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# Universal Sets of $n$ Points for 1-bend Drawings of Planar Graphs with $n$ Vertices <sup>★</sup>

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**Abstract.** This paper shows that any planar graph with  $n$  vertices can be point-set embedded with at most one bend per edge on a universal set of  $n$  points in the plane. An implication of this result is that any number of planar graphs admit a simultaneous embedding without mapping with at most one bend per edge.

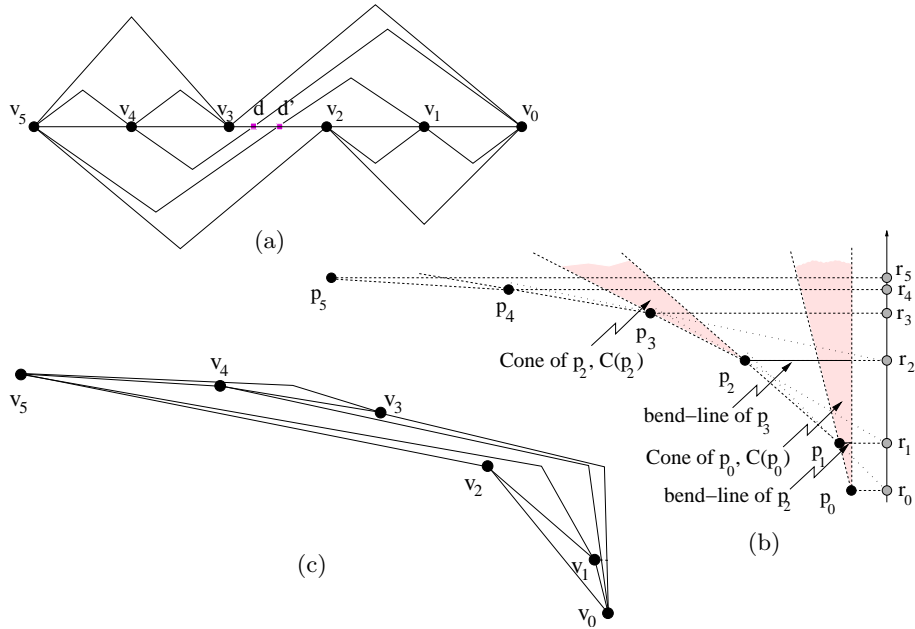
## 1 Introduction

Let  $S$  be a set of  $m$  distinct points in the plane and let  $G$  be a planar graph with  $n$  vertices ( $n \leq m$ ). A *point-set embedding* of  $G$  on  $S$  is a planar drawing of  $G$  such that each vertex is drawn as a point of  $S$  and the edges are drawn as poly-lines. The problem of computing point-set embeddings of planar graphs has a long tradition both in the graph drawing and in the computational geometry literature (see, e.g., [5, 6, 8]). Considerable attention has been devoted to the study of universal sets of points. A set  $S$  of  $m$  points is said to be  *$h$ -bend universal* for the family of planar graphs with  $n$  vertices ( $n \leq m$ ) if any graph in the family admits a point-set embedding onto  $S$  that has at most  $h$  bends along each edge.

Gritzman, Mohar, Pach and Pollack [5] proved that every set of  $n$  distinct points in the plane is 0-bend universal for the all outerplanar graphs with  $n$  vertices. De Fraysseix, Pach, and Pollack [3] and independently Schnyder [9] proved that a grid with  $O(n^2)$  points is 0-bend universal for all planar graphs with  $n$  vertices. De Fraysseix et al. [3] also showed that a 0-bend universal set of points for all planar graphs having  $n$  vertices cannot have  $n + o(\sqrt{n})$  points. This last lower bound was improved by Chrobak and Karloff [2] and later by Kurowski [7] who showed that linearly many extra points are necessary for a 0-bend universal set of points for all planar graphs having  $n$  vertices. On the other hand, if two bends along each edge are allowed, a tight bound on the size of the

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**Fig. 1.** (a) A proper monotone topological book embedding. The spine crossings  $d$  and  $d'$  are proper. (b) A necklace of six points. The cone of  $p_0$ , the cone of  $p_2$ , the bend-line of  $p_2$ , and the bend-line of  $p_3$  are highlighted. (c) A point-set embedding computed by Algorithm 1-bend Universal Drawer.

point-set is known: Kaufmann and Wiese [6] proved that every set of  $n$  distinct points in the plane is 2-bend universal for all planar graphs with  $n$  vertices.

