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## - To cite this version:

Hassan Fahs. Numerical evaluation of a non-conforming discontinuous Galerkin method on triangular meshes for solving the time-domain Maxwell equations. [Research Report] RR-6311, INRIA. 2007, pp.48. inria-00175738v2

## HAL Id: inria-00175738 <br> https://inria.hal.science/inria-00175738v2

Submitted on 2 Oct 2007

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Numerical evaluation of a non-conforming discontinuous Galerkin method on triangular meshes for solving the time-domain Maxwell equations

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$\mathrm{N}^{\circ} 6311$
August 2007
$\qquad$ Thème NUM $\qquad$


# Numerical evaluation of a non-conforming discontinuous Galerkin method on triangular meshes for solving the time-domain Maxwell equations 

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#### Abstract

We report on a detailed numerical evaluation of the non-dissipative, non-conforming discontinuous Galerkin method on triangular meshes previously introduced in [15], and further extended in [16], for solving the two-dimensional time-domain Maxwell equations. In [16], a $h p$-like DGTD (Discontinuous Galerkin Time-Domain) method is considered, where the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way. The resulting DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method combines a centered approximation for the evaluation of fluxes at the interface between neighboring elements of the mesh, with a second order leap-frog time integration scheme. Moreover, non-conforming meshes with arbitrary-level hanging nodes are allowed. Here, our objective is to assess the convergence, the stability and the efficiency of the method, but also discuss its limitations, through numerical experiments for 2D propagation problems in homogeneous and heterogeneous media with various types and locations of material interfaces.


Key-words: Maxwell's equations, time-domain, discontinuous Galerkin method, non-conforming triangular meshes, $h p$-like method.

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# Evaluation numérique d'une méthode Galerkin discontinue non-conforme en maillages triangulaires pour la résolution des équations de Maxwell instationnaires 

Résumé : Dans cette étude, on réalise une évaluation numérique détaillée de la méthode Galerkin discontinue non-conforme en maillages triangulaires pour la résolution des équations de Maxwell instationnaires précédemment introduite dans [15], puis étendue dans [16]. Dans la méthode considérée dans [16], l'ordre d'interpolation $p$ et le pas du maillage $h$ varient localement. La méthode DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ résultante combine une approximation centrée pour l'évaluation des flux aux interfaces entre éléments voisins du maillage, à un schéma d'intégration en temps de type saute-mouton. De plus, cette méthode autorise l'utilisation de maillages non-conformes possédant un nombre arbitraire de noeuds flottants. On présente ici les résultats d'une série d'expériences numériques pour évaluer la convergence, la stabilité et l'éfficacité de la méthode, mais aussi identifier ses limitations. Ces tests numériques sont effectués en deux dimensions d'espace et dans des milieux homogènes et hétérogènes avec différents types et différentes localisations d'interfaces entre matériaux.

Mots-clés : équations de Maxwell, domain temporel, méthode Galerkin discontinue, maillages triangulaires non-conformes, méthode $h p$.

## 1 Introduction

The difficulties linked to the numerical solution of the time-domain Maxwell equations find their roots in the characteristics of the underlying wave propagation problems i.e. the geometrical characteristics of the diffracting objects, the physical characteristics of the propagation medium (heterogeneity, physical dispersion and dissipation) and the characteristics of the sources (wires, etc.). Applications with such characteristics can be found throughout the applied sciences and engineering e.g. the design and optimization of antennas [7] and radars [28], the design of emerging technologies such as high speed electronics and integrated optics, and a variety of military and civilian applications [29]-[27]. Other challenging applications are addressing societal questions such as the potential adverse effects of electromagnetic waves emitted by mobile phones [31]. Such problems require high fidelity numerical solutions with a rigorous control of the numerical errors. Even for linear problems such conditions force one to look beyond standard computational techniques and seek new computational frameworks enabling the accurate, efficient, and robust modeling of wave phenomena over long simulation times in settings of realistic geometric complexity.

The simplicity and reasonable accuracy of the classical finite difference time-domain (FDTD) method [39] has propelled this method to become the method of choice among engineers and scientists solving Maxwell's equations in the time-domain. In particular the last decade has seen an increased number of applications and further developments of this method, many driven from the very influential texts by Taflove [33]. It is however clear that the original FDTD method has several limitations. Its inherent second order accuracy limits its ability to correctly represent wave motion over long distances unless the grid is prohibitively fine. Furthermore, the simplicity of the method, often recognized as its strength, also becomes its most severe restriction by prohibiting the accurate representation of problems in complex geometries. In recent years, a number of efforts aimed at addressing the shortcomings of the classical FDTD scheme, e.g. embedding schemes to overcome staircasing [13]-[37], high order finite difference schemes [33]-[19], non-conforming orthogonal FDTD methods [10]. Other techniques and improvements are discussed in [33]. Most of these methods, however, have not really penetrated into main stream user community, partly due to their complicated nature and partly because these new methods themselves often introduce other complications.

Finite element time-domain (FETD) methods can handle unstructured meshes and complex geometries but the development of such methods for solving Maxwell's equations, especially those with high order accuracy, has been relatively slow. A primary reason is the appearance of spurious, non-physical solutions when a straightforward nodal continuous Galerkin finite element scheme is used to approximate the Maxwell curl-curl equations. Bossavit made the fundamental observation that the use of special curl-conforming elements [24] would overcome the problem of spurious modes by mimicking properties of vector algebra [4]. Although very successful, such formulations are not entirely void of problems: the algebraic problems are larger than for nodal elements and the conformity requirements of the continuous Galerkin formulation makes adaptivity complex.

In an attempt to offer an alternative to the classical finite element formulation based on edge elements, we consider here discontinuous Galerkin formulations [8] based on high order nodal interpolation for solving the first order time-domain Maxwell's equations. Discontinuous Galerkin time-domain (DGTD) methods based on discontinuous finite element spaces, easily handle elements of various types and shapes, irregular non-conforming meshes [15], and even locally varying polynomial degree, and hence offer great flexibility in the mesh design, but they also lead to (block-) diagonal mass matrices and therefore yield fully explicit, inherently parallel methods when coupled with explicit time stepping [3]. In fact, for constant material coefficients, the mass matrix is diagonal for a judicious choice of (locally orthogonal) shape functions [30]. Moreover, continuity is weakly enforced across mesh interfaces by adding suitable bilinear forms (so-called numerical fluxes) to the standard variational formulations. Whereas high order discontinous Galerkin time-domain methods have been developed on conforming hexahedral [9] and tetrahedral [18] meshes, the design of non-conforming discontinuous Galerkin time-domain methods is still in its in-
fancy. In practice, the non-conformity can result from a local refinement of the mesh (i.e. $h$-refinement), of the interpolation degree (i.e. p-enrichment) or of both of them (i.e. hp-refinement).

This work is a continuation of $[16]$ where a $h p$-like DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method was introduced for solving the two-dimensional time-domain Maxwell equations on non-conforming triangular meshes. It was numerically shown that the proposed method has many advantages by comparing it with a $h$-refinement one. The main goals of this report are, on one hand, to study numerically the convergence of the $h$ - and $h p$-refinement DGTD methods for propagation problems in both homogeneous and heterogeneous media and using conforming and non-conforming meshes and, on the other hand, to compare these methods in terms of accuracy and computational costs. The rest of this report is organized as follows. In section 2 we recall the basic features of our discontinuous Galerkin time-domain formulation for solving the first order Maxwell equations in the time domain, based on totally centered numerical fluxes and a leap-frog time-integration scheme. Numerical experiments are presented in section 3 for homogeneous media and section 4 for heterogeneous domain. Finally, conclusions and future works are summarized in section 5.

## 2 The discontinuous Galerkin scheme

We shall consider the solution of the two-dimensional Maxwell equations in the TMz polarization on a bounded domain $\Omega \subset \mathbb{R}^{2}$.

$$
\left\{\begin{array}{l}
\epsilon \frac{\partial E_{z}}{\partial t}-\frac{\partial H_{y}}{\partial x}+\frac{\partial H_{x}}{\partial y}=0  \tag{2.1}\\
\mu \frac{\partial H_{x}}{\partial t}+\frac{\partial E_{z}}{\partial y}=0 \\
\mu \frac{\partial H_{y}}{\partial t}-\frac{\partial E_{z}}{\partial x}=0
\end{array}\right.
$$

where the unknowns $\mathbf{E}=\left(0,0, E_{z}\right)$ and $\mathbf{H}=\left(H_{x}, H_{y}, 0\right)$ are the electric and magnetic fields respectively. The electric permittivity $\epsilon$ and the magnetic permeability $\mu$ of the medium are assumed to be piecewise constant. Moreover, we assume that the field components as well as the material parameters $\epsilon$ and $\mu$ do not depend on the $z$ coordinate. On the boundary $\Gamma=\partial \Omega$ we use either a perfect electric conductor condition i.e. $E_{z}=0$, or a first order Silver-Müller absorbing boundary (i.e. an artificial boundary used to truncate the computational domain) condition i.e. $E_{z}=c \mu\left(n_{y} H_{x}-n_{x} H_{y}\right)$ or both of them, where $c$ is the speed of propagation and $\vec{n}=\left(n_{x}, n_{y}\right)$ denotes the unit normal vector pointing outward to $\Gamma$.

We consider a partition $\mathcal{T}_{h}$ of $\Omega$ into a set of triangles $T_{i}$ of size $h_{i}$ with boundaries $\partial T_{i}$ such that $h=$ $\max _{T_{i} \in \mathcal{T}_{h}} h_{i}$. To each $T_{i} \in \mathcal{T}_{h}$ we assign an integer $p_{i} \geq 0$ and we collect the $p_{i}$ in the vector $p=\left\{p_{i}: T_{i} \in \mathcal{T}_{h}\right\}$. $T_{i} \in \mathcal{T}_{h}$
Within this construction we admit meshes with possibly hanging nodes i.e. by allowing non-conforming (or irregular) meshes where element vertices can lie in the interior of edges of other elements (see Fig. 2.1). Each triangle $T_{i}$ is assumed to be the image, under a smooth bijective (diffeomorphic) mapping $\tau_{i}$, of a fixed reference triangle $\hat{T}=\{\hat{x}, \hat{y} \mid \hat{x}, \hat{y} \geq 0 ; \hat{x}+\hat{y} \leq 1\}$. Assuming that $T_{i}$ is a straight sided triangle defined through the coordinates of the three vertices $\mathbf{v}_{1}^{i}, \mathbf{v}_{2}^{i}$ and $\mathbf{v}_{3}^{i}$ (see Fig. 2.2), the correspondence between the two triangles $\hat{T}$ and $T_{i}$ is established through the use of the barycentric coordinates $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$. We recall that any point $\mathbf{x}^{i} \in T_{i}$ can be expressed as a convex combination of the vertices of $T_{i}$ and the mapping is simply given by $\tau_{i}:(\hat{x}, \hat{y}) \in \hat{\mathrm{T}} \rightarrow \mathbf{x}^{i}$, such that $\mathbf{x}^{i}(\hat{x}, \hat{y})=\lambda_{1} \mathbf{v}_{1}^{i}+\lambda_{2} \mathbf{v}_{2}^{i}+\lambda_{3} \mathbf{v}_{3}^{i}$, where $\lambda_{1}+\lambda_{2}+\lambda_{3}=1$ and $0 \leq\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) \leq 1$ with $\lambda_{1}=1-\hat{x}-\hat{y}, \lambda_{2}=\hat{x}$ and $\lambda_{3}=\hat{y}$.

In the following, for a given partition $\mathcal{T}_{h}$ and vector $p$, we seek approximate solutions to (2.1) in the finite dimensional subspace $V_{p}\left(\mathcal{T}_{h}\right):=\left\{v \in L^{2}(\Omega): v_{\mid T_{i}} \in \mathbb{P}^{p_{i}}\left(T_{i}\right), \forall T_{i} \in \mathcal{T}_{h}\right\}$, where $\mathbb{P}^{p_{i}}\left(T_{i}\right)$ denotes the space of nodal polynomials $\left\{\varphi_{i j}\right\}_{j=1}^{d_{i}}$ of total degree at most $p_{i}$ on the element $T_{i}$. The space $V_{p}\left(\mathcal{T}_{h}\right)$ has the dimension $d_{i}$, the local number of degrees of freedom (DOF). Note that the polynomial degree, $p_{i}$, may vary from element to element in the mesh and that a function $v_{h}^{p} \in V_{p}\left(\mathcal{T}_{h}\right)$ is discontinuous across
element interfaces. By non-conforming interface we mean an interface $a_{i k}$ which has at least one of its two vertices in a hanging node or/and $p_{\left.i\right|_{a_{i k}}} \neq p_{\left.k\right|_{a_{i k}}}$.


Figure 2.1: Irregular mesh with hanging nodes.
For each triangle $T_{i}, \epsilon_{i}$ and $\mu_{i}$ respectively denote the local constant electric permittivity and magnetic permeability. For two distinct triangles $T_{i}$ and $T_{k}$ in $\mathcal{T}_{h}$, the intersection $T_{i} \cap T_{k}$ is an (oriented) edge $s_{i k}$ which we will call interface, with oriented normal vector $\vec{n}_{i k}$ and with unitary one $\overrightarrow{\tilde{n}}_{i k}$. For the boundary interfaces, the index $k$ corresponds to a fictitious element outside the domain. Finally, we denote by $\mathcal{V}_{i}$ the set of indices of the elements which are neighbors of $T_{i}$.


Figure 2.2: Mapping between the physical triangle $T_{i}$ and the master triangle $\hat{T}$.
The DGTD method at the heart of this study is based on a leap-frog time scheme ( $E_{z}$ is computed at integer time-stations and $H_{x}$ and $H_{y}$ at half-integer time-stations) and totally centered numerical fluxes at the interface between elements. Decomposing $H_{x}, H_{y}$ and $E_{z}$ on element $T_{i}$ according to $H_{\mathbf{x}}\left(., t^{n+\frac{1}{2}}\right)=\sum_{j=1}^{d_{i}} H_{\mathbf{x}_{i j}}^{n+\frac{1}{2}} \varphi_{i j}$ and $E_{z}\left(., t^{n}\right)=\sum_{j=1}^{d_{i}} E_{z_{i j}}^{n} \varphi_{i j}$, where $\mathbf{x} \in\{x, y\}$.

Using the notations $\mathbf{E}_{z_{i}}^{n}=\left(E_{z_{i 1}}^{n}, \ldots, E_{z_{i d_{i}}}^{n}\right)^{t}$ and $\mathbf{H}_{\mathbf{x}_{i}}^{n+\frac{1}{2}}=\left(H_{\mathbf{x}_{i 1}}^{n+\frac{1}{2}}, \ldots, H_{\mathbf{x}_{i d_{i}}}^{n+\frac{1}{2}}\right)^{t}$, the DGTD-P $\mathbb{P}_{p_{i}}$ method writes:

$$
\left\{\begin{array}{l}
\epsilon_{i} \mathbb{M}_{i} \frac{\mathbf{E}_{z_{i}}^{n+1}-\mathbf{E}_{z_{i}}^{n}}{\Delta t}=-\mathbb{K}_{i}^{x} \mathbf{H}_{y_{i}}^{n+\frac{1}{2}}+\mathbb{K}_{i}^{y} \mathbf{H}_{x_{i}}^{n+\frac{1}{2}}+\sum_{k \in \mathcal{V}_{i}}\left(\mathbb{G}_{x_{i k}}^{n+\frac{1}{2}}-\mathbb{G}_{y_{i k}}^{n+\frac{1}{2}}\right), \\
\mu_{i} \mathbb{M}_{i} \frac{\mathbf{H}_{x_{i}}^{n+\frac{1}{2}}-\mathbf{H}_{x_{i}}^{n-\frac{1}{2}}}{\Delta t}=\mathbb{K}_{i}^{y} \mathbf{E}_{z_{i}}^{n}-\sum_{k \in \mathcal{V}_{i}} \mathbb{F}_{y_{i k}}^{n} \\
\mu_{i} \mathbb{M}_{i} \frac{\mathbf{H}_{y_{i}}^{n+\frac{1}{2}}-\mathbf{H}_{y_{i}}^{n-\frac{1}{2}}}{\Delta t}=-\mathbb{K}_{i}^{x} \mathbf{E}_{z_{i}}^{n}+\sum_{k \in \mathcal{V}_{i}} \mathbb{F}_{x_{i k}}^{n}
\end{array}\right.
$$

where $\mathbb{M}_{i}$ is the local mass (symmetric positive definite) matrix, and $\mathbb{K}_{i}^{\mathbf{x}}$ is the (skew-symmetric) stiffness matrix. The vector quantities $\mathbb{F}_{\mathbf{x}_{i k}}^{n}$ and $\mathbb{G}_{\mathbf{x}_{i k}}^{n+\frac{1}{2}}$ are defined as:

$$
\mathbb{F}_{\mathbf{x}_{i k}}^{n}=\mathbb{S}_{i k}^{\mathbf{x}} \mathbf{E}_{z_{k}}^{n}, \quad \mathbb{G}_{\mathbf{x}_{i k}}^{n+\frac{1}{2}}=\mathbb{S}_{i k}^{\mathbf{x}} \mathbf{H}_{\mathbf{x}_{k}}^{n+\frac{1}{2}}
$$

where $\mathbb{S}_{i k}^{\mathbf{x}}$ is the $d_{i} \times d_{k}$ interface matrix on $s_{i k}$ which verifies ${ }^{t} \mathbb{S}_{i k}^{\mathbf{x}}=-\mathbb{S}_{k i}^{\mathbf{x}}$ (if $s_{i k}$ is an internal interface) and ${ }^{t} \mathbb{S}_{i k}^{\mathbf{x}}=\mathbb{S}_{i k}^{\mathbf{x}}$ (if $s_{i k}$ is a boundary interface). Note that, for non-conforming interfaces, we calculate the matrix $\mathbb{S}_{i k}^{\mathbf{x}}$ by using a Gauss quadrature formula [16].

