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ON THE EQUATIONS OF THE MOVING CURVE IDEAL

LAURENT BUSÉ

ABSTRACT. Given a parametrization of a plane algebraic curve \mathcal{C} , some explicit adjoint linear systems on \mathcal{C} are described in terms of determinants. Moreover, some generators of the Rees algebra associated to this parametrization are presented. The main ingredient developed in this paper is a detailed study of the elimination ideal of two homogeneous polynomials in two homogeneous variables that form a regular sequence.

1. INTRODUCTION

Suppose given a rational map

$$(1.1) \quad \begin{aligned} \mathbb{P}^1 & \xrightarrow{\phi} \mathbb{P}^2 \\ (X_1 : X_2) & \mapsto (g_1 : g_2 : g_3)(X_1, X_2) \end{aligned}$$

where g_1, g_2, g_3 are homogeneous polynomials of degree $d \geq 1$ in the polynomial ring $\mathbb{K}[X_1, X_2]$ with \mathbb{K} a field. We assume that g_1, g_2, g_3 are not all zero and that the greatest common divisor of g_1, g_2, g_3 over $\mathbb{K}[X_1, X_2]$ has degree $< d$, so that the closed image of the rational map ϕ is an algebraic curve \mathcal{C} .

The geometric modeling community is very interested in the manipulation of parametrized algebraic plane curves and has developed many tools for this purpose in the last decade. One of them is what is called the *moving curve ideal* [8]. Denoting by $\underline{T} = (T_1, T_2, T_3)$ the homogeneous coordinates of $\mathbb{P}_{\mathbb{K}}^2$, a *moving curve* of degree $\nu \geq 0$ is a polynomial

$$\sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = \nu}} A_{\alpha_1, \alpha_2, \alpha_3}(X_1, X_2) T_1^{\alpha_1} T_2^{\alpha_2} T_3^{\alpha_3}$$

where $A_{\alpha_1, \alpha_2, \alpha_3}(X_1, X_2) \in \mathbb{K}[X_1, X_2]$. Such a moving curve is said to *follow the parametrization* ϕ if

$$\sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = \nu}} A_{\alpha_1, \alpha_2, \alpha_3}(X_1, X_2) g_1(X_1, X_2)^{\alpha_1} g_2(X_1, X_2)^{\alpha_2} g_3(X_1, X_2)^{\alpha_3} = 0.$$

The set of all moving curves that follow the parametrization ϕ form an ideal in the polynomial ring $\mathbb{K}[X_1, X_2][T_1, T_2, T_3]$. It is called the *moving curve ideal* of the parametrization ϕ .

From an algebraic point of view, the moving curve ideal of ϕ can be seen as the defining ideal of the Rees Algebra of the ideal $I = (g_1, g_2, g_3)$ in $\mathbb{K}[X_1, X_2]$. More precisely, $\text{Rees}_{\mathbb{K}[X_1, X_2]}(I)$ is the image of the $\mathbb{K}[X_1, X_2]$ -algebra morphism

$$\mathbb{K}[X_1, X_2][T_1, T_2, T_3] \xrightarrow{\beta} \mathbb{K}[X_1, X_2][Z] : T_i \mapsto g_i Z$$

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and the kernel of β is exactly the moving curve ideal of ϕ . (see for instance [4, Proposition 3.5] for a proof of this well-known fact).

The moving curve ideal of ϕ is canonically \mathbb{N} -graded with respect to the couple of homogeneous variables X_1, X_2 . For instance, its degree 0 part is the principal ideal of $\mathbb{K}[T_1, T_2, T_3]$ generated by an implicit equation of the curve \mathcal{C} . Recently, David Cox observed that the degree 1 part the moving curve ideal of ϕ also carries a lot of geometric properties of the curve \mathcal{C} . More precisely, he suggested [8, Conjecture 3.8] a close relation between adjoint linear systems on \mathcal{C} and moving curves following ϕ of degree 1 in X_1, X_2 and degree $d - 2$ (resp. $d - 1$) in T_1, T_2, T_3 . In this paper, we prove several results that gives a better understanding of this relation. The main contribution is to show that one can always find an adjoint linear system on \mathcal{C} of degree $d - 2$ in T_1, T_2, T_3 that belongs to the moving curve ideal of ϕ . These adjoint linear systems on \mathcal{C} are described very simply in terms of certain determinants.

In [8], as well as in the more recent works [10, 7], the question of getting a full system of generators for the moving curve ideal of ϕ is treated. In this paper, we provide new results on the character of some of the generators of the moving curve ideal. Our approach is based on another description of the moving curve ideal of ϕ in terms of an elimination ideal, that we now recall.

The first syzygy module of g_1, g_2, g_3 is known to be a free homogeneous $\mathbb{K}[X_1, X_2]$ -module of rank 2 and a basis of this syzygy module consists in two homogeneous elements (the notation \underline{X} and \underline{T} stand for the set of variables X_1, X_2 and T_1, T_2, T_3 respectively)

$$p = (p_1(\underline{X}), p_2(\underline{X}), p_3(\underline{X})), \quad q = (q_1(\underline{X}), q_2(\underline{X}), q_3(\underline{X})) \in \mathbb{K}[X_1, X_2]^3$$

of degree μ and $d - \mu$ respectively. It is a consequence of the Hilbert-Burch Theorem. Notice that the choice of p and q is not unique, but their degree is fixed and only depend on the parametrization ϕ . We will identify p, q with the two polynomials

$$(1.2) \quad \begin{aligned} p(\underline{X}, \underline{T}) &= p_1(\underline{X})T_1 + p_2(\underline{X})T_2 + p_3(\underline{X})T_3 \\ q(\underline{X}, \underline{T}) &= q_1(\underline{X})T_1 + q_2(\underline{X})T_2 + q_3(\underline{X})T_3 \end{aligned}$$

in $\mathbb{K}[\underline{X}, \underline{T}]$ that form what has been called a μ -basis of the parametrization ϕ by the geometric modeling community because of its importance to handle parametrized plane curves. As an illustration of the benefit of μ -bases, we recall the following well-known formula that relates the resultant of a μ -basis, an implicit equation $C(T_1, T_2, T_3)$ of \mathcal{C} and the degree of the parametrization (1.1):

$$\text{Res}_{X_1, X_2}(p, q) = \alpha C(T_1, T_2, T_3)^{\deg(\phi)} \in \mathbb{K}[\underline{T}]$$

where $\alpha \in \mathbb{K} \setminus \{0\}$. Here is another characterization of the moving curve ideal of ϕ .

Proposition 1.1. *Let (p, q) be a μ -basis of the parametrization ϕ . Then, the sequence (p, q) is regular in $\mathbb{K}[\underline{X}, \underline{T}]$ and the elimination ideal $((p, q) : (X_1, X_2)^\infty)$ in $\mathbb{K}[\underline{X}, \underline{T}]$ is equal to the moving curve ideal of ϕ .*

Proof. The first point follows from [2, §2.1], and the second point from [4, Proposition 3.6] for instance. \square

The paper is organized as follows. In Section 2 we give a detailed study of the inertia forms of two homogeneous polynomials in two homogeneous variables. In particular we provide in several cases a full system of generators of these inertia forms. Then, we will apply these results in Section 3 to give some explicit generators of the moving curve ideal of ϕ and even a full system of generators in some

particular cases. Finally, in Section 4 we investigate the relation between adjoint linear systems on \mathcal{C} and moving curves following ϕ of degree 1 in X_1, X_2 and degree $d - 2$ (resp. $d - 1$) in T_1, T_2, T_3 .

2. INERTIA FORMS OF TWO POLYNOMIALS IN TWO HOMOGENEOUS VARIABLES

In this section we suppose given a non-zero commutative ring A and two homogeneous polynomials

$$f_1(X_1, X_2) = U_0 X_1^{d_1} + \cdots + U_{d_1} X_2^{d_1}, \quad f_2(X_1, X_2) = V_0 X_1^{d_2} + \cdots + V_{d_2} X_2^{d_2}$$

in the (canonically graded) polynomial ring $C = A[X_1, X_2]$ with respective degree d_1, d_2 such that $1 \leq d_1 \leq d_2$. We will denote by \mathfrak{m} , resp. I , the ideal of C generated by X_1, X_2 , resp. f_1, f_2 , and by B the (canonically graded) quotient ring C/I . Also, we define the integer $\delta = d_1 + d_2 - 2 \geq 0$.

We recall that the resultant of f_1 and f_2 , denoted $\text{Res}(f_1, f_2)$, is equal to the determinant of the well-known Sylvester matrix

$$\begin{pmatrix} U_0 & & 0 & V_0 & & 0 \\ & \ddots & & & \ddots & \\ \vdots & & U_0 & \vdots & & V_0 \\ & & \vdots & & & \vdots \\ U_{d_1} & & \vdots & V_{d_2} & & \vdots \\ & \ddots & & & \ddots & \\ 0 & & U_{d_1} & 0 & & V_{d_2} \end{pmatrix}$$

$\underbrace{\hspace{10em}}_{d_2} \qquad \underbrace{\hspace{10em}}_{d_1}$

which is of size $d_1 + d_2 = \delta + 2$. The *first-order subresultants* of f_1 and f_2 correspond to some δ -minors of this Sylvester matrix. More precisely, by developing the determinant

$$\begin{vmatrix} U_0 & & 0 & V_0 & & 0 & T_0 \\ & \ddots & & & \ddots & & \vdots \\ \vdots & & U_0 & \vdots & & V_0 & \vdots \\ & & \vdots & & & \vdots & \vdots \\ U_{d_1} & & \vdots & V_{d_2} & & \vdots & \vdots \\ & \ddots & & & \ddots & & \\ 0 & & U_{d_1} & 0 & & V_{d_2} & T_\delta \end{vmatrix}$$

$\underbrace{\hspace{10em}}_{d_2-1} \qquad \underbrace{\hspace{10em}}_{d_1-1}$

along its last column, we get the polynomial

$$\sum_{i=0}^{\delta} \text{SRes}_{\delta-i}(f_1, f_2) T_i \in A[T_0, \dots, T_\delta].$$

The elements $\text{SRes}_i(f_1, f_2) \in A$, $i = 0, \dots, \delta$, are the (first-order) subresultants of f_1 and f_2 . The element $\text{SRes}_0(f_1, f_2)$ is usually called the principal subresultant.

From now on in this section, we *assume that* (f_1, f_2) *is a* C -*regular sequence*. Our aim is to give, under suitable other conditions, an explicit description of the ideal

of *inertia forms* of f_1, f_2 w.r.t. the variables X_1, X_2 , that is an explicit description of the ideal $(I :_C \mathfrak{m}^\infty)$ consisting of all the elements $f \in C$ such that there exists an integer k with the property that $\mathfrak{m}^k f \subset (f_1, f_2)$. Since f_1 and f_2 are obvious inertia forms, it is sufficient to describe the graded B -module $H_{\mathfrak{m}}^0(B) \subset B$ since we have the canonical isomorphism

$$(I : \mathfrak{m}^\infty)/I \xrightarrow{\sim} H_{\mathfrak{m}}^0(B) = \{f \in B \text{ such that } \exists k \in \mathbb{N} : \mathfrak{m}^k f = 0\}.$$

The methods we will use are inspired by [11] where such a work has been done for two generic homogeneous polynomials of the same degree, that is in the case $A = \mathbb{Z}[U_0, \dots, U_{d_1}, V_0, \dots, V_{d_2}]$ with $d_1 = d_2$.

2.1. Sylvester forms, Morley forms and an explicit duality. We gather some known properties and results on the module $H_{\mathfrak{m}}^0(B)$ we are interested in. Notice that $H_{\mathfrak{m}}^0(B)$ is a \mathbb{N} -graded B -module; for all $\nu \in \mathbb{N}$, we will denote by $H_{\mathfrak{m}}^0(B)_\nu$ its ν^{th} graded part.

2.1.1. The duality. As proved in [12, §1.5], for all $\nu > \delta$ we have $H_{\mathfrak{m}}^0(B)_\nu = 0$. If $\nu = \delta$ then it is shown that there exists an isomorphism of A -modules $A \simeq H_{\mathfrak{m}}^0(B)_\delta$. Moreover, for all $0 \leq \nu \leq \delta$ the multiplication map (defined by the B -module structure of $H_{\mathfrak{m}}^0(B)$)

$$B_{\delta-\nu} \otimes_A H_{\mathfrak{m}}^0(B)_\nu \rightarrow H_{\mathfrak{m}}^0(B)_\delta : b \otimes b' \mapsto bb'$$

induces the isomorphism of A -modules

$$(2.1) \quad H_{\mathfrak{m}}^0(B)_\nu \xrightarrow{\sim} \text{Hom}_A(B_{\delta-\nu}, H_{\mathfrak{m}}^0(B)_\delta) : b \mapsto (c \mapsto b \otimes c).$$

This duality (recall that $H_{\mathfrak{m}}^0(B)_\delta \simeq A$) has been made explicit by Jouanolou by using the *Morley forms*. Before describing these forms, we recall the construction of the *Sylvester forms* which are the first examples of simple non-trivial inertia forms.