In this paper we study the minimum size of a universal set of points for all planar graphs with  $n$  vertices under the assumption that at most one bend per edge is allowed in the point-set embedding. We prove the following theorem.

**Theorem 1.** *Let  $\mathcal{F}_n$  be the family of all planar graphs with  $n$  vertices. There exists a set of  $n$  distinct points in the plane that is 1-bend universal for  $\mathcal{F}_n$ .*

The proof is constructive; an example is shown in Figure 1. We define a set  $S$  of  $n$  points and show how to compute an embedding of any planar graph with  $n$  vertices on  $S$  such that the resulting drawing has at most one bend per edge. The drawing procedure starts by computing a special type of book embedding defined in Section 2, and then uses this book embedding to construct the point-set embedding with the algorithm described in Section 3.

Our universal set of  $n$  points can be defined either (i) with algebraic coordinates such that they are the vertices of a convex chain with unit-length edges or (ii) on a regular grid of size  $n2^n$  by  $n$ . In the former case all planar graphs of  $\mathcal{F}_n$  can be drawn on that point set with all bend-points and vertices in a square of size  $n$  by  $n$  at distance at least  $\frac{1}{2d}$  apart, where  $d$  is the maximum degree of the graph. In the latter case, the graphs can be drawn with all bend-points on the grid points of the  $n2^n$  by  $n$  grid.

We conclude this introduction by noting a result that is immediately implied by Theorem 1. Two planar graphs  $G_1$  and  $G_2$  with the same set of vertices are said to admit a *simultaneous embedding without mapping* if there exists a set of points in the plane that supports a point-set embedding of both  $G_1$  and  $G_2$  [1]. It is not known whether any two planar graphs admit a simultaneous embedding without mapping such that all edges are straight-line segments. A consequence of [5] is that a planar graph has a straight-line simultaneous embedding without mapping with any number of outerplanar graphs. A consequence of [6] is that any two planar graphs have a simultaneous embedding without mapping such that each edge is drawn with at most two bends. Theorem 1 implies the following.

**Corollary 1.** *Any number of planar graphs sharing the same vertex set admit a simultaneous embedding without mapping with at most one bend per edge.*

## 2 Monotone Topological Book Embeddings

Consider the Cartesian coordinate system  $(O, x, y)$  and let  $p, q$  be two points in the plane. We say that  $p$  is *left of*  $q$  and we denote it as  $p < q$  if the  $x$ -coordinate of  $p$  is less than the  $x$ -coordinate of  $q$ ; we shall also use the notation  $p \leq q$  to mean that either  $p$  is left of  $q$  or  $p$  coincides with  $q$ ; we define similarly  $p > q$  and  $p \geq q$ . A *spine* is a horizontal line. Let  $\ell$  be a spine and let  $p, q$  be two points of  $\ell$ . Let  $p < q$  and let  $b$  be a point of the perpendicular bisector of  $\overline{pq}$ , at positive distance from  $\ell$ . An *arc* connecting  $p$  to  $q$ , denoted as  $(p, q)$ , is a polygonal chain consisting of two segments: segment  $\overline{pb}$  and segment  $\overline{bq}$ . Point  $p$  is the *left endpoint* of  $(p, q)$ , point  $q$  is the *right endpoint* of  $(p, q)$ , and point  $b$  is the *bend-point* of  $(p, q)$ . Arc  $(p, q)$  can be either in the half-plane above the spine or in the half-plane below the spine (such half-planes are assumed to be closed sets); in the first case we say that the arc is in the *top page* of  $\ell$ , otherwise it is in the *bottom page* of  $\ell$ . From now on, when we denote an arc as  $(p, q)$  we shall implicitly assume that  $p$  is its left endpoint.