In [15]-[16], a numerical dispersion was observed when a low order conforming DGTD- $\mathbb{P}_{p}(p=0,1$ and $p_{i}=p$ everywhere) is applied. This dispersion error is not reduced notably by using a $h$-refinement strategy (i.e. by refining the mesh for a fixed $p$, yielding a locally non-conforming mesh). On the other hand, the dispersion error is minimized when a $p$-enrichment strategy (i.e. by increasing $p$ for a fixed $h$ ) is used. However, the latter approach requires a large number of DOF thus increasing substantially the computing time and memory usage. In the recent work [16] a $h p$-like DGTD method has been proposed, where we combine the $h$-refinement and $p$-enrichment strategies. This method consists in using a high polynomial degree in the coarse (i.e. not refined) mesh and a low order one in the refined region. The resulting scheme is referred to as a DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ (or $h p$-refinement) method where $p_{c}$ and $p_{f}$ are the polynomial degrees in the coarse and fine elements respectively. We have numerically demonstrated that the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method can strongly reduce, in the one hand, the dispersion error and, in the other hand, the computational cost and memory consumption compared with the $h$-refinement method. Of course, such a non-conforming scheme is a first step towards a fully $h p$-adaptive method relying on appropriate error estimators.

The stability of the DGTD- $\mathbb{P}_{p_{i}}$ method for non-conforming meshes is studied in [15] using an energetic approach. Furthermore, a theoretical proof of the convergence has been established in [18] for any interpolation degree $p \neq 0$ in the case of conforming simplicial meshes. It is shown that the convergence rate of DGTD- $\mathbb{P}_{p}$ method is $\mathcal{O}\left(T h^{\min (s, p)}\right)+\mathcal{O}\left(\Delta t^{2}\right)$, where $\Delta t$ is the time step over the interval $[0, T]$ and the solution belongs to $H^{s}(\Omega)$ with $s>1 / 2$. Our attention is turned on the validity of this result in the case of non-conforming meshes using the DGTD- $\mathbb{P}_{p}$ and DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods, and an answer is given here on the basis of numerical simulations.

## 3 Homogeneous media

In this section, we consider several wave propagation problems in homogeneous media for which analytical solutions exist. Our objectives are the following:

- to assess numerically and compare the convergence of the conforming and non-conforming DGTD methods,
- to provide insights regarding the overall performances of the $h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method,
- to compare, on one hand, the conforming DGTD- $\mathbb{P}_{p}$ method with the non-conforming one and, on the other hand, the $h p$-refinement method with the $h$-refinement one.

The above points are studied for the following test cases:

1. a concentric PEC cylinders resonator,
2. a circular PEC resonator,
3. a wedge-shaped PEC resonator.

### 3.1 Concentric PEC cylinders resonator

We consider a resonator which consists of two concentric PEC cylinders with an electromagnetic wave bouncing back and forth between the walls (see Fig. 3.1). The material is taken to be the vacuum i.e. $\epsilon=\mu=1$ (relative quantities). The radii of the two cylinders are $r_{1}=\frac{1}{6}$ and $r_{2}=\frac{1}{2}$. The exact time-domain solution of the problem is [12]-[13]:

$$
\begin{aligned}
E_{z}= & \cos (\omega t+\theta)\left[J_{1}(\omega r)+a Y_{1}(\omega r)\right] \\
H_{x}= & -\frac{1}{2} \sin (\omega t+\theta) \sin (\theta)\left[J_{0}(\omega r)-J_{2}(\omega r)+a\left(Y_{0}(\omega r)-Y_{2}(\omega r)\right)\right] \\
& -\frac{\cos (\theta)}{\omega r} \cos (\omega t+\theta)\left[J_{1}(\omega r)+a Y_{1}(\omega r)\right] \\
H_{y}= & \frac{1}{2} \sin (\omega t+\theta) \cos (\theta)\left[J_{0}(\omega r)-J_{2}(\omega r)+a\left(Y_{0}(\omega r)-Y_{2}(\omega r)\right)\right] \\
& -\frac{\sin (\theta)}{\omega r} \cos (\omega t+\theta)\left[J_{1}(\omega r)+a Y_{1}(\omega r)\right]
\end{aligned}
$$

for $r_{1}<r<r_{2}$, where $(r, \theta)=\left(\sqrt{x^{2}+y^{2}}, \arctan (y / x)\right)$ are the usual polar coordinates; $J_{n}$ and $Y_{n}$ stand for the $n$-th order Bessel functions of the first and second kind, respectively. The values of the parameters $\omega$ and $a$ are obtained by enforcing the boundary condition $E_{z}=0$ at $r=r_{1}$ and $r=r_{2}$. Then, as in [12], we set $\omega=9.813695999428405$ and $a=1.76368380110927$. First, a quasi-uniform conforming mesh is constructed (see Fig. 3.2 left). Then a non-conforming mesh is obtained by locally refining a cylindrical zone as shown on Fig. 3.2 right. Contour lines of the $E_{z}$ component at times $t=1$ and $t=10$ are shown on Fig. 3.3 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.


Figure 3.1: Concentric PEC cylinders resonator setting.

Two strategies are considered for this problem: the first one is the DGTD- $\mathbb{P}_{p}$ (or $h$-refinement) method and the second one is the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ (or $h p$-refinement) method.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. We first consider the case where the interpolation degree $p$ is uniform and the mesh size $h$ is varied. In Tab. 3.1, we summarize the CFL values evaluated numerically for some $p$ (i.e. by assessing the limit beyond which we observe a growth of the discrete energy). Tab. 3.2 and 3.3 give the $L^{2}$ error on the electric field component $E_{z}$, the convergence rate, the CPU time and the number of time steps (\# time steps) to reach $t=1$ and using sequences of conforming and non-conforming


Figure 3.2: Concentric cylinders PEC resonator.
Example of conforming (left) and non-conforming (right) triangular meshes.


Figure 3.3: Concentric cylinders PEC resonator.
Contour lines of $E_{z}$ at $t=1$ (left) and $t=10$ (right).
$D G T D-\mathbb{P}_{1}$ method using a conforming mesh with 1088 nodes and 2048 triangles.
meshes. The characteristics of these meshes are also listed in Tab. 3.2 and 3.3. The non-conforming meshes are obtained by refining (one refinement level) the cylindrical zone $1 / 3 \leq r \leq 3 / 8$. Fig. 3.4 shows the $L^{2}$ error on $E_{z}$ as a function of the square root of the number of degrees of freedom (\# DOF). We can deduce from Tab. 3.2 and 3.3 that the DGTD- $\mathbb{P}_{p}$ method converges as $h^{(2)} \forall p \geq 1$ and as $h^{(1)}$ for $p=0$, for conforming as well as non-conforming meshes.

Table 3.1: Concentric cylinders PEC resonator. Numerical CFL values for the DGTD- $\mathbb{P}_{p}$ method.

| $p$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| CFL | 1.0 | 0.3 | 0.2 | 0.1 | 0.09 |

Table 3.2: Concentric cylinders PEC resonator.
Convergence study for the h-refinement DGTD-P $\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, $C P U$ time and \# time steps are measured at $t=1$.

| \# nodes | \# triangles | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | $\begin{gathered} \text { \# time } \\ \text { steps } \\ \hline \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |
| 1088 | 2048 | 2048 | $5.24 \mathrm{E}-02$ | - | 2 | 80 |
| 4224 | 8192 | 8192 | $2.62 \mathrm{E}-02$ | 1.00 | 9 | 158 |
| 16640 | 32768 | 32768 | $1.31 \mathrm{E}-02$ | 1.00 | 74 | 313 |
| 66048 | 131072 | 131072 | $6.56 \mathrm{E}-03$ | 1.00 | 593 | 624 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |
| 288 | $512\left(\right.$ mesh-A $\left.{ }^{a}\right)$ | 1536 | $2.52 \mathrm{E}-02$ | - | 2 | 137 |
| 1088 | 2048 (mesh-B) | 6144 | $6.46 \mathrm{E}-03$ | 1.96 | 9 | 266 |
| 4224 | 8192 (mesh-C) | 24576 | $1.65 \mathrm{E}-03$ | 1.97 | 65 | 525 |
| 16640 | 32768 (mesh-D) | 98304 | $4.11 \mathrm{E}-04$ | 2.00 | 524 | 1043 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |
|  | mesh-A | 3072 | $1.44 \mathrm{E}-02$ | - | 3 | 205 |
|  | mesh-B | 12288 | $3.58 \mathrm{E}-03$ | 2.01 | 21 | 399 |
|  | mesh-C | 49152 | 8.93E-04 | 2.00 | 168 | 787 |
|  | mesh-D | 196608 | $2.23 \mathrm{E}-04$ | 2.00 | 1390 | 1564 |
| DGTD-P ${ }_{3}$ |  |  |  |  |  |  |
|  | mesh-A | 5120 | $1.42 \mathrm{E}-02$ | - | 10 | 410 |
|  | mesh-B | 20480 | $3.52 \mathrm{E}-03$ | 2.02 | 71 | 797 |
|  | mesh-C | 81920 | $8.74 \mathrm{E}-04$ | 2.01 | 565 | 1574 |
|  | mesh-D | 327680 | $2.18 \mathrm{E}-04$ | 2.01 | 4916 | 3128 |

${ }^{a}$ This means that mesh-A contains 288 nodes and 512 triangles, etc.
$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. We now consider the case where both $p$ and $h$ are locally refined. The observed numerical CFL values for the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method are summarized in Tab. 3.4. We give in Tab. 3.5, the $L^{2}$ error on $E_{z}$, the convergence rate, the CPU time and the number of time steps to reach $t=1$ using non-conforming meshes which are obtained by refining the zone $1 / 3 \leq r \leq 3 / 8$ as previously. Fig. 3.5 shows the $L^{2}$ error on $E_{z}$ as a function of the square root of the number of DOF. Clearly, the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method converges as $h^{(2)}, \forall p_{f} \neq 0$, and for $p_{f}=0$ the convergence rate is more than $\mathcal{O}\left(h^{(1.5)}\right), \forall p_{c}$.

In summary, the results given in Tab. 3.2, 3.3 and 3.5 motivates the following remarks:

- in the case of the $h$-refinement DGTD- $\mathbb{P}_{p}$ method using both conforming and non-conforming meshes, it is not necessary to increase the interpolation degree $p$ to more than 2 . Clearly, doing so

Table 3.3: Concentric cylinders PEC resonator.
Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using non-conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=1$.
Characteristics of the non-conforming meshes.

| \# nodes | $\begin{aligned} & \text { \# hanging } \\ & \text { nodes } \end{aligned}$ | \# triangles | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | $\begin{aligned} & \text { \# time } \\ & \text { steps } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |  |
| 1536 | 128 | 2816 | 2816 | $5.49 \mathrm{E}-02$ | - | 3 | 117 |
| 5888 | 256 | 11264 | 11264 | $2.75 \mathrm{E}-02$ | 1.00 | 19 | 231 |
| 23040 | 512 | 45056 | 45056 | $1.37 \mathrm{E}-02$ | 1.00 | 155 | 458 |
| 91136 | 1024 | 180224 | 180224 | $6.86 \mathrm{E}-03$ | 1.00 | 1270 | 913 |
| DGTD-P ${ }_{1}$ |  |  |  |  |  |  |  |
| 416 | 64 | $704\left(\right.$ mesh-E $\left.{ }^{a}\right)$ | 2112 | $2.27 \mathrm{E}-02$ | - | 3 | 199 |
| 1536 | 128 | 2816 (mesh-F) | 8448 | $5.83 \mathrm{E}-03$ | 1.97 | 17 | 389 |
| 5888 | 256 | 11264 (mesh-G) | 33792 | $1.48 \mathrm{E}-03$ | 1.98 | 131 | 768 |
| 23040 | 512 | 45056 (mesh-H) | 135168 | $3.73 \mathrm{E}-04$ | 1.99 | 1071 | 1526 |
| DGTD-P ${ }_{2}$ |  |  |  |  |  |  |  |
|  |  |  | 4224 | $1.42 \mathrm{E}-02$ | - | 6 | 299 |
|  |  |  | 16896 | $3.52 \mathrm{E}-03$ | 2.01 | 43 | 583 |
|  |  |  | 67584 | $8.75 \mathrm{E}-04$ | 2.01 | 347 | 1151 |
|  |  |  | 270336 | $2.18 \mathrm{E}-04$ | 2.00 | 2842 | 2289 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |  |
| mesh-E |  |  | 7040 | $1.41 \mathrm{E}-02$ | - | 18 | 597 |
| mesh-F |  |  | 28160 | $3.50 \mathrm{E}-03$ | 2.01 | 144 | 1165 |
| mesh-G |  |  | 112640 | $8.69 \mathrm{E}-04$ | 2.01 | 1148 | 2302 |
| mesh-H |  |  | 450560 | $2.17 \mathrm{E}-04$ | 2.01 | 9466 | 4578 |

${ }^{a}$ This means that mesh-E contains 416 nodes in which 64 are hanging nodes, and 704 triangles, etc.


Figure 3.4: Concentric cylinders $P E C$ resonator.
Numerical convergence of the h-refinement DGTD- $\mathbb{P}_{p}$ method.
Conforming (left) and non-conforming (right) triangular meshes.

Table 3.4: Concentric cylinders PEC resonator.
Numerical CFL values for the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.

| $p_{c}: p_{f}$ | $1: 0$ | $2: 0$ | $2: 1$ | $3: 0$ | $3: 1$ | $3: 2$ | $4: 2$ | $4: 3$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CFL | 0.5 | 0.3 | 0.3 | 0.19 | 0.19 | 0.19 | 0.1 | 0.1 |

Table 3.5: Concentric cylinders PEC resonator. Convergence study of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. Non-conforming locally refined meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=1$.

| mesh ${ }^{\text {a }}$ | $\% \mathrm{DOF}_{f}{ }^{\text {b }}$ | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 16\% | 1600 | $2.51 \mathrm{E}-02$ | - | 2 | 120 |
| mesh-F | 16\% | 6400 | $7.89 \mathrm{E}-03$ | 1.67 | 8 | 233 |
| mesh-G | 16\% | 25600 | $2.11 \mathrm{E}-03$ | 1.90 | 64 | 461 |
| mesh-H | 16\% | 102400 | $7.08 \mathrm{E}-04$ | 1.58 | 527 | 916 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 9\% | 2944 | $1.90 \mathrm{E}-02$ | - | 3 | 199 |
| mesh-F | $9 \%$ | 11776 | $7.11 \mathrm{E}-03$ | 1.42 | 21 | 389 |
| mesh-G | 9\% | 47104 | $1.91 \mathrm{E}-03$ | 1.89 | 168 | 768 |
| mesh-H | 9\% | 188416 | $7.58 \mathrm{E}-04$ | 1.33 | 1394 | 1526 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 5\% | 4736 | $1.90 \mathrm{E}-02$ | - | 7 | 315 |
| mesh-F | 5\% | 18944 | $6.29 \mathrm{E}-03$ | 1.60 | 53 | 613 |
| mesh-G | 5\% | 75776 | $2.14 \mathrm{E}-03$ | 1.56 | 421 | 1212 |
| mesh-H | 5\% | 303104 | $8.52 \mathrm{E}-04$ | 1.33 | 3445 | 2410 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh-E | 22\% | 3456 | $1.49 \mathrm{E}-02$ | - | 4 | 199 |
| mesh-F | 22\% | 13824 | $3.71 \mathrm{E}-03$ | 2.00 | 25 | 389 |
| mesh-G | 22\% | 55296 | $9.23 \mathrm{E}-04$ | 2.01 | 194 | 768 |
| mesh-H | 22\% | 221184 | $2.30 \mathrm{E}-04$ | 2.00 | 1597 | 1526 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh-E | 15\% | 5248 | $1.48 \mathrm{E}-02$ | - | 8 | 299 |
| mesh-F | 15\% | 20992 | $3.66 \mathrm{E}-03$ | 2.01 | 55 | 583 |
| mesh-G | 15\% | 83968 | $9.11 \mathrm{E}-04$ | 2.01 | 461 | 1212 |
| mesh-H | 15\% | 335872 | $2.27 \mathrm{E}-04$ | 2.00 | 3775 | 2410 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh-E | 26\% | 6016 | $1.42 \mathrm{E}-02$ | - | 8 | 299 |
| mesh-F | 26\% | 24064 | $3.53 \mathrm{E}-03$ | 2.02 | 63 | 583 |
| mesh-G | 26\% | 96256 | $8.76 \mathrm{E}-04$ | 2.01 | 526 | 1212 |
| mesh-H | 26\% | 385024 | $2.18 \mathrm{E}-04$ | 2.01 | 4262 | 2410 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh-E | 19\% | 8256 | $1.42 \mathrm{E}-02$ | - | 21 | 597 |
| mesh-F | 19\% | 33024 | $3.50 \mathrm{E}-03$ | 2.02 | 165 | 1165 |
| mesh-G | 19\% | 132096 | $8.70 \mathrm{E}-04$ | 2.01 | 1340 | 2302 |
| mesh-H | 19\% | 528384 | $2.17 \mathrm{E}-04$ | 2.01 | 10955 | 4578 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{3}$ |  |  |  |  |  |  |
| mesh-E | 28\% | 9280 | $1.42 \mathrm{E}-02$ | - | 24 | 597 |
| mesh-F | 28\% | 37120 | $3.50 \mathrm{E}-03$ | 2.02 | 188 | 1165 |
| mesh-G | 28\% | 148480 | $8.70 \mathrm{E}-04$ | 2.01 | 1501 | 2302 |
| mesh-H | 28\% | 593920 | $2.16 \mathrm{E}-04$ | 2.00 | 12254 | 4578 |

${ }^{a}$ See Tab. 3.3 for the characteristics of these meshes.
${ }^{b}$ Represents the percentage of the DOF in the fine mesh.


Figure 3.5: Concentric cylinders $P E C$ resonator.
Numerical convergence of the hp-refinement $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming triangular meshes.
does not improve the convergence rate. Furthermore, to reach a given error level, the DGTD- $\mathbb{P}_{3}$ is more expensive than the DGTD- $\mathbb{P}_{2}$ method.

