposed at the head of the proof.*

Proof. The proof is conducted by induction on the derivation and detailed in Appendix A. \square

It implies the following lemma.

Lemma 5. *If $R : P \rightarrow \varphi \in \mathcal{Th}_2$ and $\Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta$, then $\Gamma, P \Vdash_{\equiv_1}^{cf+2} \Delta$.*

Proof. By iteration of Lemma 4. The idea is to unite the decomposition of φ at the leaves of the proof: all these steps can move upward since φ is right-handed and decomposed on the left. \square

Lemmas 3 and 5 are concentrated in the following theorem, which proves that rewriting through $\mathcal{Th}_1 \cup \mathcal{Th}_2$ preserves provability in superdeduction modulo.

Theorem 8. *If $\varphi \equiv_{1,2} \psi$, then*

$$\begin{array}{l} \Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta \quad \text{if and only if} \quad \Gamma \Vdash_{\equiv_1}^{cf+2} \psi, \Delta \\ \Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta \quad \text{if and only if} \quad \Gamma, \psi \Vdash_{\equiv_1}^{cf+2} \Delta \end{array}$$

Proof. First by Lemmas 3 and 5, provability is preserved by one-step head reduction through \mathcal{Th}_2 (i.e. if $\varphi = P\sigma$ for some substitution σ and $\psi = \phi\sigma$ and $P \rightarrow \phi \in \mathcal{Th}_2$). By induction on φ , we prove that it extends to any one-step reduction ($\varphi \rightarrow_2 \psi$). Then by induction on $\varphi \equiv_{1,2} \psi$, we obtain the final result. \square

Lemma 6 (From $\Vdash_{\equiv_{1,2}}^{cf}$ to $\Vdash_{\equiv_1}^{cf+2}$). *If $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$, then $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$.*

Proof. By induction on the proof of $\Gamma \Vdash_{\equiv_{1,2}}^{cf} \Delta$: The raw deductive steps and the steps modulo \mathcal{Th}_1 are unchanged. A step

$$\frac{\Gamma \Vdash_{\equiv_{1,2}}^{cf} \psi, \Delta}{\Gamma \Vdash_{\equiv_{1,2}}^{cf} \varphi, \Delta} \psi \equiv_2 \varphi$$

is translated (by induction hypothesis) into a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} \psi, \Delta$ which is translated using Theorem 8 into a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$. \square

Let us notice that the transformation from $\Vdash_{\equiv_{1,2}}^{cf}$ to $\Vdash_{\equiv_1}^{cf+2}$ includes a *structuration* of the proof. Hence the final result may only use superdeduction inferences, as in the proof of Figure 2, therefore forcing a structured use of (superdeduction) axioms. Finally we get the equivalence of deduction modulo and superdeduction modulo:

Theorem 9. $\Gamma \Vdash_{\equiv_{\mathcal{Th}_1 \cup \mathcal{Th}_2}}^{cf} \Delta$ if and only if $\Gamma \Vdash_{\equiv_{\mathcal{Th}_1}}^{cf+\mathcal{Th}_2} \Delta$.

Proof. By Lemmas 1 and 6. \square

As a corollary we directly obtain that cut-elimination holds for deduction modulo $\mathcal{Th}_1 \cup \mathcal{Th}_2$ if and only if it holds for the superdeduction modulo system associated with $(\mathcal{Th}_1, \mathcal{Th}_2)$. In particular it holds if $(\mathcal{Th}_1, \mathcal{Th}_2)$ verifies the criterion of Theorem 5, 6 or 7.

4. A tableau method for superdeduction

An interesting use of the cut-elimination property for superdeduction modulo is to express the system through a sound and complete tableau method. In this section we

choose to extend TaMeD —Tableau Method for Deduction modulo [3, 5]— whose rules are depicted in Appendix C. We use the presentation of TaMeD of [3], where each formula φ is annotated with a set of first-order variables l , denoted φ^l . This set at least contains the free variables of φ . Besides each formula ψ of the form $\forall x.\varphi$ or $\neg(\exists x.\varphi)$ is annotated with an integer n , denoted ψ_n . This integer is the maximal number of decomposition of ψ that are allowed, in order to keep the method terminating.

The extension of TaMeD is done in the following manner. Each superdeduction rule on the left

$$\frac{(\Gamma, \Gamma_i \vdash \Delta_i, \Delta)_{1 \leq i \leq n}}{\Gamma, P \vdash \Delta} X \cap \mathcal{FV}(\Gamma, \Delta) = \emptyset$$

is translated into $\mathcal{B}_1, P_{n_P}^l \mid \mathcal{T} \xrightarrow{tab}$

$$\begin{aligned} & \mathcal{B}_1, P_{n_P-1}^l, \Gamma_1^{l \cup Y} \sigma, \neg \Delta_1^{l \cup Y} \sigma \mid \\ & \dots \mid \mathcal{B}_1, P_{n_P-1}^l, \Gamma_n^{l \cup Y} \sigma, \neg \Delta_n^{l \cup Y} \sigma \mid \mathcal{T} \quad \text{if } n_P > 0 \end{aligned}$$

where Γ_i^l and Δ_i^l stand respectively for Γ_i and Δ_i where each meta-term (such as t in the \exists_R -rule of LK) is replaced by a fresh variable. The set of these fresh variables is denoted Y . Finally σ stands for a substitution replacing each variable x_i of $X = x_1, x_2, \dots$ by a fresh skolem function $f_i(l)$. The construction of the tableau rule corresponding to the superdeduction rules on the right is done in a similar way.