2.1.2. Sylvester forms [11, §3.10]. Suppose given $\beta = (\beta_1, \beta_2) \in \mathbb{N} \times \mathbb{N}$ such that $|\beta| = \beta_1 + \beta_2 \leq d_1 - 1$. Then, f_1 and f_2 can be decomposed in C as

$$(2.2) \quad f_1(X_1, X_2) = X_1^{\beta_1+1} f_{1,1} + X_2^{\beta_2+1} f_{1,2}, \quad f_2(X_1, X_2) = X_1^{\beta_1+1} f_{2,1} + X_2^{\beta_2+2} f_{2,2}$$

where $f_{i,j} \in C$ is homogeneous of degree $d_i - \beta_j - 1$. Therefore the determinant of the matrix $(f_{i,j})_{i,j=1,2}$ is a homogeneous polynomial in C of degree $\delta - |\beta|$. The class of this determinant in B turns out to be independent of the choice of the decompositions (2.2); it is called a *Sylvester form* of f_1, f_2 and will be denoted $\text{sylv}_\beta(f_1, f_2) \in B_{\delta-|\beta|}$.

It is easy to check that, for $k = 1, 2$, we have $X_k^{\beta_k+1} \det(f_{i,j})_{i,j=1,2} \in I$. We deduce that $\text{sylv}_\beta(f_1, f_2)$ is an inertia form of f_1, f_2 w.r.t. X_1, X_2 , that is to say that

$$\text{sylv}_\beta(f_1, f_2) \in H_{\mathfrak{m}}^0(B)_{\delta-|\beta|}.$$

The Sylvester form $\text{sylv}_{(0,0)}(f_1, f_2)$, very similar to the classical Jacobian, plays a particular rôle since it makes explicit the isomorphism

$$(2.3) \quad A \xrightarrow{\sim} H_{\mathfrak{m}}^0(B)_\delta : a \mapsto a \cdot \text{sylv}_{(0,0)}(f_1, f_2).$$

Moreover, by duality we deduce that for all α, β such that $0 \leq |\alpha| = |\beta| \leq d_1 - 1$ we have

$$X^{\alpha} \text{sylv}_{\beta}(f_1, f_2) = \begin{cases} \text{sylv}_{(0,0)}(f_1, f_2) & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

in $H_{\mathfrak{m}}^0(B)_{\delta}$.

2.1.3. *Morley forms* [11, §3.11]. Introducing two new indeterminates Y_1 and Y_2 , we choose arbitrary decompositions in the polynomial ring $A[X_1, X_2, Y_1, Y_2]$:

$$(2.4) \quad f_i(X_1, X_2) - f_i(Y_1, Y_2) = (X_1 - Y_1)h_{i,1} + (X_2 - Y_2)h_{i,2}, \quad i = 1, 2.$$

The determinant of the matrix $(h_{i,j})_{i,j=1,2}$ is a polynomial in $A[X_1, X_2, Y_1, Y_2]$ which is easily seen to be homogeneous in the variables X_1, X_2 , resp. Y_1, Y_2 , of degree δ . Now, consider the ring

$$B \otimes_A B \simeq \frac{A[X_1, X_2, Y_1, Y_2]}{(f_1(X_1, X_2), f_1(Y_1, Y_2), f_2(X_1, X_2), f_2(Y_1, Y_2))}.$$

Since B is graded, $B \otimes_A B$ inherits of a canonical bi-grading (a grading w.r.t. the variables X_1, X_2 and another one w.r.t. Y_1, Y_2); we set

$$B \otimes_A B = \bigoplus_{(p,q) \in \mathbb{N} \times \mathbb{N}} B_p \otimes_A B_q = \bigoplus_{r \in \mathbb{N}} \left(\bigoplus_{p+q=r} B_p \otimes_A B_q \right) = \bigoplus_{r \in \mathbb{N}} [B \otimes_A B]_r.$$

The *Morley form* of f_1 and f_2 , denoted $\text{morl}(f_1, f_2)$, is the class of $\det(h_{i,j})_{i,j=1,2}$ in $B \otimes_A B$. It is independent of the choice of the decompositions (2.4). Since $\text{morl}(f_1, f_2) \in [B \otimes_A B]_{\delta}$, for all $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q = \delta$ we denote by $\text{morl}_{p,q}(f_1, f_2) \in B_p \otimes_k B_q$ its homogeneous component of bi-degree (p, q) .

Recall some properties of Morley forms that will be useful in the rest of this paper (see [11, §3.11] for the proofs):

- (1) Let τ be the symmetry exchanging X_i with Y_i for $i = 1, 2$. Then τ leaves $\text{morl}(f_1, f_2)$ invariant and for all couple $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q = \delta$ we have $\tau(\text{morl}_{p,q}(f_1, f_2)) = \text{morl}_{q,p}(f_1, f_2)$.
- (2) $\text{morl}_{\delta,0}(f_1, f_2) = \text{sylv}_{(0,0)}(f_1, f_2) \otimes 1 \in B_{\delta} \otimes_A B_0$ and hence, by the above property, $\text{morl}_{0,\delta}(f_1, f_2) = 1 \otimes \text{sylv}_{(0,0)}(f_1, f_2) \in B_0 \otimes_A B_{\delta}$.
- (3) The fact that $(X_i - Y_i)\text{morl}(f_1, f_2) = 0 \in [B \otimes_A B]_{\delta+1}$ implies that

$$\begin{aligned} (b \otimes 1)\text{morl}_{p,q}(f_1, f_2) &= \text{sylv}_{(0,0)}(f_1, f_2) \otimes b && \text{for all } b \in B_q \\ (1 \otimes b)\text{morl}_{p,q}(f_1, f_2) &= b \otimes \text{sylv}_{(0,0)}(f_1, f_2) && \text{for all } b \in B_p \end{aligned}$$

2.1.4. *Explicit duality*. For all $0 \leq \nu \leq \delta$, choosing an arbitrary decomposition

$$\text{morl}_{\delta-\nu,\nu}(f_1, f_2) = \sum_s x_s \otimes y_s \quad \text{with } x_s \in B_{\delta-\nu}, y_s \in B_{\nu},$$

we have the following isomorphism of A -modules [12, §3.6] (see also [13])

$$(2.5) \quad \begin{aligned} \theta_{\nu} : B_{\delta-\nu}^{\vee} = \text{Hom}_A(B_{\delta-\nu}, A) &\xrightarrow{\sim} H_{\mathfrak{m}}^0(B)_{\nu} \subset B_{\nu} \\ u &\mapsto \sum_s u(x_s)y_s. \end{aligned}$$

In particular, this isomorphism shows that it is possible to describe explicitly all the inertia forms of f_1, f_2 of degree ν if one can describe explicitly the dual of $B_{\delta-\nu}$. This is the approach we will follow hereafter. For technical reasons, our analysis is divided into the three intervals $0 \leq \nu \leq d_1 - 2$, $d_1 - 1 \leq \nu \leq d_2 - 2$ and

$d_2 - 1 \leq \nu \leq \delta$. Of course, depending on the values of d_1 and d_2 , it may happen that one or two of these intervals are empty. We recall that we always assume that $1 \leq d_1 \leq d_2$.

2.2. Inertia forms of degree $\leq d_1 - 2$. Assuming that $2 \leq d_1 \leq d_2$, we analyze the inertia forms of f_1, f_2 of degree ν such that $0 \leq \nu \leq d_1 - 2$, or equivalently such that $d_2 \leq \delta - \nu \leq \delta$.

For all integer $\nu = 0, \dots, d_1 - 1$, we introduce the new indeterminates

$$\underline{W} = (W_0, \dots, W_{\delta-2\nu+1}), \quad \underline{T} = (T_0, \dots, T_{\delta-\nu})$$

and we consider the polynomials

$$\begin{aligned} h &= W_0 X_1^{\delta-2\nu+1} + \dots + W_{\delta-2\nu+1} X_2^{\delta-2\nu+1}, \\ \varphi &= T_0 X_1^{\delta-\nu} + \dots + T_{\delta-\nu} X_2^{\delta-\nu}. \end{aligned}$$

We define the determinant $D(\nu)$ as

$$\begin{array}{cccccccc} \left| \begin{array}{cccccccc} U_0 & & 0 & & V_0 & & 0 & & W_0 & & & & 0 \\ & \ddots & & & & \ddots & & & & \ddots & & & \\ \vdots & & U_0 & & \vdots & & V_0 & & \vdots & & & & W_0 \\ & & \vdots & & V_{d_2} & & \vdots & & W_{\delta-2\nu+1} & & & & \vdots \\ U_{d_1} & & & & & & & & & & & & \\ & \ddots & & & & \ddots & & & & \ddots & & & \\ 0 & & U_{d_1} & & 0 & & V_{d_2} & & 0 & & & & W_{\delta-2\nu+1} \\ 0 & \dots & 0 & & 0 & \dots & 0 & & X_1^\nu & \dots & & & X_2^\nu \end{array} \right| \\ \underbrace{\hspace{1.5cm}}_{d_2-\nu} & & \underbrace{\hspace{1.5cm}}_{d_1-\nu} & & \underbrace{\hspace{1.5cm}}_{\nu+1} \end{array}$$

and the determinant $D_1(\nu)$ as

$$\begin{array}{cccccccc} \left| \begin{array}{cccccccc} U_0 & & 0 & & V_0 & & 0 & & T_0 & & W_0 & & & 0 \\ & \ddots & & & & \ddots & & & \vdots & & & \ddots & & \\ \vdots & & U_0 & & \vdots & & V_0 & & \vdots & & \vdots & & & W_0 \\ & & \vdots & & V_{d_2} & & \vdots & & \vdots & & W_{\delta-2\nu+1} & & & \vdots \\ U_{d_1} & & & & & & & & \vdots & & & & & \\ & \ddots & & & & \ddots & & & & & & & & \\ 0 & & U_{d_1} & & 0 & & V_{d_2} & & T_{\delta-\nu} & & 0 & & & W_{\delta-2\nu+1} \end{array} \right| \\ \underbrace{\hspace{1.5cm}}_{d_2-\nu-1} & & \underbrace{\hspace{1.5cm}}_{d_1-\nu-1} & & \underbrace{\hspace{1.5cm}}_{\nu} \end{array}$$

Notice that $D(0) = \text{Res}(f_1, f_2)$ and $D_1(0) = \sum_{i=0}^{\delta} \text{SRes}_{\delta-i} T_i$.

Lemma 2.1 ($1 \leq \nu \leq d_1 - 1$). *In $C[\underline{W}, \underline{T}]/(f_1(X), f_2(X))$ we have*

$$\varphi D(\nu) = (-1)^{d_1+1} D_1(\nu) \text{sylv}_{(0,0)}(f_1, f_2).$$

Proof. By specialization, it is sufficient to prove the claimed equality in the generic case, that is to say in the case where $A = \mathbb{Z}[U_0, \dots, U_{d_1}, V_0, \dots, V_{d_2}]$. The proof of this lemma is a straightforward extension of the proof of [11, Lemme 3.11.18.30] that we closely follow.

Denote by $(r_i)_{i=0, \dots, \delta-\nu+1}$, r the rows of the matrix defining the determinant $D(\nu)$ from top to bottom. By construction, we have

$$(2.6) \quad \left(\sum_{i=0}^{\delta-\nu+1} X_1^{\delta-\nu+1-i} X_2^i r_i \right) - hr = \\ \left(X_1^{d_2-\nu-1} f_1, \dots, X_2^{d_2-\nu-1} f_1, X_1^{d_1-\nu-1} f_2, \dots, X_2^{d_1-\nu-1} f_2, 0, \dots, 0 \right)$$

which shows that $X_1^{\delta-\nu+1} D(\nu) \in (f, g)$ and $X_2^{\delta-\nu+1} D(\nu) \in (f, g)$, i.e. that $D(\nu)$ is an inertia form of f_1, f_2 w.r.t. \mathfrak{m} in $C[\underline{W}, \underline{T}]$. Notice that $D(\nu)$ is homogeneous of degree ν in the variables X_1, X_2 .

