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*New Developments and Cosine Effect in the Viscous  
Shallow Water and Quasi-Geostrophic Equations*

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## New Developments and Cosine Effect in the Viscous Shallow Water and Quasi-Geostrophic Equations

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**Abstract:** The viscous Shallow Water Equations and Quasi-Geostrophic Equations are considered in this paper. Some new terms, related to the Coriolis force, are revealed thanks to a rigorous asymptotic analysis. After providing well-posedness arguments for the new models, the authors perform some numerical computations that confirm the role played by the cosine effect in various physical configurations.

**Key-words:** multiscale analysis, Coriolis force, cosine effect, asymptotic behavior, shallow water equation, viscosity, quasi-geostrophic limit, numerical computations.

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# Nouveaux développements et effet cosinus dans les équations de Saint-Venant visqueuses et les équations Quasi-Géostrophiques

**Résumé :** Nous considérons, dans ce papier, les équations de Saint-Venant visqueuses et les équations Quasi-Géostrophiques. De nouveaux termes, liés à la force de Coriolis, sont révélés grâce à une analyse asymptotique rigoureuse. Après avoir donné des arguments quant au caractère bien posé des nouveaux modèles, les auteurs présentent quelques calculs numériques qui confirment le rôle joué par l'effet cosinus dans diverses configurations physiques.

**Mots-clés :** analyse multi-échelles, force de Coriolis, comportement asymptotique, équation de Saint-Venant, limite quasi-géostrophique, calculs numériques.

## 1 Introduction

In this article, we aim at contributing to the improvement of the derivation of the Shallow Water System. This model, obtained from the incompressible Navier Stokes Equations with free surface under the shallow water approximation, has been studied by numerous authors, both in the inviscid [17, 16] and viscous cases [7, 11].

We consider a viscosity comparable to the aspect ratio, in order to take small scales effects into account. More precisely, in the non-dimensional formulation of the equations, the viscosity will thus be a small parameter. However, looking at the dimensional formulation, the viscosity is much greater than the usual friction in the Navier-Stokes equations for water. Doing this, we actually consider turbulence-modelled Navier-Stokes equations, which feature a turbulent viscosity.

In the asymptotic analysis that is classically performed, we show that **new terms** appear in the viscous Shallow Water Equations (SWE):

$$\begin{aligned} \partial_t H + \operatorname{div}_x(Hu) &= 0, \\ \partial_t(Hu) + \operatorname{div}_x(Hu \otimes u) + \frac{g}{2} \nabla_x H^2 &= - \left(1 + \frac{kH}{3\mu}\right)^{-1} k u + aH \nabla_x \Delta_x H \\ &\quad + 2\mu \nabla_x (H \operatorname{div}_x u) + 2\mu \operatorname{div}_x (H D_x u) + \Omega \cos \theta \nabla_x (\mathbf{u}_1 H^2) \\ &\quad + \Omega \cos \theta H^2 \mathbf{e}_1 \operatorname{div}_x \mathbf{u} - 2\Omega \sin \theta H u^\perp - 2\Omega \cos \theta H \mathbf{e}_1 \nabla_x \mathbf{b} \cdot \mathbf{u} \\ &\quad + 2\Omega \cos \theta \mathbf{u}_1 H \nabla_x \mathbf{b} + aH \nabla_x \Delta_x b - gH \nabla_x b. \end{aligned}$$

These terms, that are classically neglected, are of the order of the viscous terms. They also arise in the derivation of the viscous Quasi-Geostrophic (QG) Equations:

$$\begin{aligned} D_t \left( (\partial_{x_1}^2 + (1 + \delta^2) \partial_{x_2}^2) \psi - \frac{(2\Omega \sin \theta_0)^2}{gH_{char}} \psi + \left(1 - \frac{H_{char}}{2 \tan \theta_0} \partial_{x_2}\right) \frac{2\Omega \sin \theta_0}{H_{char}} b + \beta x_2 \right) \\ = - \frac{1}{\varepsilon L_{char}} \alpha_0 (H_{char}) \Delta \psi + \mu \Delta^2 \psi + \operatorname{curl} f, \end{aligned}$$

where  $D_t = (\partial_t + u^0 \cdot \nabla_x)$ ,  $H_{char} = \varepsilon L_{char}$  and  $\delta = \Omega \sqrt{H_{char}/g} \cos \theta_0$ .

We present in Figure 1 the hierarchy of derivations that have been obtained in the past. The left part corresponds to the classical derivation of SWE by Saint-Venant [15], together with the recent works of [7] for the viscous SWE. It is interesting to notice that (at least for the rigid lid configuration), the QG equations are the same ( $\boxed{3} = \boxed{3'}$  in the figure) when derived from the viscous Shallow Water Equations (see [5]) or from the Navier-Stokes equations ([12, 9]). The right part corresponds to the work of [10] and the present article. As Figure 1 shows, the inviscid SWE are the same, whether or not the first order approximation is made on the Coriolis force. However, and this is the main point of this work, the corresponding *viscous* equations differ from the one in [7]. Starting from the new SWE and considering the quasi-geostrophic approximation, we end up with another model (see box  $\boxed{4}$  in Figure 1).

This so-called viscous Quasi-Geostrophic Shallow Water model has been introduced by [3] and is *a priori* different from the one obtained by [5]<sup>1</sup>.

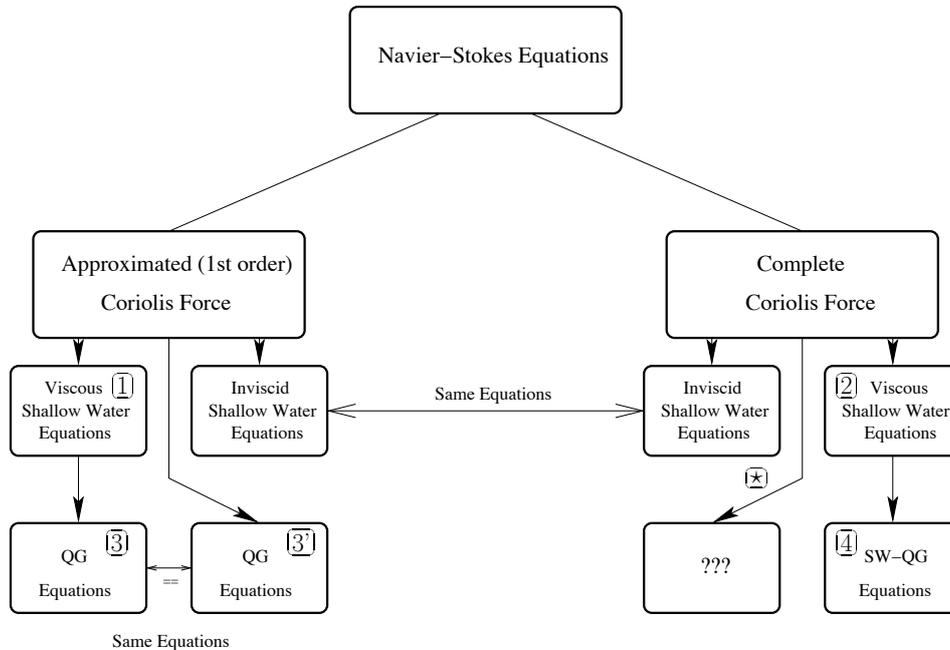


Figure 1: Hierarchy of models.

The aim of this work is to study whether this so-called "cosine effect" (see **new terms** in equations above) has to be, or not, considered in the models. In other words, and looking at Figure 1, we want to see the differences between models [3] and [4].

The article is organized as follows. In Section 2, following the idea of [7, 11], we perform the asymptotic analysis that leads to the Shallow Water Equations, and then derive the Quasi-Geostrophic limit with cosine effect. In Section 3, we show that the new model is mathematically satisfying, and provide some *a priori* estimates that ensure well-posedness of the system in the corresponding Sobolev spaces. Finally, we present some numerical simulations that compare the QG model with and without the cosine terms. These simulations show, at least for some realistic situations and as far as the long time behavior is concerned, that this cosine effect has to be taken into account.