Together with the rules of Appendix C, we obtain a sound and complete tableau method for superdeduction modulo, provided that the superdeduction modulo system enjoys cut-elimination: each use of a superdeduction tableau transformation can be translated into a use of the corresponding superdeduction inference rule and conversely. Since it also holds for the other rules of TaMeD (soundness and completeness of TaMeD [3, 5]), we consequently obtain the equivalence between the superdeduction modulo tableau method and the corresponding cut-free superdeduction modulo system.

Let us show how it operates on the example of Figure 2. The tableau rule which is associated with the \mathbb{N}_L inference rule is

$$\begin{array}{l} \mathcal{B}, \mathbb{N}(n)_n^l \mid \mathcal{T} \xrightarrow{tab} \mathcal{B}, \mathbb{N}(n)_{n-1}^l, \neg(0 \in Y)^{l \cup \{Y\}} \\ \quad \mid \mathcal{B}, \mathbb{N}(n)_{n-1}^l, \neg(\mathfrak{S}(Y))^{l \cup \{Y\}} \\ \quad \mid \mathcal{B}, \mathbb{N}(n)_{n-1}^l, (n \in Y)^{l \cup \{Y\}} \mid \mathcal{T} \end{array}$$

It corresponds to the first step of the proof of Figure 2, except that a first-order variable Y replaces the first-order constant \tilde{P} which relates to $\text{sum}(x) = x * x$. However Y may be instantiated to \tilde{P} through the extended

narrowing rule (see [3] or Appendix C)

$$\frac{\mathcal{B}, U \mid \mathcal{T} [C]}{\mathcal{T}' \mid \mathcal{T} [C \cup \{U|_w \stackrel{?}{=} l\}]} \left\{ \begin{array}{l} l \rightarrow_1 r \\ U|_w \text{ is an atomic proposition} \\ \mathcal{T}' = \text{Tab}(\mathcal{B}, U[r]_w) \end{array} \right.$$

where $U|_w$ denotes the subformula of U at some position w and $U[r]_w$ denotes U where this subformula at this position is replaced by r . We can use this TaMeD rule to instantiate Y to \tilde{P} in any $t \in Y$ which may appear, using the fact that $t \in \tilde{P} \rightarrow_1 P(t)$. The generated constraint $(U|_w \stackrel{?}{=} l)$ is then $t \in Y \stackrel{?}{=} t \in \tilde{P}$ which will always be equivalent to $Y \stackrel{?}{=} \tilde{P}$.

5. Conclusion

We have formally defined superdeduction modulo, the combination of superdeduction and deduction modulo with both inference rules systematically derived from an axiomatic theory and the ability to conduct deduction modulo computation. Then we proved that the corresponding system is sound and complete with respect to predicate logic. We also proved that cut-free deduction modulo can be translated into cut-free superdeduction modulo using permutations of the applications of the inference rules, provided that some hypothesis on the synchrony of the superdeduction axioms are verified. We show how this translation corresponds to a structuration of the proof. The inverse translation being trivial, we acquired as a corollary that cut-elimination for superdeduction modulo is equivalent to cut-elimination for deduction modulo, consequently obtaining that the numerous criteria for cut-elimination in deduction modulo also hold for superdeduction modulo. Finally we proposed a tableau method for superdeduction modulo which is sound and complete provided that the corresponding deduction system enjoys cut-elimination.

Superdeduction modulo is a promising framework for proof engineering. Our tableau method may be in particular the foundation of an automated theorem prover based on superdeduction modulo. Besides let us notice that superdeduction modulo is already the core of a small proof assistant named Lemuridæ which can be downloaded at <http://rho.loria.fr/lemuridae.html>.

Superdeduction modulo is also already used in [10] in a restricted manner since modulo is only used on first-order terms (and *not* first-order propositions). Its expressivity is nevertheless demonstrated since an encoding of functional PTS is proposed.

Filling the gap between superdeduction and deduction modulo has also be done in a related approach

by Brauner and Dowek [6]. Whereas we proved the equivalence of cut-elimination for superdeduction modulo and deduction modulo, they proved the equivalence of normalisation for deduction modulo and supernatural deduction. Superdeduction and supernatural deduction stand for the same paradigm applied to classical sequent calculus for the first and to intuitionistic natural deduction for the second. However in this latter system, permutability problems forbid the paradigm to handle disjunctions or existential quantifications. Therefore an interesting extension would be to apply the approach of [6] to (classical sequent calculus) superdeduction.

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A. Proofs of theorems and lemmas

Definition 8. Γ', Δ' is fully decomposed in some proof of $\Gamma, \Gamma' \Vdash_{\equiv_1}^{cf+2} \Delta', \Delta$ if when applying Ax-rules in this proof, no logical connector of Γ', Δ' remain (they have been fully decomposed).

Let us first prove the auxiliary lemma.

Lemma 7. A proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$ can be transformed into a proof of the same sequent where Γ, Δ is fully decomposed.