Since (f_1, f_2) is a C -regular sequence, we have the canonical isomorphism, inverse of (2.3),

$$(2.7) \quad \lambda : H_{\mathfrak{m}}^0(C[\underline{W}, \underline{T}]/(f_1, f_2))_{\delta} \xrightarrow{\sim} A[\underline{W}, \underline{T}]$$

with the property that $\lambda(b) \text{sy}lv_0(f_1, f_2) = b$ for all $b \in H_{\mathfrak{m}}^0(C[\underline{W}, \underline{T}]/(f_1, f_2))_{\delta}$. Now, consider the $A[\underline{W}, \underline{T}]$ -linear maps

$$\Lambda : A[\underline{W}, \underline{T}][X_1, X_2]_{\delta-\nu} \rightarrow A[\underline{W}, \underline{T}] : F \mapsto \lambda(FD(\nu))$$

and

$$v : A[\underline{W}, \underline{T}]^{\delta-\nu} \rightarrow A[\underline{W}, \underline{T}]^{\delta-\nu+1}$$

which is given by the matrix defining $D_1(\nu)$ after deleting its unique column depending on the T_i 's. It is clear that Λ vanishes on $(f_1, f_2)_{\delta-\nu}$, that corresponds to the first $\delta - 2\nu$ columns of the matrix of $D_1(\nu)$, and on $(h)_{\delta-\nu}$, that corresponds to the last ν columns of the matrix of $D_1(\nu)$ since $hD(\nu) \in (f, g)$ by (2.6). Therefore, $\Lambda \circ v = 0$ and hence Λ belongs to the kernel of v^{\vee} , the dual of v . Moreover, it is not hard to check that the depth of the ideal of $(\delta - \nu)$ -minors of v is at least 2 and this implies that the Buchsbaum-Rim complex associated to v^{\vee} is acyclic; it is of the form

$$0 \rightarrow A[\underline{W}, \underline{T}]^{\vee} \simeq \bigwedge^{\delta-\nu+1} (A[\underline{W}, \underline{T}]^{\delta-\nu+1})^{\vee} \rightarrow (A[\underline{W}, \underline{T}]^{\delta-\nu+1})^{\vee} \xrightarrow{v^{\vee}} (A[\underline{W}, \underline{T}]^{\delta-\nu})^{\vee}.$$

It follows that there exists an element $a \in A[\underline{W}, \underline{T}]$ such that, for all elements $F \in A[\underline{W}, \underline{T}][X_1, X_2]_{\delta-\nu}$ we have

$$\Lambda(F) = a \det \left(X_1^{d_2-\nu-2} f_1, \dots, X_2^{d_2-\nu-2} f_1, \right. \\ \left. X_1^{d_1-\nu-2} f_2, \dots, X_2^{d_1-\nu-2} f_2, F, X_1^{\nu-1} h, \dots, X_2^{\nu-1} h \right)$$

(notice that the element $(X_1^{\nu-1} h, \dots, X_2^{\nu-1} h)$ disappears in the case $\nu = 0$). In particular, $\lambda(\varphi D(\nu)) = aD_1(\nu)$ in $A[\underline{W}, \underline{T}]$, that is to say

$$\varphi D(\nu) = aD_1(\nu) \text{sy}lv_{(0,0)}(f_1, f_2) \text{ in } (A[\underline{W}, \underline{T}][X_1, X_2]/(f_1, f_2))_{\delta}.$$

By inspecting the homogeneous degrees w.r.t. $\underline{U}, \underline{V}, \underline{T}, \underline{W}$, we deduce that $a \in \mathbb{Z}$. Then, to determine a we consider the specialization

$$f_1 \mapsto X_1^{d_1}, f_2 \mapsto X_2^{d_2}, h \mapsto X_1^{d_1-\nu-1} X_2^{d_2-\nu}, \varphi \mapsto X_1^{d_1-1} X_2^{d_2-\nu-1}$$

which sends $D(\nu)$ to $(-1)^{\nu(d_1-\nu)} X_2^{\nu}$ and $D_1(\nu)$ to $(-1)^{(d_1-\nu-1)(\nu+1)}$. We conclude that

$$a = (-1)^{\nu(d_1-\nu)-(d_1-\nu-1)(\nu+1)} = (-1)^{2\nu-d_1+1} = (-1)^{d_1+1}$$

and the lemma is proved. \square

It is interesting to notice that in the case $\nu = 0$, this lemma shows that

$$\left(\sum_{i=0}^{\delta} T_i X_1^{\delta-i} X_2^i \right) \text{Res}(f_1, f_2) = (-1)^{d_1+1} \left(\sum_{i=0}^{\delta} \text{SRes}_{\delta-i}(f_1, f_2) T_i \right) \text{sylv}_{(0,0)}(f_1, f_2)$$

in $C[\underline{T}]/(f_1(X), f_2(X))$. Writing $\text{sylv}_{(0,0)}(f_1, f_2) = \sum_{i=0}^{\delta} q_i X_1^{\delta-i} X_2^i$ with $q_i \in A$, we deduce by specialization of T_i to q_i for all $i = 0, \dots, \delta$, and using the isomorphism (2.7), that

$$(2.8) \quad \text{Res}(f_1, f_2) = (-1)^{d_1+1} \sum_{i=0}^{\delta} \text{SRes}_{\delta-i}(f_1, f_2) q_i$$

in A . Observe that this is also equal to $(-1)^{d_1+1}$ times the determinant $D_1(\nu)$ where T_i is specialized to q_i for all $i = 0, \dots, \delta$.

Theorem 2.2 ($0 \leq \nu \leq d_1 - 2$). *Let $D(\nu) = \sum_{|\beta|=\nu} D_{\beta}(X_1, X_2) W^{\beta}$. The element $D_{\beta} \in C$ is an inertia form of degree ν for all β such that $|\beta| = \nu$.*

Moreover, if the inequality

$$(2.9) \quad \text{depth}_A \text{Det}_{\delta-2\nu} \left(C_{\delta-\nu-d_1} \oplus C_{\delta-\nu-d_2} \xrightarrow{\mathbf{f}=(f_1, f_2)} C_{\delta-\nu} \right) \geq \nu + 2$$

holds, then the collection of inertia forms $(D_{\beta})_{|\beta|=\nu}$ is a system of generators of $H_{\mathfrak{m}}^0(B)_{\nu}$.

Proof. The formula (2.6) shows that $\mathfrak{m}^{\delta-\nu+1} D(\nu) \subset (f_1, f_2)$ in $C[\underline{W}]$ and hence that $D_{\beta}(X_1, X_2) \in C$ is an inertia form of degree ν of f_1, f_2 w.r.t. \mathfrak{m} for all β such that $|\beta| = \nu$.

Now, let $G \in C_{\nu}$ be an inertia form of f_1, f_2 w.r.t. \mathfrak{m} of degree ν and consider the A -linear map

$$\Gamma_G : C_{\delta-\nu} \rightarrow A : F \mapsto \lambda(FG)$$

where λ is defined by (2.7). It is clear that Γ_G vanishes on $(f_1, f_2)_{\delta-\nu}$, that is to say that $\Gamma_G \circ \mathbf{f} = 0$, and hence that Γ_G belongs to the kernel of the dual \mathbf{f}^{\vee} of \mathbf{f} . Notice that the first $\delta - 2\nu$ columns of the matrix of $D_1(\nu)$ give a matrix of \mathbf{f} in appropriate monomial basis. Under the hypothesis (2.9), the Buchsbaum-Rim complex associated to \mathbf{f}^{\vee} is acyclic; it is of the form

$$\cdots \rightarrow \bigwedge^{\delta-2\nu+1} (C_{\delta-\nu})^{\vee} \xrightarrow{\varepsilon_{\nu}} (C_{\delta-\nu})^{\vee} \xrightarrow{\mathbf{f}^{\vee}} (C_{\delta-\nu-d_1})^{\vee} \oplus (C_{\delta-\nu-d_2})^{\vee}$$

where we recall that the map ε_{ν} sends the basis element

$$(X^{\alpha_1})^{\vee} \wedge \cdots \wedge (X^{\alpha_m})^{\vee}$$

where $m = \delta - 2\nu + 1$, to

$$\sum_{i=1}^m (-1)^i (X^{\alpha_i})^{\vee} \times \\ (\mathbf{f}_1^{\vee}((X^{\alpha_1})^{\vee}) \wedge \cdots \wedge \mathbf{f}_1^{\vee}((X^{\alpha_{i-1}})^{\vee}) \wedge \mathbf{f}_1^{\vee}((X^{\alpha_{i+1}})^{\vee}) \wedge \cdots \wedge \mathbf{f}_1^{\vee}((X^{\alpha_m})^{\vee})).$$

For all set $I \subset \{1, \dots, \delta - \nu + 1\}$ with cardinality $|I| = \delta - 2\nu + 1$, we denote by $D_I \in A[\underline{T}]$ the determinant of the minor of $D_1(\nu)$ corresponding to the first

$\delta - 2\nu + 1$ columns and the rows indexed by I . From the acyclicity of the above complex we deduce that there exists a collection of elements $a_I \in A$ such that

$$(2.10) \quad \lambda(\varphi G) = \sum_{\substack{I \subset \{1, \dots, \delta - \nu + 1\} \\ |I| = \delta - 2\nu + 1}} a_I D_I \in A[\underline{T}].$$

To finish the proof we distinguish the two cases $\nu = 0$ and $1 \leq \nu \leq d_1 - 2$.

If $\nu = 0$, (2.10) reduces to the equality $\lambda(\varphi G) = a D_1(\nu)$ for some $a \in A$, since $D_1(\nu)$ does not depend on \underline{W} in this case. Specializing φ to $\text{sylv}_{(0,0)}(f_1, f_2)$, we deduce that

$$G = G \lambda(\text{sylv}_{(0,0)}(f_1, f_2)) = \lambda(\text{sylv}_{(0,0)}(f_1, f_2) G) = a \text{Res}(f_1, f_2)$$

because $G \in A$ and this specialization sends $D_1(\nu)$ to $\text{Res}(f_1, f_2)$ [11, Corollaire 3.10.22]. Consequently, as $\text{Res}(f_1, f_2) = D(0) = D_{\beta=(0,0)}$ in this case, we have $G = a D(0)$, i.e. $D(0)$ is a generator of $H_{\mathfrak{m}}^0(B)_0$.

Now, assume that $1 \leq \nu \leq d_1 - 2$. By definition of $(D_I)_I$, there exists a collection of polynomials $q_I(\underline{W}) \in \mathbb{Z}[\underline{W}]$ such that

$$(-1)^{d_1+1} D_1 = \sum_{\substack{I \subset \{1, \dots, \delta - \nu + 1\} \\ |I| = \delta - 2\nu + 1}} D_I q_I(\underline{W}) \in A[\underline{T}, \underline{W}].$$

The collection $(q_I(\underline{W}))$ form a basis of the homogeneous polynomials of degree ν in the variables \underline{W} , as well as the collection $(\underline{W}^\beta)_{|\beta|=\nu}$ by [11, Remarque 3.11.18.22]. Therefore, there exists polynomials $H_I \in A[X_1, X_2]_\nu$ such that

$$D = \sum_{\substack{I \subset \{1, \dots, \delta - \nu + 1\} \\ |I| = \delta - 2\nu + 1}} H_I q_I(\underline{W}) \in A[\underline{W}]$$

and such that $(D_\beta)_{|\beta|=\nu}$ and $(H_I)_{|I|=\delta-2\nu+1}$ form two systems of generators of the same A -module. With this notation, Lemma 2.1 gives

$$\sum_{\substack{I \subset \{1, \dots, \delta - \nu + 1\} \\ |I| = \delta - 2\nu + 1}} \lambda(\varphi H_I) q_I(\underline{W}) = \sum_{\substack{I \subset \{1, \dots, \delta - \nu + 1\} \\ |I| = \delta - 2\nu + 1}} D_I q_I(\underline{W}),$$

i.e. $\lambda(\varphi H_I) = D_I$ for all $I \subset \{1, \dots, \delta - \nu + 1\}$ with $|I| = \delta - 2\nu + 1$. Therefore, we deduce from (2.10) that

$$\lambda(\varphi G) = \lambda \left(\varphi \left(\sum_I a_I H_I \right) \right).$$

Now, since λ is injective, we have $\varphi G = \varphi(\sum_I a_I H_I)$ in $H_{\mathfrak{m}}^0(B)_\delta$. By identifying the coefficients in the variables \underline{T} , we deduce that the two multiplication maps by G and $\sum_I a_I H_I$ from $B_{\delta-\nu}$ to $H_{\mathfrak{m}}^0(B)_\delta$ coincide. Therefore, by the duality (2.1) we deduce that $G = \sum_I a_I H_I$. \square

Remark 2.3. In the generic case, i.e. $A = \mathbb{Z}[U_0, \dots, U_{d_1}, V_0, \dots, V_{d_2}]$, one can show very similarly to [11, Proposition 3.11.18.19 (c)], that the inequality (2.9) holds.

Theorem 2.2 shows that $D(0) = \text{Res}(f_1, f_2)$ is a generator of $H_{\mathfrak{m}}^0(B)_0$ if the ideal of A generated by the subresultants $\text{SRes}_i(f_1, f_2)$, $i = 0, \dots, \delta$, has depth at least 2. To show that this condition can not be avoided we consider the following example: $A = \mathbb{Z}[U_0, V_2]$, $f_1 = U_0 X_1^{d_1}$ and $f_2 = V_{d_2} X_2^{d_2}$. It is easy to compute that

$\text{Res}(f_1, f_2) = U_0^{d_2} V_{d_2}^{d_1}$ and $H_m^0(B)_0 = (U_0 V_{d_2}) \subset A$. Therefore, $\text{Res}(f_1, f_2)$ is not a generator of $H_m^0(B)_0$. Also, one can check that $\text{SRes}_{d_1-1}(f_1, f_2) = U_0^{d_2-1} V_{d_2}^{d_1-1}$ and that all the other subresultants vanishes, so the ideal generated by all the subresultants has depth exactly 1.

Later on we will be concerned with the case $\nu = 1$, so we detail a little more the situation in this case. Notice that this case may occur only if $d_2 \geq d_1 \geq 3$. For simplicity, we rename the inertia forms $D_\beta(p, q)$ with $|\beta| = 1$ as $D_i(p, q)$, $i = 0, \dots, \delta - 1$ in the following way :

$$D(1) = \begin{array}{c} \left(\begin{array}{cccccc} U_0 & 0 & V_0 & 0 & W_0 & 0 \\ & \ddots & & \ddots & & \\ \vdots & U_0 & \vdots & V_0 & \vdots & W_0 \\ U_\mu & \vdots & V_{d-\mu} & \vdots & W_{\delta-1} & \vdots \\ & \ddots & & \ddots & & \\ 0 & U_\mu & 0 & V_{d-\mu} & 0 & W_{\delta-1} \\ 0 & \dots & 0 & 0 & \dots & 0 & X_1 & X_2 \end{array} \right) = \sum_{i=0}^{\delta-1} D_i(X_1, X_2) W_i. \\ \underbrace{\hspace{10em}}_{d-\mu-1} \quad \underbrace{\hspace{10em}}_{\mu-1} \quad \underbrace{\hspace{10em}}_2 \end{array}$$

Corollary 2.4. *With the above notation, for all $i = 0, \dots, \delta - 1$ we have, in A ,*

$$D_i(p, q) = X_2 \text{SRes}_{\delta-i}(p, q) - X_1 \text{SRes}_{\delta-i-1}(p, q).$$

Moreover, if the inequality

$$\text{depth}_A \text{Det}_{\delta-2} \left(\begin{array}{cccc} U_0 & 0 & V_0 & 0 \\ & \ddots & & \ddots \\ \vdots & U_0 & \vdots & V_0 \\ U_{d_1} & \vdots & V_{d_2} & \vdots \\ & \ddots & & \ddots \\ 0 & U_{d_1} & 0 & V_{d_2} \end{array} \right) \geq \nu + 2$$

$\underbrace{\hspace{10em}}_{d_2-2} \quad \underbrace{\hspace{10em}}_{d_1-2}$

holds, then the collection of inertia forms $(D_i)_{i=0, \dots, \delta-1}$ is a system of generators of $H_m^0(B)_1$.