<sup>1</sup>The direct derivation of the viscous quasi-geostrophic equations starting from the Navier-Stokes equations with a complete Coriolis force (see path  $\star$  in Figure 1) is still, to the best of our knowledge, an open problem.

## 2 Derivation of the Two New Models

This section is devoted to the new derivation of the Shallow Water and Quasi-Geostrophic models. First, we show in Section 2.1 that for some choice of the viscosity (depending on the aspect ratio) one should consider a complete Coriolis force to get the viscous Shallow Water System. We then get a new model from which we can derive, in Section 2.2, the corresponding Quasi-Geostrophic Equations<sup>2</sup>.

### 2.1 Cosine Effect on Shallow Water Equations

The two dimensional Shallow Water System is obtained from three dimensional Navier-Stokes Equations in a shallow domain. We look for the equations satisfied by the horizontal mean velocity field and the free surface.

We consider 3D Navier-Stokes Equations (NSE) for an homogenous fluid:

$$\begin{aligned} \partial_t U + \operatorname{div}(U \otimes U) &= \operatorname{div} \sigma - 2\vec{\Omega} \times U + f, \\ \operatorname{div} U &= 0, \end{aligned} \quad (1)$$

for  $(x, z)$  in  $\mathbb{T}^2 \times [b(x), h(t, x)]$ , where  $U = (u, w) \in \mathbb{R}^2 \times \mathbb{R}$  is the fluid velocity,  $\sigma$  is the stress tensor,  $2\vec{\Omega} \times U$  is the Coriolis force with  $\vec{\Omega} = \Omega(0, \cos \theta, \sin \theta)$ ,  $\theta$  represents the latitude and will be first considered as a constant. Finally  $f = -g\vec{e}_3$  is the gravity force.

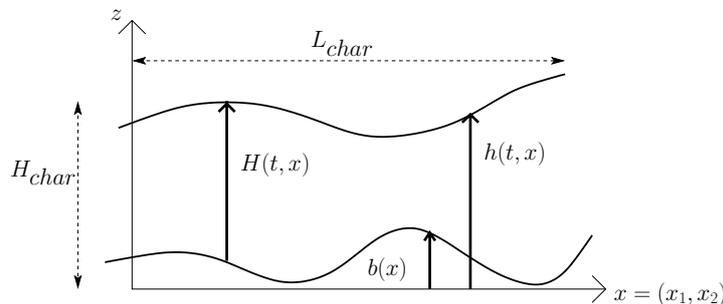


Figure 2: Notations used for the Shallow Water System

Figure 2 above describes the computational domain we consider, together with the bathymetry  $b$ , the water column height  $H$  and the free surface  $h$ . We supplement Equation (1) with the following boundary conditions:

<sup>2</sup>As we mentioned above, we are interested in **viscous** QG equations, with a small viscous term which is used to parameterize small scale effects.

- at the free surface  $z = h(t, x)$ :  
we usually neglect atmospheric pressure and take

$$\sigma n = 0.$$

Sometimes we can add surface tension effects, that is

$$\sigma n = a \kappa n,$$

where  $a$  is the capillarity coefficient,  $\kappa$  is the mean curvature,  $n$  the unit vector normal to the surface.

But in each case we know that the normal velocity in the referential linked to a particle moving on the surface is zero:

$$\partial_t h + u \cdot \nabla_x h = w.$$

- at the bottom  $z = b(x)$ :  
we have Navier type conditions

$$(\sigma n) \cdot \tau_1 \tau_1 + (\sigma n) \cdot \tau_2 \tau_2 = k U \cdot \tau_1 \tau_1 + k U \cdot \tau_2 \tau_2,$$

where  $k$  is the friction coefficient,  $(\tau_1, \tau_2)$  is a basis of the tangential surface. We add the non-penetration condition

$$-u \cdot \nabla_x b + w = 0.$$

In what follows we write these equations under their non-dimensional form, taking for  $\sigma$  the expression of the usual Cauchy stress tensor. Next we make an asymptotic development of  $U$  and study its first orders for a viscosity of order  $\varepsilon$  (geophysical flows). Note that, even if the non-dimensional viscosity is small, the corresponding viscosity in the dimensional equations would be much greater than the friction in the NSE for water, which is used to model the turbulence. We get Shallow Water Equations by an integration over the water height. We also give the Shallow Water System when the latitude is not constant.

### Dimensionless Navier-Stokes Equations

We write the Navier-Stokes system and the boundary conditions in a non-dimensionalized form, using some characteristic scales specially chosen to get the Shallow Water model.

Let us start with the 3D NSE for an homogenous fluid, with a stress tensor that reads  $\sigma = -p\text{Id} + 2\mu D(U)$ , where  $\mu$  represents the viscosity and  $D(U)$  is the symmetric part of the velocity gradient:

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + w \partial_z u &= -\nabla_x p + 2\mu \text{div}_x(D_x(u)) + \mu \partial_z^2 u + \mu \nabla_x(\partial_z w) \\ &\quad - 2\Omega \sin \theta u^\perp - 2\Omega \cos \theta w e_1 \\ \partial_t w + u \cdot \nabla_x w + w \partial_z w &= -\partial_z p + \mu \partial_z(\text{div}_x u) + \mu \Delta_x w + 2\mu \partial_z^2 w \\ &\quad + 2\Omega \cos \theta u_1 - g \\ \text{div}_x u + \partial_z w &= 0. \end{aligned}$$

Here the subscript  $x$  denotes horizontal variables,  $u_1$  the first component of the vector  $u$  and  $e_1$  the unit vector  $(1, 0)$ .

We introduce the following dimensionless variables and numbers:

$$\begin{aligned} x &= L_{char} x', & z &= H_{char} z' & \text{with } \varepsilon &= \frac{H_{char}}{L_{char}} \ll 1, \\ u &= u_{char} u', & w &= w_{char} w' & \text{with } w_{char} &= \varepsilon u_{char}, \\ \nu &= \frac{\mu}{L_{char} u_{char}}, & p &= p_{char} p' & \text{with } p_{char} &= u_{char}^2, \\ t &= \frac{L_{char}}{u_{char}} t', & Ro &= \frac{u_{char}}{2L_{char}\Omega}, & Fr &= \frac{u_{char}}{\sqrt{gH_{char}}}, \end{aligned}$$

where  $\varepsilon$  is the aspect ratio,  $Ro$  the Rossby number,  $Fr$  the Froude number. We drop the primes to rewrite the 3D NSE:

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + w \partial_z u &= -\nabla_x p + 2\nu \operatorname{div}_x(D_x(u)) + \frac{\nu}{\varepsilon^2} \partial_z^2 u + \nu \nabla_x(\partial_z w) \\ &\quad - \frac{\sin \theta}{Ro} u^\perp - \varepsilon \frac{\cos \theta}{Ro} w e_1, \end{aligned} \quad (2)$$

$$\begin{aligned} \partial_t w + u \cdot \nabla_x w + w \partial_z w &= -\frac{1}{\varepsilon^2} \partial_z p + \frac{\nu}{\varepsilon^2} \partial_z(\operatorname{div}_x u) + \nu \Delta_x w + \frac{2\nu}{\varepsilon^2} \partial_z^2 w \\ &\quad + \frac{1}{\varepsilon} \frac{\cos \theta}{Ro} u_1 - \frac{1}{\varepsilon^2 Fr^2}, \end{aligned} \quad (3)$$

$$\operatorname{div}_x u + \partial_z w = 0. \quad (4)$$

We also have to change the boundary conditions. We first replace  $\sigma$  with its expression and then change the variables:

- at the free surface  $z = h(t, x)$ :

the normal vector  $n$  is  $n = (1 + (\nabla_x h)^2)^{-\frac{1}{2}} \begin{pmatrix} -\nabla_x h \\ 1 \end{pmatrix}$ , and the curvature can be approximated by  $\kappa = \Delta_x h + O(\varepsilon^2)$ .