Proof. Let us consider a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} \Delta$. The proof is conducted by induction on the proof, the only interesting case being the Ax-rule. For instance an axiom on a conjunction

$$\text{Ax} \frac{}{\Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} \psi, \Delta} \varphi \equiv_1 \psi \equiv_1 \varphi_1 \wedge \varphi_2$$

can be turned into

$$\begin{array}{c} \text{Ax} \frac{}{\Gamma, \varphi_1, \varphi_2, \Vdash_{\equiv_1}^{cf+2} \varphi_2, \Delta} \\ \vdots \\ \text{Ax} \frac{}{\Gamma, \varphi_1, \varphi_2, \Vdash_{\equiv_1}^{cf+2} \varphi_1, \Delta} \\ \wedge_R \frac{}{\Gamma, \varphi_1, \varphi_2 \Vdash_{\equiv_1}^{cf+2} \psi, \Delta} \psi \equiv_1 \varphi_1 \wedge \varphi_2 \\ \wedge_L \frac{}{\Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} \psi, \Delta} \varphi \equiv_1 \varphi_1 \wedge \varphi_2 \end{array}$$

and similarly for other connectors. \square

Let us prove Lemma 2

Lemma 2. • If $\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$.

- If $\Gamma, A \wedge B \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma, A, B \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma, A \vee B \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$ and $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma, A \Rightarrow B \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma, B \Vdash_{\equiv_1}^{cf+2} \Delta$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$.
- If $\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \Delta$ then $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$ for some fresh variable c .
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} \neg A, \Delta$ then $\Gamma, A \Vdash_{\equiv_1}^{cf+2} \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \wedge B, \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \vee B, \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} A, B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} A \Rightarrow B, \Delta$, then $\Gamma, A \Vdash_{\equiv_1}^{cf+2} B, \Delta$.
- If $\Gamma \Vdash_{\equiv_1}^{cf+2} \forall x.Q, \Delta$ then $\Gamma \Vdash_{\equiv_1}^{cf+2} Q[c/x], \Delta$ for some fresh variable c .

Proof. Let us prove the first proposition.

If the last step of the proof of $\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta$ introduces $\varphi \equiv_1 \neg A$, then φ is not an atomic predicate, and since Th_1 is confluent (Hypothesis 2) and only rewrites

atomic propositions and first-order terms, $\varphi = \neg\psi$ with $\psi \equiv_1 A$. The decomposition of φ is consequently \neg_L , and we have a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} \psi, \Delta$ (which is also a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$ since $A \equiv_1 \psi$).

Otherwise let us proceed with an induction on the proof of $\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta$:

axiom if the proof is

$$\text{Ax} \frac{}{\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta}$$

then there exists some $\varphi \in \Gamma$ and $\psi \in \Delta$ such that $\psi \equiv_1 \varphi$ or some $\phi \equiv_1 \neg A$ appears in Δ . In the first case we can write the proof

$$\text{Ax} \frac{}{\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta}$$

In the second case the proof

$$\text{Ax} \frac{}{\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \phi, \Delta'}$$

where $\Delta = \phi, \Delta'$ can be transformed into

$$\begin{array}{c} \text{Ax} \frac{}{\Gamma, A \Vdash_{\equiv_1}^{cf+2} A, \Delta'} \\ \neg_R \frac{}{\Gamma \Vdash_{\equiv_1}^{cf+2} A, \phi, \Delta'} \phi \equiv_1 \neg A \end{array}$$

implication-right if the proof is

$$\begin{array}{c} \dots \\ \Rightarrow_R \frac{}{\Gamma, \neg A, B \Vdash_{\equiv_1}^{cf+2} C, \Delta'} \varphi \equiv_1 B \Rightarrow C \\ \Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta' \end{array}$$

then by induction hypothesis we obtain a proof of $\Gamma, B \Vdash_{\equiv_1}^{cf+2} A, C, \Delta'$ and through \Rightarrow_R we get a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \varphi, \Delta'$.

implication-left if the proof is

$$\begin{array}{c} \dots \\ \Rightarrow_L \frac{}{\Gamma, \neg A, C \Vdash_{\equiv_1}^{cf+2} \Delta} \\ \Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} B, \Delta \\ \Gamma, \neg A, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta \end{array} \varphi \equiv_1 B \Rightarrow C$$

then by induction hypothesis we obtain proofs of $\Gamma, C \Vdash_{\equiv_1}^{cf+2} A, \Delta'$ and $\Gamma \Vdash_{\equiv_1}^{cf+2} A, B, \varphi, \Delta'$. Finally through \Rightarrow_L we get a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \Delta$.

exists-right if the proof is

$$\begin{array}{c} \dots \\ \exists_R \frac{}{\Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} Q[t/x], \Delta} \varphi \equiv_1 \exists x.Q \\ \Gamma, \neg A \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta \end{array}$$

then by induction hypothesis we obtain a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, Q[t/x], \Delta$ and through \exists_R we get a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, \varphi, \Delta$.

exists-left if the proof is

$$\exists_L \frac{\dots}{\frac{\Gamma, \neg A, Q \Vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \neg A, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta}} \left\{ \begin{array}{l} x \notin \mathcal{FV}(\Gamma, \neg A, \Delta) \\ \varphi \equiv_1 \exists x.Q \end{array} \right.$$

then by induction hypothesis we obtain a proof of $\Gamma, Q \Vdash_{\equiv_1}^{cf+2} A, \Delta$ and through \exists_L we get a proof of $\Gamma, \varphi \Vdash_{\equiv_1}^{cf+2} A, \Delta$.

