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First hitting time of Double Integral Processes to curved boundaries

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Abstract: The problem of finding the first hitting time of a Double Integral Process (DIP) such as the Integrated Wiener Proces (IWP) has been a central and difficult endeavor in stochastic calculus and has applications in many fields of physics (first exit time of a particle in a noisy force field) or in biology and neuroscience (spike time of an integrate-and-fire neuron with exponentially decaying synaptic current). The only results available so far were an approximation of the stationnary mean crossing time and the distribution of the first hitting time of the IWP to a constant boundary. In this paper, we generalize those results and find an analytical formula for the first hitting time of the IWP to piecewise cubic boundaries. We use this formula to approximate the law of the first hitting time of a general DIP to a smooth curved boundary, and we provide an estimation of the convergence of this method. This approximation formula is the first analytical description of the hitting time of a DIP to a curved boundary, and allows us to infer properties of this random variable and provides a way for computing accurately its law. The accuracy of the approximation is computed in the general case for the IWP and the calculation of crossing probability can be carried out through a Monte-Carlo method.

Key-words: First hitting time, first passage time, curved boundary, integrated Wiener process, Double integral process, second-order stochastic differential equation, numerical computation.

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Premier temps d'atteinte d'un processus double-integral a une courbe

Résumé : Le problème de trouver le premier temps d'atteinte d'un processus double-integral (DIP) tel que le mouvement brownien intégré (IWP) est un probleme central et difficile dans le domaine du calcul stochastique et a des applications dans beaucoup de domaines physiques (premier temps de sortie d'une particule dans un champ de force bruité) ou en biologie et en neurosciences (temps de spike d'un neurone integre-et-tire avec des courants synaptic à décroissance exponentielle). Les seuls résultats disponibles jusqu'ici étaient la distribution du premier temps d'atteinte de l'IWP a une constante et une approximation de la moyenne de cette variable aléatoire. Dans cet article, nous généralisons ces résultats et trouvons une formule analytique pour la distribution du premier temps d'atteinte de l'IWP a une courbe cubique par morceaux. Nous utilisons cette formule pour approximer la loi de probabilité du premier temps d'atteinte d'un DIP général à une courbe lisse, et donnons une estimation de la convergence de cette méthode. Cette approximation est la première description analytique du premier temps d'atteinte d'un DIP à une courbe, et nous permet de trouver des propriétés de cette variable aléatoire et une méthode afin de la calculer efficacement. La précision de cette méthode est estimée dans le cas général pour l'IWP et le calcul numérique de cette loi peut-etre effectué en utilisant un algorithme de Monte-Carlo.

Mots-clés : premier temps d'atteinte, premier temps de passage, frontiere courbe, primitive du mouvement brownien, processus double-integral, equation differentielle stochatique du second order, calcul numérique

Introduction

First passage time problems for one-dimensional diffusion processes through a time-dependent boundary have received a lot of attention over the last three decades. Unfortunately, the evaluation of the first passage time probability distribution function (pdf) through a constant or time dependent boundary is in general an arduous task which has still not received a satisfactory solution. Analytical results are scarce and fragmentary, even if closed form solutions exist for some very particular cases. Since no analytical method seem to solve the problem, one is led either to the study of the asymptotic behavior of this function and of its moments (see e.g. [23, 24]), or to the use of somewhat *ad-hoc* numerical procedures yielding approximate evaluations of the first passage time distributions. Such procedures can be classified as follows: (i) those that are based on probabilistic approaches (see e.g. [4, 6, 7, 21, 29, 30]), and (ii) purely numerical methods, such as the widely used Monte-Carlo method which applies without any restriction, but whose results are generally too coarse (for numerical methods, see e.g. [1, 9, 11, 15]).

In two and higher dimensions, the problem is even more complex and results can hardly be found. For the simplest Double Integral Process (DIP), the Integrated Wiener Process (IWP) defined in (1.10), McKean [20] Goldman [12], Lachal [16, 17, 18] found the probability distribution of the first hitting time to a constant boundary using stochastic calculus methods. Lefebvre used the Kolmogorov (Fokker-Planck) equation to find in some special cases closed-form solutions [19]. Generalizations of these formulas to other boundaries and other kinds of processes are simply not available. In the present paper, we propose a closed-form solution for the first hitting time of the IWP to a piecewise cubic function, and apply this formula to find an approximation of the first hitting time of a DIP to any smooth curved boundary. We also provide an estimation of the rate of convergence of this approximation.

In the first section, we introduce a motivation of this study, define the Double Integral Process and prove the main properties which will be useful for us in the rest of the paper. In the second section, we study the first hitting times of the IWP and provide a closed-form formula for the first hitting time of this process to a piecewise cubic function. In the third section, we introduce the approximation method of the first hitting time of the IWP to any smooth curved boundary, and find the rate of convergence of this method. Finally in the last section we provide an approximation formula for the first hitting time of a general DIP to a curved boundary. The fifth section describes briefly a numerical Monte-Carlo algorithm which can be used to compute the probability repartition function efficiently.

1 The Double Integral Process

In this section we introduce the Double Integral Process (DIP) and prove some useful properties. But before the mathematical study of the problem, we motivate this theoretical work by a specific problem arising in neuroscience: the distribution of the spike times for an integrate-and-fire neuron with exponentially decaying synaptic currents.

1.1 Motivation

The definition of the DIP and the study of its first hitting times of curved boundaries has been motivated by numerous physical and biological problems. For instance a problem arising in neuroscience is to characterize the probability distribution of the spike (action potentials) times in presence of synaptic noise (see [10] for an introduction of the neuronal modelization of spiking neurons and [33] for review of the problem of spike time distribution).

A classical neuron model is the leaky integrate-and-fire model, where the membrane potential $V(t)$ of a neural cell integrates external inputs and the noise at the synapses, and emits a spike when the membrane potential reaches a deterministic threshold function $\theta(t)$ (which is constant in general). Hence in this model, the membrane potential is solution of the equation:

$$\tau_m dV(t) = \big(-(V(t) - V_{\text{rest}}) + I_e(t) \big)\, dt + dI_s(t) \tag{1.1}$$

In this equation τ_m is the characteristic time of integration of the membrane potential, V_{rest} is the rest potential of the neuron, I_e represents deterministic external inputs and I_s the noisy synaptic inputs (see for instance [8, 10, 33]). The simplest model of synaptic noise is a standard Brownian motion, if we neglect the integration time of the synapse. Nevertheless, real post-synaptic currents have a very short rise time and a larger decay time.

If we take into account the decay time of the synapse τ_s, then the synaptic current is solution of the stochastic differential equation:

$$\tau_s dI_s(t) = -I_s(t)dt + \sigma dW_t$$

We can integrate this system of stochastic differential equations as follows. The equation governing the membrane potential yields

$$V(t) = V_{\text{rest}}(1 - e^{-\frac{t}{\tau_m}}) + \tfrac{1}{\tau_m}\int_0^t e^{\frac{s-t}{\tau_m}} I_e(s)\, ds + \tfrac{1}{\tau_m}\int_0^t e^{\frac{s-t}{\tau_m}} I_s(s)\, ds,$$

and the synaptic current equation can be integrated as

$$I_s(t) = I_s(0)e^{-\frac{t}{\tau_s}} + \frac{\sigma}{\tau_s}\int_0^t e^{\frac{s-t}{\tau_s}} dW_s,$$

where $I_s(0)$ is a given random variable. We define $\frac{1}{\alpha} = \frac{1}{\tau_m} - \frac{1}{\tau_s}$. Replacing in the first equation $I_s(t)$ by its value in the second equation we obtain

$$\begin{aligned} V(t) = V_{\text{rest}}(1 - e^{-\frac{t}{\tau_m}}) + \tfrac{1}{\tau_m}\int_0^t e^{\frac{s-t}{\tau_m}} I_e(s)\, ds + \\ \frac{I_s(0)}{1 - \frac{\tau_m}{\tau_s}}(e^{-\frac{t}{\tau_s}} - e^{-\frac{t}{\tau_m}}) + \frac{\sigma}{\tau_m\tau_s} e^{-\frac{t}{\tau_m}} \int_0^t e^{\frac{s}{\alpha}} \left(\int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) ds \end{aligned}$$

The time of the spike emission is the first hitting time of $V(t)$ to the threshold $\theta(t)$, so the first hitting time of the stochastic process, which is a particular case of what we will call in the sequel Double Integral Process (DIP)

$$X_t = \int_0^t e^{\frac{s}{\alpha}} \left(\int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) ds \tag{1.2}$$

to the deterministic curved boundary

$$a(t) = \theta(t) - \left(V_{\text{rest}} \left(1 - e^{-\frac{t}{\tau_m}} \right) + \frac{1}{\tau_m} \int_0^t e^{\frac{s-t}{\tau_m}} I_e(s)\, ds + \frac{I_s(0)}{1-\frac{\tau_m}{\tau_s}} \left(e^{-\frac{t}{\tau_s}} - e^{-\frac{t}{\tau_m}} \right) \right).$$

1.2 Definition and main properties of DIPs

In this section, we define a class of stochastic processes including the process (1.2), and prove some useful properties of these processes.

Definition 1.1 (DIP)**.** Let $f \in \mathbb{L}^2(\mathbb{R})$ and $g \in \mathbb{L}^1(\mathbb{R})$. Let M_t be the martingale defined by $M_t := \int_0^t f(s) dW_s$.

The *double integral process (DIP)* associated to the functions f and g is defined for all t by:

$$X_t = \int_0^t g(s) M_s ds = \int_0^t g(s) \left(\int_0^s f(u) dW_u \right) ds \tag{1.3}$$

Proposition 1.1. The two-dimensional process (X_t, M_t) is a Gaussian Markov process.

Proof. First of all, note that if $\mathcal{F}_t^X$ (resp. $\mathcal{F}_t^M$) defines the canonical filtration associated to the process X (resp. M) then it is clear that $\forall t \geq 0$, $\mathcal{F}_t^X \subset \mathcal{F}_t^M$. Hence the filtration associated to the pair $(X_t, M_t)_{t \geq 0}$ is simply $(\mathcal{F}_t^M)_{t \geq 0}$, which we denote in the sequel $(\mathcal{F}_t)_{t \geq 0}$

It is also clear that M is a martingale, and satisfies the Markov property. Let $s \leq t$. We have:

$$\begin{aligned} X_t &= \int_0^t g(u) M_u du \\ &= \int_0^s g(u) M_u du + \int_s^t g(u) M_u du \end{aligned}$$

$$X_t = X_s + \int_s^t g(u)(M_u - M_s) du + M_s \int_s^t g(u) du \tag{1.4}$$

Conditionally to M_s, the process $\int_s^t g(u)(M_u - M_s) du$ is independent of $\mathcal{F}_s^M$ so the law of X_t knowing (X_s, M_s) is independent of the sigma-algebra $(\mathcal{F}_t)$, and so is M, so eventually the pair (X, M) is Markov.

The pair is clearly a Gaussian process since its two components are. Indeed, M is Gaussian as the limit of the Riemann sums of Brownian increments, which are Gaussian, and X is also the limit of Riemann sums of a Gaussian process, namely M, with the weights given by g.

□

Remark 1. In the proof of proposition 1.1, we proved also that conditionally to M_s, the increments $(X_t - X_s, M_t - M_s)$ are independent of the σ-field $\mathcal{F}_s$.