Let  $G = (V, E)$  be a planar graph. A *monotone topological book embedding* of  $G$ , denoted  $\Gamma$ , is a planar drawing such that all vertices of  $G$  are represented as points of a spine  $\ell$  and each edge is either represented as an arc in the bottom page, or as an arc in the top page, or as a poly-line that crosses the spine and consists of two consecutive arcs. Let  $e = (u, v)$  be an edge of a monotone topological book embedding that crosses the spine at a point  $d$ ; assuming that  $u$  is left of  $v$  along the spine,  $e$  is such that: (i)  $u < d < v$ , (ii) arc  $(u, d)$  is in the bottom page, and (iii) arc  $(d, v)$  is in the top page. Point  $d$  is called the *spine crossing* of  $(u, v)$ . Refer to Figure 1(a). Also, let  $u'$  be the rightmost vertex along the spine of  $\Gamma$  such that  $u' < d$  and let  $v'$  be the leftmost vertex of the spine of  $\Gamma$  such that  $d < v'$ . We say that  $u'$  and  $v'$  are the two *bounding vertices* of  $d$ . We say that  $d$  is a *proper spine crossing* if its bounding vertices  $u'$  and  $v'$  are such that  $u < u' < d < v' < v$ . (The spine crossings  $d$  and  $d'$  of Figure 1(a) are both proper and both bounded by  $v_3$  and  $v_2$ ). A monotone topological book embedding is *proper* if all of its spine crossings are proper. Di Giacomo et al. [4] proved that, for every planar graph, a monotone topological book embedding

exists and can be computed (in linear time in the size of the graph). Since an edge that crosses the spine with a non-proper spine crossing can be replaced by a single arc, we obtain the following lemma.

**Lemma 1.** *Every planar graph has a proper monotone topological book embedding which can be computed in linear time in the size of the graph.*

Let now  $\Gamma$  be a proper monotone topological book embedding of a planar graph  $G$ . If we insert a dummy vertex for each spine crossing of  $\Gamma$ , we obtain a new topological book embedding  $\Gamma'$  such that  $\Gamma'$  represents a planar subdivision  $G'$  of  $G$  obtained by splitting with a vertex some of the edges of  $G$ . We call the graph  $G'$  an *augmented form* of  $G$  and the drawing  $\Gamma'$  an *augmented topological book embedding* of  $G$ . A vertex of  $G'$  that is also a vertex of  $G$  is called a *real vertex* of  $\Gamma'$ ; a vertex of  $G'$  that corresponds to a spine crossing of  $\Gamma$  is called a *division vertex* of  $\Gamma'$ . Note that every division vertex of  $\Gamma'$  has degree two and that every edge of  $\Gamma'$  is either an arc in the top page or an arc in the bottom page. The *bounding vertices* of a division vertex  $d$  of  $\Gamma'$  are the two real vertices that form the bounding vertices of the spine crossing corresponding to  $d$  in  $\Gamma$ . The following property is a consequence of the planarity of  $\Gamma'$ .

*Property 1.* Let  $a = (u, v)$  and  $a' = (u', v')$  be two distinct arcs of  $\Gamma'$  that are in the same page and such that  $u < u'$ . Then, (i)  $u < v \leq u' < v'$  or (ii)  $u < u' < v' \leq v$ .

### 3 Proof of Theorem 1

We prove Theorem 1 by first defining a family of sets of  $n$  points in convex position (Subsection 3.1) and then by describing an algorithm that computes a point-set embedding of any planar graph with  $n$  vertices on the  $n$ -point element of the family (Subsection 3.2).

#### 3.1 Necklaces, Cones, and Bend-lines

Let  $p_0$  be any point on the  $x$ -axis strictly left of  $O$  and  $p_1$  be any point strictly in the top-left quadrant of  $p_0$ . We construct  $p_{i+2}$ , for  $0 \leq i \leq n-2$ , from  $p_i$  and  $p_{i+1}$  as follows. Let  $r_i$  be the projection of  $p_i$  on the vertical  $y$ -axis. Point  $p_{i+2}$  can be chosen anywhere on or below the line through  $r_i$  and  $p_{i+1}$  and strictly above the horizontal line through  $p_{i+1}$ . Let  $S$  be any set of  $n$  points defined by the above procedure; we call  $S$  a *necklace* of  $n$  points. See Figure 1(b).