superdeduction-right if the proof is

$$\begin{array}{c} \dots \\ \hline \Gamma, \Gamma_n, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta_n, \Delta \\ \vdots \\ \dots \\ \hline \Gamma, \Gamma_2, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta_2, \Delta \\ \vdots \\ \dots \\ \hline \Gamma, \Gamma_1, \neg A \Vdash_{\equiv_1}^{cf+2} \Delta_1, \Delta \\ \vdots \\ \dots \\ \hline R_R \frac{\dots}{\Gamma \neg A \Vdash_{\equiv_1}^{cf+2} P, \Delta} \mathcal{C} \end{array}$$

with $R : P \rightarrow \varphi \in \mathcal{Th}_2$ then by induction hypothesis we obtain proofs of the $(\Gamma, \Gamma_i \Vdash_{\equiv_1}^{cf+2} A, \Delta_i, \Delta)_i$ for $1 \leq i \leq n$ and therefore through R_R we obtain a proof of $\Gamma \Vdash_{\equiv_1}^{cf+2} A, P, \Delta$ (in particular the side condition \mathcal{C} is verified since $\mathcal{FV}(\neg A) = \mathcal{FV}(A)$).

other cases are handled the same way.

Let us prove the fifth assumption.

If the proof of $\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \Delta$ introduces $\varphi \equiv_1 \exists x.Q$, then φ is not an atomic predicate, and since \mathcal{Th}_1 is confluent (Hypothesis 2) and only rewrites atomic propositions and first-order terms, $\varphi = \exists x.\psi$ with $\psi \equiv_1 Q$. The decomposition of φ is then \exists_L , and we have a proof of $\Gamma, \psi[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$ (which is also a proof of $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$) since $Q \equiv_1 \psi$ for some fresh variable c .

Otherwise let us proceed with an induction on the proof of $\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \Delta$.

axiom if the proof is

$$\text{Ax} \frac{\dots}{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \Delta}$$

then there exists $\varphi \in \Gamma$ and $\psi \in \Delta$ such that $\varphi \equiv_1 \psi$ or some $\phi \equiv_1 \exists x.Q$ appears in Δ . In the first case we have a trivial proof of $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \Delta$. In the second case the proof is

$$\text{Ax} \frac{\dots}{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \phi, \Delta'} \phi \equiv_1 \exists x.Q$$

where $\Delta = \varphi, \Delta'$ and it can be transformed into

$$\exists_R \frac{\text{Ax} \frac{\dots}{\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} Q[c/x], \Delta'}}{\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \phi, \Delta'} \phi \equiv_1 \exists x.Q$$

implication-right if the proof is

$$\Rightarrow_R \frac{\dots}{\frac{\Gamma, \exists x.Q, B \Vdash_{\equiv_1}^{cf+2} C, \Delta'}{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta'} \varphi \equiv_1 B \Rightarrow C}$$

then by induction hypothesis we obtain a proof of $\Gamma, Q[c/x], B \Vdash_{\equiv_1}^{cf+2} C, \Delta'$ and through \Rightarrow_R we get a proof of $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta'$.

implication-left if the proof is

$$\Rightarrow_L \frac{\dots}{\frac{\Gamma, \exists x.Q, C \Vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} B, \Delta} \varphi \equiv_1 B \Rightarrow C}$$

then by induction hypothesis we obtain proofs of $\Gamma, Q[c/x], C \Vdash_{\equiv_1}^{cf+2} \Delta'$ and $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} B, \varphi, \Delta'$. Finally through \Rightarrow_L we get a proof of $\Gamma, Q[c/x], \varphi \Vdash_{\equiv_1}^{cf+2} \Delta$.

exists-right if the proof is

$$\exists_R \frac{\dots}{\frac{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} P[t/y], \Delta}{\Gamma, \exists x.Q \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta} \varphi \equiv_1 \exists y.P}$$

then by induction hypothesis we obtain a proof of $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} P[t/y], \Delta$ and through \exists_R we get a proof of $\Gamma, Q[c/x] \Vdash_{\equiv_1}^{cf+2} \varphi, \Delta$.

exists-left if the proof is

$$\exists_L \frac{\dots}{\frac{\Gamma, \exists x.Q, P \Vdash_{\equiv_1}^{cf+2} \Delta}{\Gamma, \exists x.Q, \varphi \Vdash_{\equiv_1}^{cf+2} \Delta} \left\{ \begin{array}{l} y \notin \mathcal{FV}(\Gamma, \exists x.Q, \Delta) \\ \varphi \equiv_1 \exists y.P \end{array} \right.}$$

then by induction hypothesis we obtain a proof of $\Gamma, Q[c/x], P \vdash_{\equiv_1}^{cf+2} \Delta$ and through \exists_L we get a proof of $\Gamma, Q[c/x], \varphi \vdash_{\equiv_1}^{cf+2} \Delta$ (c being a fresh variable, we may suppose it is not y and therefore $y \notin \mathcal{FV}(\Gamma, Q[c/x], \Delta)$).