Proposition 1.2. For each value of $t \geq 0$, the random variable $Y_t := (X_t, M_t)$ is a two-dimensional Gaussian variable of parameters:

$$\begin{cases} \mathbb{E}\left[Y_t\right] & = (0,0) \\ \mathbb{E}\left[Y_t^T \cdot Y_t\right] & = \begin{pmatrix} \rho_X(0,t) & C_{(X,M)}(0,t) \\ C_{(X,M)}(0,t) & \rho_M(0,t) \end{pmatrix} \end{cases} \tag{1.5}$$

where the functions $\rho_X(s,t)$, $C_{(X,M)}(s,t)$ and $\rho_M(s,t)$ are defined by:

$$\begin{cases} \rho_M(s,t) & = \int_s^t f(u)^2 du \\ \rho_X(s,t) & = 2\int_s^t g(u)\left(\int_s^u g(v)\rho_M(s,v)dv\right)du \\ C_{(X,M)}(s,t) & = \int_s^t g(u)\rho_M(s,u)du \end{cases} \tag{1.6}$$

The transition measure of the Markov process $(Y_t)_t$ has a Gaussian density w.r.t. Lebesgue's measure:

$$\mathcal{N}\left(\begin{pmatrix} x_s + m_s \int_s^t g(u)du \\ m_s \end{pmatrix}, \tilde{C}(s,t)\right) \tag{1.7}$$

where the correlation matrix $\tilde{C}(s,t)$ reads:

$$\tilde{C}(s,t) = \begin{pmatrix} \rho_X(s,t) & C_{(X,M)}(s,t) \\ C_{(X,M)}(s,t) & \rho_M(s,t)du \end{pmatrix} \tag{1.8}$$

Proof. The calculations are essentially straightforward. To compute the transition density function, we use the equation (1.4) and write:

$$\begin{pmatrix} X_t \\ M_t \end{pmatrix} = \begin{pmatrix} X_s + M_s \int_s^t g(u)du \\ M_s \end{pmatrix} + \begin{pmatrix} \int_s^t g(u)(M_u - M_s)du \\ M_t - M_s \end{pmatrix} \tag{1.9}$$

The first term in the sum in the righthand side of (1.9) is $\mathcal{F}_s$ measurable. Given $X_s = x_s$ and $M_s = m_s$, it is equal to

$$\begin{pmatrix} x_s + m_s \int_s^t g(u)du \\ m_s \end{pmatrix}$$

The second term is independent of $\mathcal{F}_s$ and is Gaussian.

Eventually, the process Y_t knowing $Y_s = (x_s, m_s)$ has the same law as the Gaussian process:

$$\mathcal{N}\left(\begin{pmatrix} x_s + m_s \int_s^t g(u)du \\ m_s \end{pmatrix}, \tilde{C}(s,t)\right)$$

□

Definition 1.2 (IWP)**.** The *Integrated Wiener Process* is a special case of the DIP where the functions f and g are identically equal to 1 :

$$X_t = \int_0^t W_s\, ds \quad M_s = W_s \tag{1.10}$$

From proposition 1.2, we know that its transition measure reads:

$$\mathbb{P}\Big[X_{t+s} \in du, W_{t+s} \in dv \Big| X_s = x, W_s = y\Big] \stackrel{\text{def}}{=} p_t(u\, v;\, x, y) du\, dv =$$
$$\frac{\sqrt{3}}{\pi t^2} \exp\Big[-\frac{6}{t^3}(u - x - ty)^2 + \frac{6}{t^2}(u - x - ty)(v - y) - \frac{2}{t}(v - y)^2\Big] du\, dv \tag{1.11}$$

Lemma 1.3. Let $(X_t)_{t\geq 0}$ be a DIP defined by (1.3). Assume that $f(s) \neq 0$ for all $s \geq 0$. The study of the hitting times of the DIP X is equivalent to the study of the simpler process:

$$\tilde{X}_t = \int_0^t \tilde{g}(s) W_s ds,$$

where $\tilde{g}$ is defined in the proof.

Proof. Let $(M_t)_t$ be the martingale defined by:

$$M_t = \int_0^t f(s) dW_s$$

Dubins-Schwarz' theorem[1] ensures us that there exists a Brownian motion $(W_t)_t$ such that almost surely

$$M_t = W_{\langle M \rangle_t}$$

We note

$$\Phi(t) = \langle M \rangle_t = \int_0^t f^2(s) ds$$

[1] Even though $\langle M \rangle_\infty \neq \infty$ because of our hypothesis on f, see [14].

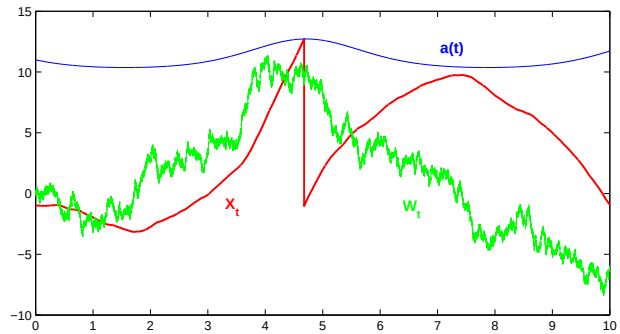


Figure 1: A sample path of the process $U_t = (X_t, W_t)$ where X is a standard IWP and W a standard Brownian motion, and a boundary curve $a(t)$. The IWP X_t is reset to 0 when it crosses the boundary.

Φ is continuous and since we assumed that $f(s) \neq 0$ for all $s \geq 0$, stricly increasing, so it is a bijection. Its derivative $\Phi'(t)$ exists and is nonzero for all $t \geq 0$. We use the change of variable $u = \Phi(s)$. We have:

$$\begin{aligned} X_t &= \int_0^t g(s) M_s \, ds \\ &\stackrel{\mathcal{L}}{=} \int_0^t g(s) W_{\Phi(s)} ds \\ &= \int_0^{\Phi^{-1}(t)} \frac{g(\Phi^{-1}(u))}{\Phi'\big(\Phi^{-1}(u)\big)} W_u \, du \end{aligned}$$

Hence the hitting time of a general DIP can be deduced from the hitting time of the process $\tilde{X}_t = X_{\Phi(t)}$ which is of type $\int_0^t \tilde{g}(s) W_s ds$, where $\tilde{g}(t) = \frac{g(\Phi^{-1}(t))}{\Phi'(\Phi^{-1}(t))}$ □

2 Hitting time of the integrated Wiener process

We consider the special case $(W_t)_{t\geq 0}$, a standard Brownian motion. We are interested in the first hitting time to a curved boundary $a(t)$ of the stochastic process:

$$X_t = \int_0^t W_s ds \tag{2.1}$$

This problem has been widely studied and has received no satisfactory solution so far. One of the main difficulties comes from the fact that the process is non Markov, so we have

to refer to the underlying Wiener process. Classical approaches based on Volterra equations or Durbin's method, work for the Brownian motion, but fail in providing a solution to this problem (see for instance [33] for a review). To achieve the program of characterizing those hitting times, we first recall existing results on the first hitting times to constant boundaries, generalize them to cubic and piecewise cubic boundaries, to end with the approximation formula for general boundaries.

2.1 First hitting time to a constant boundary

Lachal in [16] studies this problem in the case where the boundary is a constant. More precisely, in this section we study the process $U_t = (X_t + x + ty, W_t + y)$ where X_t is the standard IWP. We denote by

$$\tau_a = \inf\left\{t > 0\ ;\ X_t + x + ty = a\right\}$$

the first passage time at a of the first component of the bidimensional Markov process U_t. The work of Lachal [16] follows the work of McKean [20], where the joint law of the process (τ_a, W_{τ_a}) is computed in the case $x = a$. The result is:

$$\begin{aligned}\mathbb{P}\Big[\tau_a \in dt\ ;\ |W_{\tau_a}| \in dz\Big|U_0 = (a,y)\Big] &\stackrel{\text{def}}{=} \mathbb{P}_{(a,\,y)}(\tau_a \in dt;\, |W_{\tau_a}| \in dz)\\ &= \frac{3z}{\pi\sqrt{2}t^2}e^{-(2/t)(y^2-|y|z+z^2)}\left(\int_0^{4|y|z/t} e^{-3\theta/2}\frac{d\theta}{\sqrt{\pi\theta}}\right)\mathbb{1}_{[0,+\infty)}(z)dzdt \qquad (2.2)\end{aligned}$$

We denote this density by $m^a(t,y,z)$.

Later, Goldman in [12] computed the distribution of the random variable τ_a in the case where $x < a$ and $y \leq 0$ and obtained the formula:

$$\begin{aligned}\mathbb{P}\Big[\tau_a \in dt\Big|U_0 = (x,y)\Big] &= dt\Big[\sqrt{\frac{3}{8\pi t^3}}\big(\frac{3(a-x)}{t} - y\big)e^{-3(a-x-ty)^2/(2t^3)}\\ &+ \int_0^{+\infty} zdz\int_0^t\int_0^\infty \mathbb{P}\Big[\tau_0 \in ds\ ;\ |W_{\tau_0}| \in d\mu\Big|U_0 = (0,z)\Big]q_{t-s}(x,y;a,z)\Big] \qquad (2.3)\end{aligned}$$

where $q_t(x,y;u,v) = p_t(x,y;u,v) - p_t(x,y;u,-v)$.

Lastly, Lachal in [16] extended all these results and gave the joint distribution of the pair (τ_a, W_{τ_a}) in all cases. The quite complex formula reads:

$$\begin{aligned}\mathbb{P}_{(x,y)}\left[\tau_a \in dt\ ;\ W_{\tau_a} \in dz\right] = |z|\Big[&p_t(x,y;a,z) -\\ &\int_0^t\int_0^{+\infty} m^0(s,-|z|,\mu)p_{t-s}(x,y;a,-\varepsilon\mu)\,d\mu\,ds\Big]\mathbb{1}_A(z)dzdt \qquad (2.4)\end{aligned}$$

where $A = [0, \infty)$ if $x < a$, $A = (-\infty, 0]$ if $x > a$, $\varepsilon = \text{sign}(a - x)$ and $m^0(s, -|z|, \mu)$ is given by McKean's formula (2.2). We denote this density by $l^a_{x,y}(t, z)$.

2.2 First Hitting time to a cubic boundary

The problem we adress now is the question of finding similar formulae for more general boundaries. For the Brownian motion itself, few results are available. A formula has been found for a linear boundary using Girsanov's theorem, for a quadratic boundary using the Laplace transform characterization (see [13]). Lastly, the method of images has been shown to provide closed form results in very particular cases (see [27] for a review). The difficulty for finding closed-form characterizations of the first hitting time of the Brownian motion incited people to look for approximations. Monte-Carlo simulation is often used. Even if it can be used with no restriction, it is often considered too coarse and computationally inefficient. Furthermore, it is purely numerical and global, and does not provide any analytical information on the hitting time. For these reason, other semi-analytical methods of approximation have been developed to provide analytical approximations [5, 6, 7, 28, 34], sometimes together with error estimations [3, 25].

The problem is even more complex for the first hitting time of the integrated Wiener process.