- for a given $p$, the conforming DGTD- $\mathbb{P}_{p}$ method always requires less computing time than the nonconforming one. In fact, a comparison of the two methods from the point of view of the CPU time is rather unfair in the present case since the local refinement of the mesh is performed arbitrarily (i.e. not motivated by any physical or geometrical concern).
- in general, for a given error level $<10^{-2}$ the $h p$-refinement method is less expensive than the $h$-refinement one. For example, if we use the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{0}$ method for $p_{c}=1,2,3$ respectively, rather than the DGTD- $\mathbb{P}_{0}$ method, the gains in the number of DOF and CPU time vary from $95 \%$ to $85 \%$. Similarly, the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method can be compared with the DGTD- $\mathbb{P}_{p}$ one for $p=1,2$.
- finally, we have observed that, in the case of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method, it is not reasonable to increase the interpolation degree $p_{c}$ in the coarse mesh to more than $p_{f}+1$ since doing so does not improve the convergence rate and results in increased CPU time and memory usage.

Comparison with various FDTD methods. Here we compare the DGTD- $\mathbb{P}_{0}$ method with two different FDTD schemes:

1. the classical second order Yee scheme [39] staggered both in space and time where the numerical solution is carried out on a staircased mesh;
2. a modification of the Yee scheme (refered as $\operatorname{Ty}(2,4)$ ), see [35]-[36], and chapter 2 in [33]. This method is fourth order accurate in space and second order accurate in time. The numerical solution is also computed on a stagger staircased mesh.

The $L^{2}$ error on the $E_{z}$ component and the convergence rate of these two FDTD schemes measured at times $t=1$ and $t=10$ are given in Tab. 3.6. These results were taken from [1]-[38]. It is seen from Tab. 3.6 that the staircased FDTD approximation leads to an extremely slow convergence rate at early time $(t=1)$ and a divergent scheme at late time $(t=10)$. According to [38], this is probably not only because the staircasing misrepresents the shape of the cylinders, but also because of the fact that in this resonator case an electromagnetic wave is bouncing back and forth between the walls, so numerical errors accumulate quickly in the solution. These FDTD results are compared with the classical centered
finite volume DGTD- $\mathbb{P}_{0}$ scheme on conforming triangular meshes (see Tab. 3.6). One can see that the DGTD- $\mathbb{P}_{0}$ conserves a linear convergence rate even for a long simulation time.

Table 3.6: Concentric cylinders PEC resonator.
The $L^{2}$ error on $E_{z}$ for two different $F D T D$ schemes and for the $D G T D-\mathbb{P}_{0}$ method.

| $\mathrm{t}=1$ |  |  | $\mathrm{t}=10$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Yee scheme (staircased) cartesian grid ${ }^{a}$. |  |  |  |  |  |
| grid | $L^{2}$ error | convergence rate | grid | $L^{2}$ error | convergence rate |
| 1000 | $4.322 \mathrm{E}-01$ | - | 1000 | $5.101 \mathrm{E}-01$ | - |
| 4000 | $3.635 \mathrm{E}-01$ | 0.28 | 4000 | $4.364 \mathrm{E}-01$ | 0.23 |
| 16000 | $1.742 \mathrm{E}-01$ | 1.06 | 16000 | $6.683 \mathrm{E}-01$ | -0.61 |
| Ty(2,4) (staircased) cartesian grid ${ }^{6}$. |  |  |  |  |  |
| 1000 | $4.038 \mathrm{E}-01$ | - | 1000 | $2.642 \mathrm{E}-01$ | - |
| 4000 | $3.347 \mathrm{E}-01$ | 0.27 | 4000 | $7.079 \mathrm{E}-01$ | -1.42 |
| 16000 | $1.579 \mathrm{E}-01$ | 1.08 | 16000 | $7.243 \mathrm{E}-01$ | -0.03 |
| The classical centered finite volume DGTD- $\mathbb{P}_{0}$ scheme. |  |  |  |  |  |
| $\sqrt{\text { DOF }}$ | $L^{2}$ error | convergence rate | $\sqrt{\text { DOF }}$ | $L^{2}$ error | convergence rate |
| 90 | $2.62 \mathrm{E}-02$ | - | 90 | $4.52 \mathrm{E}-02$ | - |
| 180 | $1.31 \mathrm{E}-02$ | 0.99 | 180 | $1.89 \mathrm{E}-02$ | 1.25 |
| 360 | $6.65 \mathrm{E}-03$ | 1.00 | 360 | $8.96 \mathrm{E}-03$ | 1.08 |

[^1]
### 3.2 Circular PEC resonator

We consider a circular PEC resonator with radius $r=1 / 2$. The material is taken to be the vacuum. The exact time-domain solution of the problem is [36]:

$$
\begin{aligned}
E_{z} & =J_{1}(\omega r) \cos (\omega t+\theta) \\
H_{x} & =\frac{y}{2 r}\left[J_{0}(\omega r)-J_{2}(\omega r)\right] \cos (\omega t+\theta)-\frac{x}{\omega r^{2}} J_{1}(\omega r) \sin (\omega t+\theta) \\
H_{y} & =-\frac{x}{2 r}\left[J_{0}(\omega r)-J_{2}(\omega r)\right] \cos (\omega t+\theta)-\frac{y}{\omega r^{2}} J_{1}(\omega r) \sin (\omega t+\theta)
\end{aligned}
$$

where $\omega=14.03117333963124$, which is obtained from the PEC boundary condition $E_{z}=0$ at $r=1 / 2$. As in the previous test case, $(r, \theta)$ represent the usual polar coordinates.

Numerical simulations make use of quasi-uniform triangular meshes with possibly obtuse angles (see Fig. 3.6 left). The percentage of triangles with obtuse angles is between $45 \%$ and $50 \%$ which correspond to a maximum angle between $100^{\circ}$ and $90^{\circ}$ respectively. The non-conforming meshes are obtained by locally refining a cylindrical zone as shown on Fig. 3.6 right. Contour lines of the $E_{z}$ component at times $t=1$ and $t=10$ are shown on Fig. 3.7 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.

As for the previous test case, we apply the $h$-refinement DGTD- $\mathbb{P}_{p}$ and the $h p$-refinement DGTD$\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. For each polynomial degree $p$, we use four different conforming and non-conforming meshes whose characteristics are listed in Tab. 3.7 and 3.8. The non-conforming meshes are obtained by locally refining (one refinement level) the cylindrical zone $0.3 \leq r \leq 0.4$ of the conforming meshes. We use here the same CFL values as in the previous example (see Tab. 3.1). We


Figure 3.6: Circular PEC resonator.
Example of conforming (left) and non-conforming (right) triangular meshes.


Figure 3.7: Circular PEC resonator.
Contour lines of $E_{z}$ at $t=1$ (left) and $t=10$ (right).
DGTD- $\mathbb{P}_{1}$ method using a conforming mesh with 4201 nodes and 8280 triangles.
give in Tab. 3.7 and 3.8 the $L^{2}$ error on $E_{z}$, the convergence rate, the CPU time and the number of time steps to reach $t=0.8$. Fig. 3.8 shows the corresponding $L^{2}$ error as a function of the square root of the number of DOF. We can deduce from Tab. 3.7 and 3.8 that for conforming as well as non-conforming meshes, the DGTD- $\mathbb{P}_{p}$ method converges as $h^{(2)}$ for $p \geq 1$ and as $h^{(1)}$ for $p=0$.

Table 3.7: Circular PEC resonator.
Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=0.8$.

| \# nodes | \# triangles | \# DOF | $L^{2}$ error | convergence | CPU (s) | $\begin{aligned} & \text { \# time } \\ & \text { steps } \\ & \hline \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |
| 176 | 325 (mesh-A) | 325 | $6.73 \mathrm{E}-02$ | - | $<1$ | 48 |
| 701 | 1350 (mesh-B) | 1350 | $2.88 \mathrm{E}-02$ | 1.19 | 2 | 182 |
| 2801 | 5500 (mesh-C) | 5500 | $1.34 \mathrm{E}-02$ | 1.08 | 24 | 717 |
| 11201 | 22200 (mesh-C) | 22200 | $6.58 \mathrm{E}-03$ | 1.02 | 417 | 2856 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |
|  | mesh-A | 975 | $4.45 \mathrm{E}-02$ | - | < 1 | 160 |
|  | mesh-B | 4050 | $1.21 \mathrm{E}-02$ | 1.83 | 10 | 606 |
|  | mesh-C | 16500 | $3.07 \mathrm{E}-03$ | 1.95 | 169 | 2388 |
|  | mesh-D | 66600 | $7.76 \mathrm{E}-04$ | 1.97 | 2855 | 9518 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |
|  | mesh-A | 1950 | $1.32 \mathrm{E}-02$ | - | 2 | 239 |
|  | mesh-B | 8100 | $3.19 \mathrm{E}-03$ | 2.00 | 26 | 908 |
|  | mesh-C | 33000 | $7.89 \mathrm{E}-04$ | 1.99 | 432 | 3582 |
|  | mesh-D | 133200 | $1.96 \mathrm{E}-04$ | 1.99 | 7229 | 14277 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |
|  | mesh-A | 3250 | $1.29 \mathrm{E}-02$ | - | 7 | 478 |
|  | mesh-B | 13500 | $3.17 \mathrm{E}-03$ | 1.97 | 86 | 1816 |
|  | mesh-C | 55000 | $7.88 \mathrm{E}-04$ | 1.98 | 1450 | 7163 |
|  | mesh-D | 222000 | $1.97 \mathrm{E}-04$ | 1.99 | 24190 | 28554 |



Figure 3.8: Circular PEC resonator.
Numerical convergence of the h-refinement $D G T D-\mathbb{P}_{p}$ method. Conforming (left) and non-conforming (right) triangular meshes.

Table 3.8: Circular PEC resonator.
Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using non-conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=0.8$.

| \# nodes | $\begin{aligned} & \text { \# hanging } \\ & \text { nodes } \end{aligned}$ | \# triangles | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | $\begin{aligned} & \text { \# time } \\ & \text { steps } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |  |
| 181 | 40 | 300 (mesh-E) | 300 | $7.71 \mathrm{E}-02$ | - | <<1 | 29 |
| 681 | 80 | 1240 (mesh-F) | 1240 | $4.01 \mathrm{E}-02$ | 0.92 | 1 | 105 |
| 2641 | 160 | 5040 (mesh-G) | 5040 | $2.05 \mathrm{E}-02$ | 0.96 | 12 | 411 |
| 10401 | 320 | 20320 (mesh-H) | 20320 | $1.05 \mathrm{E}-02$ | 0.96 | 224 | 1633 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |  |
| mesh-E |  |  | 900 | $6.87 \mathrm{E}-02$ | - | $<1$ | 95 |
| mesh-F |  |  | 3720 | $1.96 \mathrm{E}-02$ | 1.77 | 6 | 350 |
| mesh-G |  |  | 15120 | $5.07 \mathrm{E}-03$ | 1.93 | 90 | 1368 |
| mesh-H |  |  | 60960 | $1.28 \mathrm{E}-03$ | 1.97 | 1484 | 5443 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |  |
| mesh-E |  |  | 1800 | $2.10 \mathrm{E}-02$ | - | 2 | 142 |
| mesh-F |  |  | 7440 | $4.99 \mathrm{E}-03$ | 2.03 | 14 | 524 |
| mesh-G |  |  | 30240 | $1.23 \mathrm{E}-03$ | 2.00 | 234 | 2052 |
| mesh-H |  |  | 121920 | $3.06 \mathrm{E}-04$ | 2.00 | 3838 | 8164 |
|  |  |  | DGTD-P ${ }_{3}$ |  |  |  |  |
| mesh-E |  |  | 3000 | $2.03 \mathrm{E}-02$ | - | 4 | 284 |
| mesh-F |  |  | 12400 | $4.96 \mathrm{E}-03$ | 1.98 | 49 | 1048 |
| mesh-G |  |  | 50400 | $1.23 \mathrm{E}-03$ | 1.99 | 782 | 4104 |
| mesh-H |  |  | 203200 | $3.06 \mathrm{E}-04$ | 1.99 | 12681 | 16327 |

$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. The observed numerical CFL values of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method are reported in Tab. 3.9. As with the $h$-refinement method, the non-conforming meshes are obtained by locally refining the cylindrical zone $0.3 \leq r \leq 0.4$. We give in Tab. 3.12 the $L^{2}$ error on $E_{z}$, the convergence rate, the CPU time and the number of time steps to reach $t=0.8$. Fig. 3.9 shows the corresponding $L^{2}$ error on $E_{z}$ as a function of the square root of the number of DOF. It is clear from Tab. 3.12 that the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ converges as $h^{(2)}, \forall p_{f} \neq 0$ while the convergence rate for $p_{f}=0$ is more than $\mathcal{O}\left(h^{(1.8)}\right), \forall p_{c}$.

Table 3.9: Circular PEC resonator.
Numerical CFL values for $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.

| $p_{c}: p_{f}$ | $1: 0$ | $2: 0$ | $2: 1$ | $3: 0$ | $3: 1$ | $3: 2$ | $4: 2$ | $4: 3$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CFL | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.08 | 0.08 |

In summary, the results given in Tab. 3.7, 3.8 and 3.12 call for the following comments:

- it is not reasonable to increase the polynomial degree in the coarse mesh to more than $p_{c}=p_{f}+1$ (see Tab. 3.10). One can clearly deduce from Tab. 3.10 that the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is the least expensive $h p$-refinement method for this problem.
- the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is less expensive than the non-conforming DGTD- $\mathbb{P}_{p}$ one. For example, we have observed (results not reported in a table here) that to obtain an error level of $0.4 \%$, on one hand, the DGTD- $\mathbb{P}_{1}$ and DGTD- $\mathbb{P}_{2}$ method require 19500 and 8454 DOF respectively while the corresponding CPU times are 160 and 29 seconds and, on the other hand, the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method needs 8100 DOF and 28 seconds (see Tab. 3.10).
- the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method can be compared with the conforming DGTD- $\mathbb{P}_{p}$ one. Indeed, the results of Tab. 3.10 together with those of Tab. 3.11 show that the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is less expensive than the DGTD- $\mathbb{P}_{1}$ and DGTD- $\mathbb{P}_{3}$ ones, but it is not much more expensive than the DGTD- $\mathbb{P}_{2}$ method. One can note here that the results of the $D G T D-\mathbb{P}_{2}: \mathbb{P}_{1}$ method remain very satisfactory despite that the space $\mathbb{P}_{1}$ is used in $53 \%$ of the triangles of the mesh.

Table 3.10: Circular PEC resonator.
\# DOF and CPU time to reach an error level of $0.4 \%$. $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method with locally refined meshes.

| DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ | $\mathbb{P}_{1}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{0}$ | $\mathbb{P}_{3}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{2}$ | $\mathbb{P}_{4}: \mathbb{P}_{2}$ | $\mathbb{P}_{4}: \mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| \# DOF | 25602 | 47088 | 8100 | 47090 | 12120 | 12150 | 17670 | 22500 |
| \# CPU (s) | 320 | 1000 | 28 | 1200 | 170 | 157 | 190 | 201 |

Table 3.11: Circular PEC resonator.
\# DOF and CPU time to reach an error level of $0.4 \%$.
$D G T D-\mathbb{P}_{p}$ method using conforming meshes.

| DGTD- $_{p}$ | DGTD- $\mathbb{P}_{1}$ | DGTD- $\mathbb{P}_{2}$ | DGTD- $\mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: |
| \# DOF | 13689 | 6402 | 10400 |
| \# CPU $(\mathrm{s})$ | 100 | 18 | 59 |



Figure 3.9: Circular PEC resonator.
Numerical convergence of the hp-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming triangular meshes.

### 3.3 Wedge-shaped PEC resonator

We consider a computational domain which is bounded by the curves $y=\tan \left(\frac{3 \pi}{7}\right) x, x^{2}+y^{2}=\frac{1}{4}$ and the $x$-axis (see Fig. 3.10). The boundaries of the sector are assumed to be perfectly conducting. The exact time-domain solution is [36]:

Table 3.12: Circular PEC resonator. Convergence study of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.