The horizontal variable  $h$  is rescaled as  $h = H_{char} h'$  and  $\kappa = \varepsilon L_{char}^{-1} \kappa'$ . We also define  $A = \varepsilon a u_{char}^{-2} L_{char}^{-1}$  to get the dimensionless conditions at the free surface:

$$p \nabla_x h + A \kappa \nabla_x h - 2\nu D_x(u) \nabla_x h + \nu \nabla_x w + \frac{\nu}{\varepsilon^2} \partial_z u = 0, \quad (5)$$

$$p + A \kappa - 2\nu \partial_z w + \varepsilon^2 \nu \nabla_x w \cdot \nabla_x h + \nu \partial_z u \cdot \nabla_x h = 0, \quad (6)$$

$$\partial_t h + u \cdot \nabla_x h = w. \quad (7)$$

- at the bottom  $z = b(x)$ :

if we write  $b = H_{char} b'$ , the non-penetration condition reads

$$-u \cdot \nabla_x b + w = 0. \quad (8)$$

The Navier condition is more involved. We choose the tangential vectors

$$\tau_1 = \frac{1}{|\nabla_x b|} \begin{pmatrix} \nabla_x^\perp b \\ 0 \end{pmatrix} \text{ and } \tau_2 = \frac{1}{\sqrt{|\nabla_x b|^2 + |\nabla_x b|^4}} \begin{pmatrix} -\nabla_x b \\ -|\nabla_x b|^2 \end{pmatrix}.$$

We define  $K = k u_{char}^{-1}$  and we get a complex expression that we do not detail here. We perform some approximations in the following and then give a simplified equality obtained from this expression.

### Hydrostatic Approximation

We now use the hydrostatic approximation, that is we suppose  $\varepsilon$  to be small. We only keep the first two orders in Equation (3) and we also drop terms in boundary conditions.

We are led to study the system:

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + w \partial_z u &= -\nabla_x p + 2\nu \operatorname{div}_x(D_x(u)) + \frac{\nu}{\varepsilon^2} \partial_z^2 u + \nu \nabla_x(\partial_z w) \\ &\quad - \frac{\sin \theta}{Ro} u^\perp - \varepsilon \frac{\cos \theta}{Ro} w e_1, \end{aligned} \quad (9)$$

$$\partial_z p = \nu \partial_z(\operatorname{div}_x u) + 2\nu \partial_z^2 w - \frac{1}{Fr^2} + \varepsilon \frac{\cos \theta}{Ro} u_1, \quad (10)$$

$$\operatorname{div}_x u + \partial_z w = 0. \quad (11)$$

We can simplify the Navier condition

$$2\varepsilon \nu D_x \cdot \nabla_x b - \varepsilon \nu \left( \frac{1}{\varepsilon^2} \partial_z u + \nabla_x w \right) = -Ku + 2\varepsilon \nu \partial_z w \nabla_x b + \varepsilon \nu (\nabla_x b \cdot \partial_z u) \nabla_x b. \quad (12)$$

At the free surface, Equation (7) is not modified but we can rewrite (5) as follows:

$$\nu \partial_z u = -\varepsilon^2 (p \nabla_x h + A \kappa \nabla_x h - 2\nu D_x(u) \nabla_x h + \nu \nabla_x w) \quad \text{for } z = h(x, t),$$

and plug it into Equation (6):

$$\begin{aligned} p + A \kappa - 2\nu \partial_z w &= -\varepsilon^2 \nu \nabla_x w \cdot \nabla_x h - \nu \partial_z u \cdot \nabla_x h \\ &= \varepsilon^2 (p \nabla_x h + A \kappa \nabla_x h - 2\nu D_x(u) \nabla_x h) \quad \text{for } z = h(x, t). \end{aligned}$$

As  $\varepsilon$  goes to zero, we get

$$p + A \kappa - 2\nu \partial_z w = 0 \quad \text{for } z = h(x, t). \quad (13)$$

We integrate the vertical derivative of the pressure (given by Equation (10)) from  $h$  to  $z$ , with  $z$  between  $b$  and  $h$ . The value of  $p$  at the free surface is given by (13), and thanks to the divergence free condition (11) we find the pressure at order  $\varepsilon$ :

$$p(t, x, z) = \frac{1}{Fr^2} (h(t, x) - z) - A \kappa - \nu (\operatorname{div}_x u)|_{z=h(t, x)} - \nu \operatorname{div}_x u + \varepsilon \frac{\cos \theta}{Ro} \int_h^z u_1. \quad (14)$$

As we are looking for equations on the mean velocity and on the evolution of the free surface, we first integrate the momentum equation (9) over the water height (between  $z = b(x)$  and  $z = h(t, x)$ ). We apply Leibniz formula and get:

$$\begin{aligned} \partial_t \int_b^h u - \partial_t h u|_{z=h} + \operatorname{div}_x \int_b^h (u \otimes u) - ((u \cdot \nabla_x h) u)|_{z=h} + ((u \cdot \nabla_x b) u)|_{z=b} \\ + (uw)|_{z=h} - (uw)|_{z=b} + \nabla_x \int_b^h p = \nabla_x h p|_{z=h} - \nabla_x b p|_{z=b} \\ + 2\nu \operatorname{div}_x \int_b^h D_x(u) - 2\nu D_x(u)|_{z=h} \nabla_x h + 2\nu D_x(u)|_{z=b} \nabla_x b + \frac{\nu}{\varepsilon^2} \partial_z u|_{z=h} \\ - \frac{\nu}{\varepsilon^2} \partial_z u|_{z=b} + \nu \nabla_x w|_{z=h} - \nu \nabla_x w|_{z=b} - \frac{\sin \theta}{Ro} \int_b^h u^\perp - \varepsilon \frac{\cos \theta}{Ro} \int_b^h w e_1. \end{aligned}$$

Then, we use boundary conditions (5), (7), (8) and (12) to simplify the expressions at the surface and at the bottom and obtain the integrated momentum equation:

$$\begin{aligned} \partial_t \int_b^h u + \operatorname{div}_x \int_b^h (u \otimes u) + \nabla_x \int_b^h p = -A \kappa \nabla_x h - \nabla_x b p|_{z=b} - \frac{K}{\varepsilon} u|_{z=b} \\ + 2\nu \operatorname{div}_x \int_b^h D_x(u) + \nu \nabla_x b (2\partial_z w + \nabla_x b \cdot \partial_z u)|_{z=b} \\ - \frac{\sin \theta}{Ro} \int_b^h u^\perp - \varepsilon \frac{\cos \theta}{Ro} \int_b^h w e_1, \end{aligned} \quad (15)$$

with a new Coriolis term (the last one).

We also want the evolution of the free surface: we integrate the divergence free equation (11) from the bottom to the surface, use Leibniz formula again and conditions at the surface (7) and at the bottom (8) to find:

$$\partial_t h(t, x) + \operatorname{div}_x \int_{b(x)}^{h(t, x)} u = 0. \quad (16)$$

In the sequel we fix the value of  $\nu$  and  $K$  with respect to  $\varepsilon$  and study the integrated momentum equation (16) and the free surface equation (16) when we approximate  $u$  at the first order and at the second order.

### Shallow Water System with a Viscosity of Order $\varepsilon$

We have already done the main assumption to get the Shallow Water System, that is the depth is small compared to the length of the domain. Now, we make some hypotheses on physical quantities.

From now on, we consider  $A$  and  $K$  of order  $\varepsilon$ :  $A = \varepsilon A_0$ , and  $K = \varepsilon K_0$ . As we mentioned in the introduction, the viscosity is also comparable to the aspect ratio:  $\nu = \varepsilon \nu_0$ .

We develop  $u, w, H, p, b$  in powers of  $\varepsilon$ , that is  $u = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$  (and so on) with  $H(t, x) = h(t, x) - b(x)$ .