In this section we apply Girsanov's theorem to transform the problem of finding a closed-form expression of the first hitting time of the IWP to a cubic function to the problem discussed in the previous section that has been solved by McKean, Goldman and Lachal [12, 16, 20]. More precisely, we prove that under a certain probability, the process $W_t + \frac{\beta}{2}t^2 + \alpha t + x$ is a Wiener process. Under this probability, the process $X_t + \frac{\beta}{6}t^3 + \frac{\alpha}{2}t^2 + tx + y$ has the law of an IWP. Hence the knowledge of the probability density function (pdf) of the first hitting time of the IWP to a constant will give us, using Girsanov's theorem, the pdf of the hitting time of the IWP to the cubic

$$t \mapsto \frac{\beta}{6}t^3 + \frac{\alpha}{2}t^2 + bt + a.$$

For the sake of generality we compute the new probability starting at a general time s at the point (x, y). The index $\{s, (x, y)\}$ denotes in the following the conditionning on the event $(X_s = x, W_s = y)$.

Theorem 2.1. *The process $W_t + \alpha(t - s) + \frac{\beta}{2}(t - s)^2 + y$, $0 \le s \le t$ is a Wiener process starting at y at time s, under the probability:*

$$\mathbb{P}^{\alpha,\beta}_{s,(x,y)}\Big|_{\mathcal{F}_{s,t}} = D^{\alpha,\beta}_{s,(x,y)}(t)\, \mathbb{P}_{s,(x,y)}\Big|_{\mathcal{F}_{s,t}} \tag{2.5}$$

where:

$$D^{\alpha,\beta}_{s,(x,y)}(t) = \exp\Big(-\frac{1}{6}\beta^2(t^3 - s^3) - \frac{1}{2}\alpha\beta(t^2 - s^2) - \frac{1}{2}\alpha^2(t - s) - (\alpha + t\beta)W_t + (\alpha + s\beta)y + \beta(X_t - x)\Big) \tag{2.6}$$

Proof. We consider the full process $U_t^{\alpha,\beta} = (X_t^{\alpha,\beta}, W_t^{\alpha,\beta}) = (X_t + x + y\,t + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3, W_t + y + \alpha(t-s) + \frac{\beta}{2}(t-s)^2) = (X_t^{\alpha,\beta}, W_t^{\alpha,\beta})$. This is a diffusion process satisfying the two-dimensional stochastic differential equation:

$$\begin{cases} dX_t^{\alpha,\beta} &= W_t^{\alpha,\beta}\,dt \\ dW_t^{\alpha,\beta} &= (\alpha + (t-s)\beta)\,dt + dW_t \end{cases} \tag{2.7}$$

to be solved for $t \geq s \geq 0$, with initial conditions $U_s^{\alpha,\beta} = (x, y)$.

We want $W_t^{\alpha,\beta}$ to be a Brownian motion under a new probability. This is a straightforward application of Girsanov's theorem (or the particular case of Cameron-Martin's formula). We define

$$L_t^{\alpha,\beta} := -\int_s^t (\alpha + \beta h) dW_h,\ 0 \leq s \leq t$$

This is a martingale for $\mathcal{F}_{s,t}$ satisfying $\left\langle L^{\alpha,\beta},\, L^{\alpha,\beta}\right\rangle_t = \int_s^t (\alpha+\beta h)^2\,dh$, therefore $\mathbb{E}[\exp\{\frac{1}{2}\left\langle L^{\alpha,\beta},\, L^{\alpha,\beta}\right\rangle_t\}] < \infty$, $0 \leq s \leq t < \infty$. We conclude from Novikov's criterion that $W_t - \left\langle W,\, L^{\alpha,\beta}\right\rangle_t = W_t^{\alpha,\beta}$ is a Brownian motion under a new probability, noted $\mathbb{P}_{s,(x,y)}^{\alpha,\beta}$.
The Radon-Nicodym derivative $D_{s,(x,y)}^{\alpha,\beta}(t)$ of this new probability with respect to the initial probability $\mathbb{P}_{s,(x,y)}$ is given by Girsanov's theorem and is equal to

$$\exp\left(L_t^{\alpha,\beta} - \frac{1}{2}\left\langle L^{\alpha,\beta},\, L^{\alpha,\beta}\right\rangle_t\right).$$

This can be written

$$\begin{aligned} D_{s,(x,y)}^{\alpha,\beta}(t) &= \left.\frac{d\mathbb{P}^{\alpha,\beta}}{d\mathbb{P}}\right|_{\mathcal{F}_{s,t}} \\ &= \exp\Big(-\frac{1}{2}\int_s^t (\alpha + h\beta)^2 dh + \int_s^t (\alpha + h\beta) dW_h\Big) \\ &= \exp\Big(-\frac{1}{2}\big(\alpha^2(t-s) + \alpha\beta(t^2 - s^2) + \frac{1}{3}\beta^2(t^3 - s^3)\big) \\ &\quad - \alpha(W_t - y) - \beta(tW_t - sy) + \beta(X_t - x)\Big) \end{aligned}$$

using Ito's formula for the process tW_t. Hence we obtain a formula equivalent to (2.6). □

Remark 2. In this proof we have seen that the IWP comes from the stochastic integration of the function $\alpha+\beta t$ with respect to the Brownian density. If we had chosen a polynomial of degree greater than 1, the integration by parts would have produced higher-order integrals of the Brownian motion that we do not want to deal with since we have no knowledge of their first hitting time. This is the reason why we study in the sequel the first hitting time of the IWP to cubic boundaries and why we cannot go further. This method does not

generalize to polynomial boundaries of degree larger than three. Anyway we show that this is sufficient to approximate the probability distribution of the first hitting time of the IWP and of other DIP to general curved boundaries, precisely by approximating these boundaries with piecewise-cubic polynomials.

We note

$$d^{\alpha,\beta}(s,x,y;t,u,v) = \exp\Big(-\frac{1}{6}\beta^2(t^3-s^3) - \frac{1}{2}\alpha\beta(t^2-s^2) - \frac{1}{2}\alpha^2(t-s) \\ - (\alpha+t\beta)v + (\alpha+s\beta)y + \beta(u-x)\Big) \quad (2.8)$$

the probability density function of the new probability w.r.t. the initial one.

Theorem 2.2. *Let τ_C be the first hitting time of the standard IWP to the cubic curve C of equation*

$$C(t-s) = a + b(t-s) + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3. \quad t \geq s$$

Under the reference probability $\mathbb{P}$, the law of the random variable (τ_C, W_{τ_C}) satisfies the equation:

$$\mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) = d^{-\alpha,-\beta}(s,x,y-b;t,a,z-b-\alpha(t-s)-\frac{\beta}{2}(t-s)^2) \\ \times \mathbb{P}_{s,(x,y-b)}(\tau_a \in dt, W_{\tau_a} + b + \alpha(\tau_a - s) + \frac{\beta}{2}(\tau_a - s)^2 \in dz) \quad (2.9)$$

The second term of the righthand side is given by Lachal's formula (2.4) *if $a \neq x$ or by McKean's formula* (2.2) *if $a = x$.*

Proof. Let $\Gamma \subset \mathbb{R}$ be a measurable set and $t \geq s \geq 0$. We have by the change of probability formula:

$$\begin{aligned}
\mathbb{P}^{\alpha,\beta}_{s,(x,y)}(\tau_a \leq t, W_{\tau_a} \in \Gamma) &= \mathbb{E}_{s,(x,y)}\left[\mathbb{1}_{s\leq\tau_a\leq t, W_{\tau_a}\in\Gamma} D^{\alpha,\beta}_{s,(x,y)}(t)\right] \\
&= \mathbb{E}_{s,(x,y)}\left[\mathbb{1}_{s\leq\tau_a\leq t, W_{\tau_a}\in\Gamma} \mathbb{E}_{s,(x,y)}\left(D^{\alpha,\beta}_{s,(x,y)}(t)\middle|\mathcal{F}_{s\vee\tau_a,\, t\wedge\tau_a}\right)\right] \\
&= \mathbb{E}_{s,(x,y)}\left[\mathbb{1}_{s\leq\tau_a\leq t, W_{\tau_a}\in\Gamma} D^{\alpha,\beta}_{s,(x,y)}(t\wedge\tau_a)\right] \\
&= \mathbb{E}_{s,(x,y)}\left[\mathbb{1}_{s\leq\tau_a\leq t, W_{\tau_a}\in\Gamma} D^{\alpha,\beta}_{s,(x,y)}(\tau_a)\right] \\
&= \int_\Gamma \int_s^t D^{\alpha,\beta}_{s,(x,y)}(t')\mathbb{P}_{s,(x,y)}\left(\tau_a \in dt'\ ;\ W_{\tau_a} \in dz\right)
\end{aligned}$$

In going from the second to the third equation we used the fact that, according to Girsanov's theorem, $D^{\alpha,\beta}_{s,(x,y)}(t)$ is a martingale.

If $x \neq a$, the last probability is given by Lachal's formula (2.4) and gives the density of the hitting time of (τ_a, W_{τ_a}) under the new probability $\mathbb{P}^{\alpha,\beta}_{s,(x,y)}$:

$$D^{\alpha,\beta}_{s,(x,y)}(t)|z|\Big[p_t(x,y;a,z)- \int_0^t \int_0^{+\infty} \mathbb{P}_{(0,-|z|)}\{\tau_0 \in ds';\ W_{\tau_0} \in d\mu\} p_{t-s'}(x,y;a,-\varepsilon\mu)\Big] \mathbb{1}_A(z)$$

where $A = [0,\infty)$ if $x < a$ and $A = (-\infty, 0]$ if $x > a$, $\varepsilon = \text{sign}(a-x)$ and $\mathbb{P}_{(0,-|z|)}$ is given by McKean's formula (2.2).

We are interested in the probability density under $\mathbb{P}$ of the first hitting time of the curve $C(t-s)$. This hitting time reads:

$$\begin{aligned}
\tau_C &= \inf\Big\{t > s,\ X_t = C(t-s) \mid X_s = x,\, W_s = y\Big\} \\
&= \inf\Big\{t > s,\ X_t - \frac{\beta}{6}(t-s)^3 - \frac{\alpha}{2}(t-s)^2 = a \mid \\
&\quad X_s = x,\, W_s = y - b\Big\} \\
&= \inf\Big\{t > s,\ X_t^{-\alpha,-\beta} = a \mid X_s^{-\alpha,-\beta} = x,\, W_s^{-\alpha,-\beta} = y - b\Big\}
\end{aligned}$$

Hence τ_C under $\mathbb{P}_{s,x,y}$ has the same law as τ_a under $\mathbb{P}^{-\alpha,-\beta}_{s,x,y-b}$. The corresponding location of $W^{-\alpha,-\beta}_{\tau_a}$ is $W_{\tau_a} + b + \alpha(t-s) + \frac{\beta}{2}(t-s)^2$.

So eventually the law of τ_C, W_{τ_C} under $\mathbb{P}$ reads:

$$\begin{aligned}
\mathbb{P}_{s,(x,y)}(\tau_C \in dt,\, W_{\tau_C} \in dz) &= d^{-\alpha,-\beta}(s,\, x,\, y-b\ ;\ t,\, a,\, z - b - \alpha(\tau_a - s) - \frac{\beta}{2}(\tau_a - s)^2) \\
&\quad \times \mathbb{P}_{s,(x,y-b)}(\tau_a \in dt,\, W_{\tau_a} + b + \alpha(\tau_a - s) + \frac{\beta}{2}(\tau_a - s)^2 \in dz)
\end{aligned}$$

which is exactly (2.9).