Non-conforming locally refined meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=0.8$.

| mesh ${ }^{a}$ | $\% \mathrm{DOF}_{f}{ }^{\text {b }}$ | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 28 \% | 580 | $8.30 \mathrm{E}-02$ | - | < 1 | 95 |
| mesh-F | 26 \% | 2440 | $1.99 \mathrm{E}-02$ | 1.98 | 4 | 350 |
| mesh-G | 26 \% | 10000 | $8.25 \mathrm{E}-03$ | 1.23 | 66 | 1368 |
| mesh-H | 25 \% | 40480 | $2.34 \mathrm{E}-03$ | 1.80 | 1123 | 5443 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 16 \% | 1000 | $1.30 \mathrm{E}-01$ | - | 1 | 142 |
| mesh-F | $15 \%$ | 4240 | $5.89 \mathrm{E}-02$ | 1.10 | 9 | 524 |
| mesh-G | 15 \% | 17440 | $2.14 \mathrm{E}-02$ | 1.43 | 146 | 2052 |
| mesh-H | $15 \%$ | 70720 | $1.85 \mathrm{E}-03$ | 3.50 | 2427 | 8164 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh-E | 10 \% | 1560 | $5.65 \mathrm{E}-02$ | - | 3 | 284 |
| mesh-F | $10 \%$ | 6640 | $1.50 \mathrm{E}-02$ | 1.83 | 26 | 1048 |
| mesh-G | $9 \%$ | 27360 | $6.39 \mathrm{E}-03$ | 1.20 | 441 | 4104 |
| mesh-H | $9 \%$ | 111040 | $1.76 \mathrm{E}-03$ | 1.84 | 7271 | 16327 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh-E | 36 \% | 1320 | $2.52 \mathrm{E}-02$ | - | 2 | 142 |
| mesh-F | $35 \%$ | 5520 | $6.09 \mathrm{E}-03$ | 1.98 | 11 | 524 |
| mesh-G | $34 \%$ | 22560 | $1.51 \mathrm{E}-03$ | 1.98 | 184 | 2052 |
| mesh-H | $34 \%$ | 91200 | $3.79 \mathrm{E}-04$ | 1.98 | 3016 | 8164 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh-E | 26 \% | 1880 | $2.40 \mathrm{E}-02$ | - | 3 | 284 |
| mesh-F | 24 \% | 7920 | $6.03 \mathrm{E}-03$ | 1.92 | 31 | 1048 |
| mesh-G | 24 \% | 32480 | $1.51 \mathrm{E}-03$ | 1.96 | 520 | 4104 |
| mesh-H | 23 \% | 131520 | $3.78 \mathrm{E}-04$ | 1.98 | 8483 | 16327 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh-E | 41 \% | 2360 | $2.03 \mathrm{E}-02$ | - | 4 | 284 |
| mesh-F | $40 \%$ | 9840 | $4.96 \mathrm{E}-03$ | 1.97 | 37 | 1048 |
| mesh-G | $38 \%$ | 40160 | $1.23 \mathrm{E}-03$ | 1.99 | 621 | 4104 |
| mesh-H | $38 \%$ | 162240 | $3.06 \mathrm{E}-04$ | 1.99 | 10041 | 16327 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh-E | 31 \% | 3060 | $2.02 \mathrm{E}-02$ | - | 5 | 354 |
| mesh-F | $30 \%$ | 12840 | $4.96 \mathrm{E}-03$ | 1.96 | 58 | 1310 |
| mesh-G | $29 \%$ | 52560 | $1.23 \mathrm{E}-03$ | 1.98 | 993 | 5130 |
| mesh-H | 29 \% | 212640 | $3.06 \mathrm{E}-04$ | 1.99 | 16239 | 20409 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{3}$ |  |  |  |  |  |  |
| mesh-E | 43 \% | 3700 | $2.03 \mathrm{E}-02$ | - | 5 | 315 |
| mesh-F | 42 \% | 15400 | $4.96 \mathrm{E}-03$ | 1.97 | 72 | 1310 |
| mesh-G | 42 \% | 62800 | $1.23 \mathrm{E}-03$ | 1.99 | 1195 | 5130 |
| mesh-H | $40 \%$ | 253600 | $3.05 \mathrm{E}-04$ | 2.00 | 20167 | 20409 |

[^2]\[

$$
\begin{aligned}
E_{z}= & J_{\nu}(\omega r) \sin (\nu \theta) \cos (\omega t) \\
H_{x}= & -\frac{y}{2 r}\left[J_{\nu-1}(\omega r)-J_{\nu+1}(\omega r)\right] \sin (\nu \theta) \sin (\omega t) \\
& -\frac{\nu x}{\omega r^{2}} J_{\nu}(\omega r) \cos (\nu \theta) \sin (\omega t) \\
H_{y}= & \frac{x}{2 r}\left[J_{\nu-1}(\omega r)-J_{\nu+1}(\omega r)\right] \sin (\nu \theta) \sin (\omega t) \\
& -\frac{\nu y}{\omega r^{2}} J_{\nu}(\omega r) \cos (\nu \theta) \sin (\omega t)
\end{aligned}
$$
\]

where $\omega=16.75883874736728$ and $\nu=\frac{14}{3}$ are obtained by enforcing the PEC condition on the boundaries. Numerical simulations make use of quasi-uniform triangular meshes with possibly obtuse angles (see Fig. 3.11 left). The percentage of triangles with obtuse angles is between $45 \%$ and $50 \%$ which correspond to a maximum angle between $94^{\circ}$ and $91^{\circ}$ respectively. The non-conforming meshes are obtained thanks to local refinements of a region situated between two arcs of the sector as shown on Fig. 3.11 right. Contour lines of the $E_{z}$ component at times $t=1$ and $t=10$ are shown on Fig. 3.12 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.


Figure 3.10: Wedge-shaped PEC resonator setting.
As for the previous test cases, we apply the $h$-refinement DGTD- $\mathbb{P}_{p}$ and $h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. For each polynomial degree $p$, we use four different conforming and non-conforming meshes whose characteristics are given in Tab. 3.13 and 3.14. The non-conforming meshes are obtained by locally refining (one refinement level) the zone of the domain bounded by the curves $\tan (9 \pi / 7 \alpha) \leq y / x \leq \tan (3(\alpha-3) \pi / 7 \alpha)$ and $1 / 5 \leq x^{2}+y^{2} \leq 3 / 10$. The integer $\alpha$ takes a value in the set $\{9,18,36,72\}$ depending on the mesh size. We summarize in Tab. 3.13 and 3.14 the $L^{2}$ error on $E_{z}$, the convergence rate, the CPU time and the number of time steps to reach $t=0.75$. Fig. 3.13 shows the corresponding $L^{2}$ error on $E_{z}$ as a function of the square root of the number of DOF. The observed numerical CFL values are given in Tab. 3.1. We can deduce from Tab. 3.13 and 3.14 that the DGTD- $\mathbb{P}_{p}$ method converges as $h^{(1)}$ if $p=0$ and as $h^{(2)}$ if $p \geq 1$ (excepted for $p=2$ using conforming meshes, where the convergence rate is close to $\left.\mathcal{O}\left(h^{(2.5)}\right)\right)$.

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Figure 3.11: Wedge-shaped PEC resonator.
Example of conforming (left) and non-conforming (right) triangular meshes.


Figure 3.12: Wedge-shaped PEC resonator.
Contour lines of $E_{z}$ at $t=1$ (left) and $t=10$ (right).
DGTD- $\mathbb{P}_{1}$ method using a conforming mesh with 862 nodes and 1640 triangles.

Tab. 3.13 and 3.14 and Fig. 3.13 call for the following remarks:

- for this problem and using conforming meshes, it is clear that the gains in CPU time and memory usage are notable when the interpolation degree is increased. For example, to obtain an error level of $0.01 \%$, the number of DOF required by the DGTD- $\mathbb{P}_{1}$, DGTD- $\mathbb{P}_{2}$ and DGTD- $\mathbb{P}_{3}$ methods is 16902,5042 and 2500 respectively while the corresponding CPU times are 1700,130 and 47 seconds (results not reported in a table below).
- using non-conforming meshes, it is not necessary to increase the interpolation degree to more than 2. For example, to achieve an error level of $0.01 \%$, the number of DOF required by the DGTD- $\mathbb{P}_{1}$, DGTD- $\mathbb{P}_{2}$ and DGTD- $\mathbb{P}_{3}$ methods is 28560,6402 and 10000 respectively while the corresponding CPU times are 2250, 160 and 420 seconds.
- the conforming $h$-refinement method is less expensive (in terms of the required number of DOF and CPU time) than the non-conforming one, however, the results obtained by using non-conforming meshes are very satisfactory despite of the presence of a large number of triangles in the refined zone.

Table 3.13: Wedge-shaped PEC resonator. Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=0.75$.

| \# nodes | \# triangles | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | $\begin{gathered} \text { \# time } \\ \text { steps } \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |
| 67 | 110 | 110 | $2.41 \mathrm{E}-02$ | - | <<1 | 69 |
| 232 | 420 | 420 | $1.21 \mathrm{E}-02$ | 1.03 | 2 | 247 |
| 862 | 1640 | 1640 | $6.22 \mathrm{E}-03$ | 0.98 | 21 | 938 |
| 3322 | 6480 | 6480 | $3.16 \mathrm{E}-03$ | 0.98 | 324 | 3656 |
| DGTD-P ${ }_{1}$ |  |  |  |  |  |  |
| 67 | 110 | 330 | $5.06 \mathrm{E}-03$ | - | 1 | 228 |
| 232 | 420 | 1260 | $1.36 \mathrm{E}-03$ | 1.96 | 13 | 822 |
| 862 | 1640 | 4920 | $2.99 \mathrm{E}-04$ | 2.22 | 169 | 3124 |
| 3322 | 6480 | 19440 | 7.98E-05 | 1.92 | 2780 | 12185 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |
| 28 | 40 | 240 | $2.99 \mathrm{E}-03$ | - | 1 | 139 |
| 86 | 144 | 864 | $8.47 \mathrm{E}-04$ | 1.97 | 5 | 450 |
| 298 | 544 | 3264 | $1.70 \mathrm{E}-04$ | 2.42 | 56 | 1608 |
| 1106 | 2112 | 12672 | $2.92 \mathrm{E}-05$ | 2.59 | 838 | 6064 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |
| 28 | 40 | 400 | $1.29 \mathrm{E}-03$ | - | 2 | 277 |
| 86 | 144 | 1440 | $1.86 \mathrm{E}-04$ | 3.02 | 15 | 899 |
| 298 | 544 | 5440 | $4.60 \mathrm{E}-05$ | 2.10 | 190 | 3216 |
| 1106 | 2112 | 21120 | $1.16 \mathrm{E}-05$ | 2.03 | 2807 | 12128 |

$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. The numerical CFL values used are still those given in Tab. 3.9. The non-conforming meshes are those already used in conjunction with the $h$-refinement method. We summarize in Tab. 3.17 the $L^{2}$ error on $E_{z}$, the convergence rate, the CPU time and the number of time steps to reach $t=0.75$. Fig. 3.14 shows the $L^{2}$ error on $E_{z}$ as a function of the square root of the number of DOF. It can be seen from Tab. 3.17 that the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method converges as $h^{(2)}, \forall p_{f} \neq 0$ and for $p_{f}=0$ the convergence rate is close to $\mathcal{O}\left(h^{(1.2)}\right), \forall p_{c} \neq 3$ (for $p_{c}=3$ this rate is close to $\left.\mathcal{O}\left(h^{(1)}\right)\right)$.
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Table 3.14: Wedge-shaped PEC resonator.
Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using non-conforming meshes.
$L^{2}$ errors, CPU times and \# time steps are measured at $t=0.75$.

| \# nodes | $\begin{gathered} \text { \# hanging } \\ \text { nodes } \end{gathered}$ | \# triangles | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | $\begin{gathered} \text { \# time } \\ \text { steps } \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |  |
| 64 | 8 | 99 | 99 | $3.03 \mathrm{E}-02$ | - | <<1 | 54 |
| 277 | 28 | 486 | 486 | $1.11 \mathrm{E}-02$ | 1.26 | 2 | 202 |
| 1135 | 68 | 2124 | 2124 | $4.83 \mathrm{E}-03$ | 1.13 | 22 | 804 |
| 4579 | 148 | 8856 | 8856 | $2.25 \mathrm{E}-03$ | 1.07 | 398 | 3210 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |  |
| 64 | 8 | 99 | 297 | $7.81 \mathrm{E}-03$ | - | 2 | 179 |
| 277 | 28 | 486 | 1458 | $1.68 \mathrm{E}-03$ | 1.93 | 12 | 673 |
| 1135 | 68 | 2124 | 6372 | $4.05 \mathrm{E}-04$ | 1.93 | 192 | 2679 |
| 4579 | 148 | 8856 | 26568 | $8.08 \mathrm{E}-05$ | 2.26 | 3329 | 10700 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |  |
| 46 | 4 | $69\left(\operatorname{mesh}^{a}\right)$ | 414 | $1.92 \mathrm{E}-03$ | - | 3 | 217 |
| 209 | 20 | 362 (mesh2) | 2172 | $3.27 \mathrm{E}-04$ | 2.14 | 19 | 789 |
| 871 | 52 | 1620 (mesh3) | 9720 | $6.72 \mathrm{E}-05$ | 2.11 | 328 | 3128 |
| 3539 | 116 | 6824 (mesh4) | 40944 | $1.57 \mathrm{E}-05$ | 2.03 | 5617 | 12487 |
| DGTD-P ${ }_{3}$ |  |  |  |  |  |  |  |
| 46 | 4 | 69 | 690 | $1.03 \mathrm{E}-03$ | - | 4 | 434 |
| 209 | 20 | 362 | 3620 | $2.43 \mathrm{E}-04$ | 1.74 | 64 | 1577 |
| 871 | 52 | 1620 | 16200 | $6.29 \mathrm{E}-05$ | 1.81 | 1125 | 6256 |
| 3539 | 116 | 6824 | 68240 | $1.53 \mathrm{E}-05$ | 1.96 | 19067 | 24973 |

${ }^{a}$ This means that the mesh1 contains 46 nodes in which 4 are hanging nodes and 69 triangles, etc.


Figure 3.13: Wedge-shaped PEC resonator.
Numerical convergence of the $h$-refinement $D G T D-\mathbb{P}_{p}$ method.
Conforming (left) and non-conforming (right) triangular meshes.

Table 3.15: Wedge-shaped PEC resonator. \# DOF and CPU time to reach an error level of $0.05 \%$. $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method with locally refined meshes.

| DGTD- $_{p_{c}}: \mathbb{P}_{p_{f}}$ | $\mathbb{P}_{1}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{0}$ | $\mathbb{P}_{3}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{2}$ | $\mathbb{P}_{4}: \mathbb{P}_{2}$ | $\mathbb{P}_{4}: \mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| \# DOF | 14502 | 22500 | 1296 | 57600 | 1530 | 1440 | 2010 | 2100 |
| \# CPU (s) | 1600 | 2500 | 10 | 20000 | 19 | 15 | 25 | 27 |

Table 3.16: Wedge-shaped PEC resonator.
\# DOF and CPU time to reach an error level of $0.05 \%$. $D G T D-\mathbb{P}_{p}$ method using conforming meshes.

| DGTD- $_{p}$ | DGTD- $\mathbb{P}_{1}$ | DGTD- $\mathbb{P}_{2}$ | DGTD- $\mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: |
| \# DOF | 3135 | 1308 | 760 |
| \# CPU $(\mathrm{s})$ | 70 | 12 | 6 |

A detailed analysis of Tab. 3.17 leads to the following remarks:

- it is not necessary to increase the polynomial degree in the coarse mesh to more than $p_{c}=p_{f}+1$ (see Tab. 3.15). One can clearly deduce from Tab. 3.15 that the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is the least expensive $h p$-refinement method for this problem.
- the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is less expensive than the non-conforming DGTD- $\mathbb{P}_{p}$ methods. For example, we have observed (results not reported in a table here) that to obtain an error level of $0.05 \%$, on one hand, the DGTD- $\mathbb{P}_{1}$ and DGTD- $\mathbb{P}_{2}$ methods require 5184 and 1482 DOF respectively while the corresponding CPU times are 130 and 13 seconds and, on the other hand, the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method needs 1296 DOF and 10 seconds (see Tab. 3.15).
- the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method can be compared with the conforming DGTD- $\mathbb{P}_{p}$ one. Indeed, the results of Tab. 3.15 together with those of Tab. 3.16 show that the DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ method is less expensive than the DGTD- $\mathbb{P}_{1}$ and DGTD- $\mathbb{P}_{2}$ methods, but it is not much more expensive than the DGTD- $\mathbb{P}_{3}$ method. One can note here that the results of the $\mathrm{DGTD}-\mathbb{P}_{2}: \mathbb{P}_{1}$ method remain very satisfactory despite that the space $\mathbb{P}_{1}$ is used in $38 \%$ of the triangles of the mesh.


## 4 Heterogeneous media

This section is devoted to the numerical evaluation of the conforming and non-conforming DGTD methods in the context of wave propagation problems in heterogeneous media. As in the previous section, we have selected test cases for which analytical solution are available:

1. a rectangular PEC resonator with one material interface,
2. a dielectric in a PEC cavity with two material interfaces,
3. a dielectric cylinder illuminated by a plane wave.

### 4.1 Rectangular PEC resonator with one material interface

We consider a problem in which a dielectric of relative permittivity $\epsilon_{2}$ occupying the spatial region $[0,0.5] \times[0,1]$, is enclosed by PEC-bounded $[0,1.25] \times[0,1]$ domain. The exact time-domain solution is [41]:

Table 3.17: Wedge-shaped PEC resonator. Convergence study of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.