Proof. This is a straightforward computation from the definitions. \square

2.3. Inertia forms of degree $\geq d_1 - 1$ and $\leq d_2 - 2$. Choosing a decomposition (2.4), we set

$$\det (h_{i,j}(X_1, X_2, Y_1, Y_2))_{i,j=1,2} = \sum_{0 \leq |\beta| \leq \delta} q_\beta(X) Y^\beta.$$

For all integers ν such that $d_1 - 1 \leq \nu \leq d_2 - 2$, we have $d_1 \leq \delta - \nu \leq d_2 - 1$ and we consider the $(\delta - \nu + 1) \times (\delta - \nu - d_1 + 2)$ -matrix

$$M_\nu = \begin{pmatrix} U_0 & & 0 & & \\ & \ddots & & & \vdots \\ \vdots & & U_0 & & q_\beta(X) \\ & & & \ddots & \vdots \\ U_{d_1} & & & & \\ & & & \ddots & \\ & & & & U_{d_1} \end{pmatrix}.$$

$\underbrace{\hspace{10em}}_{\delta - \nu - d_1 + 1} \quad \underbrace{\hspace{2em}}_1$

built as follows: the left block of M_ν is the matrix of the multiplication map $C_{\delta - \nu - d_1} \rightarrow C_{\delta - \nu} : p \mapsto pf_1$ in the monomial bases ordered with the lexicographical order $Y_1 \succ Y_2$; the last column contains the coefficients of $\sum_{|\beta| = \delta - \nu} q_\beta(X) Y^\beta$ in the same ordered monomial basis $(Y^\beta)_{|\beta| = \delta - \nu}$.

Set $m = \delta - \nu - d_1 + 2$ and denote by $\Delta_{\alpha_1, \dots, \alpha_m}$, with $Y^{\alpha_1} \succ \dots \succ Y^{\alpha_m}$ and $|\alpha_i| = \delta - \nu$ for all $i = 1, \dots, m$, the determinant of the m -minor of M_ν corresponding to the rows of M_ν indexed by $(Y^{\alpha_i})_{i=1, \dots, m}$.

Theorem 2.5 ($d_1 - 1 \leq \nu \leq d_2 - 2$). *The minors $\Delta_{\alpha_1, \dots, \alpha_m} \in A[X_1, X_2]$ are independent of the choice of the decomposition (2.4) modulo the ideal (f_1, f_2) and are inertia forms of f_1, f_2 w.r.t. \mathfrak{m} of degree $\nu = |\alpha_1| = \dots = |\alpha_m|$.*

Moreover, if the inequality

$$(2.11) \quad \text{depth}_A(U_0, \dots, U_{d_1}) \geq d_1 + 1$$

holds, then the collection of minors $(\Delta_{\alpha_1, \dots, \alpha_m})_{Y^{\alpha_1} \succ \dots \succ Y^{\alpha_m}}$ is a system of generators of the A -module $H_{\mathfrak{m}}^0(B)_\nu$.

Proof. Let $(h_{i,j})_{i,j=1,2}$ and $(h'_{i,j})_{i,j=1,2}$ be two decompositions (2.4) and set

$$\det(h_{i,j}) = \sum_{|\beta| \leq \delta} q_\beta(X) Y^\beta, \quad \det(h'_{i,j}) = \sum_{|\beta| \leq \delta} q'_\beta(X) Y^\beta.$$

For any choice of sequences $(\alpha_1, \dots, \alpha_m)$ such that $Y^{\alpha_1} \succ \dots \succ Y^{\alpha_m}$ and $|\alpha_i| = \delta - \nu$ for all $i = 1, \dots, m$, we will denote by $\Delta_{\alpha_1, \dots, \alpha_m}$ and $\Delta'_{\alpha_1, \dots, \alpha_m}$ the determinants associated to the decompositions $(h_{i,j})_{i,j=1,2}$ and $(h'_{i,j})_{i,j=1,2}$ respectively.

By Section 2.1.3, we know that

$$\det(h_{i,j}) - \det(h'_{i,j}) = \sum_{|\beta| \leq \delta} (q_\beta(X) - q'_\beta(X)) Y^\beta \in (f_1(Y), f_2(Y)) \frac{A[X, Y]}{(f_1(X), f_2(X))}$$

and, by taking homogeneous components for $0 \leq \nu \leq \delta$, that

$$(2.12) \quad \sum_{|\beta| = \delta - \nu} (q_\beta(X) - q'_\beta(X)) Y^\beta \in (f_1(Y), f_2(Y))_{\delta - \nu} B[Y].$$

Assume now that $d_1 \leq \delta - \nu \leq d_2 - 1$. Since $f_1(Y)$ is not a zero-divisor in $B[Y]$, we have the exact sequence

$$(2.13) \quad 0 \rightarrow B[Y]_{\delta - \nu - d_1} \xrightarrow{\times f_1} B[Y]_{\delta - \nu} \rightarrow (B[Y]/(f_1(Y)))_{\delta - \nu} \rightarrow 0$$

and (2.12) implies that we also have the exact sequence

$$(2.14) \quad B[Y]_{\delta-\nu-d_1} \oplus B \xrightarrow{\tilde{M}_\nu} B[Y]_{\delta-\nu} \rightarrow B[Y](f_1(Y))_{\delta-\nu} \rightarrow 0,$$

\tilde{M}_ν being defined as the matrix M_ν where each element $q_\beta(X)$ in the last column is replaced by the difference $q_\beta(X) - q'_\beta(X)$ respectively. Therefore, the comparison of (2.13) and (2.14) shows, by invariance of Fitting ideals, that the class $\Delta_{\alpha_1, \dots, \alpha_m} - \Delta'_{\alpha_1, \dots, \alpha_m}$ in $B = A[X]/(f_1(X), f_2(X))$ is null. We deduce that $\Delta_{\alpha_1, \dots, \alpha_m}$ is independent of the choice of the decomposition (2.4) modulo (f_1, f_2) , as claimed.

Since $d_1 \leq \delta - \nu \leq d_2 - 1$, we have the exact sequence of A -modules

$$0 \rightarrow C_{\delta-\nu-d_1} \xrightarrow{\mathbf{f}_1} C_{\delta-\nu} \rightarrow B_{\delta-\nu} \rightarrow 0,$$

where \mathbf{f}_1 denotes the multiplication by f_1 and we deduce, by duality, that we have the exact sequence of A -modules

$$0 \rightarrow B_{\delta-\nu}^\vee \rightarrow C_{\delta-\nu}^\vee \xrightarrow{\mathbf{f}_1^\vee} C_{\delta-\nu-d_1}^\vee.$$

In particular, $B_{\delta-\nu}^\vee$ is isomorphic to the kernel of \mathbf{f}_1^\vee .

Now, consider the Buchsbaum-Rim complex associated to \mathbf{f}_1^\vee ; it is of the form

$$(2.15) \quad \cdots \rightarrow \bigwedge^m (C_{\delta-\nu}^\vee) \xrightarrow{\varepsilon_\nu} C_{\delta-\nu}^\vee \xrightarrow{\mathbf{f}_1^\vee} C_{\delta-\nu-d_1}^\vee \rightarrow 0$$

(recall $m = \delta - \nu - d_1 + 2$). Since it is a complex, the image of ε_ν is contained in $B_{\delta-\nu}^\vee$ and hence we can consider the composition map

$$\theta_\nu \circ \varepsilon_\nu : \bigwedge^m (C_{\delta-\nu}^\vee) \rightarrow H_m^0(B)_\nu.$$

Choosing a decomposition (2.4) and setting $\det(h_{i,j})_{i,j=1,2} = \sum_{|\beta| \leq \delta} q_\beta(X) Y^\beta$, we deduce that $\theta_\nu \circ \varepsilon_\nu$ sends the basis element $(Y^{\alpha_1})^\vee \wedge \cdots \wedge (Y^{\alpha_m})^\vee$, with $Y^{\alpha_1} \succ \cdots \succ Y^{\alpha_m}$, to the determinant $\Delta_{\alpha_1, \dots, \alpha_m}$ up to sign. Therefore, all the determinants $\Delta_{\alpha_1, \dots, \alpha_m}$ are inertia forms of degree ν , as claimed. Moreover, if (2.11) holds then (2.15) is acyclic since

$$(U_0, \dots, U_{d_1})^{\delta-\nu-d_1+1} = \text{Det}_{\delta-\nu-d_1+1} \left(C_{\delta-\nu-d_1} \xrightarrow{\times f_1} C_{\delta-\nu} \right).$$

Therefore

$$\text{depth}_A \left(\text{Det}_{\delta-\nu-d_1+1} \left(C_{\delta-\nu-d_1} \xrightarrow{\times f_1} C_{\delta-\nu} \right) \right) \geq d_1 + 1$$

and it follows that the image of ε_ν is exactly $B_{\delta-\nu}$. We hence deduce that $\theta_\nu \circ \varepsilon_\nu$ is surjective. \square

The case $d_1 = 1$ is particularly interesting because then the matrix M_ν is square for all ν (such that $d_1 - 1 = 0 \leq \nu \leq d_2 - 2 = \delta - 1$). Therefore, if (2.11) holds then $H_m^0(B)_\nu$ is a free A -module of rank 1 generated by $\det(M_\nu)$. In other words, for all $0 \leq \nu \leq d_2 - 2$ we have the isomorphism

$$A \xrightarrow{\sim} H_m^0(B)_\nu : a \mapsto a \det(M_\nu).$$

For the sake of completeness, we give an alternative construction of a system of generators of $H_m^0(B)$ in this case. It has been discovered independently in [7] and [10].

Since $d_1 = 1$, we have $f_1 = U_0X_1 + U_1X_2$. We know by (2.3) that the Sylvester form $h_\delta(f_1, f_2) = \text{sylv}_{(0,0)}(f_1, f_2) \in H_{\mathfrak{m}}^0(B)_\delta$ is a generator of $H_{\mathfrak{m}}^0(B)_\delta$. Now, for all integer $i = 1, \dots, \delta$ we define $h_{\delta-i}$ by induction with the formula

$$h_{\delta-i}(f_1, f_2) = \text{sylv}_{(0,0)}(f_1, h_{\delta-i+1}(f_1, f_2)) \in B_{\delta-i}.$$

Proposition 2.6 ($d_1 = 1, 0 \leq \nu \leq d_2 - 2 = \delta - 1$). *With the above notation, $h_\nu(f_1, f_2)$ is an inertia form of f_1, f_2 of degree ν and is equal to $\det(M_\nu)$ up to sign in $H_{\mathfrak{m}}^0(B)_\nu$. In particular, if $\text{depth}_A(U_0, U_1) \geq 2$ then h_ν is a generator of $H_{\mathfrak{m}}^0(B)_\nu$.*

Proof. By specialization, it is sufficient to prove the claimed equality in the generic case, so we assume that $A = \mathbb{Z}[U_0, U_1, V_0, \dots, V_{d_2}]$. In this case, $\text{depth}_A(U_0, U_1) \geq 2$ and hence $\det(M_\nu)$ is a generator of $H_{\mathfrak{m}}^0(B)_\nu$. By construction, we have, for all $\nu = 0, \dots, \delta - 1$,

$$(X_1, X_2)h_\nu \subset (f_1, h_{\nu+1}) \subset A[X_1, X_2].$$

Therefore, since $\text{sylv}_{(0,0)}(f_1, f_2) \in H_{\mathfrak{m}}^0(B)$, we deduce that h_ν , for all $\nu = 0, \dots, \delta$, is an inertia form of f_1, f_2 . Moreover, h_ν is homogeneous of degree ν in X_1, X_2 , hence $h_\nu \in H_{\mathfrak{m}}^0(B)_\nu$, and by construction h_ν and $\det(M_\nu)$ are both homogeneous of degree $\delta - \nu + 1$, resp. 1, in the variables U_0, U_1 , resp. V_0, \dots, V_{d_2} . It follows that there exists $a_\nu \in \mathbb{Z}$ such that $h_\nu = a_\nu \det(M_\nu)$ for all $\nu = 0, \dots, \delta - 1$. Finally, to prove that $a_\nu = \pm 1$ for all ν we observe that the specialization sending f_1 to X_1 and f_2 to $X_2^{d_2}$ sends h_ν to X_2^ν , for all $\nu = 0, \dots, \delta - 1$. The last statement of this corollary is contained in Theorem 2.5.

Observe that, as a consequence of this proof, $\det(M_0)$ and h_0 are both equal to $\text{Res}(f_1, f_2)$ up to sign. \square

Going back to the general setting of this paragraph, we now examine in more detail the case $\nu = 1$ to make a link with subresultants. This case may occur only if $d_1 = 1$ or $d_1 = 2$. Since we have already studied the case $d_1 = 1$ above, we concentrate on the case $d_1 = 2$.