If $x \neq a$, this formula reads:

$$\begin{aligned}
\mathbb{P}_{s,(x,y)}(\tau_C \in dt,\, W_{\tau_C} \in dz) &= d^{-\alpha,-\beta}(s,\, x,\, y-b\ ;\ t,\, a,\, z - b - \alpha(t-s) - \frac{\beta}{2}(t-s)^2) \\
&\quad \times l^a_{x,y-b}(t-s,\, z - b - \alpha(t-s) - \frac{\beta}{2}(t-s)^2)\, dt\, dz \qquad (2.10)
\end{aligned}$$

where $l^\alpha_{x,y}$ is Lachal's density (2.4).

If $x = a$ the same calculus using McKean's formula (2.2) gives us the formula of the hitting time probability density using the same method:

$$\mathbb{P}_{s,(x,y)}(\tau_C \in dt,\, W_{\tau_C} \in dz) = d^{-\alpha,-\beta}(s,\, x,\, y-b\,;\, t,\, a,\, z-b-\alpha(t-s)-\frac{\beta}{2}(t-s)^2)$$
$$\times\, m(t-s,\, y-b,\, z-b-\alpha(t-s)-\frac{\beta}{2}(t-s)^2)\, dt\, dz \quad (2.11)$$

□

3 Approximation of hitting times of the IWP to a general boundary

In this section we provide the formula of the first hitting time of the IWP to a piecewise cubic function and use it to compute an approximation formula of the law of the first hitting time of the IWP to a general smooth curved boundary. We also provide the convergence rate of this approximation. This approach in inspired the works [2, 26, 28, 35] on the first hitting time of the Brownian motion to a general boundary. In [28, 35], the authors use the formula of the first hitting time of the Brownian motion to an affine boundary to derive an approximation of the first hitting time of the Brownian motion to a curved boundary, approximated by a piecewise affine boundary. The rate of approximation is computed in [2]. This approach is here applied to our problem for the IWP. Here the boundary is approximated by a cubic spline, for which we compute an explicit formula for the pdf of the first hitting time of the IWP. We then compute the convergence speed to the first hitting time of the IWP to a general boundary of the first hitting time of a cubic spline approximation.

3.1 First hitting time to a continuous piecewise cubic function

In this section we consider the first hitting time of an IWP to a continuous piecewise cubic function $C(t)$ defined on the interval $[0,T]$:

$$C(t) = \sum_{i=0}^{n-1} \left(a_i + b_i(t-t_i) + \frac{\alpha_i}{2}(t-t_i)^2 + \frac{\beta_i}{6}(t-t_i)^3 \right) \mathbb{1}_{[t_i,t_{i+1})}(t) \quad (3.1)$$

The coefficients $\{(a_i, b_i, \alpha_i, \beta_i),\ i = 1 \dots n\}$ are constant on each interval $[t_i, t_{i+1})$, $i = 1, \cdots, n-1$. The continuity assumption [1] requires that

$$\forall i \in \{1, \dots, n-1\}\ \ a_{i+1} = a_i + b_i(t_{i+1}-t_i) + \frac{\alpha_i}{2}(t_{i+1}-t_i)^2 + \frac{\beta_i}{6}(t_{i+1}-t_i)^3$$

[1] This continuity assumption is not essential. Nevertheless we limit ourselves to a continuous boundary because it is sufficient to find good approximations of the first hitting time pdf with continuous functions, since we prove in theorem 3.2 that for Lipschitz continuous boundaries there exists a density for the first hitting time. If the boundary was not continuous, then the density function of the first hitting times would have atoms at the points of discontinuity of the boundary. This could be handled at the cost of an unnecessary increase in technical difficulty.

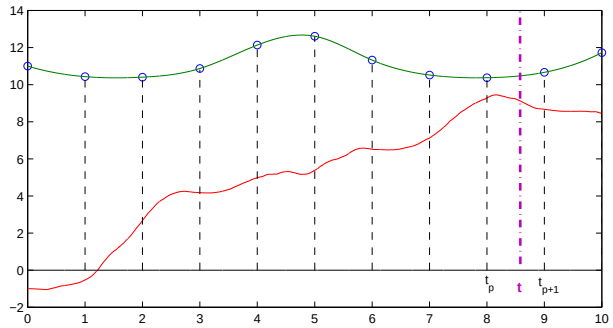


Figure 2: Principle of the proof of theorem 3.1: probability that the first hitting time is greater than $t \in [t_p, t_{p+1})$.

We denote by $(U_t)_{t\geq 0}$ the two-dimensional process $(X_t, W_t)_{t\geq 0}$ and assume that the starting point U_0 is fixed:

$$X_0 = x, W_0 = y$$

We recall that the process $(U_t)_t$ is strongly Markovian with transition density (1.11). We denote by τ_C^s the first hitting time of the process $(X_t)_{t\geq s}$ to the curve C before the time T:

$$\tau_C^s := \inf\{t > s;\ X_t = C(t)\}$$

Let us fix $t \in [0, T[$, and note p the index of the bin containing t (i.e. $t \in [t_p, t_{p+1})$). The principle of the proof is to use the strong Markov property of $(U_t)_t$ to express $\mathbb{P}(\tau_C^0 \geq t|U_0)$ recursively as an integral of a product of $p+1$ terms. p of these terms are related to the results of section 2.2 and their analytical expression is obtained from Theorem 2.2, see Fig. 2.

We know that τ_C^s is a stopping time under the filtration associated to U, which is strongly Markovian. The event $\{U_{t_1} = u_1, \tau_C^0 \geq t_1,\ U_0\}$ is in $\mathcal{F}^{U_{t_1}}$ therefore $\mathbb{P}\left(\tau_C^0 \geq t \middle| U_{t_1} = u_1, \tau_C^0 \geq t_1,\ U_0\right) = \mathbb{P}\left(\tau_C^{t_1} \geq t \middle| U_{t_1} = u_1\right)$. It follows that

$$\begin{aligned}\mathbb{P}\left(\tau_C^0 \geq t \middle| U_0\right) &= \int^{(2)} \mathbb{P}\left(\tau_C^0 \geq t \middle| U_{t_1} = u_1, \tau_C^0 \geq t_1,\ U_0\right) \mathbb{P}\left(U_{t_1} \in du_1,\ \tau_C^0 \geq t_1 \middle| U_0\right) \\ &= \int^{(2)} \mathbb{P}\left(\tau_C^{t_1} \geq t \middle| U_{t_1} = u_1\right) \mathbb{P}\left(U_{t_1} \in du_1,\ \tau_C^0 \geq t_1 \middle| U_0\right)\end{aligned}$$

The first term in this integral is similar to the lefthand side of the equation. By an immediate recursion we get:

$$
\begin{aligned}
\mathbb{P}\left(\tau_C^0 \geq t \middle| U_0\right) &= \int^{(4)} \mathbb{P}\left(\tau_C^{t_2} \geq t \middle| U_{t_2} = u_2\right) \\
&\quad \times \mathbb{P}\left(U_{t_2} \in du_2,\ \tau_C^{t_1} \geq t_2 | U_{t_1} = u_1\right) \\
&\quad \times \mathbb{P}\left(U_{t_1} \in du_1,\ \tau_C^0 \geq t_1 \middle| U_0\right) \\
\dots \\
&= \int^{(2p)} \mathbb{P}\left(\tau_C^{t_p} \geq t \middle| U_{t_p} = u_p\right) \\
&\quad \times \mathbb{P}\left(U_{t_p} \in du_p,\ \tau_C^{t_{p-1}} \geq t_p | U_{t_{p-1}} = u_{p-1}\right) \\
&\quad \times \mathbb{P}\left(U_{t_{p-1}} \in du_{p-1},\ \tau_C^{t_{p-2}} \geq t_{p-1} | U_{t_{p-2}} = u_{p-2}\right) \\
&\quad \times \dots \\
&\quad \times \mathbb{P}\left(U_{t_1} \in du_1,\ \tau_C^0 \geq t_1 \middle| U_0\right) \qquad (3.2)
\end{aligned}
$$

where $\int^{(N)}$ denotes an integral on $\mathbb{R}^N$. Note that the integration variables $(u_i)_{\{i=1,\dots p\}}$ are two-dimensional.

The terms in the product (3.2) are of the same kind. Their expression is given by theorem 3.1.

Indeed, since

$$
\{U_{t_k} \in du_k,\ \tau_C^{t_{k-1}} \geq t_k\} = \{U_{t_k} \in du_k\} \setminus \left\{U_{t_k} \in du_k,\ \tau_C^{t_{k-1}} < t_k\right\},
$$

where $\setminus$ is the set difference, we have

$$
\begin{aligned}
&\mathbb{P}\Big(U_{t_k} \in du_k,\ \tau_C^{t_{k-1}} \geq t_k | U_{t_{k-1}} = u_{k-1}\Big) \\
&\quad = \mathbb{P}\Big(U_{t_k} \in du_k | U_{t_{k-1}} = u_{k-1}\Big) - \mathbb{P}\Big(U_{t_k} \in du_k,\ \tau_C^{t_{k-1}} \leq t_k | U_{t_{k-1}} = u_{k-1}\Big) \\
&\quad = \mathbb{P}\Big(U_{t_k} \in du_k | U_{t_{k-1}} = u_{k-1}\Big) - \int_{t_{k-1}}^{t_k} \mathbb{P}\Big(U_{t_k} \in du_k, \tau_C^{t_{k-1}} \in ds | U_{t_{k-1}} = u_{k-1}\Big) \\
&\quad = \mathbb{P}\Big(U_{t_k} \in du_k | U_{t_{k-1}} = u_{k-1}\Big) \\
&\qquad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \mathbb{P}\Big(U_{t_k} \in du_k | \tau_C^{t_{k-1}} = s,\ W_s = y,\ U_{t_{k-1}} = u_{k-1}\Big) \\
&\qquad \times \mathbb{P}\Big(\tau_C^{t_{k-1}} \in ds,\ W_s \in dy \Big| U_{t_{k-1}} = u_{k-1}\Big) \\
&\quad = \Big(p_{t_k - t_{k-1}}(u_k; u_{k-1}) \\
&\qquad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s}(u_k; C(s), y) \mathbb{P}\left(\tau_C^{t_{k-1}} \in ds,\ W_s \in dy \Big| U_{t_{k-1}} = u_{k-1}\right)\Big) du_k
\end{aligned}
$$

where $p_t(x, y; u, v)$ is the transition density function (1.11) of the process U. The curve C on the interval $[t_{k-1}, t_k)$ is a fixed cubic function; The hitting time of the IWP starting at u_{k-1} to C has a known density computed in section 2.2 and the term we are interested in can be deduced from the expression we derived previously.

Hence we have proved the following theorem:

Theorem 3.1. *The law of the first hitting time of the IWP to a continuous piecewise cubic boundary is given by the formula:*

$$
\begin{aligned}
\mathbb{P}\Big(\tau_C^0 \geq t \Big| U_0\Big) = \int^{(2p)} \mathbb{P}\big(\tau_C^{t_p} \geq t | U_{t_p} = u_p\big) \prod_{k=1}^{p} \Bigg(& p_{t_k - t_{k-1}}(u_k; u_{k-1}) \\
& - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s}(u_k; C(s), y) \mathbb{P}\left(\tau_C^{t_{k-1}} \in ds,\ W_s \in dy \Big| U_{t_{k-1}}\right) \Bigg) du_k \qquad (3.3)
\end{aligned}
$$

3.2 Approximation of the first hitting time to a general boundary

In this section we derive an approximation of the first hitting time before a given time T of the IWP to a general smooth boundary using the results of the previous section.