Non-conforming locally refined meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=0.75$.

| mesh ${ }^{a}$ | $\% \mathrm{DOF}_{f}{ }^{\text {b }}$ | \# DOF | $L^{2}$ error | convergence rate | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh1 | $4 \%$ | 191 | $9.76 \mathrm{E}-03$ | - | 1 | 145 |
| mesh2 | 15\% | 830 | $2.52 \mathrm{E}-03$ | 1.84 | 6 | 526 |
| mesh3 | 20\% | 3452 | $1.23 \mathrm{E}-03$ | 1.00 | 81 | 2086 |
| mesh4 | 23\% | 14072 | $5.28 \mathrm{E}-04$ | 1.21 | 1426 | 8325 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh1 | 2\% | 374 | $4.68 \mathrm{E}-03$ | - | 2 | 217 |
| mesh2 | 8\% | 1532 | $2.07 \mathrm{E}-03$ | 1.16 | 15 | 789 |
| mesh3 | 11\% | 6200 | $1.02 \mathrm{E}-03$ | 1.00 | 210 | 3128 |
| mesh4 | 13\% | 24944 | $4.45 \mathrm{E}-04$ | 1.20 | 3544 | 12487 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| mesh1 | 1\% | 618 | $3.99 \mathrm{E}-03$ | - | 4 | 434 |
| mesh2 | 5\% | 2468 | $2.05 \mathrm{E}-03$ | 0.97 | 44 | 1577 |
| mesh3 | 7\% | 9864 | $1.06 \mathrm{E}-03$ | 0.95 | 675 | 6256 |
| mesh4 | 8\% | 39440 | $6.01 \mathrm{E}-04$ | 0.82 | 11159 | 24973 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh1 | $6 \%$ | 390 | $1.96 \mathrm{E}-03$ | - | 2 | 217 |
| mesh2 | 21\% | 1788 | $3.58 \mathrm{E}-04$ | 2.23 | 16 | 789 |
| mesh3 | 28\% | 7608 | $8.28 \mathrm{E}-05$ | 2.02 | 258 | 3128 |
| mesh4 | $31 \%$ | 31344 | $1.90 \mathrm{E}-05$ | 2.08 | 4415 | 12487 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| mesh1 | $4 \%$ | 634 | $1.19 \mathrm{E}-03$ | - | 5 | 434 |
| mesh2 | 14\% | 2724 | $2.71 \mathrm{E}-04$ | 2.03 | 47 | 1577 |
| mesh3 | 19\% | 11272 | $7.22 \mathrm{E}-05$ | 1.86 | 771 | 6256 |
| mesh4 | 21\% | 45840 | $1.73 \mathrm{E}-05$ | 2.04 | 12836 | 24973 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh1 | 7\% | 658 | $1.03 \mathrm{E}-03$ | - | 4 | 434 |
| mesh2 | 25\% | 3108 | $2.42 \mathrm{E}-04$ | 1.86 | 53 | 1577 |
| mesh3 | 32\% | 13384 | $6.25 \mathrm{E}-05$ | 1.86 | 913 | 6256 |
| mesh4 | $35 \%$ | 55440 | $1.46 \mathrm{E}-05$ | 2.04 | 15583 | 24973 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| mesh1 | 5\% | 963 | $1.00 \mathrm{E}-03$ | - | 7 | 542 |
| mesh2 | 18\% | 4278 | $2.47 \mathrm{E}-04$ | 1.88 | 92 | 1971 |
| mesh3 | 24\% | 17964 | $6.18 \mathrm{E}-05$ | 1.93 | 1555 | 7820 |
| mesh4 | 26\% | 73560 | $1.46 \mathrm{E}-05$ | 2.05 | 25687 | 31250 |
| DGTD- $\mathbb{P}_{4}: \mathbb{P}_{3}$ |  |  |  |  |  |  |
| mesh1 | 8\% | 995 | $1.01 \mathrm{E}-03$ | - | 7 | 542 |
| mesh2 | 27\% | 4790 | $2.46 \mathrm{E}-04$ | 1.80 | 103 | 1971 |
| mesh3 | 34\% | 20780 | $6.16 \mathrm{E}-05$ | 1.89 | 1763 | 7820 |
| mesh4 | 37\% | 86360 | $1.45 \mathrm{E}-05$ | 2.03 | 30003 | 31250 |

[^3]

Figure 3.14: Wedge-shaped PEC resonator.
Numerical convergence of the hp-refinement $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming triangular meshes.

$$
\begin{aligned}
& E_{z}=\left\{\begin{array}{lll}
\sin \left(a_{1} x\right) \sin (\omega t) \sin (b y), & 0 \leq x \leq \frac{1}{2}, & 0 \leq y \leq 1, \\
\cos \left(a_{2} x\right) \sin (\omega t) \sin (b y), & \frac{1}{2} \leq x \leq \frac{5}{4}, & 0 \leq y \leq 1,
\end{array}\right. \\
& H_{x}=\left\{\begin{array}{lll}
\frac{b}{\omega} \sin \left(a_{1} x\right) \cos (\omega t) \cos (b y), & 0 \leq x \leq \frac{1}{2}, & 0 \leq y \leq 1, \\
\frac{b}{\omega} \cos \left(a_{2} x\right) \cos (\omega t) \cos (b y), & \frac{1}{2} \leq x \leq \frac{5}{4}, & 0 \leq y \leq 1,
\end{array}\right. \\
& H_{y}=\left\{\begin{array}{cll}
-\frac{a_{1}}{\omega} \cos \left(a_{1} x\right) \cos (\omega t) \sin (b y), & 0 \leq x \leq \frac{1}{2}, & 0 \leq y \leq 1, \\
\frac{a_{2}}{\omega} \sin \left(a_{2} x\right) \cos (\omega t) \sin (b y), & \frac{1}{2} \leq x \leq \frac{5}{4}, & 0 \leq y \leq 1,
\end{array}\right.
\end{aligned}
$$

where $a_{1}^{2}+b^{2}=\epsilon_{2} \omega^{2}, a_{2}^{2}+b^{2}=\epsilon_{1} \omega^{2}, \sin \left(\frac{a_{1}}{2}\right)=\cos \left(\frac{a_{2}}{2}\right)$ and $\cos \left(\frac{5 a_{2}}{4}\right)=0$. The values of the parameters appearing in these relations are defined by imposing the PEC condition on the boundary $x=5 / 4$ and by ensuring the continuity of $E_{z}$ across the material interface $x=1 / 2$. As in [41]-[40], these parameter values are chosen as $\epsilon_{1}=1, \epsilon_{2}=2, a_{1}=3 \pi, a_{2}=2 \pi, b=\pi$, and $\omega=\sqrt{5} \pi$. The wavelength is $\lambda^{\epsilon_{1}}=0.268$ in the air zone and $\lambda^{\epsilon_{2}}=0.189$ in the dielectric zone, i.e. $\lambda^{\epsilon_{1}} \simeq 1.42 \lambda^{\epsilon_{2}}$. Across the dielectric interface (i.e. the air/dielectric interface), the tangential components of the electromagnetic field $E_{z}$ and $H_{y}$ are continuous as well their first $y$ derivative. Furthermore, the first $x$ derivative of $E_{z}$ is continuous while that of $H_{y}$ is discontinuous. An example plot of analytical solution is shown on Fig. 4.1.

Numerical simulations make use of uniform triangular meshes such that the material interface $x=1 / 2$ is aligned with the grid, i.e. the intersection between the interface $x=1 / 2$ and $\operatorname{int}\left(T_{i}\right)$ (the interior of $T_{i}$ ) is empty, $\forall$ triangle $T_{i}$ of the mesh (see Fig. 4.2 left). Hence, the mesh shown on Fig. 4.2 right is not allowed in the present modeling. One can note here that some FDTD schemes may not need this condition on the material interface [41]. Contour lines of the $E_{z}$ component at times $t=1$ and $t=10$ are shown on Fig. 4.3 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.


Figure 4.1: Rectangular PEC resonator with one material interface.
Plots of the analytical solution along the line $y=3 / 4$ at time $t=0.75$.


Figure 4.2: Rectangular PEC resonator with one material interface. Example of two possible configurations of a conforming triangular mesh. The left figure shows the mesh used in the present study and the right figure shows a situation of mesh wich is not allowed in our modeling. The blue line represents the material interface.


Figure 4.3: Rectangular PEC resonator with one material interface. Contour lines of $E_{z}$ at $t=1$ (left) and $t=10$ (right). DGTD- $\mathbb{P}_{1}$ method using a conforming mesh with 4941 nodes and 9600 triangles.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. The non-conforming meshes are obtained by locally refining (one refinement level) the material region i.e. the zone $[0,0.5] \times[0,1]$. Tab. 4.1 and 4.2 give the $L^{2}$ error on $E_{z}$ and $(\mathbf{E}, \mathbf{H})=\left(E_{z}, H_{x}, H_{y}\right)$, the convergence rate, the CPU time and the number of time steps to reach $t=1$ using conforming and non-conforming meshes. Fig. 4.4 shows the corresponding $L^{2}$ error on the electromagnetic field $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. The numerical CFL values used here are given in Tab. 3.1. We can deduce from these results that the convergence rate of the DGTD- $\mathbb{P}_{p}$ method is close to $\mathcal{O}\left(h^{(1)}\right), \forall p$. Moreover, it is not necessary to increase the interpolation degree $p$ to more than one since doing so does not result in a higher convergence rate and the method becomes too much expensive for $p \geq 2$ (see Tab. 4.5). It is clear that the presence of a material interface influences the convergence rate of the DGTD- $\mathbb{P}_{p}$ method but this behaviour does not depend on the type of material. We validate this by studying the convergence of the conforming DGTD- $\mathbb{P}_{p}$ method using three different materials. Results in Tab. 4.3 show that the convergence rate is always close to $\mathcal{O}\left(h^{(1)}\right)$. This confirms that the convergence rate depends only on the regularity of the solution. It is expected that the use of different locations of material interfaces would make the situation worse. This phenomenon has been observed in [21] and in the more recent work [5] for the Maxwell eigenvalue problem (see Remark 1). This is also observed in [14] for the time-harmonic Maxwell equations using high order continuous and discontinuous Galerkin methods.

Remark 1 In [5] a checkerboard pattern of materials inside a PEC cavity is studied using a high order discontinuous Galerkin method. It is observed that the convergence rate is less than linear, whatever the interpolation degree is, despite that without material interfaces (i.e. in homogeneous media) the convergence rate is close to $\mathcal{O}\left(h^{(2 p)}\right)$. The authors in [5] do not elaborate on this fact. Moreover, this problem was also studied in [21] using high order nodal $h$ - and p-refinement discontinuous Galerkin method with central fluxes. As in [5], the convergence of the nodal discontinuous Galerkin method turned sour in the presence of material interfaces. According to the authors of [21] the degradation of the convergence is due to the use of centered fluxes at the material interfaces (or at discontinuities). The authors indicate that by defining material properties at material interfaces as the average of cell values (i.e. by using a regularization technique [34]) should improve the situation.

Table 4.1: Rectangular PEC resonator with one material interface. Convergence study for the $h$-refinement $D G T D-\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=1$.

| \# DOF | $\begin{aligned} & \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | convergence rate on ( $\mathbf{E}, \mathbf{H}$ ) | CPU (s) | $\begin{aligned} & \text { \# time } \\ & \text { steps } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD-P ${ }_{0}$ |  |  |  |  |  |  |
| 4000 | $1.40 \mathrm{E}-02$ | - | $3.26 \mathrm{E}-02$ | - | 1 | 57 |
| 16000 | $6.73 \mathrm{E}-03$ | 1.06 | $1.91 \mathrm{E}-02$ | 0.77 | 10 | 114 |
| 64000 | $3.33 \mathrm{E}-03$ | 1.02 | $9.06 \mathrm{E}-03$ | 1.08 | 79 | 227 |
| 256000 | $1.66 \mathrm{E}-03$ | 1.01 | $4.11 \mathrm{E}-03$ | 1.14 | 644 | 453 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |
| 12000 | $5.30 \mathrm{E}-02$ | - | $7.18 \mathrm{E}-02$ | - | 8 | 189 |
| 48000 | $2.72 \mathrm{E}-02$ | 0.96 | $3.63 \mathrm{E}-02$ | 0.98 | 58 | 378 |
| 192000 | $1.38 \mathrm{E}-02$ | 0.99 | $1.86 \mathrm{E}-02$ | 0.97 | 483 | 755 |
| 768000 | $6.91 \mathrm{E}-03$ | 0.99 | $9.51 \mathrm{E}-03$ | 0.97 | 3861 | 1509 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |
| 14400 | $6.88 \mathrm{E}-02$ | - | $9.21 \mathrm{E}-02$ | - | 9 | 220 |
| 57600 | $3.47 \mathrm{E}-02$ | 0.99 | $4.76 \mathrm{E}-02$ | 0.95 | 65 | 439 |
| 230400 | $1.74 \mathrm{E}-02$ | 1.00 | $2.37 \mathrm{E}-02$ | 1.01 | 535 | 878 |
| 921600 | $8.68 \mathrm{E}-03$ | 1.00 | $1.20 \mathrm{E}-02$ | 0.98 | 4503 | 1755 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |
| 24000 | $6.89 \mathrm{E}-02$ | - | $9.38 \mathrm{E}-02$ | - | 27 | 439 |
| 96000 | $3.47 \mathrm{E}-02$ | 0.99 | $4.76 \mathrm{E}-02$ | 0.98 | 220 | 878 |
| 384000 | $1.74 \mathrm{E}-02$ | 1.00 | $2.41 \mathrm{E}-02$ | 0.98 | 1837 | 1755 |
| 1536000 | $8.68 \mathrm{E}-03$ | 1.00 | $1.20 \mathrm{E}-02$ | 1.01 | 14917 | 3510 |

Table 4.2: Rectangular PEC resonator with one material interface.
Convergence study for the $h$-refinement $D G T D-\mathbb{P}_{p}$ method using non-conforming meshes. $L^{2}$ error, $C P U$ time and \# time steps are measured at $t=1$.

| \# DOF | $\begin{aligned} & \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | convergence rate $\text { on }(\mathbf{E}, \mathbf{H})$ | CPU (s) | \# time <br> steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |
| 5280 | $1.02 \mathrm{E}-02$ | - | $2.02 \mathrm{E}-02$ | - | 2 | 88 |
| 21120 | $4.93 \mathrm{E}-03$ | 1.05 | $1.09 \mathrm{E}-02$ | 0.88 | 21 | 176 |
| 84480 | $2.45 \mathrm{E}-03$ | 1.01 | $5.15 \mathrm{E}-03$ | 1.09 | 166 | 351 |
| 337920 | $1.22 \mathrm{E}-03$ | 1.00 | $2.74 \mathrm{E}-03$ | 0.91 | 1350 | 702 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |
| 15840 | $6.71 \mathrm{E}-02$ | - | $9.06 \mathrm{E}-02$ | - | 15 | 293 |
| 63360 | $3.43 \mathrm{E}-02$ | 0.97 | $4.76 \mathrm{E}-02$ | 0.93 | 122 | 585 |
| 253440 | $1.73 \mathrm{E}-02$ | 0.99 | $2.41 \mathrm{E}-02$ | 0.98 | 1004 | 1170 |
| 1013760 | $8.66 \mathrm{E}-03$ | 1.00 | $1.20 \mathrm{E}-02$ | 1.00 | 8272 | 2340 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |
| 9900 | $1.08 \mathrm{E}-01$ | - | $1.46 \mathrm{E}-01$ | - | 7 | 250 |
| 39600 | $5.54 \mathrm{E}-02$ | 0.96 | $7.63 \mathrm{E}-02$ | 0.94 | 51 | 500 |
| 158400 | $2.78 \mathrm{E}-02$ | 0.99 | $3.88 \mathrm{E}-02$ | 0.96 | 425 | 1000 |
| 633600 | $1.39 \mathrm{E}-02$ | 1.00 | $1.95 \mathrm{E}-02$ | 0.99 | 3462 | 2000 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |
| 16500 | $1.08 \mathrm{E}-01$ | - | $1.46 \mathrm{E}-01$ | - | 20 | 500 |
| 66000 | $5.54 \mathrm{E}-02$ | 0.96 | $7.62 \mathrm{E}-02$ | 0.94 | 172 | 1000 |
| 264000 | $2.78 \mathrm{E}-02$ | 0.99 | $3.88 \mathrm{E}-02$ | 0.98 | 1410 | 2000 |
| 1056000 | $1.39 \mathrm{E}-02$ | 1.00 | $1.95 \mathrm{E}-02$ | 0.99 | 12051 | 4000 |

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Figure 4.4: Rectangular PEC resonator with one material interface.
Numerical convergence of the $h$-refinement $D G T D-\mathbb{P}_{p}$ method.
Conforming (left) and non-conforming (right) triangular meshes.

Table 4.3: Rectangular PEC resonator with one material interface for different types of materials. Convergence study for the $D G T D-\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=1$.

| \# DOF | $\begin{aligned} & \epsilon_{2}=2.125 \\ & \omega=\sqrt{40} \pi \end{aligned}$ |  | $\begin{gathered} \epsilon_{2}=3.25 \\ \omega=\sqrt{20} \pi \end{gathered}$ |  | $\begin{aligned} \epsilon_{2} & =6.625 \\ \omega & =\sqrt{8} \pi \end{aligned}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \end{gathered}$ | $\begin{gathered} \hline \text { convergence rate } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \end{gathered}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | $\begin{gathered} \hline \text { convergence rate } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \end{gathered}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \end{gathered}$ | $\begin{gathered} \hline \text { convergence rate } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \end{gathered}$ |
| 4000 | DGTD- $\mathbb{P}_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  |
|  | $1.06 \mathrm{E}-01$ | - | $1.28 \mathrm{E}-01$ | - | $1.90 \mathrm{E}-01$ | - |
| 16000 | $5.41 \mathrm{E}-02$ | 0.97 | $6.41 \mathrm{E}-02$ | 1.00 | $9.25 \mathrm{E}-02$ | 1.04 |
| 64000 | $2.66 \mathrm{E}-02$ | 1.02 | $3.20 \mathrm{E}-02$ | 1.00 | $4.63 \mathrm{E}-02$ | 1.00 |
| 256000 | $1.19 \mathrm{E}-02$ | 1.15 | $1.51 \mathrm{E}-02$ | 1.08 | $2.30 \mathrm{E}-02$ | 1.00 |
| 12000 | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  |
|  | $2.21 \mathrm{E}-01$ | - | $2.72 \mathrm{E}-01$ | - | $3.86 \mathrm{E}-01$ | - |
| 48000 | $1.23 \mathrm{E}-01$ | 0.84 | $1.48 \mathrm{E}-01$ | 0.87 | $2.07 \mathrm{E}-01$ | 0.90 |
| 192000 | $6.62 \mathrm{E}-02$ | 0.90 | $7.87 \mathrm{E}-02$ | 0.92 | $1.06 \mathrm{E}-01$ | 0.96 |
| 768000 | $3.47 \mathrm{E}-02$ | 0.93 | $4.09 \mathrm{E}-02$ | 0.94 | $5.39 \mathrm{E}-02$ | 0.98 |
| 14400 | DGTD- $\mathbb{P}_{2}$ |  | DGTD- $\mathbb{P}_{2}$ |  | DGTD- $\mathbb{P}_{2}$ |  |
|  | $3.45 \mathrm{E}-01$ | - | $3.84 \mathrm{E}-01$ | - | $4.44 \mathrm{E}-01$ | - |
| 57600 | $1.78 \mathrm{E}-01$ | 0.95 | $2.03 \mathrm{E}-01$ | 0.92 | $2.38 \mathrm{E}-01$ | 0.90 |
| 230400 | $8.87 \mathrm{E}-02$ | 1.01 | $1.01 \mathrm{E}-01$ | 1.01 | $1.21 \mathrm{E}-01$ | 0.97 |
| 921600 | $4.49 \mathrm{E}-02$ | 0.98 | $5.13 \mathrm{E}-02$ | 0.98 | $6.12 \mathrm{E}-02$ | 0.98 |
|  | DGTD-P ${ }_{3}$ |  | DGTD- $\mathbb{P}_{3}$ |  | DGTD- $\mathbb{P}_{3}$ |  |
| 24000 | $3.46 \mathrm{E}-01$ | - | $3.93 \mathrm{E}-01$ | - | $4.47 \mathrm{E}-01$ | - |
| 96000 | $1.79 \mathrm{E}-01$ | 0.96 | $2.03 \mathrm{E}-01$ | 0.95 | $2.38 \mathrm{E}-01$ | 0.91 |
| 384000 | $8.87 \mathrm{E}-02$ | 1.01 | $1.03 \mathrm{E}-01$ | 0.98 | $1.21 \mathrm{E}-01$ | 0.97 |
| 1536000 | $4.49 \mathrm{E}-02$ | 0.98 | $5.15 \mathrm{E}-02$ | 1.00 | $6.12 \mathrm{E}-02$ | 0.98 |

$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. Here the non-conforming meshes are obtained by locally refining (one refinement level) the zone $[0,0.55] \times[0,1]$. Tab. 4.6 gives the $L^{2}$ error on $E_{z}$ and $(\mathbf{E}, \mathbf{H})=$ $\left(E_{z}, H_{x}, H_{y}\right)$, the convergence rate, the CPU time and the number of time steps to reach $t=1$. Fig. 4.5 shows the corresponding $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. The numerical CFL values are given in Tab. 4.4. It is found from Tab. 4.6 that the convergence rate of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method is close to $\mathcal{O}\left(h^{(1)}\right), \forall p_{c}, p_{f}$. One can also note that to obtain a given error level the DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ is the least expensive of all $h p$-refinement and $h$-refinement methods (see Tab. 4.5).