We want an expression of  $u^0$ , the first order of the velocity. We use the horizontal momentum equation (9) and the boundary conditions (5) and (12). We obtain

$$\partial_z^2 u = O(\varepsilon), \quad (\partial_z u)|_{z=b} = O(\varepsilon), \quad (\partial_z u)|_{z=h} = O(\varepsilon),$$

which means that at the first order  $u$  does not depend on  $z$ :  $u^0(t, x, z) = u^0(t, x)$ . As we are looking for the dynamics of  $u^0$ , we study the previous equations at the first order. We can already rewrite the evolution equation (16)

$$\partial_t H^0 + \operatorname{div}_x(H^0 u^0) = 0. \quad (17)$$

Then we have the expression of  $p$  at the first order with (14):

$$p^0(t, x, z) = Fr^{-2}(h - z).$$

We replace this value in the integrated momentum equation (16) and find the equality:

$$\begin{aligned} \partial_t(H^0 u^0) + \operatorname{div}_x(H^0 u^0 \otimes u^0) + \frac{1}{2Fr^2} \nabla_x(H^0)^2 = \\ - \frac{1}{Fr^2} H^0 \nabla_x b^0 - K_0 u^0 - \frac{\sin \theta}{Ro} H^0 u^{0\perp}. \end{aligned} \quad (18)$$

Equations (17) - (18) form the Shallow Water System at the first order in non-dimensional variables.

If we go back to dimensional variables we have the Shallow Water System at the first order for a viscosity of order  $\varepsilon$ :

$$\partial_t H + \operatorname{div}_x(Hu) = 0, \quad (19)$$

$$\partial_t(Hu) + \operatorname{div}_x(Hu \otimes u) + \frac{g}{2} \nabla_x H^2 = -gH \nabla_x b - ku - 2\Omega \sin \theta H u^\perp. \quad (20)$$

But there are no viscosity effects in this system: we are led to study the equations we can obtain at the second order to make the viscous terms appear in our Shallow Water System.

We will denote by a bold letter the approximation of the variables at order  $\varepsilon$  (for example,  $\mathbf{u}^1 = u^0 + \varepsilon u^1$ ) and by a bar the mean value on the water height defined by:

$$\bar{u}(t, x) := \frac{1}{H(t, x)} \int_b^h u \, dz.$$

Let us rewrite the divergence condition at the second order

$$\partial_t \mathbf{H}^1 + \operatorname{div}_x(\mathbf{H}^1 \bar{\mathbf{u}}^1) = O(\varepsilon^2). \quad (21)$$

As before, we rewrite the momentum equation (9) but at order  $\varepsilon$ , and with the Shallow Water System at the first order (17) - (18) we obtain an expression of the second derivative of  $u$ :

$$\frac{\nu}{\varepsilon^2} \partial_z^2 u = -\frac{K_0}{H^0} u^0 + O(\varepsilon).$$

We integrate this equality from  $b$  to  $z$  (for  $z$  between  $b(x)$  and  $h(t, x)$ ) with the boundary condition (12). We integrate it again from  $b$  to  $z$  to find an approximation of  $u$  at the second order:

$$\begin{aligned} u &= \mathbf{u}^1|_{z=b} + \varepsilon \frac{K_0}{\nu_0} u^0 \int_b^z \left(1 - \frac{s - b^0}{H^0}\right) ds + O(\varepsilon^2) \\ &= \mathbf{u}^1|_{z=b} \left(1 + \varepsilon \frac{K_0}{\nu_0} (z - b^0) \left(1 - \frac{z - b^0}{2H^0}\right)\right) + O(\varepsilon^2). \end{aligned}$$

With this expression, we can compute the mean value of  $u$ :

$$\bar{u} = \mathbf{u}^1|_{z=b} \left(1 + \varepsilon \frac{K_0}{\nu_0} \frac{H^0}{3}\right) + O(\varepsilon^2).$$

One can notice that the mean value of  $u^2$  is equal to the square of the mean value of  $u$  up to  $O(\varepsilon^2)$ . We have the same property for  $u \otimes u$ . This is used in the sequel.

We also have the value of  $p$  at the second order with Equation (14) and we can replace it in the integrated momentum equation (16). Using again the divergence free condition (11) to express  $w^0$  as a function of  $h^0$  and  $u^0$ , we get

$$\begin{aligned} &\partial_t(\mathbf{H}^1 \bar{u}) + \operatorname{div}_x(\mathbf{H}^1 \bar{u} \otimes \bar{u}) + \frac{1}{2Fr^2} \nabla_x(\mathbf{H}^1)^2 - \varepsilon A_0 \nabla_x(H^0 \Delta_x h^0) \\ &- 2\varepsilon \nu_0 \nabla_x(H^0 \operatorname{div}_x u^0) - \varepsilon \frac{\cos \theta}{2Ro} \nabla_x(u_1^0 (H^0)^2) = -K_0 \mathbf{u}^1|_{z=b} \\ &- \nabla_x b \left( \frac{\mathbf{H}^1}{Fr^2} - \varepsilon A_0 \Delta_x h^0 - 2\varepsilon \nu_0 \operatorname{div}_x u^0 - \varepsilon \frac{\cos \theta}{Ro} u_1^0 H^0 \right) - \varepsilon A_0 \Delta_x h^0 \nabla_x h^0 \\ &+ \varepsilon \nu_0 \nabla_x b^0 (2\partial_z w^0 + \nabla_x b^0 \cdot \partial_z u^0)|_{z=b} + 2\varepsilon \nu_0 \operatorname{div}_x(H^0 D_x u^0) - \frac{\sin \theta}{Ro} \mathbf{H}^1 \bar{u}^\perp \\ &+ \varepsilon \frac{\cos \theta}{2Ro} (H^0)^2 e_1 \operatorname{div}_x u^0 - \varepsilon \frac{\cos \theta}{Ro} H^0 e_1 \nabla_x b^0 \cdot u^0 + O(\varepsilon^2). \end{aligned}$$

The velocity at the bottom  $u|_{z=b}$  is given by

$$\mathbf{u}^1|_{z=b} = \bar{u} \left(1 + \frac{\varepsilon K_0}{\nu_0} \frac{H^0}{3}\right)^{-1} + O(\varepsilon^2),$$

which leads to:

$$\begin{aligned}
\partial_t(\mathbf{H}^1 \bar{u}) + \operatorname{div}_x(\mathbf{H}^1 \bar{u} \otimes \bar{u}) + \frac{1}{2Fr^2} \nabla_x(\mathbf{H}^1)^2 &= -K_0 \bar{u} \left(1 + \frac{\varepsilon K_0 \mathbf{H}^1}{\nu_0} \frac{1}{3}\right)^{-1} \\
&+ \varepsilon A_0 \mathbf{H}^1 \nabla_x \Delta_x \mathbf{H}^1 + \varepsilon A_0 \mathbf{H}^1 \nabla_x \Delta_x b + 2\varepsilon \nu_0 \nabla_x(\mathbf{H}^1 \operatorname{div}_x \bar{u}) \\
&+ 2\varepsilon \nu_0 \operatorname{div}_x(\mathbf{H}^1 D_x \bar{u}) + \varepsilon \frac{\cos \theta}{2Ro} \nabla_x(\bar{u}_1 (\mathbf{H}^1)^2) + \varepsilon \frac{\cos \theta}{2Ro} (\mathbf{H}^1)^2 e_1 \operatorname{div}_x \bar{u} \\
&- \varepsilon \frac{\cos \theta}{Ro} \mathbf{H}^1 e_1 \nabla_x b \cdot \bar{u} - \frac{\sin \theta}{Ro} \mathbf{H}^1 \bar{u}^\perp - \nabla_x b \left(\frac{\mathbf{H}^1}{Fr^2} - \varepsilon \frac{\cos \theta}{Ro} \bar{u}_1 \mathbf{H}^1\right) + O(\varepsilon^2).
\end{aligned} \tag{22}$$

Equations (21)-(22) form the Shallow Water System at the second order in non-dimensional variables, with new cosine terms.

Finally, let us go back to the dimensional form to get the viscous Shallow Water System at the second order:

$$\partial_t H + \operatorname{div}_x(Hu) = 0, \tag{23}$$

$$\begin{aligned}
\partial_t(Hu) + \operatorname{div}_x(Hu \otimes u) + \frac{g}{2} \nabla_x H^2 &= -\alpha_0(H) u + aH \nabla_x \Delta_x H \\
&+ 2\mu \nabla_x(H \operatorname{div}_x u) + 2\mu \operatorname{div}_x(H D_x u) + \Omega \cos \theta \nabla_x(u_1 H^2) \\
&+ \Omega \cos \theta H^2 e_1 \operatorname{div}_x u - 2\Omega \sin \theta H u^\perp - 2\Omega \cos \theta H e_1 \nabla_x b \cdot u \\
&+ 2\Omega \cos \theta u_1 H \nabla_x b + aH \nabla_x \Delta_x b - gH \nabla_x b,
\end{aligned} \tag{24}$$

where  $\alpha_0(H) = k / \left(1 + \frac{kH}{3\mu}\right)$ .