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuously differentiable function. Let also $T > 0$ and

$$0 = t_0 < t_1 < \ldots < t_n = T$$

be a partition, noted π, of the interval $[0, T]$. We denote by $\delta(\pi)$ the mesh step defined as:

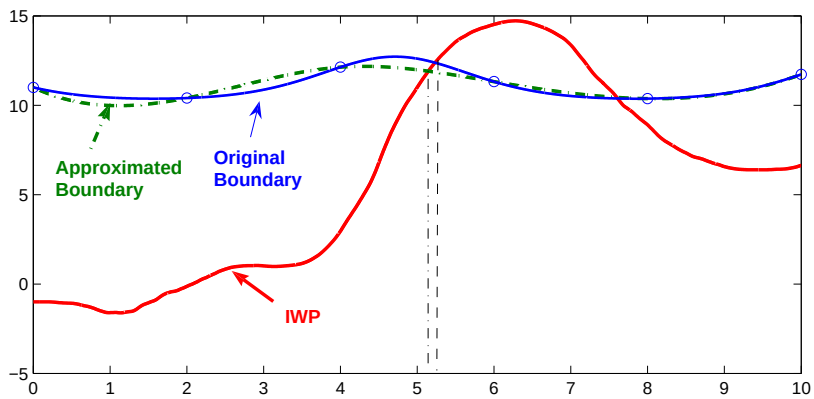


Figure 3: Principle of the approximation method: approximating the boundary by a regular piecewise cubic function

$$\delta(\pi) = \max\{t_{i+1} - t_i,\ i = 0 \dots n-1\}$$

The principle of the method is to approximate the first hitting time of the IWP to the boundary f by the first hitting time of the IWP to a smooth piecewise cubic function C_π (see fig. 3). The constraints we impose to C_π is to pass through the control points $\{(t_i,\ f(t_i)),\ i = 1 \dots n\}$ and to be at least continuously differentiable. There are several ways for defining it, see, e.g., [32]. We assume for simplicity, but it is not essential here, that f is either C^2 or C^4. One of the most popular interpolation schemes in the second case is provided by the cubic spline that yields a C^2 interpolation of f which is an approximation of order four, i.e.

$$\sup_{t\in[0,T]} |f(t) - C_\pi(t)| \leq K(f)\delta(\pi)^4, \tag{3.4}$$

where $K(f)$ is a function of f only.

$C_\pi(t)$ is therefore given by (3.1), where the coefficients a_i, b_i, α_i, and β_i are functions of f and provided by the particular interpolation scheme one uses, see, e.g., [32].

We first prove the following

Theorem 3.2. *The first hitting time of the IWP to a Lipschitz continuous boundary has a density with respect to Lebesgue's measure.*

Proof. We assume that f is Lispschitz continuous. Let L denote the Lipschitz continuity constant of f, we have:

$$f(s+h) - f(s) \geq -Lh \ \ \forall\, 0 < s < s+h < t$$

For a fixed $t \in [0,T]$, we introduce the boundary $f_t(s) := f(t) + L(t-s)$. Let now

$$\begin{aligned}\tau_t &:= \inf\{s>0;\ X_s > f_t(s) | X_0 = x_0, W_0 = y_0\}\\ &= \inf\{s>0;\ X_s + Ls > f(t) + Lt | X_0 = x_0, W_0 = y_0\}\\ &= \inf\{s>0;\ X_s > f(t) + Lt | X_0 = x_0, W_0 = y_0 + L\}\end{aligned}$$

Obviously, we have:

$$\mathbb{P}(\tau_f \in (t, t+h)) \leq \mathbb{P}(\tau_t \in (t, t+h)) \tag{3.5}$$

and hence the stopping time τ_f has a density $p(t)$ with respect to Lebesgue's measure. From Lachal's formula we have moreover:

$$p(t) \leq \int_{\mathbb{R}} l_{x_0, y_0+L}^{f(t)+Lt}(t,z)dz$$

□

We now relate the first hitting time to the cubic approximation C_π to that to the general boundary f.

Theorem 3.3. *The first hitting time of the IWP to the curve C_π before T converges in law to the first hitting time of the IWP to the curve f before T.*

Furthermore, if f is C^2, then this convergence is the same order as the approximation of f by the cubic function C_π. More precisely, for a real function g, if $P(T,g)$ denotes the probability $\mathbb{P}\Big(X_t \geq g(t)$ for some $t \in [0,T]\Big)$, there exists a constant $\tilde{K}(f,T)$ depending on the function f and the time T such that:

$$\boxed{|P(T, C_\pi) - P(T,f)| \leq \tilde{K}(f,T)\ \|f - C_\pi\|_{\infty,T}} \tag{3.6}$$

where $\|g\|_{\infty,T} = \sup_{s\in[0,T]} |g(s)|$ is the uniform norm on $[0,T]$.

To prove this theorem, we use the following lemma giving the density of the random variable $\sup_{s\in[0,t]} X_s$.

Lemma 3.4. Let $t > 0$ be a fixed real and S_t be the random variable defined by:

$$S_t = \sup_{s\in[0,t]} X_s$$

Then the law of this random variable is characterized by:

$$\begin{aligned}\mathbb{P}_{0,x,y}\Big(S_t \in da\Big) = -da \int_{-\infty}^{\infty}\int_0^t |z| \Big\{ &\frac{\partial p_s(x,y;a,z)}{\partial a}\\ &- \int_0^s \int_0^{+\infty} \mathbb{P}_{(0,-|z|)}\{\tau_0 \in du;\ W_{\tau_0} \in d\mu\} \frac{\partial p_{s-u}(x,y;a,-\varepsilon\mu)}{\partial a}\Big\} \mathbb{1}_A(z)dzdt\end{aligned} \tag{3.7}$$

Proof of lemma 3.4. We have:

$$\begin{aligned}\mathbb{P}_{0,x,y}\Big(S_t \geq a\Big) &= \mathbb{P}_{0,x,y}\Big(\tau_a \leq t\Big)\\ &= \int_0^t \int_{-\infty}^{\infty} l^a_{x,y}(t,z)dtdz\end{aligned}$$

where $l^a_{x,y}(t,z)$ is Lachal's density. From the expression of this density and its dependency in a, we can see that the random variable S_t has a density with respect to Lebesgue's measure, and this measure has the expression (3.7) (this formula is obtained using Lebesgue's theorem of derivation under the sum sign). □

Proof of theorem 3.3. This result comes directly from the existence of a smooth density of the random variable S_t, the uniform convergence of the curve C_π to f and the smoothness of the function f. Let U_0 be the initial conditions: $(X_0 = x, W_0 = y)$, such that $x < f(0)$.

We have:

$$P(T, f+\varepsilon) \leq P(T,f) \leq P(T, f-\varepsilon)$$

Assume that f_ε is a uniform approximation of f such that

$$\|f_\varepsilon - f\|_{\infty,T} \leq \varepsilon.$$

We have:

$$\begin{aligned}|P(T,f_\varepsilon) - P(T,f)| &\leq P(T,f-\varepsilon) - P(T,f+\varepsilon)\\ &= P(T,f-\varepsilon) - P(T,f) + P(T,f) - P(T,f+\varepsilon)\\ &= \mathbb{P}\Big(-\varepsilon \leq \sup_{s\in[0,t]}\big(X_s - f(s)\big) \leq 0\Big) + \mathbb{P}\Big(0 \leq \sup_{s\in[0,t]}\big(X_s - f(s)\big) \leq \varepsilon\Big)\\ &\stackrel{\text{def}}{=} \Delta_{-\varepsilon}(f) + \Delta_\varepsilon(f) \qquad (3.8)\end{aligned}$$

We have:

$$\begin{aligned}\Delta_\varepsilon(f) &= \mathbb{P}\Big(0 \leq \sup_{s\in[0,t]}\big(X_s - f(s)\big) \leq \varepsilon \Big| U_0\Big)\\ &= \int_0^t \int_{\mathbb{R}} \mathbb{P}\left(\tau_f \in ds,\ W_{\tau_f} \in dz \Big| U_0\right) \mathbb{P}\left(\sup_{v\in[s,t]}(X_v - f(v)) \leq \varepsilon \Big| X_s = f(s),\ W_s = z\right)\\ &= \int_0^t \int_{f'(s)}^{\infty} \mathbb{P}\left(\tau_f \in ds,\ W_{\tau_f} \in dz \Big| U_0\right) \mathbb{P}\left(\sup_{v\in[s,t]}(X_v - f(v)) \leq \varepsilon \Big| X_s = f(s),\ W_s = z\right)\end{aligned}$$

The last equality is a consequence of the fact that $\mathbb{P}\left(W_{\tau_f} > f'(\tau_f)\right) = 1$, i.e., $z \geq f'(s)$ almost surely. We can conclude to the convergence of the approximation as an application

of Lebesgue's theorem. Indeed, let $s \in [0,t]$ and $z > f'(s)$. Then the process $(X_t - f(t))_{t \geq s}$ conditioned by $\{X_s = f(s), W_s = z\}$ is a differentiable process starting from 0 with a strictly positive derivative at 0 implying that

$$\mathbb{P}\left(\sup_{v \in [s,t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s),\ W_s = z \right) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

Furthermore, $\mathbb{P}\left(\sup_{v \in [s,t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s),\ W_s = z\right) \leq 1$ which is integrable under the measure $\mathbb{P}\left(\tau_f \in ds,\ W_{\tau_f} \in dz \middle| U_0\right)$ so Lebesgue's theorem applies and we have the expected result:

$$\Delta_\varepsilon(f) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

The same argument applies for the term $\Delta_{-\varepsilon}(f)$ of equation (3.8). Indeed, we can bound this probability for $\varepsilon \leq \frac{f(0)-x}{2}$ by:

$$\begin{aligned}
\Delta_{-\varepsilon}(f) &= \mathbb{P}\Big(-\varepsilon \leq \sup_{s \in [0,t]} X_s - f(s) \leq 0 \Big| X_0 = x, W_0 = y \Big) \\
&= \mathbb{P}\Big(0 \leq \sup_{s \in [0,t]} X_s - f(s) \leq \varepsilon \Big| X_0 = x + \varepsilon, W_0 = y \Big) \\
&\leq \mathbb{P}\Big(0 \leq \sup_{s \in [0,t]} X_s - f(s) \leq \varepsilon \Big| X_0 = \frac{x + f(0)}{2}, W_0 = y \Big) \qquad (3.9) \\
&\underset{\varepsilon \to 0}{\longrightarrow} 0
\end{aligned}$$

We obtain a stronger result if the boundary f is C^2. Lemma 3.4 ensures us that the random variable $\sup_{s \in [0,t]} X_s$ has a smooth density with respect to Lebesgue's measure. From Girsanov's theorem, under the probability $\mathbb{Q}$ defined by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\Big[\int_0^t f''(s)\, dW_s - \frac{1}{2} \int_0^t f''(s)^2\, ds \Big] \overset{\text{def}}{\equiv} L_s$$

the process $\sup_{v \in [s,t]} (X_v - f(v))$ has the law of the sup over $[s,t]$ of an IWP, which is given by (3.7), let us note it $p_L(s)$. For $z > f'(s)$ the probability that appears at the righthand side of (3.9) is equal to

$$\begin{aligned}
\int_0^\varepsilon p_L(s) \mathbb{E}_{\mathbb{Q}}(L_s^{-1})\, ds &= \int_0^\varepsilon p_L(s) \mathbb{E}_{\mathbb{Q}}(\tilde{L}_s) \exp\Big[\int_0^s f''(u)^2\, du \Big]\, ds \\
&= \int_0^\varepsilon p_L(s) \exp\Big[\int_0^s f''(u)^2\, du \Big]\, ds,
\end{aligned}$$

since $\tilde{L}_s = \exp\Big[-\int_0^t f''(s)\, dW_s - \frac{1}{2} \int_0^t f''(s)^2\, ds \Big]$ is a martingale. The last integral is $O(\varepsilon)$.