Table 4.4: Rectangular PEC resonator with one material interface.
Numerical CFL conditions for the $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.

| $p_{c}: p_{f}$ | $1: 0$ | $2: 0$ | $3: 0$ | $2: 1$ | $3: 1$ | $3: 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CFL | 0.7 | 0.4 | 0.25 | 0.3 | 0.25 | 0.2 |

Table 4.5: Rectangular PEC resonator with one material interface.
\# DOF and CPU time to reach an error level of $2 \%$ on the electromagnetic field $(\mathbf{E}, \mathbf{H})$. $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ and DGTD- $\mathbb{P}_{p}$ methods.

| DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ | $\mathbb{P}_{1}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{0}$ | $\mathbb{P}_{3}: \mathbb{P}_{0}$ | $\mathbb{P}_{2}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{1}$ | $\mathbb{P}_{3}: \mathbb{P}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| \# DOF | 4092 | 7750 | 6990 | 93024 | 122500 | 184920 |
| $\mathrm{CPU}(\mathrm{s})$ | 2 | 5 | 5 | 185 | 290 | 485 |


| DGTD- $\mathbb{P}_{p}$ on conforming meshes. | $\mathbb{P}_{0}$ | $\mathbb{P}_{1}$ | $\mathbb{P}_{2}$ | $\mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: | :---: |
| \# DOF | 14161 | 164025 | 324900 | 562500 |
| CPU $(\mathrm{s})$ | 8 | 380 | 900 | 3150 |


| DGTD- $_{p}$ on non-conforming meshes. | $\mathbb{P}_{0}$ | $\mathbb{P}_{1}$ | $\mathbb{P}_{2}$ | $\mathbb{P}_{3}$ |
| :---: | :---: | :---: | :---: | :---: |
| \# DOF | 5280 | 360000 | 592800 | 1000000 |
| CPU $(\mathrm{s})$ | 2 | 1730 | 3120 | 10800 |



Figure 4.5: Rectangular PEC resonator with one material interface. Numerical convergence of the hp-refinement $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. Non-conforming triangular meshes.

Table 4.6: Rectangular PEC resonator with one material interface.
Convergence study of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming locally refined meshes.
$L^{2}$ error, $C P U$ time and \# time steps are measured at $t=1$.

| \# DOF | $\begin{aligned} & \text { error } \\ & \text { on } \mathbf{E} \\ & \hline \end{aligned}$ | convergence rate on $\mathbf{E}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \\ \hline \hline \end{gathered}$ | convergence rate $\text { on }(\mathbf{E}, \mathbf{H})$ | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| 2580 | $1.29 \mathrm{E}-02$ | - | $2.77 \mathrm{E}-02$ | - | 1 | 72 |
| 10320 | $5.08 \mathrm{E}-03$ | 1.35 | $1.09 \mathrm{E}-02$ | 1.34 | 6 | 143 |
| 41280 | $2.24 \mathrm{E}-03$ | 1.18 | $5.31 \mathrm{E}-03$ | 1.04 | 52 | 286 |
| 165120 | $1.06 \mathrm{E}-03$ | 1.07 | $2.95 \mathrm{E}-03$ | 0.85 | 429 | 572 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| 3840 | $1.82 \mathrm{E}-02$ | - | $3.06 \mathrm{E}-02$ | - | 2 | 125 |
| 15360 | $8.86 \mathrm{E}-03$ | 1.04 | $1.35 \mathrm{E}-02$ | 1.18 | 14 | 250 |
| 61440 | $3.58 \mathrm{E}-03$ | 1.30 | $5.77 \mathrm{E}-03$ | 1.23 | 115 | 500 |
| 245760 | $1.38 \mathrm{E}-03$ | 1.37 | $2.59 \mathrm{E}-03$ | 1.16 | 928 | 1000 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{0}$ |  |  |  |  |  |  |
| 5520 | $1.18 \mathrm{E}-02$ | - | $2.31 \mathrm{E}-02$ | - | 4 | 200 |
| 22080 | $4.83 \mathrm{E}-03$ | 1.29 | $1.02 \mathrm{E}-02$ | 1.19 | 28 | 400 |
| 88320 | $2.21 \mathrm{E}-03$ | 1.13 | $4.92 \mathrm{E}-03$ | 1.05 | 232 | 800 |
| 353280 | $1.06 \mathrm{E}-03$ | 1.06 | $2.53 \mathrm{E}-03$ | 0.96 | 1895 | 1600 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| 6480 | $5.36 \mathrm{E}-02$ | - | $7.28 \mathrm{E}-02$ | - | 3 | 167 |
| 25920 | $2.74 \mathrm{E}-02$ | 0.97 | $3.68 \mathrm{E}-02$ | 0.99 | 25 | 334 |
| 103680 | $1.38 \mathrm{E}-02$ | 0.99 | $1.90 \mathrm{E}-02$ | 0.95 | 214 | 667 |
| 414720 | $6.92 \mathrm{E}-03$ | 1.00 | $9.37 \mathrm{E}-03$ | 1.02 | 1792 | 1334 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |  |  |
| 8160 | $5.36 \mathrm{E}-02$ | - | $7.43 \mathrm{E}-02$ | - | 4 | 200 |
| 32640 | $2.74 \mathrm{E}-02$ | 0.97 | $3.84 \mathrm{E}-02$ | 0.95 | 40 | 400 |
| 130560 | $1.38 \mathrm{E}-02$ | 0.99 | $1.95 \mathrm{E}-02$ | 0.98 | 317 | 800 |
| 522240 | $6.92 \mathrm{E}-03$ | 1.00 | $9.79 \mathrm{E}-03$ | 0.99 | 2537 | 1600 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |  |  |
| 12120 | $5.53 \mathrm{E}-02$ | - | $7.63 \mathrm{E}-02$ | - | 7 | 250 |
| 48480 | $2.78 \mathrm{E}-02$ | 0.99 | $3.88 \mathrm{E}-02$ | 0.98 | 63 | 500 |
| 193920 | $1.39 \mathrm{E}-02$ | 1.00 | $1.95 \mathrm{E}-02$ | 0.99 | 523 | 1000 |
| 775680 | $6.94 \mathrm{E}-03$ | 1.00 | $9.79 \mathrm{E}-03$ | 1.00 | 4270 | 2000 |

### 4.2 Dielectric in a PEC cavity with two material interfaces

In this problem, a lossless dielectric with a relative permittivity $\epsilon_{2}$ is enclosed by air in the $x$ direction. The media are nonmagnetic, and homogeneuous along $y$ direction. The computational domain $\Omega=$ $[-1,1] \times[-1,1]$ is bounded by PEC walls. The permittivity is given as $\epsilon=\epsilon_{1}$ if $\frac{1}{2} \leq|x| \leq 1$ and $|y| \leq 1$, and $\epsilon=\epsilon_{2}$ if $|x| \leq \frac{1}{2}$ and $|y| \leq 1$, where $\epsilon_{1}=1$ and $\epsilon_{2}=2.25$. The exact time-domain solution is [41]:

$$
\begin{aligned}
& E_{z}=\left\{\begin{array}{lr}
\sin \left(\frac{\omega_{2}}{2}\right) \sin \left(\omega_{1}(x+1)\right) \sin \left(\omega_{y} y\right) \cos (\omega t), & -1 \leq x<-\frac{1}{2}, \quad|y| \leq 1 \\
-\sin \left(\frac{\omega_{1}}{2}\right) \sin \left(\omega_{2} x\right) \sin \left(\omega_{y} y\right) \cos (\omega t), & -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad|y| \leq 1 \\
\sin \left(\frac{\omega_{2}}{2}\right) \sin \left(\omega_{1}(x-1)\right) \sin \left(\omega_{y} y\right) \cos (\omega t), \quad \frac{1}{2} \leq x \leq 1, \quad|y| \leq 1
\end{array}\right. \\
& H_{x}= \begin{cases}-\frac{\omega_{y}}{\omega} \sin \left(\frac{\omega_{2}}{2}\right) \sin \left(\omega_{1}(x+1)\right) \cos \left(\omega_{y} y\right) \sin (\omega t), & -1 \leq x<-\frac{1}{2}, \quad|y| \leq 1 \\
\frac{\omega_{y}}{\omega} \sin \left(\frac{\omega_{1}}{2}\right) \sin \left(\omega_{2} x\right) \cos \left(\omega_{y} y\right) \sin (\omega t), & -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad|y| \leq 1 \\
-\frac{\omega_{y}}{\omega} \sin \left(\frac{\omega_{2}}{2}\right) \sin \left(\omega_{1}(x-1)\right) \cos \left(\omega_{y} y\right) \sin (\omega t), \quad \frac{1}{2} \leq x \leq 1, \quad|y| \leq 1\end{cases} \\
& H_{y}=\left\{\begin{array}{lll}
\frac{\omega_{1}}{\omega} \sin \left(\frac{\omega_{2}}{2}\right) \cos \left(\omega_{1}(x+1)\right) \sin \left(\omega_{y} y\right) \sin (\omega t), & -1 \leq x<-\frac{1}{2}, \quad|y| \leq 1 \\
-\frac{\omega_{2}}{\omega} \sin \left(\frac{\omega_{1}}{2}\right) \cos \left(\omega_{2} x\right) \sin \left(\omega_{y} y\right) \sin (\omega t), & -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad|y| \leq 1 \\
\frac{\omega_{1}}{\omega} \sin \left(\frac{\omega_{2}}{2}\right) \cos \left(\omega_{1}(x-1)\right) \sin \left(\omega_{y} y\right) \sin (\omega t), & \frac{1}{2} \leq x \leq 1, & |y| \leq 1
\end{array}\right.
\end{aligned}
$$

where $\omega_{1}^{2}+\omega_{y}^{2}=\epsilon_{1} \omega^{2}$ and $\omega_{2}^{2}+\omega_{y}^{2}=\epsilon_{2} \omega^{2}$. The value of $\omega_{y}$ is determined according to the relation:

$$
\sqrt{\epsilon_{2} \omega^{2}-\omega_{y}^{2}} \tan \left(\frac{\sqrt{\epsilon_{1} \omega^{2}-\omega_{y}^{2}}}{2}\right)=\sqrt{\epsilon_{1} \omega^{2}-\omega_{y}^{2}} \tan \left(-\frac{\sqrt{\epsilon_{2} \omega^{2}-\omega_{y}^{2}}}{2}\right)
$$

As in [41] we choose $\omega_{y}=2 \pi$ to satisfy the PEC conditions on $y= \pm 1$ wich leads to $\omega=$ 9.07716175885174. The wavelength $\lambda^{\epsilon_{1}}=0.208$ in the air zone and $\lambda^{\epsilon_{2}}=0.138$ in the material region, i.e. $\lambda^{\epsilon_{1}} \simeq 1.5 \lambda^{\epsilon_{2}}$. Note that across the dielectric interface (i.e. at $x= \pm 1 / 2$ ), the $E_{z}$ and $H_{x}$ components, their time derivative and their first $y$ derivative are continuous while their first $x$ derivative are discontinuous. Furthermore, the $H_{y}$ component and all its derivatives are discontinuous. An example plot of the analytical solution is shown on Fig. 4.6. Contour lines of the $H_{y}$ and $E_{z}$ components at times $t=1$ and $t=10$ are shown on Fig. 4.7 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. The non-conforming meshes are obtained by locally refining (one refinement level) the material region i.e. the zone $[-0.5,0.5] \times[-1,1]$. Tab. 4.8 and 4.9 give the $L^{2}$ error on $E_{z}, H_{y}$ and $(\mathbf{E}, \mathbf{H})$, the convergence rate, the CPU time and the number of time steps to reach $t=0.65$ using conforming and non-conforming meshes. Fig. 4.8 shows the corresponding $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. The convergence rate of the DGTD- $\mathbb{P}_{p}$ method is close to $\mathcal{O}\left(h^{(1)}\right)$ for $p=0,1$, and close to $\mathcal{O}\left(h^{(1 / 3)}\right)$ for $p=2,3$.
$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. Similarly to the $h$-refinement method, the non-conforming meshes are obtained by locally refining (one refinement level) the region $[-0.5,0.5] \times[-1,1]$. Tab. 4.10


Figure 4.6: Dielectric in a PEC cavity with two material interfaces.
Plots of the analytical solution along the line $y=\frac{1}{3}$ at time $t=0.75$.
gives the $L^{2}$ error on $E_{z}, H_{y}$ and $(\mathbf{E}, \mathbf{H})$, the convergence rate, the CPU time and the number of time steps to reach $t=0.65$. Fig. 4.9 shows the corresponding $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. The numerical CFL values are given in Tab. 4.4. It can be seen from Tab. 4.10 that the overall convergence rate of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ is close to $\mathcal{O}\left(h^{(1)}\right)$, excepted for the DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ method for which the convergence rate is close to $\mathcal{O}\left(h^{(1 / 3)}\right)$.

Despite the slow convergence rate of the DGTD- $\mathbb{P}_{p}($ for $p \geq 2)$ and DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}\left(\right.$ for $\left.p_{c} \geq 3, p_{f} \geq 2\right)$ methods, the role of the high order DGTD method remains very important to reduce the numerical dispersion even in the presence of material interfaces. To illustrate this point, we plot on Fig 4.10 and 4.11 the time evolution of the $L^{2}$ error on the electromagnetic field $(\mathbf{E}, \mathbf{H})$ for calculations based on the DGTD- $\mathbb{P}_{p}$ and DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods using conforming and non-conforming meshes with 6 and 4 points per $\lambda^{\epsilon_{1}}$ respectively.

Next, we study the numerical convergence of the DGTD- $\mathbb{P}_{p}$ method for different materials and using conforming meshes. In Tab 4.7 we summarize the relative permettivity $\epsilon_{r}=\epsilon_{2}$ for several materials. Our objective here, is to understand the influence of the material type on the convergence rate. We give in Tab 4.11 and 4.12 the $L^{2}$ error on the $E_{z}$ component and the corresponding convergence rate for some materials. We can conclude that:

- the DGTD- $\mathbb{P}_{0}$ method converges as $h^{(1)}$ in the heterogeneuous and homogeneuous cases.
- in the homogeneuous case, the DGTD- $\mathbb{P}_{1}$ converges as $h^{(2)}$, but the convergence rate is reduced to $\mathcal{O}\left(h^{(1)}\right)$ in the heterogeneous case.
- for $p \geq 2$, the DGTD- $\mathbb{P}_{p}$ methods converge as $h^{(2)}$ in the homogeneous case. This convergence rate is reduced to $\mathcal{O}\left(h^{(1)}\right)$ or less than linear in the heterogeneous case.


### 4.3 Scattering of a dielectric cylinder

We consider now a typical problem, in which a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The geometry of the scenario is shown in Fig. 4.12.

We assume that the cylinder is illuminated by a monochromatic plane wave of the form:


Figure 4.7: Dielectric in a PEC cavity with two material interfaces.
Contour lines of $H_{y}$ (top) and $E_{z}$ (bottom) at $t=1$ (left) and $t=10$ (right). DGTD- $\mathbb{P}_{1}$ method using a conforming mesh with 10201 nodes and 20000 triangles.