So we assume that $f_1 = U_0X_1^2 + U_1X_1X_2 + U_2X_2^2$. For simplicity, we rename the inertia forms of degree $\nu = 1$, that is $\Delta_{\alpha_1, \dots, \alpha_{\delta-1}}(f_1, f_2)$ with $|\alpha_i| = \delta - 1 = d_2 - 1$. In this case, we have to consider the maximal $(d_2 - 1)$ -minors of the $(d_2 - 1) \times (d_2)$ -matrix

$$M_1 = \underbrace{\begin{pmatrix} U_0 & & 0 & & \\ & \ddots & & & \vdots \\ U_1 & & U_0 & & \\ & \ddots & & & q_\beta(X) \\ U_2 & & U_1 & & \vdots \\ & \ddots & & & \\ & & U_2 & & \end{pmatrix}}_{d_2-2} \underbrace{\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}}_1.$$

We define the inertia forms Δ_i , $i \in \{1, \dots, d-2\}$ by the formula ($\delta = d_2$ here)

$$\underbrace{\begin{pmatrix} U_0 & 0 & & T_0 \\ & \ddots & & \vdots \\ U_1 & U_0 & & \vdots \\ & \ddots & q_\beta(X) & \vdots \\ U_2 & U_1 & & \vdots \\ & \ddots & & \vdots \\ & & U_2 & T_{\delta-1} \end{pmatrix}}_{M_1} \underbrace{\quad}_1 = \sum_{i=0}^{\delta-1} \Delta_{\delta-i} T_i.$$

Lemma 2.7 ($d_1 = 2, \nu = 1$). *For all $i = 0, \dots, \delta - 1$, we have, in A ,*

$$\Delta_i = X_2 \text{SRes}_{i+1}(f_1, f_2) - X_1 \text{SRes}_i(f_1, f_2).$$

Proof. We claim that this is a consequence of Lemma 2.1 applied with

$$\varphi = \sum_{|\beta|=\delta-1} q_\beta(Y) X^\beta = \text{morl}_{\delta-1,1}(f_1, f_2).$$

Indeed, denoting $D(X)$ the determinant $D(1)$ in this case, the properties of Morley's forms imply that

$$\varphi D(X) = D(X) \text{morl}_{\delta-1,1}(f_1, f_2) = D(Y) \text{sylv}_{(0,0)}(f_1, f_2)(X)$$

in $C[\underline{W}][X, Y]/(f_1(X), f_2(X))(X)$. Then, Lemma 2.1 shows that

$$\varphi D(X) = -D_1(Y) \text{sylv}_{(0,0)}(f_1, f_2),$$

so by comparison of these two equalities and duality it follows that $D(Y) = D_1(Y)$ in $C[\underline{W}][Y]$. Therefore,

$$\sum_{i=0}^{\delta-1} D_i(Y) W_i = D(Y) = D_1(Y) = \sum_{i=0}^{\delta-1} \Delta_{\delta-i}(Y) W_i$$

and hence $\Delta_i = D_{\delta-i}$ for all $i = 0, \dots, \delta - 1$, as claimed. \square

2.4. Inertia forms of degree $\geq d_2 - 1$. This last case is the easiest one. For all integer ν such that $d_2 - 1 \leq \nu \leq \delta$, we have $0 \leq \delta - \nu \leq d_1 - 1$. Since I is generated in degree at least d_1 , we have a canonical isomorphism $C_{\delta-\nu} \simeq B_{\delta-\nu}$ and hence $C_{\delta-\nu}^\vee \simeq B_{\delta-\nu}^\vee$. Therefore, the morphism (2.5) is completely explicit as it is easy to find a basis of the A -module $C_{\delta-\nu}^\vee$; for instance the dual of the monomial basis $(X^\alpha)_{|\alpha|=\delta-\nu}$ of $C_{\delta-\nu}$.

Lemma 2.8. *For all $\nu \in \mathbb{N}$ such that $0 \leq \delta - \nu \leq d_1 - 1$, the following equality holds in $B_{\delta-\nu} \otimes_A B_\nu = C_{\delta-\nu} \otimes_A B_\nu$*

$$\text{morl}_{\delta-\nu,\nu}(f_1, f_2) = \sum_{|\alpha|=\delta-\nu} X^\alpha \otimes \text{sylv}_\alpha(f_1, f_2).$$

Proof. The proof of [11, Proposition 3.11.13] works verbatim. \square

It follows that for all integers ν such that $d_2 - 1 \leq \nu \leq \delta$, the isomorphism (2.5) is given by

$$\begin{aligned} \theta_\nu : C_{\delta-\nu}^\vee &\xrightarrow{\sim} H_{\mathfrak{m}}^0(B)_\nu \\ (X^\alpha)^\vee &\mapsto \text{sylv}_\alpha(f_1, f_2) \end{aligned}$$

and we have the

Theorem 2.9 ($d_2 - 1 \leq \nu \leq \delta$). *The collection of all the Sylvester forms of degree $\delta - \nu$, that is $(\text{sylv}_\alpha(f_1, f_2))_{|\alpha|=\delta-\nu}$, yields a A -basis of $H_{\mathfrak{m}}^0(B)_\nu$.*

Finally, as we did in the previous sections we explicit the case $\nu = 1$ for further purposes. Here, the only interesting situation occurs when $d_1 = d_2 = 2$ and we have

(2.16)

$$\begin{aligned} \text{sylv}_{0,1}(f_1, f_2) &= \begin{vmatrix} U_0 & U_2 \\ V_0 & V_2 \end{vmatrix} X_1 + \begin{vmatrix} U_1 & U_2 \\ V_1 & V_2 \end{vmatrix} X_2 = \text{SRes}_2(f_1, f_2)X_2 - \text{SRes}_1(f_1, f_2)X_1, \\ \text{sylv}_{1,0}(f_1, f_2) &= \begin{vmatrix} U_0 & U_1 \\ V_0 & V_1 \end{vmatrix} X_1 + \begin{vmatrix} U_0 & U_2 \\ V_0 & V_2 \end{vmatrix} X_2 = \text{SRes}_0(f_1, f_2)X_1 - \text{SRes}_1(f_1, f_2)X_2 \end{aligned}$$

3. EQUATIONS OF THE MOVING CURVE IDEAL

We take again the parametrization (1.1)

$$\begin{aligned} \mathbb{P}_{\mathbb{K}}^1 &\xrightarrow{\phi} \mathbb{P}_{\mathbb{K}}^2 \\ (X_1 : X_2) &\mapsto (g_1 : g_2 : g_3)(X_1, X_2). \end{aligned}$$

Without loss of generality, we will assume hereafter that *the greatest common divisor of g_1, g_2, g_3 over $\mathbb{K}[X_1, X_2]$ is a non-zero constant in \mathbb{K}* . Moreover, we will restrict our study to the case of interest where the algebraic curve \mathcal{C} , image of ϕ , has degree at least 2.

Let $p(\underline{X}, \underline{T}), q(\underline{X}, \underline{T})$ be a μ -basis of the parametrization ϕ , where p , resp. q , has degree μ , resp. $d - \mu$, in the variables X_1, X_2 . By Proposition 1.1, the moving curve ideal of ϕ is equal to the ideal of inertia forms of p, q with respect to the ideal (X_1, X_2) . Therefore, the results developed in Section 2 can be used to give some of the generators of the moving curve ideal of ϕ , and sometimes a whole system of generators. To proceed, we set

$$(3.1) \quad \begin{aligned} p &= U_0(\underline{T})X_1^\mu + \cdots + U_\mu(\underline{T})X_2^\mu, \\ q &= V_0(\underline{T})X_1^{d-\mu} + \cdots + V_{d-\mu}(\underline{T})X_2^{d-\mu}, \end{aligned}$$

where U_0, \dots, U_μ and $V_0, \dots, V_{d-\mu}$ are linear forms in $A = \mathbb{K}[T_1, T_2, T_3]$ and assume, without loss of generality, that $1 \leq \mu \leq d - \mu$. We also define $C = A[X_1, X_2]$, $\mathfrak{m} = (X_1, X_2)$, $\delta = \mu + (d - \mu) - 2 = d - 2$ and consider the graded quotient ring $B = C/(p, q)$.

3.1. The case $\mu = 1$. From Section 2, we have the following list of inertia forms of p and q :

$$(3.2) \quad \begin{cases} \det(M_\nu(p, q)) \in H_{\mathfrak{m}}^0(B)_\nu & \text{for } 0 \leq \nu \leq \delta - 1, \text{ see Theorem 2.5,} \\ \text{sylv}_{(0,0)}(p, q) \in H_{\mathfrak{m}}^0(B)_\delta & \text{for } \nu = \delta, \text{ see Theorem 2.9.} \end{cases}$$

It turns out that this collection of inertia forms always give a system of generators of the moving curve ideal of ϕ , as conjectured in [10, Conjecture 4.8] and proved in [7, Theorem 2.2].

Proposition 3.1 ($\mu = 1$). *The two polynomials p, q and the collection of inertia forms (3.2) form a system of generators of the moving curve ideal of ϕ .*

Proof. According to Theorem 2.5 and Theorem 2.9, we only have to prove that $\text{depth}_A(U_0, U_1) \geq 2$. This inequality is a direct consequence of the facts that $\text{deg}(\mathcal{C}) \geq 2$ and that p is by definition a syzygy of minimal degree of g_1, g_2, g_3 . \square

3.2. The case $\mu = 2$. From Section 2, we have the following list of inertia forms of p and q :

$$(3.3) \quad \begin{cases} \text{Res}(p, q) & \text{for } \nu = 0, \text{ see Theorem 2.2,} \\ \Delta_{\alpha_1, \dots, \alpha_{\delta-\nu}}(p, q), |\alpha_i| = \delta - \nu & \text{for } 1 \leq \nu \leq \delta - 2, \text{ see Theorem 2.5,} \\ \text{sylv}_\alpha(p, q), |\alpha| = \delta - \nu & \text{for } \delta - 1 \leq \nu = \delta, \text{ see Theorem 2.9.} \end{cases}$$

Under suitable conditions, this collection of inertia forms gives a system of generators of the moving curve ideal of ϕ . The following proposition can be seen as a generalization of [10, Proposition 4.9] which deals with the case $d = 5$.

Proposition 3.2 ($\mu = 2$). *If $\text{deg}(\phi) = 1$ and $d = 4$ then p, q and the collection of inertia forms (3.3) form a system of generators of the moving curve ideal of ϕ . Moreover, the same result holds if $\text{deg}(\phi) = 1$, $d > 4$ and $V(U_0, U_1, U_2) = \emptyset \subset \mathbb{P}_{\mathbb{K}}^2$.*

Proof. If $\text{deg}(\phi) = 1$ then Equation (2.8) implies that

$$\text{depth}(\text{SRes}_0(p, q), \dots, \text{SRes}_\delta(p, q)) \geq 2.$$

Indeed, $\text{Res}(p, q)$ is an implicit equation of the curve \mathcal{C} and it is irreducible. Therefore, $\text{Res}(p, q)$ is a generator of $H_m^0(B)_0$ by Theorem 2.2.

Now, since $V(U_0, U_1, U_2) = \emptyset$ we deduce that $\text{depth}_{\mathbb{K}[\underline{T}]}(U_0, U_1, U_2) \geq 3$ and hence, from Theorem 2.5, that the collection of inertia forms $\Delta_{\alpha_1, \dots, \alpha_{\delta-\nu}}(p, q)$, $|\alpha_i| = \delta - \nu$ is a system of generators of $H_m^0(B)_\nu$ for all $1 \leq \nu \leq \delta - 2$. Finally, Theorem 2.9 shows that $\text{sylv}_\alpha(p, q)$, $|\alpha| = \delta - \nu$, is a system of generators of $H_m^0(B)_\nu$ for $\nu = \delta - 1$ and $\nu = \delta$. \square

Two comments are in order here. First, the hypothesis $V(U_0, U_1, U_2) = \emptyset$, implicitly assumed in [10, Proposition 4.9], is not superfluous since otherwise there exist some counter examples. Also, mention that this latter condition corresponds to the geometric property that there is no singular point on the curve \mathcal{C} of multiplicity $d - 2$, the maximum possible value for a singular point on \mathcal{C} in this case by [14, Theorem 3].

Secondly, we showed that if $\text{deg}(\phi) = 1$, i.e. ϕ is birational onto \mathcal{C} , then the greatest common divisor of the subresultants $\text{SRes}_i(p, q)$, $i = 0, \dots, d - 2$ is a non-zero constant. We can actually prove along the same line in [3] that this is an equivalence. Moreover, in this case the inertia forms $X_1 \text{SRes}_i(p, q) - X_2 \text{SRes}_{i+1}(p, q)$, $i = 0, \dots, d - 3$, yield rational maps from \mathbb{P}^2 to \mathbb{P}^1 that all induce the inverse of the parametrization ϕ .