So eventually, using again the same bound as (3.9), we have the expected result: there exists a constant $\tilde{\tilde{K}}(f,T)$ such that

$$|P(T,f_\varepsilon) - P(T,f)| \leq \tilde{\tilde{K}}(f,T)\varepsilon$$

Finally, if f is C^4 and C_π is a cubic spline interpolation of f we have the convergence estimation (3.4) which yields

$$\begin{aligned} |P(T,C_\pi) - P(T,f)| &\leq 2\tilde{\tilde{K}}(f,T)\|C_\pi - f\|_\infty \\ &\leq 2\tilde{K}(f,T)\delta(\pi)^4 \end{aligned}$$

□

4 Approximation of hitting times of a general DIP to a general boundary

In this section we derive an approximation formula for the probability density function of the first hitting time of a general double integral process to a general smooth boundary. Here again the idea is to use the formulas we obtained in the section 2 to build approximations on a partition of a given time interval $[0,T]$ (see Fig.4).

Thanks to lemma 1.3, we can restrict ourselves to the first hitting time of a process Y defined by:

$$Y_t = \int_0^t g(s)W_s\,ds \tag{4.1}$$

where $g(\cdot)$ is a continuously differentiable function and W a standard Brownian motion. Let π be as before a partition of the interval $[0,T]$ with n intervals:

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T$$

We denote by g^π the piecewise constant approximation of g defined by:

$$g^\pi(t) = \sum_{i=0}^{n-1} g(t_i)\mathbb{1}_{[t_i,t_{i+1})}(t), \tag{4.2}$$

and by Y^π the associated DIP:

$$Y_t^\pi = \int_0^t g^\pi(s)W_s\,ds. \tag{4.3}$$

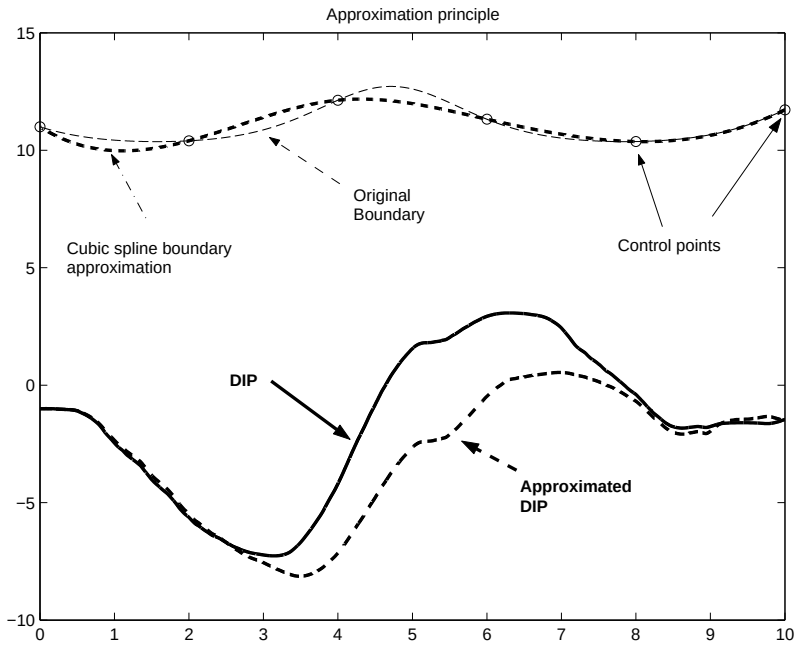


Figure 4: Approximation principle for the DIP: The boundary is approximated by a smooth piecewise cubic boundary while the process is approximated by a continuous piecewise IWP process

Proposition 4.1. The process Y_t^π converges almost surely to the process Y_t. Furthermore, there exists a real positive process Z_t such that:

$$\sup_{0\le s\le t} |Y_s^\pi - Y_s| \le \delta(\pi)\ Z_t \tag{4.4}$$

Proof.

$$\begin{aligned}\sup_{0\le s\le t} |Y_s^\pi - Y_s| &= \sup_{0\le s\le t} \left| \int_0^s (g^\pi(u) - g(u)) W_u\, du \right| \\ &\le \sup_{0\le s\le t} \int_0^s |g^\pi(u) - g(u)||W_u|\, du \\ &\le \left(\sup_{0\le s\le t} |g^\pi(u) - g(u)| \right) \sup_{0\le s\le t} \int_0^s |W_u|\, du\end{aligned}$$

We assumed that g was continuously differentiable on $\mathbb{R}^+$, so it is uniformly Lipschitz on $[0,T]$. We note

$$\|g'\|_{\infty,T} = \sup_{0\le s\le T} |g'(s)|$$

the uniform Lipschitz constant. We eventually have:

$$\sup_{0\le s\le t} |Y_s^\pi - Y_s| \le \delta(\pi)\ \|g'\|_{\infty,T} \sup_{0\le s\le t} \int_0^s |W_u|\, du$$

The process $Z_t := \sup_{0\le s\le t} \int_0^s |W_u|\, du$ is almost surely finite, thus the process Y^π converges almost surely to Y when $\delta(\pi) \to 0$ and we have the upper bound (4.4). □

Let now f be a smooth function (at least two times continuously differentiable) and C^π the approximating function (3.1).

Theorem 4.2. *The first hitting time τ^π of the process Y^π to the curve C^π converges in law to the first hitting time τ_f of the process Y to the curve f.*

Proof. A sufficient condition for this convergence is the convergence in law of the process $\sup_{s\in[0,t]}(Y_s^\pi - C_s^\pi)$ to the process $\sup_{s\in[0,t]}\big(Y_s - f(s)\big)$, and this convergence is a direct consequence of the calculations above. Here we only assume that f is continuously differentiable (the corresponding piecewise cubic approximation would be for example the Hermite cubic approximation, see [32]).

In this case, C_t^π converges to f linearly, i.e., there exists a constant $K(f) > 0$ depending on f such that:

$$\sup_{t\in[0,T]} |f(t) - C_\pi(t)| \le K(f)\delta(\pi) \tag{4.5}$$

We have:

$$\sup_{s\in[0,t]}\Big(Y_s^\pi - C_s^\pi\Big) \le \sup_{s\in[0,t]}\Big(Y_t^\pi - Y_t\Big) + \sup_{s\in[0,t]}\Big(f(t) - C_t^\pi\Big) + \sup_{s\in[0,t]}\Big(Y_t - f(t)\Big)$$
$$\le \delta(\pi)Z_t + K(f)\delta(\pi) + \sup_{s\in[0,t]}\Big(Y_t - f(t)\Big)$$

Writing the same estimation on $\sup_{s\in[0,t]}\Big(Y_s - f(s)\Big)$ yields:

$$\Big|\sup_{s\in[0,t]}\Big(Y_s^\pi - C_s^\pi\Big) - \sup_{s\in[0,t]}\Big(Y_t - f(t)\Big)\Big| \le \delta(\pi)\Big(Z_t + K(f)\Big)$$

Hence we have the expected result:

$$\mathbb{P}\Big(\tau^\pi \le t\Big) = \mathbb{P}\Big(\sup_{s\in[0,t]}\big(Y_s^\pi - C_s^\pi\big) > 0\Big)$$
$$\underset{\delta(\pi)\to 0}{\longrightarrow} \mathbb{P}\Big(\sup_{s\in[0,t]}\big(Y_s - f(s)\big) > 0\Big) = \mathbb{P}\Big(\tau_f \le t\Big)$$

□

We now compute the formula of the probability density function of the first crossing time of Y^π to the curve C^π, as we did in the section 3.

We consider π a fixed partition of the interval $[0,T]$. We note τ_π^s the first hitting time after time s of the process Y^π to the curve C^π and U^π the two-dimensional process (Y_t^π, W_t). The same proof as in section 3 gives us a formula analog to (3.2):

$$\begin{aligned}\mathbb{P}\Big(\tau_\pi^0 \ge t\Big|U_0^\pi\Big) = \int^{(2p)} & \mathbb{P}\Big(\tau_\pi^{t_p} \ge t\Big|U_{t_p}^\pi = u_p\Big)\\ & \times \mathbb{P}\Big(U_{t_p}^\pi \in du_p,\ \tau_\pi^{t_{p-1}} \ge t_p|U_{t_{p-1}}^\pi = u_{p-1}\Big)\\ & \times \mathbb{P}\Big(U_{t_{p-1}}^\pi \in du_{p-1},\ \tau_\pi^{t_{p-2}} \ge t_{p-1}|U_{t_{p-2}}^\pi = u_{p-2}\Big)\\ & \times \dots\\ & \times \mathbb{P}\Big(U_{t_1}^\pi \in du_1\Big|U_0^\pi\Big)\end{aligned} \tag{4.6}$$

Here again the terms in the recursion formula are of the same kind, and the only quantity we need to calculate are the conditional probabilities $\mathbb{P}\Big(U_{t_k}^\pi \in du_k,\ \tau_\pi^{t_{k-1}} \ge t_k|U_{t_{k-1}}^\pi = u_{k-1}\Big)$ for $k = 1, \cdots, p$.