superdeduction-right if the proof is

$$\begin{array}{c}
 \dots \\
 \hline
 \Gamma, \Gamma_n, \exists \vdash_{\equiv_1}^{cf+2} \Delta_n, \Delta \\
 \vdots \\
 \dots \\
 \hline
 \Gamma, \Gamma_2, \neg A \vdash_{\equiv_1}^{cf+2} \Delta_2, \Delta \\
 \vdots \\
 \dots \\
 \hline
 \Gamma, \Gamma_1, \neg A \vdash_{\equiv_1}^{cf+2} \Delta_1, \Delta \\
 \vdots \\
 \dots \\
 \hline
 R_R \frac{\Gamma, \exists x.Q \vdash_{\equiv_1}^{cf+2} P, \Delta}{\Gamma, \exists x.Q \vdash_{\equiv_1}^{cf+2} P, \Delta} \mathcal{C}
 \end{array}$$

with $R : P \rightarrow \varphi \in \mathcal{Th}_2$ then by induction hypothesis we obtain proofs of the $(\Gamma, \Gamma_i, Q[c/x] \vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$ for $1 \leq i \leq n$ and therefore through R_R we obtain a proof of $\Gamma, Q[c/x] \vdash_{\equiv_1}^{cf+2} P, \Delta$ (in particular the side condition \mathcal{C} is verified since c is a fresh variable and consequently we may suppose that it is different from all the variables appearing in \mathcal{C}).

other cases are handled the same way.

Other assumptions are proved the same way. \square

Before proving Lemma 3, we prove the following lemma.

Lemma 8. *If the list of sequents $(\Gamma_i \vdash \Delta_i)_i$ is a step of the decomposition of $\vdash \phi$ where ϕ is right-handed, then for all i , Δ_i is right-handed and Γ_i is left-handed.*

Proof. By routine inspection of the rules that may be used to decompose $\vdash \phi$:

Base case The sequent $\vdash \phi$ obviously verifies the condition : its left-hand context is left-handed and its right-hand context is right-handed.

Axiom The rule

$$\text{Ax} \frac{}{\Gamma, A \vdash A, \Delta}$$

has no premise and therefore preserves the conditions on the sequents

ImPLY-left Let us consider the rule

$$\Rightarrow_L \frac{\Gamma, B \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}$$

If Γ and $A \Rightarrow B$ are left-handed and Δ is right-handed, then B is left-handed and A is right-handed. Consequently both sequents $\Gamma, B \vdash \Delta$ and $\Gamma \vdash A, \Delta$ verify the conditions.

ImPLY-right Let us consider the rule

$$\Rightarrow_R \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}$$

If Γ is left-handed and $A \Rightarrow B, \Delta$ is right-handed, then B is right-handed and A is left-handed. Therefore the sequent $\Gamma, A \vdash B, \Delta$ verify the condition.

Exists-left Let us consider the rule

$$\exists_L \frac{\Gamma, Q \vdash \Delta}{\Gamma, \exists x.Q \vdash \Delta} \quad x \notin \mathcal{FV}(\Gamma, \Delta)$$

If $\Gamma, \exists x.Q$ is left-handed and Δ is right-handed, then Q is left-handed and therefore the sequent $\Gamma, Q \vdash \Delta$ verify the condition.

Exists-right Let us consider the rule

$$\exists_R \frac{\Gamma \vdash Q[t/x], \Delta}{\Gamma \vdash \exists x.Q, \Delta}$$

The formula $\exists x.Q$ cannot be right-handed and consequently this case is impossible.

Other cases are handled the same way. \square

Then we prove the following lemma.

Lemma 9. *If there is a proof of $\Gamma \vdash_{\equiv_1}^{cf+2} \phi, \Delta$ and $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$ is a list of sequents representing a step of the decomposition of ϕ in $\Gamma \vdash_{\equiv_1}^{cf+2} \phi, \Delta$, then for all i , there is a proof of $\Gamma, \Gamma_i \vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta$.*

Proof. By induction on the number of logical connectors of ϕ that are yet decomposed in $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$.

- If none of the logical connectors of ϕ is decomposed, then the list of sequents contains only $\Gamma \vdash_{\equiv_1}^{cf+2} \phi, \Delta$ and therefore by hypothesis there is a proof of $\Gamma \vdash_{\equiv_1}^{cf+2} \phi, \Delta$.
- If the step $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{cf+2} \Delta_i, \Delta)_i$ is obtained from previous step $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{cf+2} \Delta'_k, \Delta)_k$ using the rule

axiom then all the sequents $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$ are already in the list $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta'_k, \Delta)_k$. Therefore by induction hypothesis, for all i there exists a proof of $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$.

implication left then all the sequents $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$ are already in the list $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta'_k, \Delta)_k$ except some sequents $\Gamma, \Gamma', \psi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ and $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta', \Delta$ that comes from decomposing $\phi \equiv_1 \varphi \Rightarrow \psi$ in $\Gamma, \Gamma', \phi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$. From induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of $\Gamma, \Gamma', \phi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ which is also a proof of $\Gamma, \Gamma', \varphi \Rightarrow \psi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ since $\phi \equiv_1 \varphi \Rightarrow \psi$. By Lemma 2, we obtain proofs of $\Gamma, \Gamma', \psi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ and $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta', \Delta$ and consequently we have proofs of all sequents $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta)$.

implication right then all the sequents $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$ are already in the list $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta'_k, \Delta)_k$ except some sequent $\Gamma, \Gamma', \varphi \vdash_{\equiv_1}^{\mathcal{E}f+2} \psi, \Delta', \Delta$ that comes from decomposing $\phi \equiv_1 \varphi \Rightarrow \psi$ in $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \phi, \Delta', \Delta$. From induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \phi, \Delta', \Delta$ which is also a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi \Rightarrow \psi, \Delta', \Delta$ since $\phi \equiv_1 \varphi \Rightarrow \psi$. By Lemma 2, we obtain a proof of $\Gamma, \Gamma', \varphi \vdash_{\equiv_1}^{\mathcal{E}f+2} \psi, \Delta', \Delta$ and consequently we have proofs of all sequents $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta)$.