3.3. The case $\mu \geq 3$. From Section 2, we have the following list of inertia forms of p and q :

$$(3.4) \quad \begin{cases} D_\beta(p, q), |\beta| = \nu & \text{for } 0 \leq \nu \leq \mu - 2, \text{ Theorem 2.2,} \\ \Delta_{\alpha_1, \dots, \alpha_m}(p, q), |\alpha_i| = \delta - \nu & \text{for } \mu - 1 \leq \nu \leq d - \mu - 2, \text{ Theorem 2.5,} \\ \text{sylv}_\alpha(p, q), |\alpha| = \delta - \nu & \text{for } d - \mu - 1 \leq \nu = \delta, \text{ Theorem 2.9.} \end{cases}$$

By Theorem 2.9, we know that the inertia forms $\text{sylv}_\alpha(p, q), |\alpha| = \delta - \nu$, form a system of generators for $H_m^0(B)_\nu$ for $d - \mu - 1 \leq \nu = \delta$. Also, by Theorem 2.2, the collection of inertia forms $D_\beta(p, q), |\beta| = \nu$, form a system of generators of $H_m^0(B)_\nu$ for all $\nu \leq \mu - 2$ providing

$$(3.5) \quad \text{depth}_A \left(C_{d-2\mu} \oplus A \xrightarrow{(p \ q)} C_{d-\mu} \right) \geq \mu$$

(in particular ϕ has to be birational onto \mathcal{C}). The latter inequality can only be satisfied if $\mu = 3$ since $\text{depth}_A(T_1, T_2, T_3) = 3$. By Theorem 2.5, the collection of inertia forms $\Delta_{\alpha_1, \dots, \alpha_m}(p, q)$, with $m = \delta - \nu - \mu + 2$ and $|\alpha_i| = \delta - \nu$, is a system of generators for $H_m^0(B)_\nu$ for $\mu - 1 \leq \nu \leq d - \mu - 2$ if the inequality $\text{depth}_A(U_0, \dots, U_\mu) \geq \mu + 1 \geq 4$ holds. But such an inequality never holds in A .

3.4. Inertia forms of degree 1. We finally gather the results concerning the inertia forms of degree 1 for we will handle them in the next section.

If $\mu = 1$ we have seen that $H_m^0(B)_1$ is isomorphic to A and hence generated by a unique determinant, or equivalently by an iterated Sylvester form.

If $\mu \geq 2$ the inertia forms of degree 1 that we have explicated are always built from subresultants. More precisely, we proved that for all $i = 0, \dots, \delta - 1 = d - 3$, the polynomials

$$X_1 \text{SRes}_i(p, q) - X_2 \text{SRes}_{i+1}(p, q)$$

are inertia forms of degree 1. Moreover, they generate $H_m^0(B)_1$

- if $\mu = 2$ and $d = 4$ (see Equations (2.16)),
- if $\mu = 2, d \geq 5$ and $V(U_0, U_1, U_2) = \emptyset \subset \mathbb{P}^2$,
- if $\mu \geq 3$ and

$$\text{depth}_A \text{Det}_{d-4} \left(\begin{array}{cccc} U_0 & 0 & V_0 & 0 \\ & \ddots & & \ddots \\ \vdots & U_0 & \vdots & V_0 \\ U_\mu & \vdots & V_{d-\mu} & \vdots \\ & \ddots & & \ddots \\ 0 & U_\mu & 0 & V_{d-\mu} \end{array} \right) \geq 3,$$

$\underbrace{\hspace{10em}}_{d-\mu-2} \quad \underbrace{\hspace{10em}}_{\mu-2}$

(notice that the above matrix has $d - 2$ rows).

As we will mention in the next section, there are examples that show that such conditions are necessary.

4. ADJOINT LINEAR SYSTEMS

As in Section 3, we take again the parametrization (1.1)

$$\begin{aligned} \mathbb{P}_{\mathbb{K}}^1 &\xrightarrow{\phi} \mathbb{P}_{\mathbb{K}}^2 \\ (X_1 : X_2) &\mapsto (g_1 : g_2 : g_3)(X_1, X_2) \end{aligned}$$

and assume moreover that \mathbb{K} is an algebraically closed field. We recall that $p(\underline{X}, \underline{T})$, $q(\underline{X}, \underline{T})$ denote a μ -basis of the parametrization ϕ , where p , resp. q , has degree μ , resp. $d - \mu$, in the variables X_1, X_2 and $1 \leq \mu \leq d - \mu$.

Since \mathcal{C} is a rational plane curve, it is well-known that the genus of \mathcal{C} is null, that is to say that

$$(4.1) \quad \frac{(d-1)(d-2)}{2} = \sum_{\mathfrak{p} \in \text{Sing}(\mathcal{C})} \frac{m_{\mathfrak{p}}(m_{\mathfrak{p}}-1)}{2}$$

where the sum is over all the singular points, ordinary as well as infinitely near, of \mathcal{C} and $m_{\mathfrak{p}}$ denotes the multiplicity of \mathcal{C} at \mathfrak{p} (see for instance [1, Lectures 18 & 19]). To distinguish the infinitely near singularities of \mathcal{C} , we call an ordinary singularity of \mathcal{C} a usual singular point of \mathcal{C} in the $(T_1 : T_2 : T_3)$ -projective plane.

Definition 4.1. Given an ordinary singular point \mathfrak{p} of \mathcal{C} with multiplicity $m_{\mathfrak{p}}$, an algebraic curve \mathcal{D} is said to be *adjoint* to \mathcal{C} at \mathfrak{p} if \mathcal{D} has multiplicity at least $m_{\mathfrak{p}} - 1$ at \mathfrak{p} and if at all singular point \mathfrak{p}' of \mathcal{C} infinitely near to \mathfrak{p} with multiplicity $m_{\mathfrak{p}'}$, the curve \mathcal{D} has multiplicity at least $m_{\mathfrak{p}'} - 1$.

An algebraic curve \mathcal{D} is said to be *adjoint* to \mathcal{C} if it is adjoint to \mathcal{C} at every ordinary singularity of \mathcal{C} .

The curve \mathcal{C} being rational, it can be shown that curves adjoint to \mathcal{C} of degree $\leq d - 3$ do not exist, whereas curves adjoint to \mathcal{C} of degree $\geq d - 2$ are guaranteed to exist. Of course, the character of curves adjoint to \mathcal{C} of degree $d - 2$ and $d - 1$ is particularly interesting.

An *adjoint linear system* on \mathcal{C} of degree m is a one-parameter family of curves adjoint to \mathcal{C} of degree m . It is hence of the form $X_1 D_1(\underline{T}) + X_2 D_2(\underline{T})$ where D_1, D_2 are homogeneous polynomials in $\mathbb{K}[\underline{T}]$ of degree m . Recently, David Cox noticed that moving curves of ϕ following \mathcal{C} of degree $d - 2$ (resp. $d - 1$) that are linear in X_1, X_2 sometimes give adjoint linear systems on \mathcal{C} (we refer the reader to [8, Conjecture 3.8 and Remark 3.9] for precise statements). Denote by $\mathcal{L}_{d-2}(\phi)$ (resp. $\mathcal{L}_{d-1}(\phi)$) the finite \mathbb{K} -submodule of the moving curve ideal of ϕ consisting in moving curves of degree $d - 2$ (resp. $d - 1$) that are linear in X_1, X_2 . In what follows, using the results of Section 3 we determine explicit moving curves in $\mathcal{L}_{d-2}(\phi)$ and $\mathcal{L}_{d-1}(\phi)$ that give adjoint linear systems on \mathcal{C} . Point out that a similar study could be done for moving curves of degree 2, 3, etc in X_1, X_2 using the same approach. However, we will stick to the case of moving curves linear in X_1, X_2 because of its geometric content.

According to the notation of Section 3, the study of $\mathcal{L}_{d-2}(\phi)$ and $\mathcal{L}_{d-1}(\phi)$ relies on the study of the elements in $H_{\mathfrak{m}}^0(B)_1$, where $A = \mathbb{K}[\underline{T}]$, $B = A[X_1, X_2]/(p, q)$ and $\mathfrak{m} = (X_1, X_2)$, that are homogeneous of degree $d - 2$ or $d - 1$ in the variables T_1, T_2, T_3 . In the rest of the paper, we will always assume that $d \geq 3$ and that $\deg(\phi) = 1$, that is ϕ is birational onto \mathcal{C} . These two conditions are not restrictive. Indeed, a rational curve of degree ≤ 2 has no singular point and, as a consequence

of Luröth's Theorem, any rational curve can be properly re-parametrized. The reason for assuming $\deg(\phi) = 1$ is because in this case the inertia forms of degree 0 of p, q with respect to the ideal (X_1, X_2) are generated by $\text{Res}(p, q) \in \mathbb{K}[\underline{T}]$ which is irreducible and homogeneous of degree d . Therefore, the inertia forms of degree 0 of p, q w.r.t. X_1, X_2 do not contribute to $\mathcal{L}_{d-2}(\phi)$ or $\mathcal{L}_{d-1}(\phi)$.

We begin with the simple case $\mu = 1$ before turning to the case $\mu \geq 2$ for which we will need to give another characterization of adjoint curves.

4.1. The case $\mu = 1$. The μ -basis associated to the parametrization ϕ of \mathcal{C} is of the form

$$\begin{aligned} p(\underline{X}, \underline{T}) &= \sum_{i=1}^3 p_i(\underline{X})T_i = U_0(\underline{T})X_1 + U_1(\underline{T})X_2, \\ q(\underline{X}, \underline{T}) &= \sum_{i=1}^3 q_i(\underline{X})T_i = V_0(\underline{T})X_1^{d-1} + V_1(\underline{T})X_1^{d-2}X_2 + \cdots + V_{d-1}(\underline{T})X_2^{d-1}. \end{aligned}$$

Lemma 4.2 ($\mu = 1, d \geq 3$). *The curve \mathcal{C} has a unique singular point \mathfrak{p} , ordinary as well as infinitely near, of multiplicity $d - 1$. Moreover, $U_0(\mathfrak{p}) = U_1(\mathfrak{p}) = 0$.*

Proof. Since $d \geq 3$, Equation (4.1) implies that there exists at least one singular point on \mathcal{C} . By a suitable linear change of coordinates one may assume that this point is at the origin: $\mathfrak{p} = (0 : 0 : 1)$. Then, we claim that $p_3 = 0$. If this is true, clearly $U_0(\mathfrak{p}) = U_1(\mathfrak{p}) = 0$ and we will have

$$C(T_1, T_2, T_3) = \text{Res}(p, q) = \text{Res}(p_1T_1 + p_2T_2, q)$$

that shows that $\text{ord}_{\mathfrak{p}}C(T_1, T_2, T_3) \geq d - 1$, i.e. \mathfrak{p} is a singular point of multiplicity $\geq d - 1$. Then, it will follow by (4.1) that \mathfrak{p} has multiplicity exactly $d - 1$ and that it is the unique singular point of \mathcal{C} .

To prove that $p_3 = 0$ we proceed by contradiction and assume that $p_3 \neq 0$. Since $C(\mathfrak{p}) = 0$, we deduce that $\text{Res}(p_3, q_3) = 0$ and hence that p_3 divides q_3 . Therefore, by a change of μ -basis if necessary, we can assume that $q_3 = 0$. But then, by inspecting the Sylvester matrix of p and q we have

$$C(T_1, T_2, 1) = \text{Res}(p, q) = \text{Res}(p_3, q_1T_1 + q_2T_2) + R(T_1, T_2)$$

where $\text{ord}_{\mathfrak{p}}R(T_1, T_2) \geq 2$. Since \mathfrak{p} is a singular point, the term $\text{Res}(p_3, q_1T_1 + q_2T_2)$ must vanish, that is p_3 must divide $q = q_1T_1 + q_2T_2$, a contradiction with the fact that the couple (p, q) is a μ -basis of ϕ . \square

In Section 2.3 we have defined the matrix

$$M_1(p, q) = \underbrace{\begin{pmatrix} U_0 & & 0 & & \vdots \\ & \ddots & & & \\ U_1 & & U_0 & & q_\beta(X) \\ & \ddots & & & \\ 0 & & U_1 & & \vdots \end{pmatrix}}_{d-3} \underbrace{\begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}}_1$$

where $\sum_{|\beta|=d-3} q_\beta(\underline{X})Y^\beta = \text{mor}_{d-3,1}(p, q)$, and we have proved that

$$\det(M_1(p, q)) \in \mathcal{L}_{d-1}(\phi).$$

The two following propositions recover [7, Theorem 3.2] – see also Proposition 2.6.