Note that in the interval $[t_k, t_{k+1})$, the process Y_t^π reads:

$$Y_t^\pi = Y_{t_i}^\pi + g(t_k)(X_t - X_{t_k}), \tag{4.7}$$

and that $(X_t - X_{t_k})_{t \geq t_k}$ is simply an IWP starting from 0. We then have

$$\begin{aligned}
&\mathbb{P}\Big(U^\pi_{t_k} \in du_k,\ \tau^{t_{k-1}}_\pi \geq t_k | U^\pi_{t_{k-1}} = u_{k-1}\Big) \qquad (4.8)\\
&= \mathbb{P}\Big(U^\pi_{t_k} \in du_k | U^\pi_{t_{k-1}} = u_{k-1}\Big) - \mathbb{P}\Big(U^\pi_{t_k} \in du_k,\ \tau^{t_{k-1}}_\pi \leq t_{p-1} | U^\pi_{t_{k-1}} = u_{k-1}\Big)\\
&= \mathbb{P}\Big(U^\pi_{t_k} \in du_k | U^\pi_{t_{k-1}} = u_{k-1}\Big) - \int_{t_{k-1}}^{t_k} \mathbb{P}\Big(U^\pi_{t_k} \in du_k, \tau^{t_{k-1}}_\pi \in ds | U^\pi_{t_{k-1}} = u_{k-1}\Big)\\
&\quad \times \mathbb{P}\Big(U^\pi_{t_k} = du_k | U^\pi_{t_{k-1}} = u_{k-1}\Big)\\
&\quad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \mathbb{P}\Big(U^\pi_{t_k} \in du_k | \tau^{t_{k-1}}_\pi = s,\ W_s = y,\ U^\pi_{t_{k-1}} = u_{k-1}\Big)\\
&\quad \times \mathbb{P}\Big(\tau^{t_{k-1}}_\pi \in ds,\ W_s \in dy |\ U^\pi_{t_{k-1}} = u_{k-1}\Big)\\
&= du_k\Big(\tilde{p}(t_k,\ u_k; t_{k-1},\ u_{k-1}) - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \tilde{p}(t_k,\ u_k; s, (C^\pi(s), y))\\
&\quad \times \mathbb{P}\Big(\tau^{t_{k-1}}_\pi \in ds,\ W_s \in dy |\ U^\pi_{t_{k-1}} = u_{k-1}\Big)\Big) \qquad (4.9)
\end{aligned}$$

where $\tilde{p}(t, x, y; s, u, v)$ is the transition function of the process U^π (for $t \geq s$). This function can be deduced from (1.11) and (4.7) for s and t in the same bin $[t_k, t_{k+1})$ and reads:

$$\begin{aligned}
\tilde{p}(t, x, y; s, u, v) &= \mathbb{P}\Big(Y^\pi_t = x, W_t = y \Big| Y^\pi_s = u, W_s = v\Big)\\
&= \mathbb{P}\Big(Y^\pi_s + g(t_k)(X_t - X_s) = x, W_t = y \Big| Y^\pi_s = u, W_s = v\Big)\\
&= \mathbb{P}\Big(X_t - X_s = \frac{x-u}{g(t_k)}, W_t = y \Big| Y^\pi_s = u, W_s = v\Big)\\
&= \mathbb{P}\Big(X_t - X_s = \frac{x-u}{g(t_k)}, W_t - W_s = y - v \Big| Y^\pi_s = u, W_s = v\Big)
\end{aligned} \qquad (4.10)$$

We have seen in a remark in section 1.2 that conditionally to W_s the increments of the two-dimensional process are independent of $\mathcal{F}_s$, so we have:

$$\begin{aligned}
\tilde{p}(t, x, y; s, u, v) &= \mathbb{P}\Big(X_{t-s} = \frac{x-u}{g(t_k)}, W_{t-s} = y - v \Big| X_0 = 0, W_0 = 0\Big)\\
&= p_{t-s}\left(\frac{x-u}{g(t_k)}, y - v; 0, 0\right) \qquad (4.11)
\end{aligned}$$

Hence the general term (4.9) of the expansion reads:

$$dx_k\, dy_k \Big\{ p_{t_k - t_{k-1}} \left(\frac{x_k - x_{k-1}}{g(t_{k-1})}, y_k - y_{k-1}; 0, 0 \right) - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s} \left(\frac{x_k - C^\pi(s)}{g(t_{k-1})}, y_k - y; 0, 0 \right) \mathbb{P}\Big(\tau^\pi \in ds,\ W_s \in dy |\ U^\pi_{t_{k-1}} = u_{k-1} \Big) \Big\} \quad (4.12)$$

The last thing to compute is $\mathbb{P}\Big(\tau^\pi \in ds,\ W_s \in dy |\ U^\pi_{t_{k-1}} = u_{k-1} \Big)$ the law of the hitting time τ^π in a given bin $[t_k, t_{k-1})$, which appears in the expression (4.9). This law can be deduced from that of the first hitting time of the IWP using the formula (4.7).

Indeed, to compute this probability, we use the fact that conditionally on the event $\{U^\pi_{t_{k-1}} = u_{k-1}, t \geq t_{k-1}\}$ we have:

$$\begin{aligned}
\tau^\pi &= \inf\{ s > t_{k-1},\ Y^\pi_s = C^\pi(s) | U^\pi_{t_{k-1}} = u_{k-1} \} \\
&= \inf\{ s > t_{k-1},\ Y^\pi_{t_{k-1}} + g(t_{k-1})(X_s - X_{t_{k-1}}) = C^\pi(s) | U^\pi_{t_{k-1}} = u_{k-1} \} \\
&= \inf\{ s > t_{k-1},\ X_s - X_{t_{k-1}} = \frac{C^\pi(s) - Y^\pi_{t_{k-1}}}{g(t_{k-1})} | U^\pi_{t_{k-1}} = u_{k-1} \} \\
&= \inf\{ s > 0,\ \widetilde{X}_s = \frac{C^\pi(s) - Y^\pi_{t_{k-1}}}{g(t_{k-1})} | \widetilde{X}_0 = 0, \widetilde{W}_0 = u_{t_{k-1},2} \}
\end{aligned}$$

where $(\widetilde{X}_t, \widetilde{W}_t)_t$ is a standard IWP and $u_{t_{k-1},2}$ is the second component of $U^\pi_{t_{k-1}}$. Hence we have:

$$\mathbb{P}\Big(\tau^\pi \in ds,\ W_s \in dy |\ U^\pi_{t_{k-1}} = u_{k-1} \Big) = \mathbb{P}_{s,(0,u_{t_{k-1},2})} \Big(\tau_{(C - Y^\pi_{t_{k-1}})/g(t_{k-1})} \in ds,\ W_s \in dy \Big) \quad (4.13)$$

and the last expression is given by (2.10) and (2.11).

We have proved the following

Theorem 4.3. *Let g be a Lipschitz continuous real function, $T > 0$ and π a partition of the interval $[0, T]$*

$$0 = t_0 < t_1 < \ldots < t_n = T$$

Let f be a continuously differentiable function. The first hitting time τ^π of the approximated process Y^π defined by (4.3) to a cubic spline approximation of f on the partition π, denoted by f^π, satisfies the equation:

$$\mathbb{P}(\tau^\pi \geq T | U_0) = \int^{(2n)} \prod_{k=1}^{n} \Big\{ p_{t_k - t_{k-1}} \Big(\frac{x_k - x_{k-1}}{g(t_{k-1})}, y_k - y_{k-1}; 0, 0 \Big) - \int_{t_k - 1}^{t_k} \int_{\mathbb{R}} p_{t_k - s} \Big(\frac{x_k - f^\pi(s)}{g(t_{k-1})}, y_k - y; 0, 0 \Big) \mathbb{P}_{s,(0,y)} (\tau_{(C - x_{k-1})/g(t_{k-1})} \in ds,\ W_s \in dy) \Big\} dx_k dy_k \quad (4.14)$$

where $\mathbb{P}(\tau_C \in ds,\ W_s \in dy)$ *is given by equations* (2.11) *or* (2.10).

5 Numerical Evaluation

5.1 Algorithm

In this section we propose an algorithm to evaluate the approximation formula we derived in the previous section. The expressions we found for the first hitting time involve integrals on $\mathbb{R}^{2n}$ when there are $n+1$ points in the mesh, which have no closed-form expression. Numerical computation of these integrals can be quite intricate and time consuming, so a numerical approximation is needed and another approximation is done besides (3.6) and theorem 4.2. The principle of the numerical approximation we propose is to express this integral as an expectation over a certain probability measure and to use a Monte-Carlo algorithm to compute this expectation. The accuracy of this approximation can be assessed through standard procedures for Monte Carlo simulations [22, 31].

Corollary 5.1 (of theorem 3.1)**.** Let $(X_t, W_t)_{t\geq 0}$ be a standard IWP-Brownian motion pair and f a smooth boundary function. The law of the first hitting time τ of X to f satisfies for $t \in [t_{p-1}, t_p)$:

$$\mathbb{P}\Big(\tau_C \geq t \Big| U_0\Big) = \mathbb{E}\Big[h_p(\pi, t, X_{t_1}, W_{t_1}, \ldots, X_{t_{p-1}}, W_{t_{p-1}}, X_t, W_t)\Big| U_0\Big] \tag{5.1}$$

where the function h_p is defined for $t \in [t_{p-1}, t_p)$ by:

$$\begin{aligned} h_p(\pi, t, x_1, y_1, \ldots, x, y) := \prod_{k=1}^{p-1} \Bigg(1 - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \frac{p_{t_k - s}(x_k, y_k, C(s), z)}{p_{t_k - t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} \\ \times \mathbb{P}\Big(\tau_C \in ds,\ W_{\tau_C} \in dz \Big| X_{t_{k-1}} = x_{k-1}, W_{t_{k-1}} = y_{k-1}\Big)\Bigg) \\ \times \Bigg(1 - \int_{t_{p-1}}^{t} \int_{\mathbb{R}} \frac{p_{t_{p-1} - s}(x, y, C(s), z)}{p_{t - t_{p-1}}(x, y, x_{p-1}, y_{p-1})} \mathbb{P}\Big(\tau_C \in ds,\ W_{\tau_C} \in dz \Big| X_{t_{p-1}} = x_{p-1}, W_{t_{p-1}} = y_{p-1}\Big)\Bigg) \end{aligned} \tag{5.2}$$

The same method can be applied for the DIP for a given function g.

Corollary 5.2 (of theorem 4.3)**.** Let g be a Lipschitz continuous real function, $(X_t, W_t)_{t\geq 0}$ be a standard IWP-Brownian motion pair, $T > 0$ and π a partition of the interval $[0, T]$

$$0 = t_0 < t_1 < \ldots < t_n = T$$

Let f be a continuously differentiable function. The first hitting time τ^π of the approximated process Y^π defined by (4.3) to a cubic spline approximation of f on the partition π, denoted f^π, can be computed as the expectation:

$$\mathbb{P}\Big(\tau^\pi \geq t \Big| U_0\Big) = \mathbb{E}\Big[h_p^{g,\pi}(t, X_{t_1}, Wt_1, \ldots, X_t, W_t)\Big| U_0\Big] \tag{5.3}$$

where the function $h_p^{g,\pi}$ is defined by:

$$\begin{aligned} h_p^{g,\pi}(x_1, y_1 \ldots, x, y) := \prod_{k=1}^{p-1} \Bigg\{ & \frac{p_{t_k - t_{k-1}}\Big(\frac{x_k - x_{k-1}}{g(t_{k-1})}, y_k - y_{k-1}; 0, 0\Big)}{p_{t_k - t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \\ & \frac{p_{t_k - s}\Big(\frac{x_k - f^\pi(s)}{g(t_{k-1})}, y_k - z; 0, 0\Big)}{p_{t_k - t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} \mathbb{P}_{s,(0,y_s)}(\tau_{(C - x_{k-1})/g(t_{k-1})} \in ds,\ W_s \in dz) \Bigg\} \\ & \times \Bigg(\frac{p_{t - t_{p-1}}\big(\frac{x - x_{p-1}}{g(t_{p-1})}, y - y_{p-1}; 0, 0\big)}{p_{t - t_{p-1}}(x, y, x_{p-1}, y_{p-1})} \\ & - \int_{t_{p-1}}^{t} \int_{\mathbb{R}} \frac{p_{t-s}\Big(\frac{x - f^\pi(s)}{g(t_{p-1})}, y - z; 0, 0\Big)}{p_{t - t_{p-1}}(x, y, x_{p-1}, y_{p-1})} \mathbb{P}_{s,(0,z)}(\tau_{(C - x_{p-1})/g(t_{p-1})} \in ds,\ W_s \in dz) \Bigg) \end{aligned} \tag{5.4}$$

where $\mathbb{P}(\tau_C \in ds,\ W_s \in dy_s)$ is given by equations (2.11) or (2.10).