forall right then all the sequents $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$ are already in the list $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta'_k, \Delta)_k$ except some sequent $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \psi[c/x], \Delta', \Delta$ that comes from decomposing $\varphi \equiv_1 \forall x.\psi$ in $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta', \Delta$ where c is a fresh variable. By induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta', \Delta$ which also is a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \forall x.\psi, \Delta', \Delta$ since $\varphi \equiv_1 \forall x.\psi$. By Lemma 2, we obtain a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \psi[c/x], \Delta', \Delta$ and hence we have proofs of all sequents $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta)$.

exists right this case is contradictory by Lemma 8.

forall left this case is contradictory by Lemma 8.

forall right then all the sequents $\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta$

Δ_i, Δ are already in the list $(\Gamma, \Gamma'_k \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta'_k, \Delta)_k$ except some sequent $\Gamma, \Gamma', \psi[c/x] \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ that comes from decomposing $\varphi \equiv_1 \exists x.\psi$ in $\Gamma, \Gamma', \varphi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ where c is a fresh variable. By induction hypothesis, there exists proofs of all sequents from the previous step. Therefore there exists a proof of $\Gamma, \Gamma', \varphi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ which also is a proof of $\Gamma, \Gamma', \exists x.\psi \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ since $\varphi \equiv_1 \exists x.\psi$. By Lemma 2, we obtain a proof of $\Gamma, \Gamma', \psi[c/x] \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ and hence we have proofs of all sequents $(\Gamma, \Gamma_i \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta_i, \Delta)$.

other cases are handled the same way. \square

Let us prove now Lemma 3.

Lemma 3. *If $R : P \rightarrow \varphi \in Th_2$ and $\Gamma \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta$, then there exists (cut-free) proofs of each premise of the introduction of P on the right. Therefore $\Gamma \vdash_{\equiv_1}^{\mathcal{E}f+2} P, \Delta$.*

Proof. Since there exists a proof of $\Gamma \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta$, by Lemma 9 and since the premises of the R_R rule represent a step (the last one) of the decomposition of φ in $\Gamma \vdash_{\equiv_1}^{\mathcal{E}f+2} \varphi, \Delta$, by Lemma 9 there exists proofs of each premise (where the variables appearing in the side condition of R_R are replaced by fresh variables: consequently this side condition is verified). Therefore through R_R we obtain a proof of $\Gamma \vdash_{\equiv_1}^{\mathcal{E}f+2} P, \Delta$. \square

Let us prove now (an extension of) Lemma 4.

Lemma 4. *Let us suppose that Γ' is right-handed and that Δ' is left-handed. Then a proof of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$ can be transformed into a proof of the same sequent where either no formula of Γ, Δ is decomposed, either Γ' and Δ' are not decomposed at the head of the proof. Besides this transformation preserves the full decomposition of Γ', Δ' (Definition 8). Finally if the proof changes, the maximal number of steps decomposing formulae of Γ, Δ in a path from a decomposition of Γ', Δ' to a leaf is strictly decreasing or during this transformation.*

Proof. The proof is conducted by induction on the derivation of $\Gamma, \Gamma' \vdash_{\equiv_1}^{\mathcal{E}f+2} \Delta', \Delta$. We suppose we are not in the case where no formula of Γ, Δ is decomposed and that Γ', Δ' are decomposed at the head of the proof. This decomposition can be $\neg_L, \wedge_L, \vee_L, \Rightarrow_L, \forall_L$ or $\neg_R, \wedge_R, \vee_R, \Rightarrow_R, \exists_R$. It cannot be \exists_L or \forall_R since Γ' is right-handed and Δ' is left-handed. Therefore we deal with all these cases.

Disjunction on the left Let us assume that the proof is

$$\frac{\dots \quad \frac{\Gamma, \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}{\Gamma, \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\Gamma, \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}$$

with $\phi_1 \vee \phi_2, \Gamma'' = \Gamma'$. By induction hypothesis, there exists a proof of $\Gamma, \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta$ and a proof of $\Gamma, \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta$ where either no formula of Γ, Δ is decomposed, either respectively $\phi_1, \Gamma'', \Delta'$ and $\phi_2, \Gamma'', \Delta'$ are not decomposed at the head of the proof. Besides the maximal number of steps decomposing formulae of Γ, Δ in a path from a decomposition of Γ', Δ' to a leaf is (non-strictly) decreasing during this transformation. These proofs either start with a decomposition of a formula of Γ, Δ , either do not decompose Γ, Δ at all. If both proofs do not start with a decomposition of a formula of Γ, Δ , then we are done since we are therefore in the case where no formula of Γ, Δ is decomposed. Let us assume then that one of these proofs starts with a decomposition of Γ, Δ . In particular the path containing the maximal number of decompositions of Γ, Δ from a decomposition of Γ', Δ' to a leaf which is in the first premise is contained in a proof which starts with a decomposition of Γ, Δ . In fact both proofs can contain such a path. We will assume that only the first proof contains such a path. The case where only the second contains such a path is symmetrical. The case where both contain such a path is handled by combining the two previous cases. Now if only the first one contains such a path, we assume that it starts with a decomposition of $\psi_1 \vee \psi_2$ in $\Gamma = \psi_1 \vee \psi_2, \Gamma'''$ (other connectors are handled similarly). The proof is

$$\frac{\frac{\frac{\dots \quad \frac{\psi_2, \Gamma''', \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}{[\pi_2]} \quad \dots}{\psi_1, \Gamma''', \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1 \vee \psi_2, \Gamma''', \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1 \vee \psi_2, \Gamma''', \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1 \vee \psi_2, \Gamma''', \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}$$