Table 4.7: Dielectric in a PEC cavity with two material interfaces. Relative permittivities for different materials.

| Material/medium | $\epsilon_{r}$ |
| :---: | :---: |
| Air | 1.0 |
| Teflon | 2.1 |
| Polyethylene | 2.25 |
| Polystyrene | 2.6 |
| Ice at $0^{\circ} \mathrm{C}$ | 3.2 |
| Pyrex | 4.7 |

Table 4.8: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$. Convergence study for the h-refinement DGTD- $\mathbb{P}_{p}$ method using conforming meshes.
$L^{2}$ error, $C P U$ time and \# time steps are measured at $t=0.65$.
$\left.\begin{array}{ccccccccc}\text { \# DOF } & \begin{array}{c}\text { error } \\ \text { on } E_{z}\end{array} & \begin{array}{c}\text { convergence } \\ \text { rate on } E_{z}\end{array} & \begin{array}{c}\text { error } \\ \text { on } H_{y}\end{array} & \begin{array}{c}\text { convergence } \\ \text { rate on } H_{y}\end{array} & \begin{array}{c}\text { error } \\ \text { on }(\mathbf{E}, \mathbf{H})\end{array} & \begin{array}{c}\text { convergence } \\ \text { rate on }(\mathbf{E}, \mathbf{H})\end{array} & \begin{array}{c}\text { CPU (s) }\end{array} & \begin{array}{c}\text { \# time } \\ \text { steps }\end{array} \\ \hline \hline & & & & & \text { DGTD- } \mathbb{P}_{0}\end{array}\right]$

Table 4.9: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$. Convergence study for the h-refinement $D G T D-\mathbb{P}_{p}$ method using non-conforming meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=0.65$.

| \# DOF | $\begin{aligned} & \text { error } \\ & \text { on } F_{\text {a }} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{gathered} \text { error } \\ \text { on } H_{y} \\ \hline \end{gathered}$ | convergence <br> rate on $H_{y}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | convergence <br> rate on ( $\mathbf{E}, \mathbf{H}$ ) | CPU (s) | \# time <br> steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |  |  |  |  |
| 2880 | $1.92 \mathrm{E}-02$ | - | $2.38 \mathrm{E}-02$ | - | $3.62 \mathrm{E}-02$ | - | $<1$ | 23 |
| 11520 | $9.25 \mathrm{E}-03$ | 1.06 | $1.25 \mathrm{E}-02$ | 0.93 | $1.84 \mathrm{E}-02$ | 0.98 | 2 | 45 |
| 46080 | $4.62 \mathrm{E}-03$ | 1.00 | $6.08 \mathrm{E}-03$ | 1.04 | $9.01 \mathrm{E}-03$ | 1.03 | 18 | 89 |
| 184320 | $2.31 \mathrm{E}-03$ | 1.00 | $2.73 \mathrm{E}-03$ | 1.15 | $4.18 \mathrm{E}-03$ | 1.11 | 142 | 177 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |  |  |  |  |
| 8640 | $1.12 \mathrm{E}-02$ | - | $1.48 \mathrm{E}-02$ | - | $2.22 \mathrm{E}-02$ | - | <2 | 74 |
| 34560 | $3.85 \mathrm{E}-03$ | 1.54 | $6.49 \mathrm{E}-03$ | 1.19 | $8.90 \mathrm{E}-03$ | 1.32 | 13 | 148 |
| 138240 | $1.99 \mathrm{E}-03$ | 0.95 | $3.00 \mathrm{E}-03$ | 1.12 | $4.28 \mathrm{E}-03$ | 1.06 | 105 | 295 |
| 552960 | $1.01 \mathrm{E}-03$ | 0.98 | $1.50 \mathrm{E}-03$ | 1.00 | $2.22 \mathrm{E}-03$ | 0.95 | 824 | 589 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |  |  |  |  |
| 48000 | $1.56 \mathrm{E}-03$ | - | $1.46 \mathrm{E}-03$ | - | $2.33 \mathrm{E}-03$ | - | 18 | 184 |
| 192000 | $1.22 \mathrm{E}-03$ | 0.36 | $1.28 \mathrm{E}-03$ | 0.27 | $1.88 \mathrm{E}-03$ | 0.31 | 148 | 368 |
| 768000 | $1.08 \mathrm{E}-03$ | 0.17 | $1.08 \mathrm{E}-03$ | 0.16 | $1.69 \mathrm{E}-03$ | 0.15 | 1218 | 736 |
| 3072000 | $6.58 \mathrm{E}-04$ | 0.72 | $7.11 \mathrm{E}-04$ | 0.61 | $1.08 \mathrm{E}-03$ | 0.65 | 10110 | 1472 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |  |  |  |  |
| 80000 | $1.56 \mathrm{E}-03$ | - | $1.44 \mathrm{E}-03$ | - | $2.30 \mathrm{E}-03$ | - | 65 | 368 |
| 320000 | $1.22 \mathrm{E}-03$ | 0.35 | $1.20 \mathrm{E}-03$ | 0.26 | $1.88 \mathrm{E}-03$ | 0.30 | 596 | 736 |
| 1280000 | $1.08 \mathrm{E}-03$ | 0.17 | $1.08 \mathrm{E}-03$ | 0.16 | $1.69 \mathrm{E}-03$ | 0.15 | 5487 | 1472 |
| 5120000 | $6.58 \mathrm{E}-04$ | 0.72 | $6.39 \mathrm{E}-04$ | 0.76 | $1.00 \mathrm{E}-03$ | 0.75 | 51103 | 2944 |



Figure 4.8: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$.
Numerical convergence of the h-refinement $D G T D-\mathbb{P}_{p}$ method.
Conforming (left) and non-conforming (right) triangular meshes.

Table 4.10: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$.
Convergence study of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming locally refined meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=0.65$.

| \# DOF | $\begin{aligned} & \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{gathered} \text { error } \\ \text { on } H_{y} \end{gathered}$ | convergence rate on $H_{y}$ | $\begin{gathered} \text { error } \\ \text { on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | $\begin{gathered} \text { convergence } \\ \text { rate on }(\mathbf{E}, \mathbf{H}) \end{gathered}$ | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |  |  |  |  |
| 4032 | $1.49 \mathrm{E}-02$ | - | $2.37 \mathrm{E}-02$ | - | $3.31 \mathrm{E}-02$ | - | < 1 | 32 |
| 16128 | $6.34 \mathrm{E}-03$ | 1.23 | $1.28 \mathrm{E}-02$ | 0.88 | $1.72 \mathrm{E}-02$ | 0.95 | 4 | 64 |
| 64512 | $3.46 \mathrm{E}-03$ | 0.87 | $6.28 \mathrm{E}-03$ | 1.03 | $8.46 \mathrm{E}-03$ | 1.02 | 34 | 127 |
| 258048 | $1.87 \mathrm{E}-03$ | 0.89 | $3.10 \mathrm{E}-03$ | 1.02 | $4.21 \mathrm{E}-03$ | 1.01 | 280 | 253 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{0}$ |  |  |  |  |  |  |  |  |
| 5760 | $1.50 \mathrm{E}-02$ | - | $1.83 \mathrm{E}-02$ | - | $2.73 \mathrm{E}-02$ | - | 1 | 56 |
| 23040 | $5.04 \mathrm{E}-03$ | 1.57 | $7.79 \mathrm{E}-03$ | 1.23 | $1.10 \mathrm{E}-02$ | 1.31 | 8 | 111 |
| 92160 | $3.01 \mathrm{E}-03$ | 0.74 | $4.25 \mathrm{E}-03$ | 0.87 | $5.92 \mathrm{E}-03$ | 0.89 | 69 | 221 |
| 368640 | $1.75 \mathrm{E}-03$ | 0.78 | $2.46 \mathrm{E}-03$ | 0.79 | $3.39 \mathrm{E}-03$ | 0.81 | 498 | 442 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |  |  |  |  |
| 10368 | $1.32 \mathrm{E}-02$ | - | $1.15 \mathrm{E}-02$ | - | $1.93 \mathrm{E}-02$ | - | 2 | 74 |
| 41472 | $3.29 \mathrm{E}-03$ | 2.00 | $4.30 \mathrm{E}-03$ | 1.42 | $5.54 \mathrm{E}-03$ | 1.80 | 16 | 148 |
| 165888 | $1.77 \mathrm{E}-03$ | 0.89 | $2.19 \mathrm{E}-03$ | 0.97 | $3.01 \mathrm{E}-03$ | 0.88 | 126 | 295 |
| 663552 | $9.37 \mathrm{E}-04$ | 0.92 | $1.18 \mathrm{E}-03$ | 0.89 | $1.75 \mathrm{E}-03$ | 0.78 | 933 | 589 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |  |  |  |  |
| 12672 | $1.31 \mathrm{E}-02$ | - | $1.19 \mathrm{E}-02$ | - | $1.96 \mathrm{E}-02$ | - | 3 | 89 |
| 50688 | $2.94 \mathrm{E}-03$ | 2.16 | $2.92 \mathrm{E}-03$ | 2.03 | $4.06 \mathrm{E}-03$ | 2.27 | 21 | 177 |
| 202752 | $1.65 \mathrm{E}-03$ | 0.83 | $1.63 \mathrm{E}-03$ | 0.85 | $2.37 \mathrm{E}-03$ | 0.78 | 175 | 353 |
| 811008 | $9.37 \mathrm{E}-04$ | 0.82 | $1.06 \mathrm{E}-03$ | 0.61 | $1.62 \mathrm{E}-03$ | 0.55 | 1402 | 706 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |  |  |  |  |
| 54400 | $1.56 \mathrm{E}-03$ | - | $1.44 \mathrm{E}-03$ | - | $2.31 \mathrm{E}-03$ | - | 21 | 184 |
| 217600 | $1.22 \mathrm{E}-03$ | 0.35 | $1.21 \mathrm{E}-03$ | 0.26 | $1.88 \mathrm{E}-03$ | 0.30 | 172 | 368 |
| 870400 | $1.08 \mathrm{E}-03$ | 0.17 | $1.08 \mathrm{E}-03$ | 0.15 | $1.69 \mathrm{E}-03$ | 0.15 | 1392 | 736 |
| 3481600 | $6.56 \mathrm{E}-04$ | 0.72 | $7.02 \mathrm{E}-04$ | 0.63 | $1.07 \mathrm{E}-03$ | 0.66 | 11270 | 1472 |



Figure 4.9: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$.
Numerical convergence of the hp-refinement $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming triangular meshes.

Table 4.11: Dielectric in a PEC cavity with two material interfaces for different types of materials. Convergence study of the DGTD- $\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=0.65$.

| \# DOF | $\begin{aligned} & \epsilon_{2}=1.00 \\ & \omega=\sqrt{5} \pi \\ & \hline \end{aligned}$ |  | $\begin{gathered} \epsilon_{2}=1.25 \\ \omega=6.574295470 \end{gathered}$ |  | $\begin{gathered} \epsilon_{2}=2.10 \\ \omega=9.217644926 \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{aligned} & \hline \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{aligned} & \hline \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{aligned} & \hline \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ |
|  | DGTD- $\mathbb{P}_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  |
| 3200 | $2.71 \mathrm{E}-02$ | - | $1.74 \mathrm{E}-02$ | - | $1.56 \mathrm{E}-02$ | - |
| 12800 | $1.40 \mathrm{E}-02$ | 0.95 | $9.15 \mathrm{E}-03$ | 0.93 | $7.21 \mathrm{E}-03$ | 1.11 |
| 51200 | $7.14 \mathrm{E}-03$ | 0.97 | $4.69 \mathrm{E}-03$ | 0.96 | $3.59 \mathrm{E}-03$ | 1.01 |
| 204800 | $3.60 \mathrm{E}-03$ | 0.99 | $2.38 \mathrm{E}-03$ | 0.98 | $1.79 \mathrm{E}-03$ | 1.00 |
| 9600 | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  |
|  | $1.26 \mathrm{E}-02$ | - | $2.14 \mathrm{E}-02$ | - | $7.34 \mathrm{E}-03$ | - |
| 38400 | $3.19 \mathrm{E}-03$ | 1.98 | $1.16 \mathrm{E}-02$ | 0.90 | $4.67 \mathrm{E}-03$ | 0.65 |
| 153600 | $8.64 \mathrm{E}-04$ | 1.89 | $6.10 \mathrm{E}-03$ | 0.93 | $3.02 \mathrm{E}-03$ | 0.63 |
| 614400 | $2.02 \mathrm{E}-04$ | 2.10 | $3.14 \mathrm{E}-03$ | 0.96 | $1.70 \mathrm{E}-03$ | 0.83 |
| 19200 | DGTD- $\mathbb{P}_{2}$ |  | DGTD- $\mathbb{P}_{2}$ |  |  |  |
|  | 5.34E-04 | - | $2.47 \mathrm{E}-02$ | - | $\frac{\text { DGTD- } \mathbb{P}_{2}}{4.17 \mathrm{E}-03}$ |  |
| 76800 | $1.26 \mathrm{E}-04$ | 2.09 | $1.26 \mathrm{E}-02$ | 0.97 | $4.88 \mathrm{E}-03$ | -0.2 |
| 307200 | $3.14 \mathrm{E}-05$ | 2.00 | $6.39 \mathrm{E}-03$ | 0.99 | $3.13 \mathrm{E}-03$ | 0.64 |
| 1228800 | $7.74 \mathrm{E}-06$ | 2.02 | $3.21 \mathrm{E}-03$ | 0.99 | $1.73 \mathrm{E}-03$ | 0.85 |
|  | DGTD- $\mathbb{P}_{3}$ |  | DGTD-P ${ }_{3}$ |  | DGTD-P ${ }_{3}$ |  |
| 32000 | $1.22 \mathrm{E}-04$ | - | $2.49 \mathrm{E}-02$ | - | $4.16 \mathrm{E}-03$ | - |
| 128000 | $3.11 \mathrm{E}-05$ | 1.97 | $1.27 \mathrm{E}-02$ | 0.97 | $4.88 \mathrm{E}-03$ | -0.2 |
| 512000 | $7.63 \mathrm{E}-06$ | 2.03 | $6.39 \mathrm{E}-03$ | 0.99 | $3.13 \mathrm{E}-03$ | 0.64 |
| 2048000 | $1.65 \mathrm{E}-06$ | 2.21 | $3.21 \mathrm{E}-03$ | 0.99 | $1.73 \mathrm{E}-03$ | 0.85 |



Figure 4.10: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$. Time evolution of the $L^{2}$ error using the $h$-refinement $D G T D-\mathbb{P}_{p}$ method. Conforming (top) and non-conforming (bottom) triangular meshes.


Figure 4.11: Dielectric in a PEC cavity with two material interfaces with $\epsilon_{2}=2.25$.
Time evolution of the $L^{2}$ error using the hp-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method.
Non-conforming triangular meshes.

Table 4.12: Dielectric in a PEC cavity with two material interfaces for different types of materials. Convergence study of the DGTD- $\mathbb{P}_{p}$ method using conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=0.65$.

| \# DOF | $\begin{gathered} \epsilon_{2}=2.60 \\ \omega=8.819946882 \end{gathered}$ |  | $\begin{gathered} \epsilon_{2}=3.20 \\ \omega=6.480642472 \end{gathered}$ |  | $\begin{gathered} \epsilon_{2}=4.70 \\ \omega=8.987555688 \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{aligned} & \hline \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ | $\begin{aligned} & \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence <br> rate on $E_{z}$ | $\begin{aligned} & \text { error } \\ & \text { on } E_{z} \end{aligned}$ | convergence rate on $E_{z}$ |
|  | DGTD-P ${ }_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  | DGTD- $\mathbb{P}_{0}$ |  |
| 3200 | $4.04 \mathrm{E}-03$ | - | $3.67 \mathrm{E}-02$ | - | $9.60 \mathrm{E}-03$ | - |
| 12800 | $1.89 \mathrm{E}-03$ | 1.10 | $1.73 \mathrm{E}-02$ | 1.09 | $3.89 \mathrm{E}-03$ | 1.30 |
| 51200 | $9.40 \mathrm{E}-04$ | 1.01 | $8.50 \mathrm{E}-03$ | 1.02 | $1.84 \mathrm{E}-03$ | 1.07 |
| 204800 | $4.70 \mathrm{E}-04$ | 1.00 | $4.24 \mathrm{E}-03$ | 1.00 | $9.07 \mathrm{E}-04$ | 1.02 |
|  | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  | DGTD- $\mathbb{P}_{1}$ |  |
| 9600 | $4.96 \mathrm{E}-03$ | - | $1.04 \mathrm{E}-01$ | - | $9.84 \mathrm{E}-03$ | - |
| 38400 | $1.53 \mathrm{E}-03$ | 1.69 | $5.75 \mathrm{E}-02$ | 0.85 | $1.47 \mathrm{E}-03$ | 2.74 |
| 153600 | $5.12 \mathrm{E}-04$ | 1.58 | $3.01 \mathrm{E}-02$ | 0.93 | $5.73 \mathrm{E}-04$ | 1.36 |
| 614400 | $1.91 \mathrm{E}-04$ | 1.43 | $1.54 \mathrm{E}-02$ | 0.97 | $2.95 \mathrm{E}-04$ | 0.96 |
|  | DGTD- $\mathbb{P}_{2}$ |  | DGTD- $\mathbb{P}_{2}$ |  | DGTD- $\mathbb{P}_{2}$ |  |
| 19200 | $4.43 \mathrm{E}-03$ | - | $1.11 \mathrm{E}-01$ | - | $7.40 \mathrm{E}-03$ | - |
| 76800 | $1.37 \mathrm{E}-03$ | 1.69 | $5.96 \mathrm{E}-02$ | 0.90 | $4.15 \mathrm{E}-04$ | 4.16 |
| 307200 | $4.70 \mathrm{E}-04$ | 1.54 | $3.07 \mathrm{E}-02$ | 0.96 | $6.36 \mathrm{E}-04$ | -0.6 |
| 1228800 | $1.82 \mathrm{E}-04$ | 1.37 | $1.55 \mathrm{E}-02$ | 0.98 | $5.21 \mathrm{E}-04$ | 0.29 |
|  | DGTD- $\mathbb{P}_{3}$ |  | DGTD- $\mathbb{P}_{3}$ |  | DGTD- $\mathbb{P}_{3}$ |  |
| 32000 | $4.43 \mathrm{E}-03$ | - | $1.11 \mathrm{E}-01$ | - | $7.39 \mathrm{E}-03$ | - |
| 128000 | $1.37 \mathrm{E}-03$ | 1.69 | $5.96 \mathrm{E}-02$ | 0.90 | $4.14 \mathrm{E}-04$ | 4.16 |
| 512000 | $4.71 \mathrm{E}-04$ | 1.54 | $3.07 \mathrm{E}-02$ | 0.96 | $6.36 \mathrm{E}-04$ | -0.6 |
| 2048000 | $1.82 \mathrm{E}-04$ | 1.37 | $1.56 \mathrm{E}-02$ | 0.98 | $5.21 \mathrm{E}-04$ | 0.29 |



Figure 4.12: Scattering of a dielectric cylinder setting.