We now have the viscous terms we couldn't find at the first order. The cosine part of the Coriolis force should be taken into account.

**Remark 1** *If the latitude is not constant, the only difference with the previous development is that the term that reads  $\Omega \cos \theta \nabla_x(u_1 H^2)$  in the constant case must be replaced by  $\Omega \nabla_x(\cos \theta u_1 H^2)$ , the other ones remaining unchanged.*

## 2.2 Quasi-Geostrophic Equations

The Quasi-Geostrophic model is used for modeling and forecasting mid-latitude oceanic and atmospheric circulation (see [4]). It is obtained from the Shallow Water System assuming that the Rossby and Froude numbers are very small.

### Derivation of the Quasi-Geostrophic Equations with Cosine Effect

We consider the dimensionless Shallow Water System (21) - (22), that depends on the aspect ratio  $\varepsilon$ . We add in these equations a source term  $\tilde{f}$  that could be obtained directly in the

previous derivation.

We write an asymptotic development  $u = u^0 + \eta u^1 + \dots$ ,  $H = 1 + \eta FH^1 + \dots$  where we suppose  $Ro = \eta$ ,  $Fr^2 = F\eta^2$ ,  $b = \eta \tilde{b}$ , with  $\eta \ll 1$ ,  $F = (2\Omega L_{char})^2 / (gH_{char})$  and  $\varepsilon$  fixed. Let us underline that, from now on,  $\varepsilon$  is a fixed parameter, and the Rossby number  $\eta$  is the *asymptotic parameter*, meant to go to zero. Doing this, we are not considering a crossed asymptotic  $(\varepsilon, \eta) \rightarrow 0$ , but we let  $\eta$  go to zero as  $\varepsilon$  remains fixed. In other words, referring to Figure 1, we are interested in path  $\boxed{2}$  -  $\boxed{4}$ , that could be different from path  $\boxed{\star}$ . This is beyond the scope of this paper and is left to further studies. We also use the beta-plane approximation around the latitude  $\theta_0$  (see [13]).

We then have to study the following equations:

$$\partial_t H + \operatorname{div}_x(Hu) = 0, \quad (25)$$

$$\begin{aligned} \partial_t(Hu) + \operatorname{div}_x(Hu \otimes u) + \frac{1}{F\eta^2} H \nabla_x H &= -\frac{\sin \theta_0}{\eta} H u^\perp - \beta x_2 H u^\perp \\ + \varepsilon \frac{\cos \theta_0}{2\eta} e_1 H^2 \operatorname{div}_x u - \frac{\varepsilon}{2} \tan \theta_0 \beta x_2 e_1 H^2 \operatorname{div}_x u + \varepsilon \frac{\cos \theta_0}{2\eta} \nabla_x (H^2 u_1) \\ - \frac{\varepsilon}{2} \tan \theta_0 \beta \nabla_x (H^2 u_1 x_2) - \tilde{\alpha}_0(H) u + 2\nu \operatorname{div}_x(HD(u)) + 2\nu \nabla_x(H \operatorname{div} u) \\ + A H \nabla_x \Delta_x H + A H \nabla_x \Delta_x b - \varepsilon \frac{\cos \theta_0}{\eta} H e_1 \nabla_x b \cdot u + \varepsilon \tan \theta_0 \beta x_2 H e_1 \nabla_x b \cdot u \\ + \varepsilon \frac{\cos \theta_0}{\eta} u_1 H \nabla_x b - \varepsilon \tan \theta_0 \beta x_2 u_1 H \nabla_x b - \frac{1}{F\eta^2} H \nabla_x b + H \tilde{f} + O(\eta). \end{aligned} \quad (26)$$

At the first and second orders, Equation (25) gives:

$$\operatorname{div}_x u^0 = 0, \quad \text{and} \quad F \partial_t H^1 + \operatorname{div}_x u^1 + F \nabla_x H^1 \cdot u^0 = 0.$$

We also look at the first and second orders of the momentum equation (26). The first order gives

$$\nabla_x H^1 + \left( \sin \theta_0 - \varepsilon \frac{\cos \theta_0}{2} \partial_{x_2} \right) u^{0\perp} + \frac{\nabla_x \tilde{b}}{F} = 0.$$

Then we take the *curl* (i.e.  $-\partial_{x_2}$  of the first component +  $\partial_{x_1}$  of the second one) of the second order and get

$$\begin{aligned} (\partial_t + u^0 \cdot \nabla_x)(\operatorname{curl} u^0) &= -\tilde{\alpha}_0(1) \operatorname{curl} u^0 + \nu \Delta(\operatorname{curl} u^0) + \sin \theta_0 F(\partial_t H^1 + u^0 \cdot \nabla_x H^1) \\ &\quad - \sin \theta_0 F u^0 \cdot \nabla_x H^1 - \beta u_2^0 + \varepsilon F \frac{\cos \theta_0}{2} \partial_{x_2} (\partial_t H^1 + u^0 \cdot \nabla_x H^1) \\ &\quad - \nabla_x^\perp H^1 \cdot \nabla_x \tilde{b} + \varepsilon \cos \theta_0 \partial_{x_2} (u^0 \cdot \nabla_x \tilde{b}) + \varepsilon \cos \theta_0 \nabla_x^\perp u_1^0 \cdot \nabla_x \tilde{b} + \operatorname{curl} \tilde{f}. \end{aligned}$$

We note that

$$\sin \theta_0 u^0 \cdot \nabla_x H^1 - \varepsilon \frac{\cos \theta_0}{2} (\partial_{x_2} u^0) \cdot \nabla_x H^1 + \nabla_x^\perp H^1 \cdot \frac{\nabla_x \tilde{b}}{F} = 0,$$

and

$$(\partial_{x_2} u^0) \cdot \nabla_x \tilde{b} + \nabla_x^\perp u_1^0 \cdot \nabla_x \tilde{b} = 0.$$

We define  $\psi$  by  $u^0 = \nabla_x^\perp \psi$ , and consequently  $H^1 = (\sin \theta_0 - \varepsilon \cos \theta_0 \partial_{x_2}/2) \psi - \tilde{b}/F$ , and find

$$\begin{aligned} (\partial_t + u^0 \cdot \nabla_x) \left( \left( \partial_{x_1}^2 + \left( 1 + \varepsilon^2 F \frac{\cos^2 \theta_0}{4} \right) \partial_{x_2}^2 \right) \psi - \sin^2 \theta_0 F \psi \right. \\ \left. + \left( \sin \theta_0 - \varepsilon \frac{\cos \theta_0}{2} \partial_{x_2} \right) \tilde{b} + \beta x_2 \right) = -\tilde{\alpha}_0(1) \Delta \psi + \nu \Delta^2 \psi + \text{curl} \tilde{f}. \end{aligned} \quad (27)$$

Equation (27) is the non-dimensional Quasi-Geostrophic Equation obtained from the viscous Shallow Water System at the second order with variable topography. The new terms (due to the Coriolis force) are  $\varepsilon^2 F \cos^2 \theta_0 \partial_{x_2}^2 \psi/4$  and  $\varepsilon \cos \theta_0 \partial_{x_2} \tilde{b}/2$ . The “unusual”  $\sin \theta_0$  coefficient is linked to an “unusual” Rossby number expression that arises in  $F$  and  $\tilde{b}$ ; in the “usual” case, the term  $\sin \theta_0$  is replaced by 1, and  $\cos \theta_0$  by  $1/\tan \theta_0$ .