Then this proof can be transformed into

$$\frac{\frac{\frac{\frac{\dots \quad \frac{\psi_1, \psi_1 \vee \psi_2, \Gamma''', \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}{[\pi_3]} \quad \dots}{\psi_1, \psi_1 \vee \psi_2, \Gamma''', \phi_1, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1, \psi_1 \vee \psi_2, \Gamma''', \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1, \psi_1 \vee \psi_2, \Gamma''', \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta} \quad \dots}{\psi_1 \vee \psi_2, \Gamma''', \phi_1 \vee \phi_2, \Gamma'' \mid_{\equiv_1}^{cf+2} \Delta', \Delta}$$

Here the maximal number of steps decomposing formulae of Γ, Δ in a path from a decomposition of Γ', Δ' to a leaf is strictly decreasing during this transformation since this unique path was in π_1 and containing the decomposition of $\psi_1 \vee \psi_2$ and now is still in π_1 (which is unchanged) but does not contain the decomposition of $\psi_1 \vee \psi_2$ anymore.

Finally it is easy to check that the full decomposition of Γ', Δ' is preserved. \square

It implies the following lemma.

Lemma 10. *If ϕ is right-handed, then a proof of $\Gamma, \phi \mid_{\equiv_1}^{cf+2} \Delta$ can be transformed into a proof of the same sequent where all the logical connectors of ϕ are decomposed and either no formula of Γ, Δ is decomposed, either they are entirely decomposed before decomposing ϕ .*

Proof. First of all, by Lemma 7, there exists a proof of $\Gamma, \phi \mid_{\equiv_1}^{cf+2} \Delta$ which fully decomposes ϕ . Let us consider this proof now. By induction on the couple composed by the maximal number of steps decomposing formulae of Γ, Δ in a path from a decomposition of ϕ to a leaf and the size of the proof, ordered lexically: since ϕ is right-handed, by Lemma 4 there exists a proof of $\Gamma, \phi \mid_{\equiv_1}^{cf+2} \Delta$ where all the logical connectors of ϕ are decomposed and either no formula of Γ, Δ is decomposed, either ϕ is not decomposed at the head of the proof. Besides this proof still fully decomposes ϕ . In the first case, we consequently obtain the proof we were looking for. In the

second case, the proof is

$$\frac{\frac{\vdots}{[\pi_1]} \quad \dots \quad \frac{\vdots}{[\pi_n]}}{\Gamma_1, \phi \stackrel{cf+2}{\equiv_1} \Delta_1 \quad \dots \quad \Gamma_n, \phi \stackrel{cf+2}{\equiv_1} \Delta_n} \Gamma, \phi \stackrel{cf+2}{\equiv_1} \Delta$$

Either this proof is equal to the initial one, either Lemma 4 has changed it. In both cases we can apply the induction hypothesis on the proofs of the premises since

- in the first case, the maximal number of steps decomposing formulæ of Γ, Δ in a path from a decomposition of ϕ to a leaf is equal and the size of the proof is strictly decreasing;
- in the second case, the maximal number of steps decomposing formulæ of Γ, Δ in a path from a decomposition of ϕ to a leaf is strictly decreasing.

Therefore by induction hypothesis applied on the proofs π_i for each i , there exists a proof π'_i of $\Gamma_i, \phi \stackrel{cf+2}{\equiv_1} \Delta_i$ where either no formula of Γ_i, Δ_i is decomposed, either they are entirely decomposed before decomposing ϕ . Finally we can construct the proof

$$\frac{\frac{\vdots}{[\pi'_1]} \quad \dots \quad \frac{\vdots}{[\pi'_n]}}{\Gamma_1, \phi, \stackrel{cf+2}{\equiv_1} \Delta_1 \quad \dots \quad \Gamma_n, \phi, \stackrel{cf+2}{\equiv_1} \Delta_n} \Gamma, \phi \stackrel{cf+2}{\equiv_1} \Delta$$

In this proof, all the logical connectors of ϕ are decomposed and Γ and Δ are entirely decomposed before decomposing ϕ . \square

It allows us to prove Lemma 5.

Lemma 5. *If $R : P \rightarrow \varphi \in Th_2$ and $\Gamma, \varphi \stackrel{cf+2}{\equiv_1} \Delta$, then $\Gamma, P \stackrel{cf+2}{\equiv_1} \Delta$.*

Proof. Since $R : P \rightarrow \varphi \in Th_2$, φ is right-handed, by Lemma 10 we obtain a proof of $\Gamma, \varphi \stackrel{cf+2}{\equiv_1} \Delta$ where all the logical connectors of φ are decomposed and either no formula of Γ, Δ is decomposed, either they are entirely decomposed before decomposing φ . The proof is conducted by induction on the decompositions of Γ, Δ .

Let us suppose first that no formula of Γ, Δ is decomposed. Since all the logical connectors of φ are decomposed in the proof, it only consists of all these decompositions. φ is right-handed, therefore we may suppose that the decompositions are contraction-free. Consequently the proof can directly be replaced by one single inference rule R_L : we obtain a proof of $\Gamma, P \stackrel{cf+2}{\equiv_1} \Delta$.