Proposition 4.3 ($\mu = 1, d = 3$). *The element p is a \mathbb{K} -basis of $\mathcal{L}_1(\phi)$ and gives an adjoint linear system on \mathcal{C} . Moreover, any element in $\mathcal{L}_2(\phi)$, which is \mathbb{K} -generated by T_1p, T_2p, T_3p and $\text{sylv}_{(0,0)}(p, q)$, gives an adjoint linear system on \mathcal{C} .*

Proof. From Section 3.1, we know that $\mathcal{L}_1(\phi) = \langle p \rangle_{\mathbb{K}}$ and that

$$\mathcal{L}_2(\phi) = \langle T_1p, T_2p, T_3p, \text{sylv}_{(0,0)}(p, q) \rangle_{\mathbb{K}}.$$

By Lemma 4.2, \mathfrak{p} is the unique singular point of \mathcal{C} , it has multiplicity 2 and $U_0(\mathfrak{p}) = U_1(\mathfrak{p}) = 0$. Moreover, $\text{sylv}_{(0,0)}(p, q) \in (U_0, U_1)$ by construction and hence it also vanishes at \mathfrak{p} . \square

Proposition 4.4 ($\mu = 1, d > 3$). *The element $\det(M_1(p, q)) \in \mathcal{L}_{d-1}(\phi)$ gives an adjoint linear system on \mathcal{C} .*

Proof. By construction $q_\beta \in (U_0, U_1)$. Therefore, from the definition of $M_1(p, q)$ we deduce that

$$\det(M_1(p, q)) \in (U_0, U_1)^{d-2}.$$

Now, if \mathfrak{p} be a singular point of \mathcal{C} , then by Lemma 4.2 \mathfrak{p} is unique with multiplicity $d - 1$ and $U_0(\mathfrak{p}) = U_1(\mathfrak{p}) = 0$. Therefore, $\det(M_1(p, q))$ vanishes at \mathfrak{p} with multiplicity at least $d - 2$. Notice that $\det(M_1(p, q)) \neq 0$ for it is a generator of $H_m^0(B)_1$ which can not be zero since $\text{Res}(p, q) \in H_m^0(B)_0$ is non-zero. \square

In general, if $d > 3$ and $\mu = 1$ an element of $\mathcal{L}_{d-2}(\phi) \oplus_{\mathbb{K}} \mathcal{L}_{d-1}(\phi)$ is not an adjoint linear system on \mathcal{C} , but one can always find one, namely $\det(M_1(p, q))$. Indeed, from Section 3.1 we have

$$\mathcal{L}_{d-2} = \langle (\underline{T}^\alpha p)_{|\alpha|=d-3} \rangle_{\mathbb{K}} \quad \text{and} \quad \mathcal{L}_{d-1} = \langle (\underline{T}^\alpha p)_{|\alpha|=d-2}, \det(M_1(p, q)) \rangle_{\mathbb{K}}.$$

So the element p of the μ -basis has degree $\mu = 1$ and hence contribute to $\mathcal{L}_{d-2}(\phi) \oplus_{\mathbb{K}} \mathcal{L}_{d-1}(\phi)$ without producing an adjoint linear system on \mathcal{C} (see e.g. [8, Example 3.7]).

4.2. Adjoint and polar curves. Instead of using directly Definition 4.1 to prove that a certain curve \mathcal{D} is adjoint to \mathcal{C} , we will use a property of adjoint curves that allows to prove that \mathcal{D} is adjoint to \mathcal{C} by looking at the intersection of \mathcal{C} and \mathcal{D} at all the ordinary singularities. This approach has the advantage to avoid the consideration of the infinitely near singularities of \mathcal{C} through a desingularization process of \mathcal{C} . To state this property, we first need to fix some notation.

Given two plane curves \mathcal{D} and \mathcal{D}' that intersect in a finite set of points, we denote by $\text{mult}_{\mathfrak{p}}(\mathcal{D}, \mathcal{D}')$ the intersection multiplicity of \mathcal{D} and \mathcal{D}' at the point \mathfrak{p} , and by $\text{mult}_{\mathfrak{p}}(\mathcal{D})$ the multiplicity of \mathcal{D} at \mathfrak{p} . Recall that

$$\text{mult}_{\mathfrak{p}}(\mathcal{D}) = \min_{\mathcal{L} \text{ line through } \mathfrak{p}} \text{mult}_{\mathfrak{p}}(\mathcal{D}, \mathcal{L})$$

where the minimum is taken over all the lines \mathcal{L} passing through the point \mathfrak{p} , and also that

$$\text{mult}_{\mathfrak{p}}(\mathcal{D}) = \text{ord}_{\mathfrak{p}} D(T_1, T_2, T_3)$$

where $D(T_1, T_2, T_3)$ is an implicit equation of \mathcal{D} .

Given $\gamma = (\alpha(x), \beta(x))$ a branch of \mathcal{D} centered at \mathfrak{p} , the intersection multiplicity of γ and \mathcal{D}' at \mathfrak{p} is defined as

$$\text{mult}_{\mathfrak{p}}(\gamma, \mathcal{D}') = \text{ord}_{\mathfrak{p}} D'(\alpha(x), \beta(x)).$$

Then, the multiplicity of γ at \mathfrak{p} is

$$\text{mult}_{\mathfrak{p}}(\gamma) = \min_{\mathcal{L} \text{ line through } \mathfrak{p}} \text{mult}_{\mathfrak{p}}(\gamma, \mathcal{L})$$

and $\text{mult}_{\mathfrak{p}}(\mathcal{D})$ is equal to the sum of the multiplicities of the branches of \mathcal{D} .

Definition 4.5. Suppose given a curve $\mathcal{D} \subset \mathbb{P}^2$ with equation $D(T_1, T_2, T_3) = 0$ and a point $\mathfrak{q} = (\mathfrak{q}_1 : \mathfrak{q}_2 : \mathfrak{q}_3) \in \mathbb{P}^2$. The *polar curve* of \mathcal{C} w.r.t. \mathfrak{q} is the curve defined by the equation

$$\mathfrak{q}_1 \frac{\partial D}{\partial T_1} + \mathfrak{q}_2 \frac{\partial D}{\partial T_2} + \mathfrak{q}_3 \frac{\partial D}{\partial T_3} = 0.$$

We can now state the following characterization of adjoint curves.

Proposition 4.6 ([6, Theorem 6.3.1]). *Let $\mathcal{C} \subset \mathbb{P}^2$ be a curve, $\mathfrak{q} \in \mathbb{P}^2$ be a point not lying on \mathcal{C} and $\mathcal{P}_{\mathfrak{q}}$ be the polar curve of \mathcal{C} w.r.t. \mathfrak{q} . A curve $\mathcal{D} \subset \mathbb{P}^2$ is adjoint to \mathcal{C} if*

$$\text{mult}_{\mathfrak{p}}(\gamma, \mathcal{D}) \geq \text{mult}_{\mathfrak{p}}(\gamma, \mathcal{P}_{\mathfrak{q}}) - \text{mult}_{\mathfrak{p}}(\gamma, \mathcal{L}_{\mathfrak{q}}) + 1$$

for all singular point \mathfrak{p} of \mathcal{C} and all branch γ of \mathcal{C} centered at \mathfrak{p} , where $\mathcal{L}_{\mathfrak{q}}$ denotes the line joining the points \mathfrak{p} and \mathfrak{q} .

It should be noticed that the quantity $\text{mult}_{\mathfrak{p}}(\gamma, \mathcal{P}_{\mathfrak{q}}) - \text{mult}_{\mathfrak{p}}(\gamma, \mathcal{L}_{\mathfrak{q}})$ is independent of the choice of the point $\mathfrak{q} \notin \mathcal{C}$. We also recall that a birational parametrization of a plane algebraic curve gives naturally parametrizations for all the branch curves. In particular, the number of irreducible branches at a singular point \mathfrak{p} is in correspondence with the number of its distinct pre-images under the parametrization (see for instance [6, Proposition 3.7.8]).

4.3. The case $\mu \geq 2$. An implicit equation of the curve \mathcal{C} of degree d is given by $\text{Res}(p, q) \in \mathbb{K}[\underline{T}]$, where (p, q) is a μ -basis. The next result shows that the first-order subresultants $\text{SRes}_i(p, q)$, $i = 0, \dots, d-2$, define curves of degree $d-2$ that are adjoint to \mathcal{C} (notice that since $2 \leq \mu \leq d-\mu$, we must have $d \geq 4$). To prove this, we will need the following lemma that can be found in [5, Lemma 5.1]; we include the proof for the convenience of the reader.

Lemma 4.7. *Suppose given two homogeneous polynomials*

$$\begin{aligned} g_1(X_1, X_2) &= a_{d_1} X_1^{d_1} + a_{d_1-1} X_1^{d_1-1} X_2 + \dots + a_1 X_1 X_2^{d_1-1} + a_0 X_2^{d_1}, \\ g_2(X_1, X_2) &= b_{d_2} X_1^{d_2} + b_{d_2-1} X_1^{d_2-1} X_2 + \dots + b_1 X_1 X_2^{d_2-1} + b_0 X_2^{d_2}, \end{aligned}$$

of degree $d_1, d_2 \geq 2$, respectively, and with coefficients a_i 's and b_j 's in $R[T]$ where R is a commutative ring. Then, we have the following equality in $R[T, x]$:

$$\begin{aligned} \frac{\partial \text{Res}(g_1, g_2)}{\partial T} = \\ (-1)^{d_1+d_2} \left| \begin{array}{cc} \frac{\partial g_1}{\partial T}(x, 1) & \frac{\partial g_2}{\partial T}(x, 1) \\ \frac{\partial g_1}{\partial X_1}(x, 1) & \frac{\partial g_2}{\partial X_1}(x, 1) \end{array} \right| \text{SRes}_0(g_1, g_2) \text{ modulo } (g_1(x, 1), g_2(x, 1)). \end{aligned}$$

Proof. Consider the polynomials

$$\begin{aligned} g_1(X_1 + xX_2, X_2) &= a_{d_1}^x X_1^{d_1} + a_{d_1-1}^x X_1^{d_1-1} X_2 + \dots + a_1^x X_1 X_2^{d_1-1} + a_0^x X_2^{d_1}, \\ g_2(X_1 + xX_2, X_2) &= b_{d_2}^x X_1^{d_2} + b_{d_2-1}^x X_1^{d_2-1} X_2 + \dots + b_1^x X_1 X_2^{d_2-1} + b_0^x X_2^{d_2}, \end{aligned}$$

where the a_i^x 's and the b_j^x 's are polynomials in $R[T, x]$. By the base change formula for subresultants [9], we have the equalities in $R[T, x]$:

$$\begin{aligned} \operatorname{Res}(g_1(X_1 + xX_2, X_2), g_2(X_1 + xX_2, X_2)) &= \operatorname{Res}(g_1(X_1, X_2), g_2(X_1, X_2)), \\ \operatorname{SRes}_0(g_1(X_1 + xX_2, X_2), g_2(X_1 + xX_2, X_2)) &= \operatorname{SRes}_0(g_1(X_1, X_2), g_2(X_1, X_2)). \end{aligned}$$

Therefore, expanding the determinant

$$\operatorname{Res}(g_1(X_1 + xX_2, X_2), g_2(X_1 + xX_2, X_2)) = \begin{vmatrix} a_{d_1}^x & 0 & \cdots & 0 & b_{d_2}^x & 0 & 0 \\ a_{d_1-1}^x & a_{d_1}^x & & \vdots & b_{d_2-1}^x & \ddots & 0 \\ \vdots & & \ddots & 0 & \vdots & & b_{d_2}^x \\ a_0^x & & & a_{d_1}^x & b_1^x & & b_{d_2-1}^x \\ 0 & a_0^x & & a_{d_1-1}^x & b_0^x & & \vdots \\ \vdots & & \ddots & \vdots & 0 & \ddots & b_1^x \\ 0 & \cdots & 0 & a_0^x & 0 & 0 & b_0^x \end{vmatrix}$$

with respect to its two last rows, we get

$$\operatorname{Res}(g_1, g_2) = (-1)^{d_1+d_2} \begin{vmatrix} a_0^x & b_0^x \\ a_1^x & b_1^x \end{vmatrix} \operatorname{SRes}_0(g_1, g_2) + (a_0^x)^2 \Delta_1 + a_0^x b_0^x \Delta_2 + (b_0^x)^2 \Delta_3,$$

where Δ_i ($i = 1, 2, 3$) are polynomials in the a_i^x 's and b_j^x 's. Taking the derivative with respect to the variable T , we deduce that

$$(4.2) \quad \frac{\partial \operatorname{Res}(g_1, g_2)}{\partial T} = (-1)^{d_1+d_2} \begin{vmatrix} \partial_2 a_0^x & \partial_2 b_0^x \\ a_1^x & b_1^x \end{vmatrix} \operatorname{SRes}_0(g_1, g_2) \text{ modulo } (a_0^x, b_0^x)$$

in $R[T, x]$. But it is easy to check that

$$a_0^x = g_1(x, 1), \quad b_0^x = g_2(x, 1), \quad a_1^x = \frac{\partial g_1}{\partial X_1}(x, 1), \quad b_1^x = \frac{\partial g_2}{\partial X_1}(x, 1)$$

and therefore to deduce that (4.2) is the claimed equality. \square

Theorem 4.8 ($\mu \geq 2, d \geq 4$). *For all $i = 0, \dots, d-2$, the equation $\operatorname{SRes}_i(p, q) = 0$ defines a plane curve which is adjoint to \mathcal{C} .*

Proof. Recall that the curve \mathcal{C} of degree d is parametrized by the generically injective rational map

$$\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^2 : (X_1 : X_2) \mapsto (g_1 : g_2 : g_3)(X_1, X_2)$$

and that (p, q) stands for a μ -basis of ϕ . Hereafter, we will denote by \mathcal{D} the curve defined by $\operatorname{SRes}_0(p, q) = 0$. It is well defined by properties of the first principal subresultant and the μ -basis. We first prove that the curve \mathcal{D} of equation $\operatorname{SRes}_0(p, q) = 0$ is adjoint to \mathcal{C} by using the characterization of adjoint curves given in Proposition 4.6.

So let \mathfrak{p} be a singular point of \mathcal{C} and γ be an irreducible branch of \mathcal{C} centered at \mathfrak{p} . By a linear change of coordinates in \mathbb{P}^2 , we can assume that \mathfrak{p} is the origin $(0 : 0 : 1)$ and that the point $\mathfrak{q} = (0 : 1 : 0)$ does not belong to \mathcal{C} . Also, by a linear change of coordinates in \mathbb{P}^1 , we can assume that $\phi(0 : 1) = \mathfrak{p}$; turning to the affine parameter $(x : 1) \in \mathbb{P}^1$ with $x \in \mathbb{K}$, we have $\phi(0) = \mathfrak{p}$ and we can assume that

$$\operatorname{ord}_{\mathfrak{p}} \phi_1(x) \leq \operatorname{ord}_{\mathfrak{p}} \phi_2(x), \quad \operatorname{ord}_{\mathfrak{p}} \phi_3(x) \neq 0$$

where $(\phi_1(x) : \phi_2(x) : \phi_3(x))$ is a local parametrization of γ .