Let us remove this problem coming back to dimensional variables. We get:

$$\begin{aligned} D_t \left( (\partial_{x_1}^2 + (1 + \delta^2) \partial_{x_2}^2) \psi - \frac{(2\Omega \sin \theta_0)^2}{g H_{char}} \psi + \left( 1 - \frac{H_{char}}{2 \tan \theta_0} \partial_{x_2} \right) \frac{2\Omega \sin \theta_0}{H_{char}} b + \beta x_2 \right) \\ = -\frac{1}{\varepsilon L_{char}} \alpha_0(H_{char}) \Delta \psi + \mu \Delta^2 \psi + \text{curl} f, \end{aligned} \quad (28)$$

where  $D_t = (\partial_t + u^0 \cdot \nabla_x)$ ,  $H_{char} = \varepsilon L_{char}$  and  $\delta = \Omega \sqrt{H_{char}/g} \cos \theta_0$ .

We add to this equation the following boundary conditions:

$$\psi = 0 \text{ and } \Delta \psi = 0 \text{ on } \partial \mathcal{D},$$

which respectively express the non-penetration condition and the slip condition.

**Remark 2** *We can already notice that the cosine term has two different contributions. First, the laplacian is modified in the second direction by the small coefficient  $\delta$ . The other change is on the topography coefficient: we see the derivative of the topography in the second variable.*

### 3 Mathematical Properties of the QG Equations

The new terms in the QG model do not raise any additional mathematical difficulties. Hereafter, we proceed as in [2] to obtain *a priori* estimates that ensure well-posedness of the model (thanks to Galerkin method), and insist on the computations that are specific to the new model. The reader is referred to [2] for further details.

We consider a rectangular domain  $\mathcal{D}$ , multiply Equation (28) successively by  $\psi$  and  $\Delta \psi + \delta^2 \partial_{x_2}^2 \psi$ , and integrate over  $\mathcal{D}$ . In the integrations by parts, the integrated term vanishes as

we have the boundary conditions  $\psi = 0$  and  $\Delta\psi = 0$  on  $\partial\mathcal{D}$ .

To simplify the notations, we write (28) as

$$D_t \left( (\partial_{x_1}^2 + (1 + \delta^2) \partial_{x_2}^2) \psi - C_H \psi + B(x_1, x_2) + \beta x_2 \right) = -\alpha \Delta \psi + \mu \Delta^2 \psi + \operatorname{curl} f,$$

or, if  $J$  represents the jacobian operator,

$$\begin{aligned} \partial_t \left( (\Delta + \delta^2 \partial_{x_2}^2) \psi - C_H \psi \right) + J \left( \psi, (\Delta + \delta^2 \partial_{x_2}^2) \psi - C_H \psi + B(x_1, x_2) + \beta x_2 \right) \\ = -\alpha \Delta \psi + \mu \Delta^2 \psi + \operatorname{curl} f. \end{aligned} \quad (29)$$

We obtain after multiplication of (29) by  $\psi$ :

$$\begin{aligned} \int_{\mathcal{D}} \partial_t \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi - C_H \psi \right) \cdot \psi &= \alpha \int_{\mathcal{D}} (\nabla \psi)^2 + \mu \int_{\mathcal{D}} (\Delta \psi)^2 + \int_{\mathcal{D}} \operatorname{curl} f \cdot \psi, \\ \frac{d}{dt} \left( \|\nabla \psi\|_{L^2(\mathcal{D})}^2 + \delta^2 \|\partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2 + C_H \|\psi\|_{L^2(\mathcal{D})}^2 \right) &+ 2\alpha \int_{\mathcal{D}} (\nabla \psi)^2 \\ + 2\mu \int_{\mathcal{D}} (\Delta \psi)^2 &\leq 2 \left| \int_{\mathcal{D}} \operatorname{curl} f \cdot \psi \right| \leq \frac{C^2}{2\alpha} \|\operatorname{curl} f\|_{L^2(\mathcal{D})}^2 + \alpha \|\nabla \psi\|_{L^2(\mathcal{D})}^2. \end{aligned}$$

We finally get the inequality

$$\begin{aligned} \frac{d}{dt} \left( \|\nabla \psi\|_{L^2(\mathcal{D})}^2 + \delta^2 \|\partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2 + C_H \|\psi\|_{L^2(\mathcal{D})}^2 \right) + \alpha \|\nabla \psi\|_{L^2(\mathcal{D})}^2 \\ + 2\mu \|\Delta \psi\|_{L^2(\mathcal{D})}^2 \leq \frac{C^2}{\alpha} \|\operatorname{curl} f\|_{L^2(\mathcal{D})}^2, \end{aligned}$$

which ensure the following bounds:

$$\nabla \psi \in L^\infty(0, T; (L^2(\mathcal{D}))^2), \quad \psi \in L^\infty(0, T; L^2(\mathcal{D})), \quad \Delta \psi \in L^2(0, T; L^2(\mathcal{D})). \quad (30)$$

Similarly, multiplying (29) by  $\Delta \psi + \delta^2 \partial_{x_2}^2 \psi$  we get:

$$\begin{aligned} \int_{\mathcal{D}} \partial_t \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi - C_H \psi \right) \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right) \\ + \int_{\mathcal{D}} J \left( \psi, \beta x_2 + B \right) \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right) &= -\alpha \int_{\mathcal{D}} \Delta \psi \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right) \quad (31) \\ + \mu \int_{\mathcal{D}} \Delta^2 \psi \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right) &+ \int_{\mathcal{D}} \operatorname{curl} f \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right). \end{aligned}$$

Let us write some properties of the different terms of this equality:

$$\int_{\mathcal{D}} \left( \Delta \psi + \delta^2 \partial_{x_2}^2 \psi \right)^2 = \|\Delta \psi\|_{L^2(\mathcal{D})}^2 + \|\delta^2 \partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 + 2\delta^2 \int_{\mathcal{D}} \Delta \psi \partial_{x_2}^2 \psi \quad (32)$$

$$= \|\Delta \psi\|_{L^2(\mathcal{D})}^2 + \|\delta^2 \partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 + 2\delta^2 \|\nabla \partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2, \quad (33)$$

as the integrated term vanishes with the boundary conditions. We also have

$$2 \int_{\mathcal{D}} \Delta \psi \partial_{x_2}^2 \psi \leq 2 \|\Delta \psi\|_{L^2(\mathcal{D})} \|\partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}, \quad (34)$$

$$\|\partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 \leq \|\partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 + \|\partial_{x_1}^2 \psi\|_{L^2(\mathcal{D})}^2 + \|\partial_{x_1} \partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2 = \|\Delta \psi\|_{L^2(\mathcal{D})}^2, \quad (35)$$

and consequently

$$\begin{aligned} \int_{\mathcal{D}} (\Delta \psi + \delta^2 \partial_{x_2}^2 \psi)^2 &\leq (1 + 2\delta^2) \|\Delta \psi\|_{L^2(\mathcal{D})}^2 + \delta^4 \|\partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 \\ &\leq (1 + \delta^2)^2 \|\Delta \psi\|_{L^2(\mathcal{D})}^2. \end{aligned} \quad (36)$$

In addition, one should note the equality:

$$\begin{aligned} \int_{\mathcal{D}} \Delta^2 \psi (\Delta \psi + \delta^2 \partial_{x_2}^2 \psi) &= - \int_{\mathcal{D}} (\nabla \Delta \psi)^2 + \delta^2 \int_{\mathcal{D}} \Delta^2 \psi \partial_{x_2}^2 \psi \\ &= - \int_{\mathcal{D}} (\nabla \Delta \psi)^2 - \delta^2 \int_{\mathcal{D}} (\nabla \partial_{x_2}^2 \psi)^2 - \delta^2 \int_{\mathcal{D}} (\nabla \partial_{x_1} \partial_{x_2} \psi)^2. \end{aligned} \quad (37)$$

Finally we study the jacobian term:

$$\begin{aligned} \int_{\mathcal{D}} J(\psi, \beta x_2 + B) (\Delta \psi + \delta^2 \partial_{x_2}^2 \psi) \\ \leq C (\beta + \|\nabla B\|_{L^\infty(\mathcal{D})}) \|\nabla \psi\|_{L^2(\mathcal{D})} \|\Delta \psi\|_{L^2(\mathcal{D})}. \end{aligned} \quad (38)$$