Let us suppose now that some formula of Γ, Δ is decomposed (before decomposing φ). The proof is

$$\frac{\Gamma_1, \varphi \stackrel{cf+2}{\equiv_1} \Delta_1 \quad \dots \quad \Gamma_n, \varphi \stackrel{cf+2}{\equiv_1} \Delta_n}{\Gamma, \varphi \stackrel{cf+2}{\equiv_1} \Delta}$$

By induction hypothesis, we obtain proofs of the $\Gamma_i, P \stackrel{cf+2}{\equiv_1} \Delta_i$ for all $1 \leq i \leq n$. Consequently we have a proof of $\Gamma, P \stackrel{cf+2}{\equiv_1} \Delta$. \square

B. A proof in Superdeduction Modulo

The proof of Figure 2 where computational steps are detailed is written in Figure 3.

C. TaMeD

Tableau form transformation $\text{Tab}(\mathcal{T})$

$$\begin{aligned} \mathcal{B}, (\varphi_1 \vee \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l \mid \mathcal{B}, \varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, \neg(\varphi_1 \wedge \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \neg\varphi_1^l \mid \mathcal{B}, \neg\varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, (\varphi_1 \wedge \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l, \varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, \neg(\varphi_1 \vee \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \neg\varphi_1^l, \neg\varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, (\varphi_1 \Rightarrow \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \neg\varphi_1^l \mid \mathcal{B}, \varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, \neg(\varphi_1 \Rightarrow \varphi_2)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \varphi_1^l, \neg\varphi_2^l \mid \mathcal{T} \\ \mathcal{B}, \neg\perp^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B} \mid \mathcal{T} \\ \mathcal{B}, \top^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B} \mid \mathcal{T} \\ \mathcal{B}, \neg\neg\varphi^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \varphi^l \mid \mathcal{T} \\ \mathcal{B}, \perp^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{T} \\ \mathcal{B}, \neg\top^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{T} \\ \mathcal{B}, (\forall x.P)_{n_x}^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, (\forall x.P)_{n_x-1}^l, P^{l,x} \mid \mathcal{T} \\ &\quad (x \text{ fresh and } n_x > 1) \\ \mathcal{B}, \neg(\exists x.P)_{n_x}^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, \neg(\exists x.P)_{n_x-1}^l, \neg P^{l,x} \mid \mathcal{T} \\ &\quad (x \text{ fresh and } n_x > 1) \\ \mathcal{B}, (\exists x.P)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, (P[f(l)/x])^l \mid \mathcal{T} \\ &\quad (f \text{ fresh skolem symbol}) \\ \mathcal{B}, \neg(\forall x.P)^l \mid \mathcal{T} &\xrightarrow{\text{tab}} \mathcal{B}, (\neg P[f(l)/x])^l \mid \mathcal{T} \\ &\quad (f \text{ fresh skolem symbol}) \end{aligned}$$

TaMeD rules

$$\frac{\mathcal{B}, \varphi_1, \neg\varphi_2 \mid \mathcal{T} \quad [C]}{\mathcal{T} \quad [C \cup \{\varphi_1 \stackrel{?}{=} \varphi_2\}]} \quad \frac{\mathcal{B}, U \mid \mathcal{T} \quad [C]}{\mathcal{T}' \mid \mathcal{T} \quad [C \cup \{U|_w \stackrel{?}{=} l\}]}$$

if $l \rightarrow_1 r$, $U|_w$ is an atomic proposition and $\mathcal{T}' = \text{Tab}(\mathcal{B}, U[r]_w)$

$$\begin{array}{c}
\text{Ax} \frac{\overline{S(m + (m + \text{sum}(m))) = S(m + (m + (m * m)))} \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + (m * m))) \\
\vdots \\
\text{Ax} \frac{\overline{S(m + (m + \text{sum}(m))) \in A} \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) \in A} \\
=_{\text{R}} \frac{\vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + \text{sum}(m)))}{\vdots} \\
=_{\text{L}} \frac{\overline{\text{sum}(m) = m * m} \vdash_{\equiv_1}^{+2} S(m + (m + \text{sum}(m))) = S(m + (m + (m * m)))}{\vdots} \\
\equiv \frac{\overline{\text{sum}(m) = m * m} \vdash_{\equiv_1}^{+2} \text{sum}(S(m)) = S(m) * S(m)}{\mathfrak{H}_R \frac{\vdash_{\equiv_1}^{+2} \mathfrak{H}(\tilde{P})}{\vdots}} \\
\vdots \\
\text{Ax} \frac{\overline{0 \in A} \vdash_{\equiv_1}^{+2} 0 \in A} \\
=_{\text{R}} \frac{\vdash_{\equiv_1}^{+2} 0 = 0}{\vdots} \\
\equiv \frac{\vdash_{\equiv_1}^{+2} \text{sum}(0) = 0 * 0}{\vdots} \\
\equiv \frac{\vdash_{\equiv_1}^{+2} 0 \in \tilde{P}}{\vdots} \\
\text{Ax} \frac{\overline{\text{sum}(n) = n * n} \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n} \\
\equiv \frac{n \in \tilde{P} \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n}{\vdots} \\
\text{N}(n) \vdash_{\equiv_1}^{+2} \text{sum}(n) = n * n
\end{array}$$

Figure 3. A proof where computational steps are detailed