The polar curve \mathcal{P}_q of \mathcal{C} with respect to q is the curve of equation

$$\frac{\partial \text{Res}(p, q)}{\partial T_2}(T_1, T_2, T_3) = 0$$

and therefore

$$\text{mult}_{\mathbf{p}}(\mathcal{P}_q, \gamma) = \text{ord}_{\mathbf{p}} \left(\frac{\partial \text{Res}(p, q)}{\partial T_2}(\phi_1(x), \phi_2(x), \phi_3(x)) \right).$$

Similarly, the line \mathcal{L}_q joining the points \mathbf{p} and q is the line of equation $T_1 = 0$ and hence

$$\text{mult}_{\mathbf{p}}(\mathcal{L}, \gamma) = \text{ord}_{\mathbf{p}} \phi_1(x).$$

Also, we have

$$\text{mult}_{\mathbf{p}}(\mathcal{D}_q, \gamma) = \text{ord}_{\mathbf{p}} (\text{SRes}_0(p, q)(\phi_1(x), \phi_2(x), \phi_3(x))).$$

Now, set

$$J(x) = \left| \begin{array}{cc} p_2(x) & q_2(x) \\ \sum_{i=1}^3 p'_i(x)\phi_i(x) & \sum_{i=1}^3 q'_i(x)\phi_i(x) \end{array} \right| \in \mathbb{K}[x]$$

where the notation $f'(x)$ stands for the derivative of the polynomial $f(x)$ with respect to the variable x . Since (p, q) is a μ -basis of ϕ , we have $\sum_{i=1}^3 p_i(x)\phi_i(x) = 0$ and $\sum_{i=1}^3 q_i(x)\phi_i(x) = 0$. Therefore, Lemma 4.7, shows that

$$\text{mult}_{\mathbf{p}}(\mathcal{D}, \gamma) = \text{mult}_{\mathbf{p}}(\mathcal{P}_q, \gamma) - \text{ord}_{\mathbf{p}} J(x).$$

Since $\phi_1(x) = \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p_3(x) & q_3(x) \end{array} \right|$, we have

$$\phi'_1(x) = \left| \begin{array}{cc} p'_2(x) & q'_2(x) \\ p_3(x) & q_3(x) \end{array} \right| + \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p'_3(x) & q'_3(x) \end{array} \right|.$$

Furthermore, the equality $\sum_{i=1}^3 p_i(x)\phi_i(x) = 0$ shows that $\text{ord}_{\mathbf{p}} p_3(x) \geq \text{ord}_{\mathbf{p}} \phi_1(x)$ for $\text{ord}_{\mathbf{p}} \phi_2(x) \geq \text{ord}_{\mathbf{p}} \phi_1(x)$. Similarly, $\text{ord}_{\mathbf{p}} q_3(x) \geq \text{ord}_{\mathbf{p}} \phi_1(x)$ and we deduce that

$$\text{ord}_{\mathbf{p}} \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p'_3(x) & q'_3(x) \end{array} \right| = \text{ord}_{\mathbf{p}} \phi_1(x) - 1.$$

But

$$J(x) = \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p'_1(x) & q'_1(x) \end{array} \right| \phi_1(x) + \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p'_2(x) & q'_2(x) \end{array} \right| \phi_2(x) + \left| \begin{array}{cc} p_2(x) & q_2(x) \\ p'_3(x) & q'_3(x) \end{array} \right| \phi_3(x)$$

and since $\text{ord}_{\mathbf{p}}(\phi_3) = 0$ we get $\text{ord}_{\mathbf{p}} J(x) = \text{ord}_{\mathbf{p}} \phi_1(x) - 1$. Finally, we deduce that

$$\text{mult}_{\mathbf{p}}(\mathcal{D}, \gamma) = \text{mult}_{\mathbf{p}}(\mathcal{P}, \gamma) - \text{mult}_{\mathbf{p}}(\mathcal{L}_q, \gamma) + 1$$

that proves that \mathcal{D} is adjoint to \mathcal{C} by Proposition 4.6.

To finish the proof, we need to consider the equations $\text{SRes}_i(p, q) = 0$ for all $i = 1, \dots, d-2$. As observed in Section 3, under the hypothesis $\mu \geq 2$ the polynomials $X_1 \text{SRes}_i(p, q) - X_2 \text{SRes}_{i+1}(p, q)$ are inertia forms of (p, q) with respect to the ideal (X_1, X_2) . It follows that for all $i = 1, \dots, d-2$, there exist an integer N_i such that

$$X_2^{N_i} (X_1 \text{SRes}_i(p, q) - X_2 \text{SRes}_{i+1}(p, q)) \in (p, q) \subset \mathbb{K}[\underline{X}][\underline{T}].$$

We deduce by specialization of this relationship that

$$x \text{SRes}_i(p, q)(\phi_1(x), \phi_2(x), \phi_3(x)) - \text{SRes}_{i+1}(p, q)(\phi_1(x), \phi_2(x), \phi_3(x)) = 0$$

in $\mathbb{K}[x]$. It follows that, for all $i = 1, \dots, d-2$, the equation $\text{SRes}_i(p, q)$ define an algebraic curve, say \mathcal{D}_i , such that

$$\text{mult}_p(\mathcal{D}_i, \gamma) = \text{mult}_p(\mathcal{D}, \gamma) + i$$

and the claimed theorem is proved. \square

We are now ready to state results precisizing the relation between adjoint linear systems on \mathcal{C} and moving curves following ϕ of degree 1 in X_1, X_2 and degree $d-2$ (resp. $d-1$) in T_1, T_2, T_3 (see [8, Conjecture 3.8]).

Corollary 4.9 ($\mu = 2, d = 4$). *Any non-zero element in $\mathcal{L}_2(\phi)$ or $\mathcal{L}_3(\phi)$ gives an adjoint linear system on \mathcal{C} .*

Proof. By Section 3, if $d = 4$ we known that the two moving curves

$$\text{sylv}_{0,1}(p, q) = \begin{vmatrix} U_0 & U_2 \\ V_0 & V_2 \end{vmatrix} X_1 + \begin{vmatrix} U_1 & U_2 \\ V_1 & V_2 \end{vmatrix} X_2 = \text{SRes}_2(p, q)X_2 - \text{SRes}_1(p, q)X_1$$

$$\text{sylv}_{1,0}(p, q) = \begin{vmatrix} U_0 & U_1 \\ V_0 & V_1 \end{vmatrix} X_1 + \begin{vmatrix} U_0 & U_2 \\ V_0 & V_2 \end{vmatrix} X_2 = \text{SRes}_0(p, q)X_1 - \text{SRes}_1(p, q)X_2$$

form a system of generators of $\mathcal{L}_2(\phi)$ over \mathbb{K} and that

$$\mathcal{L}_3(\phi) = \left\langle \left(T_i \text{sylv}_{(0,1)}(p, q) \right)_{i=1,2,3}, \left(T_i \text{sylv}_{(1,0)}(p, q) \right)_{i=1,2,3} \right\rangle_{\mathbb{K}}.$$

Therefore, the claimed result follows from that fact that $\text{SRes}_i(p, q) = 0$ defines a curve adjoint to \mathcal{C} for all $i = 0, 1, 2$. \square

The case $\mu = 2, d > 4$ is more intricate. Following Section 2.3, we have to consider the $(d-2) \times (d-3)$ -matrix

$$M_1(p, q) = \underbrace{\begin{pmatrix} U_0 & & 0 & & \\ & \ddots & & & \vdots \\ U_1 & & U_0 & & \\ & \ddots & & & q_\beta(X) \\ U_2 & & U_1 & & \vdots \\ & \ddots & & & \\ & & U_2 & & \end{pmatrix}}_{d-4} \underbrace{\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}}_1.$$

Its first $(d-4)$ columns are the coefficients of the polynomials

$$X_1^{d-5}p, X_1^{d-6}X_2p, \dots, X_1X_2^{d-6}p, X_2^{d-5}p$$

in the monomial basis $X_1^{d-3}, X_1^{d-2}X_2, \dots, X_2^{d-3}$ and its last column contains the coefficients of the polynomials $\sum_{|\beta|=d-3} q_\beta(X)Y^\beta$ in the monomial basis $Y_1^{d-3}, Y_1^{d-2}Y_2, \dots, Y_2^{d-3}$. We proved in Theorem 2.5 that the maximal minors Δ_i , $i \in \{1, \dots, d-2\}$, of $M_1(p, q)$ obtained by removing the i^{th} row are elements in $\mathcal{L}_2(\phi)$. Moreover, we proved in Lemma 2.7 that

$$\Delta_i = X_1 \text{SRes}_i(p, q) - X_2 \text{SRes}_{i+1}(p, q)$$

for all $i = 1, \dots, d-2$. We deduce the

Corollary 4.10 ($\mu = 2, d > 4$). *All the determinants $\Delta_i, i \in \{1, \dots, d-2\}$ give an adjoint linear system on \mathcal{C} . Moreover, if $V(U_0, U_1, U_2) = \emptyset \subset \mathbb{P}_{\mathbb{K}}^2$ then any non-zero element in $\mathcal{L}_2(\phi)$ or $\mathcal{L}_3(\phi)$ gives an adjoint linear system on \mathcal{C} .*

Notice that the hypothesis on the depth of the ideal (U_0, U_1, U_2) can not be avoided since XiaHong Jia found an example of an element in $\mathcal{L}_3(\phi)$, with $\mu = 2, d = 6$, which is not an adjoint linear system on \mathcal{C} . For this example, p does not depend on the variable T_3 and hence $\text{depth}_{\mathbb{K}[T]}(U_0, U_1, U_2) \leq 2$. Also, as mentioned earlier, notice that this condition corresponds to the absence of a singularity of multiplicity $d - 2$, the maximum possible value.

Finally, let us consider the case $\mu \geq 3$. Since $1 \leq \mu \leq d - \mu$, we must have $d \geq 6$. In Section 2.2 we have identified the collection of degree 1 inertia forms $(D_i(X_1, X_2))_{i=0, \dots, d-3}$ that satisfy to the equalities (see Corollary 2.4)

$$D_i = X_2 \text{SRes}_{d-2-i}(p, q) - X_1 \text{SRes}_{d-3-i}(p, q).$$

Corollary 4.11 ($\mu \geq 3$). *The polynomials $D_i(X_1, X_2), i = 0, \dots, d-3$, give adjoint linear systems on \mathcal{C} . Moreover, if*

$$(4.3) \quad V \left(\text{Det}_{d-4} \left(C_{d-\mu-3} \oplus C_{\mu-3} \xrightarrow{(p \ q)} C_{d-3} \right) \right) = \emptyset \subset \mathbb{P}_{\mathbb{K}}^2$$

then any element in $\mathcal{L}_{d-2}(\phi)$ or $\mathcal{L}_{d-1}(\phi)$ give an adjoint linear system on \mathcal{C} .

Point out that XiaHong Jia found an example of an element in $\mathcal{L}_{d-2}(\phi) \oplus_{\mathbb{K}} \mathcal{L}_{d-1}(\phi)$, with $\mu = 3, d = 7$, which is not an adjoint linear system of \mathcal{C} . For this example the inequality (4.3) does not hold (more precisely, p does not depend on the variable T_3 for this example).

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REFERENCES

- [1] Shreeram S. Abhyankar. *Algebraic geometry for scientists and engineers*, volume 35 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [2] J. F. Andrade, A. Simis, and W. Vasconcelos. On the grade of some ideals. *Manuscripta Math.*, 34(2-3):241–254, 1981.
- [3] Laurent Busé and Carlos D'Andrea. Inversion of parameterized hypersurfaces by means of subresultants. In *ISSAC 2004*, pages 65–71. ACM, New York, 2004.
- [4] Laurent Busé and Jean-Pierre Jouanolou. On the closed image of a rational map and the implicitization problem. *J. Algebra*, 265(1):312–357, 2003.
- [5] Laurent Busé and Bernard Mourrain. Explicit factors of some iterated resultants and discriminants. *Mathematics of Computations*, 2007. To appear, arXiv:math.AC/??
- [6] Eduardo Casas-Alvero. *Singularities of plane curves*, volume 276 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.
- [7] D. Cox, J. Hoffman, and H. Wang. Syzygies and the Rees algebra. Preprint, 2007.
- [8] David Cox. The moving curve ideal and the rees algebra. Accepted for publication in TCS, 2007.
- [9] Hoon Hong. Subresultants under composition. *Journal of Symbolic Computation*, 23(4):355–365, 1997.

- [10] J. Hong, A. Simis, and W. Vasconcelos. On the homology of two-dimensional elimination. Preprint, arXiv:math.AC:0704.0808v1, 2007.
- [11] J. P. Jouanolou. Formes d'inertie et résultant: un formulaire. *Adv. Math.*, 126(2):119–250, 1997.
- [12] Jean-Pierre Jouanolou. Résultant anisotrope, compléments et applications. *Electron. J. Combin.*, 3(2):Research Paper 2, approx. 91 pp. (electronic), 1996. The Foata Festschrift.
- [13] Jean-Pierre Jouanolou. An explicit duality for quasi-homogeneous ideals. Preprint, arXiv:math.AC/0607626, 2007.
- [14] Ning Song, Falai Chen, and Ron Goldman. Axial moving lines and singularities of rational planar curves. *Comp. Aided Geom. Design*, 24(4):200–209, 2007.

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