Gathering Inequalities (32) to (38), Equation (31) becomes:

$$\begin{aligned} \frac{d}{dt} \left( \|\Delta \psi + \delta^2 \partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 + C_H \delta^2 \|\partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2 + C_H \|\nabla \psi\|_{L^2(\mathcal{D})}^2 \right) \\ + 2\mu \|\nabla \Delta \psi\|_{L^2(\mathcal{D})}^2 + 2\mu \delta^2 \left( \|\nabla \partial_{x_2}^2 \psi\|_{L^2(\mathcal{D})}^2 + \|\nabla \partial_{x_1} \partial_{x_2} \psi\|_{L^2(\mathcal{D})}^2 \right) \leq g(t) \end{aligned}$$

where  $g$  reads

$$\begin{aligned} g(t) = C (\beta + \|\nabla B\|_{L^\infty(\mathcal{D})}) \|\nabla \psi\|_{L^2(\mathcal{D})} \|\Delta \psi\|_{L^2(\mathcal{D})} \\ + \alpha \|\Delta \psi\|_{L^2(\mathcal{D})}^2 + \|\operatorname{curl} f\|_{L^2(\mathcal{D})} \|\Delta \psi\|_{L^2(\mathcal{D})}. \end{aligned}$$

Thanks to estimates (30),  $g$  is in  $L^1(0, T)$ , which gives

$$\begin{aligned} \Delta \psi &\in L^\infty(0, T; L^2(\mathcal{D})), & \partial_{x_2}^2 \psi &\in L^\infty(0, T; L^2(\mathcal{D})), \\ \nabla \Delta \psi &\in L^2(0, T; (L^2(\mathcal{D}))^2), & \nabla \partial_{x_2}^2 \psi &\in L^2(0, T; (L^2(\mathcal{D}))^2), \\ \nabla \partial_{x_1} \partial_{x_2} \psi &\in L^2(0, T; (L^2(\mathcal{D}))^2). \end{aligned}$$

Using classical techniques, we obtain existence of solutions to the Quasi-Geostrophic Equation (28) verifying  $\psi \in \mathcal{C}([0, T]; H^3(\mathcal{D}) \cap H_0^1(\mathcal{D})) \cap L^2(0, T; H^4(\mathcal{D}) \cap H_0^1(\mathcal{D}))$  and  $\Delta \psi \in \mathcal{C}([0, T]; H_0^1(\mathcal{D})) \cap L^2(0, T; H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}))$  (for detailed proof, the reader is referred to [2] and [6] for instance).

## 4 Numerical Results: Cosine Effects on the Gulf Stream

We now present the numerical results that show that the cosine effect, at least in some physical configurations, cannot be neglected. These simulations are performed using finite differences in space, with an explicit Leap-Frog scheme in time. One step every hundred, the Leap-Frog step is replaced by an Euler explicit one, in order to avoid stability problems. To compare the numerical simulations of the QG model with and without the new terms, we use and modify an existing solver proposed by [14].

### 4.1 Methodology

Equation (28) can be rewritten in a more "physical" way, that is:

$$\begin{aligned} \frac{\partial \phi}{\partial t} + J \left( \psi, \phi + \left( \text{Id} - \frac{H_{char}}{2 \tan \theta_0} \partial_{x_2} \right) \frac{2\Omega \sin \theta_0}{H_{char}} b + \beta x_2 \right) \\ = - \frac{1}{\varepsilon L_{car}} \alpha_0(H_{car}) \Delta \psi + \mu \Delta^2 \psi + \text{curl} f, \end{aligned}$$

where  $J$  is the jacobian, and  $\phi$  a potential vorticity which reads

$$\phi = \left( \Delta + \delta^2 \partial_{x_2}^2 - \frac{(2\Omega \sin \theta_0)^2}{g H_{char}} \text{Id} \right) \psi. \quad (39)$$

The scheme used to solve the Quasi-Geostrophic Equation is the following: suppose that we know all the quantities at time  $t_n$ ,  $\phi^n$ ,  $\psi^n$  and the diverse coefficients. Then we can compute the term  $fl^n$  which represents the time derivative of  $\phi$  at  $t_n$ , namely:

$$\begin{aligned} fl^n = J \left( \phi^n + \left( \text{Id} - \frac{H_{char}}{2 \tan \theta_0} \partial_{x_2} \right) \frac{2\Omega \sin \theta_0}{H_{char}} b + \beta x_2, \psi^n \right) \\ - \frac{1}{\varepsilon L_{car}} \alpha_0(H_{car}) \Delta \psi^n + \mu \Delta^2 \psi^n + \text{curl} f^n. \end{aligned}$$

So we get  $\phi$  at time  $t_{n+1}$  and obtain  $\psi^{n+1}$  thanks to Equation (39). If we start with an initial condition identically equal to zero for example, we are able to plot the stream function at any time.

This is the main idea of the numerical scheme. The principal difficulty is the resolution of the equation that gives  $\phi^{n+1}$  as a function of  $\psi^{n+1}$ . Note that if  $\delta = 0$ , we are led to invert a laplacian. The methodology we used is the one developed in [18], that is a combination of symmetries and fast Fourier transforms. More precisely, as we have Dirichlet boundary conditions, we define two quantities  $\zeta^{n+1}$  and  $\xi^{n+1}$  by  $[0; \phi^{n+1}; 0; \phi_\sigma^{n+1}]$  (resp.  $[0; \psi^{n+1}; 0; \psi_\sigma^{n+1}]$ ) where  $f_\sigma$  is the symmetric vector of  $(-f)$  in the first variable  $x_1$ . We compute the Fourier transforms  $\hat{\zeta}$  and  $\hat{\xi}$  of  $\zeta$  and  $\xi$  in the  $x_1$  direction and we get a system in  $x_2$ . The matrix  $M$  of this system ( $M\hat{\xi} = \hat{\zeta}$ ) is tridiagonal and consequently can easily be inverted. The last step is to calculate the inverse of the Fourier transform to get  $\psi^{n+1}$  from  $\phi^{n+1}$ .

## 4.2 Choice of the Parameters

We consider that  $\mathcal{D}$  is a square basin of length  $L = 4.000\text{km}$  which is  $5.000\text{m}$  deep with a  $100 \times 100$  points grid, and thus the aspect ratio is  $\varepsilon = 1.25 \times 10^{-3}$ . We fix physical parameters close to the one in the North Atlantic ocean (see [8]). We check that we are in the case studied at the beginning, that is the non-dimensional viscosity is of order of the aspect ratio. We choose the forcing term (the wind) as  $-10^{-2} \sin(2\pi x_2/L)$  and we let the model run over nearly 1600 years (note that we have  $T = O(\varepsilon^{-1})$ ) to ensure that the convergence error is small compared to the cosine effect. We then compare the results obtained without the cosine terms and the ones with the new expressions, with a flat or a varying bottom. To determine how to do these comparisons, we first look at the evolution of the energy: it indicates that the system is chaotic, and thus we consider the time-average of  $\psi$ , and not the values of  $\psi$ . Consequently we present the corresponding figures in the following parts.

## 4.3 Flat Bottom

As we have noticed before, we have two correlated contributions of the cosine effect. The idea is to disconnect these terms: we take a flat bottom and the second term (depending on the variations of the topography) vanishes. We cannot really discern the differences between the stream function without the cosine effect and the new result, so we only give the first one and their difference.

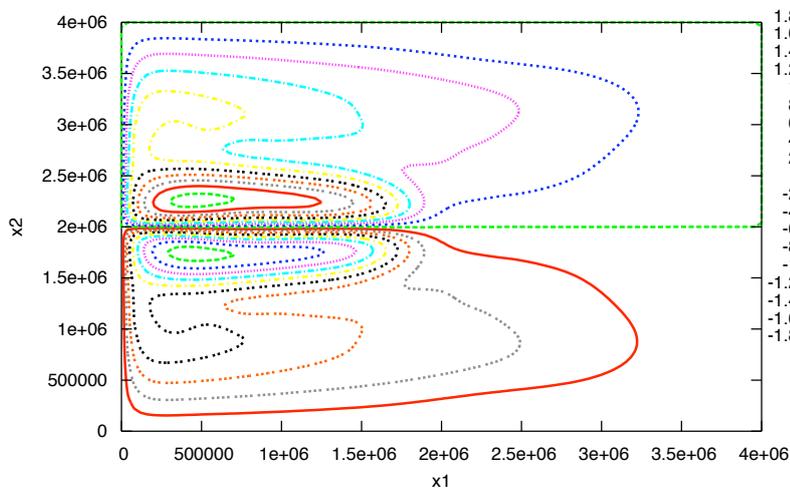


Figure 3: Time-average of the stream function over 1600 years. The upper part is negative so the velocity is in the counterclockwise direction.