The *cone* of  $p_0$ , denoted as  $C(p_0)$ , is the wedge with apex  $p_0$  and bounded by the vertical half-line above  $p_0$  and by the ray emanating from  $p_0$  and through  $p_1$ . The *cone* of  $p_i$  ( $1 \leq i \leq n-2$ ), denoted as  $C(p_i)$ , has  $p_i$  as its apex and is bounded by two rays emanating from  $p_i$  with directions  $\overrightarrow{p_{i-1}p_i}$  and  $\overrightarrow{p_i p_{i+1}}$ . In what follows we assume that  $C(p_i)$  is an open set ( $0 \leq i \leq n-1$ ).

The *bend-line* of  $p_i$  ( $i > 1$ ) is the relatively-open horizontal segment from  $p_{i-1}$  to the vertical line through  $p_0$ . The following properties follow from the definition of a necklace and can be proved with elementary geometric arguments. Let  $S = \{p_0, p_1, \dots, p_{n-1}\}$  be a necklace of  $n$  points and let  $CH(S)$  be its convex hull. Note that  $p_0, \dots, p_{n-1}$  are ordered from right to left, *i.e.*,  $p_{n-1} < \dots < p_0$ .

*Property 2.* Let  $p_h < p_t$  ( $t > 1$ ) be two points of  $S$  and let  $q$  be a point on the bend-line of  $p_t$ . Segments  $\overline{p_h q}$  and  $\overline{p_t p_{t-1}}$  intersect in their relative interior.

*Property 3.* Let  $p_{h'} \leq p_h < p_t$  ( $t > 1$ ) be three points of  $S$  and let  $q' < q$  be two points on the bend-line of  $p_t$ . Segments  $\overline{p_h q}$  and  $\overline{p_{h'} q'}$  do not intersect each other.

### 3.2 Computing 1-bend point-set embeddings

We describe a drawing algorithm, called **1-bend Universal Drawer**, that receives as input a planar graph  $G$  with  $n$  vertices and a necklace  $S$  of  $n$  points and returns a point-set embedding of  $G$  on  $S$  such that every edge of  $G$  is drawn with at most one bend. Algorithm **1-bend Universal Drawer** consists of the following steps.

Step 1: Compute a proper monotone topological book embedding  $\Gamma$  of  $G$  and the corresponding augmented proper topological book embedding  $\Gamma'$ . Let  $\ell$  be the spine of  $\Gamma'$ . Label the real vertices of  $\Gamma'$  on  $\ell$  by  $v_{n-1}, \dots, v_0$  in that order from left to right (*i.e.*,  $v_i < v_{i-1}$ ). Map each real vertex  $v_i$  to point  $p_i$  of the necklace ( $0 \leq i \leq n-1$ ).

Step 2: Draw the bends of the arcs of the top page of  $\Gamma'$  as follows. For each vertex  $v_i$  of  $\Gamma'$  mapped to point  $p_i$  ( $0 \leq i \leq n-1$ ) do the following. Let  $a_{i0}, a_{i1}, \dots, a_{i(k-1)}$  be the sequence of arcs in the top page of  $\Gamma'$  whose right endpoint is  $v_i$ ; assume that  $a_{i0}, a_{i1}, \dots, a_{i(k-1)}$  are encountered in this order when going clockwise around  $v_i$  by starting the tour from a point on  $\ell$  slightly to the left of  $v_i$ . For each  $a_{ij}$  ( $0 \leq j \leq k-1$ ) do:

- Draw a ray  $r_{ij}$  emanating from  $p_i$  such that: (i)  $r_{ij}$  is inside the cone  $C(p_i)$  of  $p_i$ , and (ii)  $r_{i(j+1)}$  is to the right of  $r_{ij}$  ( $0 \leq j \leq k-2$ ).
- Let  $v_h$  be the left endpoint of  $a_{ij}$  in  $\Gamma'$  and  $b_{ij}$  the bend-point of  $a_{ij}$ . If  $v_h$  is a real vertex of  $\Gamma'$ , draw  $b_{ij}$  at the intersection point,  $q$ , between  $r_{ij}$  and the bend-line of  $p_h$  (through  $p_{h-1}$ ).<sup>4</sup> Else, if  $v_h$  is a division vertex of  $\Gamma'$  and the two real vertices bounding  $v_h$  in  $\Gamma'$  are  $v_t$  and  $v_{t-1}$ , draw  $b_{ij}$  at the intersection point,  $q$ , between  $r_{ij}$  and the bend-line of  $p_t$ .