$$
E_{z}^{\mathrm{inc}}=\exp \left(-\mathrm{i}\left(k_{1} x-\omega t\right)\right), \quad H_{y}^{\mathrm{inc}}=-\exp \left(-\mathrm{i}\left(k_{1} x-\omega t\right)\right)
$$

where $k_{1}=\omega \sqrt{\epsilon_{1} \mu_{1}}$. In this case, the exact solution of the scattering problem is given by:

$$
E_{z}(x, y, t)=E_{z}(r, \theta, t)=\exp (\mathrm{i} \omega t) \begin{cases}\sum_{n=-\infty}^{\infty} C_{n}^{\text {tot }} J_{n}\left(k_{2} r\right) \exp (\mathrm{i} n \theta), & r \leq r_{0} \\ \sum_{n=-\infty}^{\infty}\left(\mathrm{i}^{-n} J_{n}\left(k_{1} r\right)+C_{n}^{\text {scat }} H_{n}^{(2)}\left(k_{1} r\right)\right) \exp (\mathrm{i} n \theta), & r>r_{0}\end{cases}
$$

where $J_{n}$ and $H_{n}^{(2)}$ represent the $n$-th order Bessel function of the first kind and the Hankel function of the second kind, respectively, and $k_{2}=\omega \sqrt{\epsilon_{2} \mu_{2}}$ is the propagation constant for homogeneuous, lossless dielectric medium. As usual, $(r, \theta)=\left(\sqrt{x^{2}+y^{2}}, \arctan (y / x)\right)$ represent the usual polar coordinates. The expansion coefficients for the total field interior to the cylinder are:

$$
C_{n}^{\mathrm{tot}}=\mathrm{i}^{-n} \frac{\left(k_{1} / \mu_{1}\right) J_{n}^{\prime}\left(k_{1} r_{0}\right) H_{n}^{(2)}\left(k_{1} r_{0}\right)-\left(k_{1} / \mu_{1}\right) H_{n}^{(2)^{\prime}}\left(k_{1} r_{0}\right) J_{n}\left(k_{1} r_{0}\right)}{\left.\left(k_{2} / \mu_{2}\right) J_{n}^{\prime}\left(k_{2} r_{0}\right) H_{n}^{(2)}\left(k_{1} r_{0}\right)-\left(k_{1} / \mu_{1}\right)\right) H_{n}^{(2)^{\prime}}\left(k_{1} r_{0}\right) J_{n}\left(k_{2} r_{0}\right)}
$$

The corresponding coefficients for the scattered field are:

$$
C_{n}^{\text {scat }}=\mathrm{i}^{-n} \frac{\left(k_{1} / \mu_{1}\right) J_{n}^{\prime}\left(k_{1} r_{0}\right) J_{n}\left(k_{2} r_{0}\right)-\left(k_{2} / \mu_{2}\right) J_{n}^{\prime}\left(k_{2} r_{0}\right) J_{n}\left(k_{1} r_{0}\right)}{\left(k_{2} / \mu_{2}\right) J_{n}^{\prime}\left(k_{2} r_{0}\right) H_{n}^{(2)}\left(k_{1} r_{0}\right)-\left(k_{1} / \mu_{1}\right) H_{n}^{(2)^{\prime}}\left(k_{1} r_{0}\right) J_{n}\left(k_{2} r_{0}\right)}
$$

Using Maxwell's equations (2.1), one can recover the solutions for the magnetic field components. Then, the angular component of the total magnetic field is:

$$
H_{\theta}(r, \theta, t)=-\exp (\mathrm{i} \omega t) \begin{cases}\frac{-\mathrm{i} k_{2}}{\omega \mu_{2}} \sum_{n=-\infty}^{\infty} C_{n}^{\mathrm{tot}} J_{n}^{\prime}\left(k_{2} r\right) \exp (\mathrm{i} n \theta), & r \leq r_{0} \\ \frac{-\mathrm{i} k_{1}}{\omega \mu_{1}} \sum_{n=-\infty}^{\infty}\left(\mathrm{i}^{-n} J_{n}^{\prime}\left(k_{1} r\right)+C_{n}^{\mathrm{scat}} H_{n}^{(2)^{\prime}}\left(k_{1} r\right)\right) \exp (\mathrm{i} n \theta), & r>r_{0}\end{cases}
$$

and the radial component is:

$$
H_{r}(r, \theta, t)=-\exp (\mathrm{i} \omega t) \begin{cases}\frac{\mathrm{i}}{\omega \mu_{2} r} \sum_{n=-\infty}^{\infty} \mathrm{i} n C_{n}^{\mathrm{tot}} J_{n}\left(k_{2} r\right) \exp (\mathrm{i} n \theta), & r \leq r_{0} \\ \frac{\mathrm{i}}{\omega \mu_{1} r} \sum_{n=-\infty}^{\infty} \mathrm{i} n\left(\mathrm{i}^{-n} J_{n}\left(k_{1} r\right)+C_{n}^{\mathrm{scat}} H_{n}^{(2)}\left(k_{1} r\right)\right) \exp (\mathrm{i} n \theta), & r>r_{0}\end{cases}
$$

Similarly to the previous test cases, we study the convergence of the DGTD- $\mathbb{P}_{p}$ and DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods for solving this scattering problem using conforming and non-conforming meshes. To this end, we consider a situation in which $\mu_{1}=\mu_{2}=\epsilon_{1}=1$, i.e. the material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius $r_{0}=0.6$ and bounds a material of relative permittivity $\epsilon_{2}=2.25$. The angular frequency is $\omega=2 \pi$ and the computational domain $\Omega$ is chosen as a cylinder of radius one, centered at $(0,0)$. In this special case, all three field components are continuous across the material interface. The first derivative of $E_{z}$ is also continuous across the interface, but first derivatives of $H_{x}$ and $H_{y}$ are discontinuous. Regarding the boundary condition at the artificial boundary of the computational domain, we use a first order Silver-Müller absorbing boundary condition. Contour lines of the $E_{z}$ and $H_{y}$ components at times $t=1$ and $t=10$ are shown on Fig. 4.13 for a calculation based on the conforming DGTD- $\mathbb{P}_{1}$ method.

This problem has been studied in [33] in a scattered field/total field formulation using a FDTD method, and in [20]-[17]-[32] using a pseudospectral time-domain method with a stabilized PML absorbing boundary condition. It is also solved in [6] using a central finite difference scheme away from the material interfaces and upwinding technique with jump conditions near the interfaces and exact boundary conditions. Cai et al [23] developed a discontinuous Galerkin method for this problem. This method combined upwinding fluxes with a Runge-Kutta time scheme and the fields are approximated using Legendre polynomials for rectangular elements and standard nodal polynomials for triangular elements. Recently, Deng and Cai [11] propose an extension of [23] to a higher order method based on orthogonal non-polynomial nodal basis on triangles.
$h$-refinement DGTD- $\mathbb{P}_{p}$ method. We first construct three conforming meshes whose characteristics are summarized in Tab. 4.13. Then, non-conforming meshes are obtained by locally refining (one refinement level) the dielectric region of the conforming meshes. The characteristics of the resulting non-conforming meshes are summarized in Tab. 4.14. Tab. 4.15 and 4.16 give the $L^{2}$ error on ( $\mathbf{E}, \mathbf{H}$ ), the convergence rate, the CPU time and the number of time steps to reach $t=1$ using conforming and non-conforming meshes. Fig. 4.14 shows the corresponding $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. It can be deduced from Tab. 4.15 and 4.16 that the convergence rate of the DGTD- $\mathbb{P}_{p}$ method is close to $\mathcal{O}\left(h^{(1)}\right)$ for $p=0,1$, and very slow for $p=2,3$. It is clear also that it is not necessary to increase the interpolation degree to more than 1 , since the convergence rate is not improved and this results in increased CPU time and memory usage.

Table 4.13: Scattering of a dielectric cylinder. Characteristics of the three conforming meshes.

|  | $\#$ nodes | $\#$ triangles |
| :---: | :---: | :---: |
| mesh-C1 | 101 | 190 |
| mesh-C2 | 401 | 780 |
| mesh-C3 | 1601 | 3160 |

$h p$-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. We still make use of the non-conforming meshes given in Tab. 4.14. Tab. 4.17 gives the $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$, the convergence rate, the CPU time and the number


Figure 4.13: Scattering of a dielectric cylinder.
Contour lines of $E_{z}$ (top), $H_{x}$ (middle) and $H_{y}$ (bottom) at $t=1$ (left) and $t=10$ (right).
$D G T D-\mathbb{P}_{1}$ method using a conforming mesh with 4251 nodes and 8415 triangles.

Table 4.14: Scattering of a dielectric cylinder. Characteristics of the three non-conforming meshes.

|  | \# total nodes | \# fine triangles | \# coarse triangles | \# hanging nodes |
| :--- | :---: | :---: | :---: | :---: |
| mesh-NC1 | 261 | 400 | 90 | 20 |
| mesh-NC2 | 1021 | 1600 | 380 | 40 |
| mesh-NC3 | 4041 | 6400 | 1560 | 80 |

Table 4.15: Scattering of a dielectric cylinder.
Convergence study for the h-refinement DGTD- $\mathbb{P}_{p}$ method using conforming meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=1$.

| \# DOF | error of (E,H) | convergence rate on ( $\mathbf{E}, \mathbf{H}$ ) | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |
| 190 | $1.45 \mathrm{E}-00$ | - | $<1$ | 23 |
| 780 | $6.43 \mathrm{E}-01$ | 1.15 | 4 | 71 |
| 3160 | $3.22 \mathrm{E}-01$ | 0.99 | 44 | 262 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |
| 570 | $9.39 \mathrm{E}-01$ | - | 4 | 76 |
| 2340 | $3.84 \mathrm{E}-01$ | 1.26 | 30 | 236 |
| 9480 | $1.86 \mathrm{E}-01$ | 1.03 | 427 | 874 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |
| 1140 | $3.47 \mathrm{E}-01$ | - | 7 | 113 |
| 4680 | $1.62 \mathrm{E}-01$ | 1.07 | 85 | 354 |
| 18960 | $1.51 \mathrm{E}-01$ | 0.11 | 1272 | 1310 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |
| 1900 | $2.64 \mathrm{E}-01$ | - | 23 | 226 |
| 7800 | $1.60 \mathrm{E}-01$ | 0.70 | 283 | 708 |
| 31600 | $1.51 \mathrm{E}-01$ | 0.09 | 4239 | 2620 |

Table 4.16: Scattering of a dielectric cylinder.
Convergence study for the h-refinement DGTD- $\mathbb{P}_{p}$ method using non-conforming meshes. $L^{2}$ error, CPU time and \# time steps are measured at $t=1$.

| \# DOF | error of ( $\mathbf{E}, \mathbf{H}$ ) | convergence rate on ( $\mathbf{E}, \mathbf{H}$ ) | CPU (s) | \# time steps |
| :---: | :---: | :---: | :---: | :---: |
| DGTD- $\mathbb{P}_{0}$ |  |  |  |  |
| 490 | $1.22 \mathrm{E}-00$ | - | 2 | 46 |
| 1980 | $5.71 \mathrm{E}-01$ | 1.09 | 7 | 82 |
| 7960 | $3.05 \mathrm{E}-01$ | 0.90 | 88 | 262 |
| DGTD- $\mathbb{P}_{1}$ |  |  |  |  |
| 1470 | $8.77 \mathrm{E}-01$ | - | 9 | 151 |
| 5940 | $3.25 \mathrm{E}-01$ | 1.42 | 65 | 272 |
| 23880 | $1.73 \mathrm{E}-01$ | 0.90 | 845 | 874 |
| DGTD- $\mathbb{P}_{2}$ |  |  |  |  |
| 2940 | $3.23 \mathrm{E}-01$ | - | 27 | 226 |
| 11880 | $1.62 \mathrm{E}-01$ | 0.99 | 193 | 408 |
| 47760 | $1.51 \mathrm{E}-01$ | 0.10 | 2506 | 1310 |
| DGTD- $\mathbb{P}_{3}$ |  |  |  |  |
| 4900 | $2.61 \mathrm{E}-01$ | - | 87 | 452 |
| 19800 | $1.61 \mathrm{E}-01$ | 0.70 | 633 | 816 |
| 79600 | $1.51 \mathrm{E}-01$ | 0.09 | 8264 | 2620 |



Figure 4.14: Scattering of a dielectric cylinder.
Numerical convergence of the $h$-refinement $D G T D-\mathbb{P}_{p}$ method.
Conforming (left) and non-conforming (right) triangular meshes.
of time steps to reach $t=1$. Fig. 4.15 shows the corresponding $L^{2}$ error on $(\mathbf{E}, \mathbf{H})$ as a function of the square root of the number of DOF. It can be seen from Tab. 4.17 that the overall convergence rate of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method is less than linear excepted for the $\operatorname{DGTD}-\mathbb{P}_{1}: \mathbb{P}_{0}$ where the convergence rate is close to $\mathcal{O}\left(h^{(1)}\right)$.

Table 4.17: Scattering of a dielectric cylinder.
Convergence study of the $D G T D-\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method using non-conforming meshes.
$L^{2}$ error, CPU time and \# time steps are measured at $t=1$.
\# DOF $\quad$ error of ( $\mathbf{E}, \mathbf{H}$ ) convergence rate on ( $\mathbf{E}, \mathbf{H}$ ) CPU (s) \# time steps

| DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 670 | $7.58 \mathrm{E}-01$ | - | 5 | 113 |
| 2740 | $3.17 \mathrm{E}-01$ | 1.23 | 28 | 204 |
| 11080 | $1.88 \mathrm{E}-01$ | 0.75 | 363 | 655 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |
| 940 | $4.82 \mathrm{E}-01$ | - | 13 | 226 |
| 3880 | $2.39 \mathrm{E}-01$ | 0.99 | 89 | 408 |
| 15760 | $1.73 \mathrm{E}-01$ | 0.46 | 1153 | 1310 |
| DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ |  |  |  |  |
| 1740 | $4.16 \mathrm{E}-01$ | - | 12 | 151 |
| 7080 | $1.78 \mathrm{E}-01$ | 1.21 | 87 | 272 |
| 28560 | $1.51 \mathrm{E}-01$ | 0.23 | 1134 | 874 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{1}$ |  |  |  |  |
| 2100 | $3.66 \mathrm{E}-01$ | - | 24 | 226 |
| 8600 | $1.76 \mathrm{E}-01$ | 1.04 | 176 | 408 |
| 34800 | $1.52 \mathrm{E}-01$ | 0.21 | 4566 | 2620 |
| DGTD- $\mathbb{P}_{3}: \mathbb{P}_{2}$ |  |  |  |  |
| 3300 | $2.63 \mathrm{E}-01$ | - | 33 | 226 |
| 13400 | $1.60 \mathrm{E}-01$ | 0.71 | 239 | 408 |
| 54000 | $1.51 \mathrm{E}-01$ | 0.08 | 6096 | 2620 |



Figure 4.15: Scattering of a dielectric cylinder.
Numerical convergence of the hp-refinement DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method. Non-conforming triangular meshes.

## 5 Concluding remarks

In this work, we have presented the results of a detailed numerical evaluation of DGTD- $\mathbb{P}_{p}$ and DGTD$\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods developed for the numerical solution of the two-dimensional time-domaine Maxwell equations on conforming and non-conforming triangular meshes. For this purpose, we have considered wave propagation problems in homogeneous and heterogeneous media. The DGTD- $\mathbb{P}_{p}$ and DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ methods were implemented in Fortran 77 and the computation was performed on a PC workstation equipped with an Intel Pentium M 1.7 GHz processor. The following conclusions can be drawn from these numerical experiments:

1. an optimal convergence rate is obtained for the DGTD- $\mathbb{P}_{p}(p=0,1)$ method in both homogeneous and heterogeneous media. For $p \geq 2$ the convergence rate is close to $\mathcal{O}\left(h^{(2)}\right)$ in homogeneous case but it is linear or less in heterogeneous case.
2. the convergence rate of the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method for homogeneous media is close to $\mathcal{O}\left(h^{(2)}\right), \forall p_{c}, \forall p_{f} \neq 0$ and for $p_{f}=0$ this convergence rate is close to $\mathcal{O}\left(h^{(3 / 2)}\right)$. In heterogeneous media, the convergence rate of this method is linear or less, and in some particular cases we can obtain a convergence more than linear.
3. in homogeneous media, the DGTD- $\mathbb{P}_{2}$ and DGTD- $\mathbb{P}_{2}: \mathbb{P}_{1}$ methods are the most accurate and the least expensive $h$ - and $h p$-refinement methods respectively. In heterogeneous media and to reach a given error level, the DGTD- $\mathbb{P}_{0}$ and DGTD- $\mathbb{P}_{1}: \mathbb{P}_{0}$ are the least expensive $h$ - and $h p$-refinement methods. In general, the DGTD- $\mathbb{P}_{p_{c}}: \mathbb{P}_{p_{f}}$ method is less expensive than the DGTD- $\mathbb{P}_{p}$ method.
4. to obtain a high convergence rate for high order interpolation ( $p \geq 2$ ), a high order time scheme like a fourth order leap-frog scheme must be used (see [25] for a 1D example). A special treatment of the numerical fluxes at material interfaces by using a regularization technique [2]-[22] may also improve the accuracy of the proposed method.
5. to reduce the costs of the method for a high order interpolation, three possible approaches can be used separately or in combination:

- the use of hierarchical or orthogonal nodal basis functions can reduce the cost of memory consumption and can also improve the convergence rate of the $p$-enrichment method,
- the use of a local time stepping strategy [26],
- the use of an a posteriori error estimator which remains mandatory to reduce the costs of the method on non-conforming locally refined meshes.

Acknowledgment. The author would like to express his heartfelt gratitude to Stéphane Lanteri for many helpful discussions during the course of this work.

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[^1]:    ${ }^{a}$ Results were taken from [1]-[38].
    ${ }^{b}$ Results were taken from [1].

[^2]:    ${ }^{a}$ See Tab 3.8 for the characteristics of these meshes.
    ${ }^{b}$ Represents the percentage of the DOF in the fine mesh.

[^3]:    ${ }^{a}$ See Tab. 3.14 for the characteristics of the meshes.
    ${ }^{b}$ Represents the percentage of the DOF in the fine mesh.