The values of the time-average of the stream function are presented in Figure 3: we have to keep in mind that its bounds are  $\pm 1.8 * 10^5$ . We can notice that the velocity is similar to the one in the gulf stream: an counterclockwise current in the north and a clockwise current in the southern part of the domain. In Figure 4, we plot the difference between the time-averages of the new stream function and this function without the cosine term: it represents the cosine effect with a flat bottom (the influence of the  $\delta$  term). We obtain a difference located in the jet, which conserves quite well the symmetry. The extreme values of this difference are  $1.4 * 10^4$ , that is the  $\delta$  effect is of about ten percents.

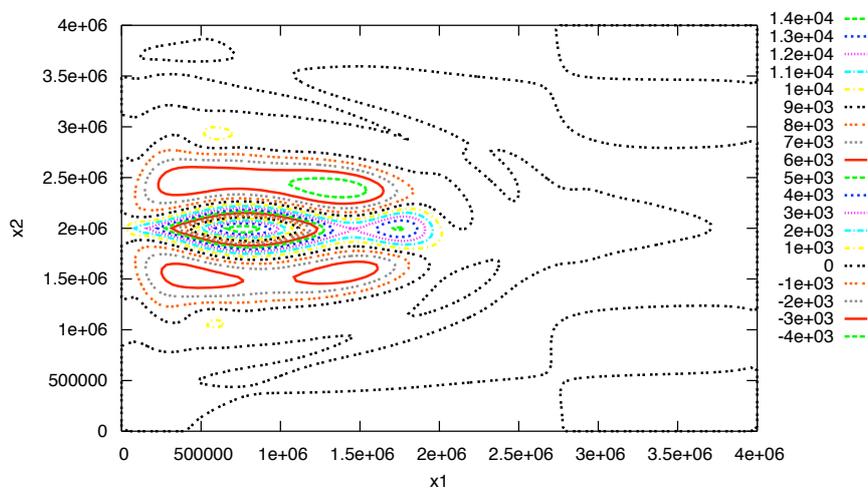


Figure 4: Time-average of the difference between the stream function with the cosine terms and the stream function without these terms, over 1600 years.

**Remark 3** *These results hold for long times; at a few years scale, the difference is not significant. We could then say that the  $\delta$  term do not have any influence for the meteorology, but it should be taken into account in climatology.*

#### 4.4 Varying Bottom

Let us now consider the cosine effect, with a bottom depending on the second variable  $x_2$ . We use a bottom resembling to the Mid-Atlantic ridge, as we can see in Figure 5. As for the flat bottom, we present in Figure 6 the time-average of the stream function, solution of the QG model without any cosine effect. Figure 7 plots the difference between the two models, with and without the cosine terms.

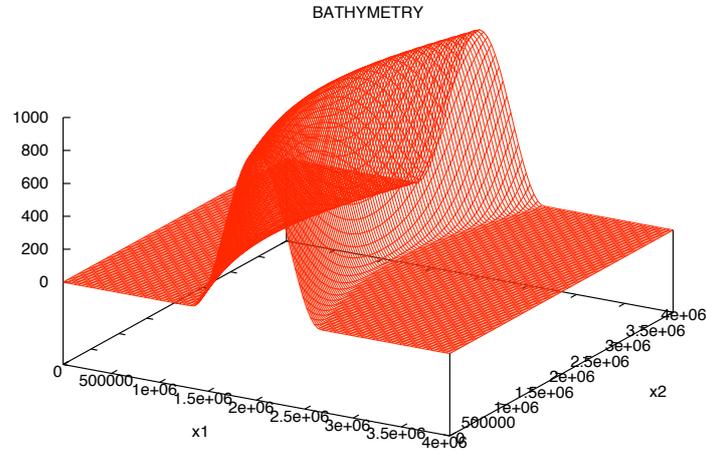


Figure 5: An example of bottom.

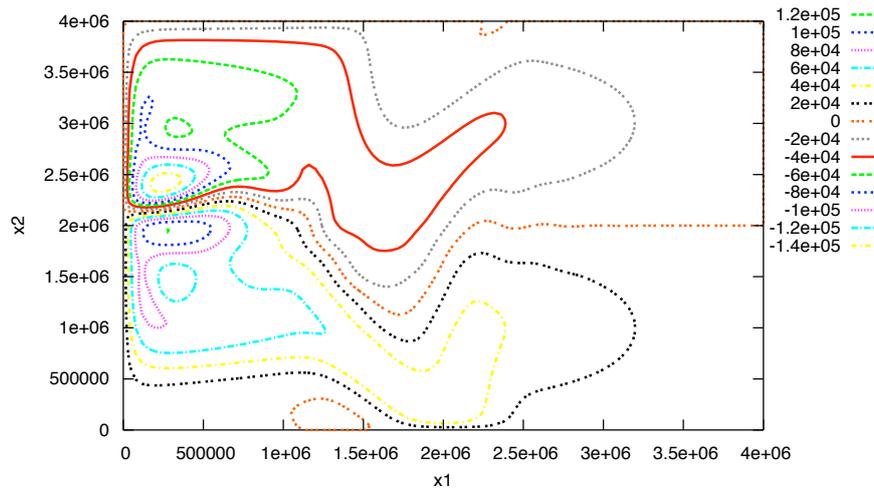


Figure 6: Time-average of the stream function over 1600 years with a varying bottom. The upper part is negative so the velocity is in the counterclockwise direction.

The jet is confined in the left part of the domain, the currents are stopped by the topography. We find a difference that is no more significant. Due to the orientation of the ridge, the cosine effect increases the currents.

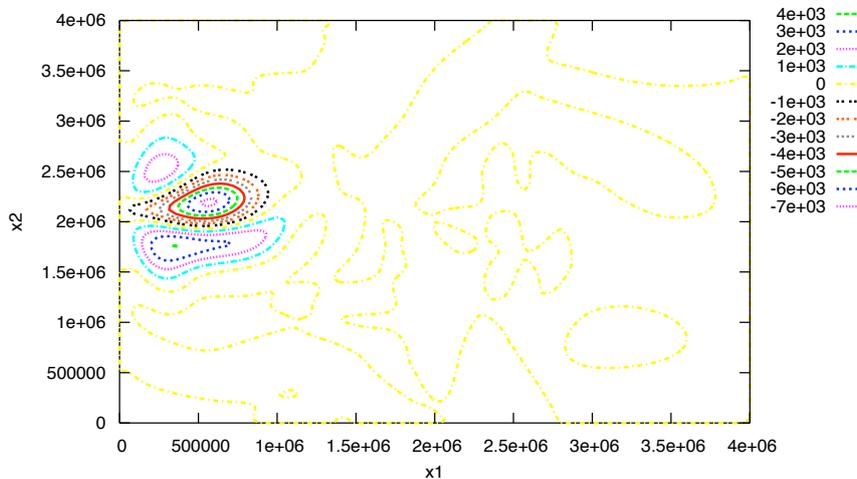


Figure 7: Time-average of the difference between the stream function with the cosine terms and the stream function without these terms, over 1600 years, with a varying bottom.

**Remark 4** *Note that these results for a flat or a varying bottom can be confirmed by the study of the energy: we have a difference due to the  $\delta$  effect on the energy spectrum that disappears when we take the topography represented on Figure 5.*

## 5 Conclusion

We proved that the cosine terms that appear in the derivation of the viscous Shallow Water Equation has to be taken into account. To the best of our knowledge, this was not the case in past works. From SWE, new Quasi-Geostrophic Equations are obtained. Depending on the topography we consider, numerical experiments show that this effect is non trivial and can be significant.

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