Step 3: Draw the division vertices of  $\Gamma'$  as follows. For each division vertex  $d$  of  $\Gamma'$ , do the following. Let  $(v_i, d)$  and  $(d, v_j)$  be the two arcs of  $\Gamma'$  sharing  $d$  such that  $(v_i, d)$  is in the bottom page and  $(d, v_j)$  is in the top page. Let  $q$  be the point computed in Step 2 such that  $q$  represents the bend of  $(d, v_j)$ . Draw  $d$  at the intersection point between  $\overline{p_i q}$  and  $CH(S)$ .

Step 4: Draw the arcs of  $\Gamma'$  as follows. For each arc  $(u, v)$  of  $\Gamma'$  do the following. Let  $p_u, p_v$  be the points representing  $u$  and  $v$  along  $CH(S)$ .

- If  $(u, v)$  is an arc in the bottom page, draw it as the chord  $\overline{p_u p_v}$ .

<sup>4</sup> If  $p_h$  and  $p_i$  are consecutive vertices of  $S$  ( $h-1 = i$ ), the ray  $r_{ij}$  and the bend-line of  $p_h$  do not intersect, though their closures intersect at  $p_i$ . For consistency, we draw  $b_{ij}$  at this intersection point  $q = p_i$ . In Step 4, the arc  $(v_h, v_i)$  is drawn as the poly-line consisting of segment  $\overline{p_h q}$  followed by  $\overline{qp_i}$ , which is reduced to point  $p_i$ .

- If  $(u, v)$  is an arc in the top page of  $\Gamma'$ , let  $q$  be the point computed at Step 2 that represents the bend-point of  $(u, v)$ . Draw  $(u, v)$  as the poly-line consisting of segment  $\overline{p_u q}$  followed by  $\overline{q p_v}$ .

Step 5: Let  $\hat{\Gamma}$  be the drawing computed at the end of Step 4. Compute a drawing of  $G$  by removing from  $\hat{\Gamma}$  those points that represent the division vertices of  $\Gamma'$ .

The proof of Theorem 1 is now completed by showing that Algorithm **1-bend Universal Drawer** correctly computes a point-set embedding of  $G$  on  $S$  such that each edge has at most one bend. The idea is to show that the drawing computed at the end of Step 5 maintains the topology of  $\Gamma$  and that the geometric properties of the proper monotone topological book embedding and of the necklace make it possible to point-set embed the graph without edge-crossings and with at most one bend per edge. In particular, we show that  $\hat{\Gamma}$  is a planar drawing by exploiting Properties 1-3; the proof is however omitted here due to lack of space.

Observe that every real vertex of  $\Gamma'$  is drawn as a point of  $S$  in  $\hat{\Gamma}$ . Since  $\hat{\Gamma}$  does not have edge crossings, removing the division vertices from  $\hat{\Gamma}$  gives a point-set embedding of  $G$  on  $S$ . Also, by construction, the two edges incident on a division vertex of  $\hat{\Gamma}$  form a flat angle, and thus removing the division vertices from  $\hat{\Gamma}$  does not increase the number of bends. It follows that the drawing computed by Algorithm **1-bend Universal Drawer** is a point-set embedding of  $G$  on  $S$  such that each edge has at most one bend. Therefore, any necklace of  $n$  vertices is a 1-bend universal set for all planar graphs having  $n$  vertices, which concludes the proof of Theorem 1. We omit here the proofs on the size of the drawings.

## 4 Conclusion

We leave as an open problem to find a universal point-set for one-bend drawing of planar graphs in a polynomial-size regular grid.

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