Hence the problem is now reduced to the computation of the expectation of a certain function of the Gaussian random vector $(X_0, W_0, X_{t_1}, W_{t_1}, \ldots, X_{t_n}, W_{t_n})$. This vector is Gaussian of mean 0 and covariance matrix defined by blocks as:

$$K(t_1, \ldots, t_n) = \begin{pmatrix} \frac{t_j^2}{6}(3t_i - t_j) & \frac{t_j}{2}(2t_i - t_j) \\ \frac{t_j^2}{2} & t_j \end{pmatrix}_{(i,j) \in \{0 \ldots n\},\, j \leq i}, \tag{5.5}$$

The Monte-Carlo algorithm we use to compute the expected probability is the following:

(i). Compute the square root $K(t_1, \ldots, t_n)^{1/2}$ of the covariance matrix (5.5) (using for instance a Cholesky decomposition)

(ii). Generate an i.i.d. sample $u = [u_1, u_2, \ldots, u_{2n}]^T$ from the normal standard distribution $\mathcal{N}(0, 1)$.

(iii). Compute the transformation $x = K(t_1, \ldots, t_n)^{1/2} \cdot u$

(iv). Calculate $h_n(x)$ or $h_n^{g,\pi}(x)$

(v). Repeat steps (ii)-(iv) N times and calculate the frequency

$$P_N = \frac{1}{N} \sum_{\text{over the realisations}} h_n(x)(\text{resp. } h_n^{g,\pi}(x))$$

The probability $\mathbb{P}(\tau \geq T)$ is then estimated by P_N. The standard error of this estimator is given by :

$$\mathcal{E}(N) = \sqrt{\frac{\sum \left[h(x) - P_N\right]^2}{N(N-1)}} \tag{5.6}$$

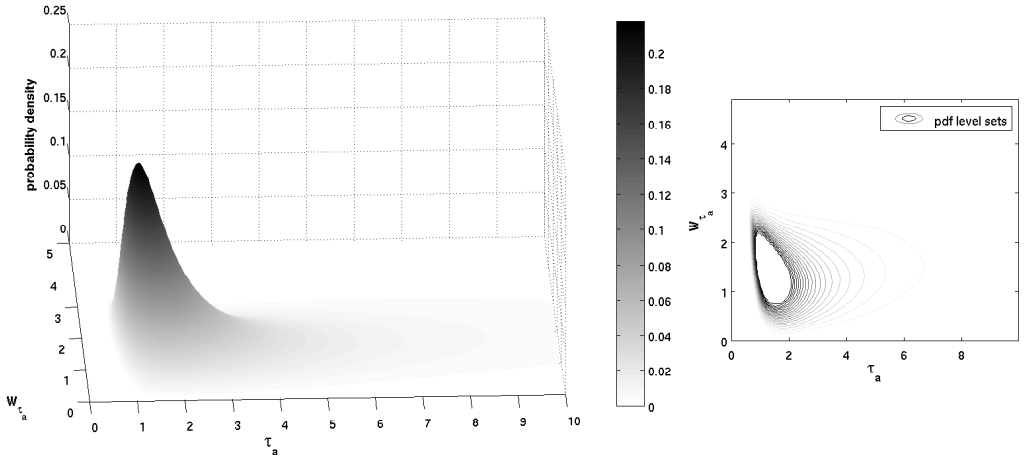
5.2 Numerical Results

Lachal's formula has been implemented using Gauss integration method. This method is very interesting for computing the double integral of Lachal's formula. It allows us to control the precision of the approximation using standard methods (see e.g. [32]). With this method we obtain the two-dimensional joint probability density function of (τ_a, W_{τ_a}) conditioned on the starting point. Figure 5 represents the dependence of this law on the starting point. Note that the probability that the hitting time τ_a is strictly less than infinity is always one, as proved by [20]. The process almost surely crosses any given constant. Computation times for a precision set to 10^{-6} are around 0.1ms.

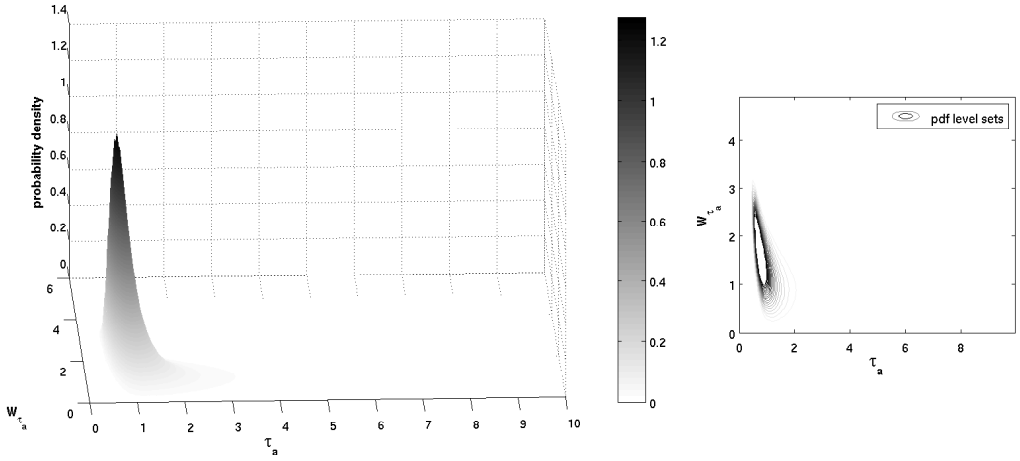
The formula we obtained for a cubic density is simply a transformation of Lachal's density using formulas (2.10) and (2.11). It is clear that now the crossing probability will not always be equal to one, for instance when $\beta > 0$. Figure 6 illustrates this property and we give a numerical estimation of the probability of crossing the boundary. This probability can be computed using the formulas we obtained, but it is not in the scope of the present paper.

Finally, our last formula can be implemented using different Monte-Carlo algorithms. Here we only get the inverse cumulative distribution function of the first hitting time. Computation times are quite large, but the main interest of this technique is not computational. Figure 7 present some results obtained for the sinusoidal functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ functions and a comparison between the results obtained using formula (5.3) and a direct integration of Lachal's formula in the case where f and g are identically equal to 1 and the boundary is constant equal to 1. In these cases, computation times are reasonable since there is no need to have a large number of points in our partition.

Finally, in table 1 we present results obtained with formula (5.3) for the first hitting time of the IWP to the sinusoidal function $t \mapsto 1 - sin(\frac{2t}{\pi})$ in the interval $[0, 1]$. We discretized the interval with 5 points (hence the approximation is of order 10^{-3}). Monte-Carlo calculations are quite long to obtain the same order of approximation (10^6 trials for points in $\mathbb{R}^{10}$ per point of the curve). Indeed, the computation of the function inside the integral on $\mathbb{R}^{2p}$ involves the computation of a fourth order integral which we compute using a Gauss method of the same order of approximation, which takes quite a long computation time. Hence we only show a table of results.

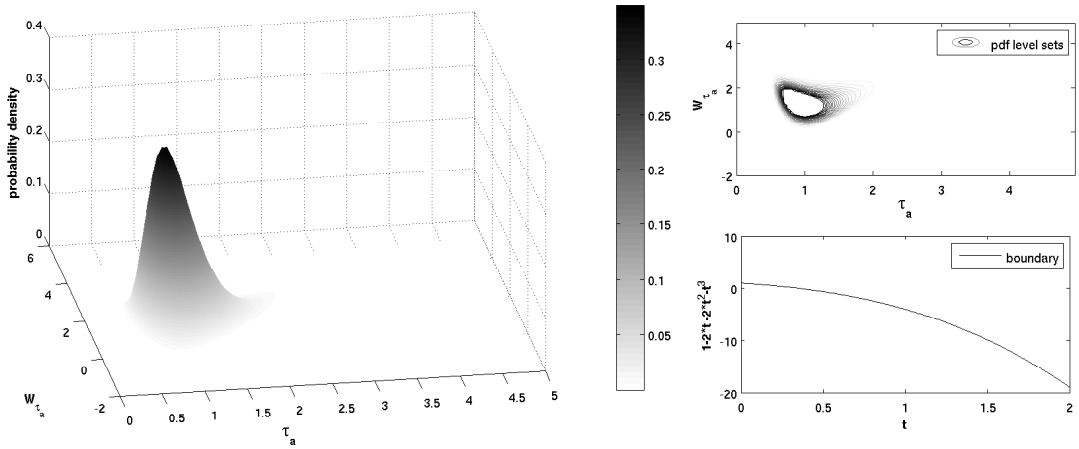


(a) Lachal's probability density function for $X_0 = 0$, $W_0 = 0$, $a = 1$

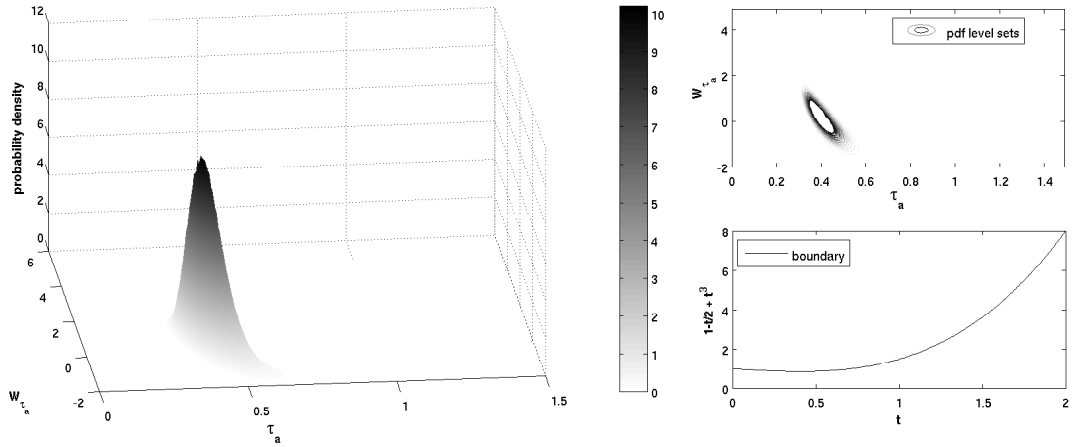


(b) Lachal's probability density function for $X_0 = 0$, $W_0 = 1$, $a = 1$

Figure 5: Probabilty density function of the first hitting time of the IWP to the constant $a = 1$ for different initial conditions. Level sets show how the pdf is distributed. The uppermost half level sets are not diplayed for the sake of visibility.

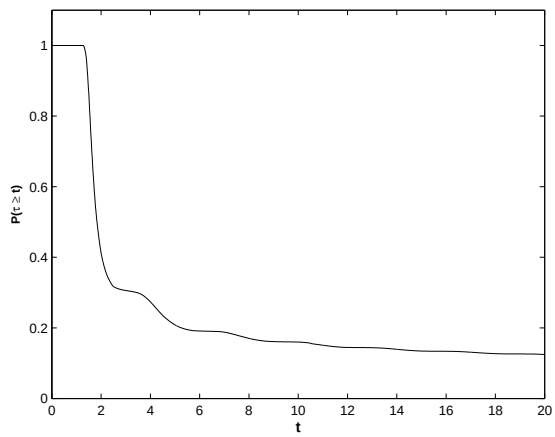


(a) Probability density function of the first hitting time of the IWP to the cubic curve: $t \mapsto 1-2t-2t^2-t^3$ with the intial condition $X_0 =$ The total mass is 1 in this case.

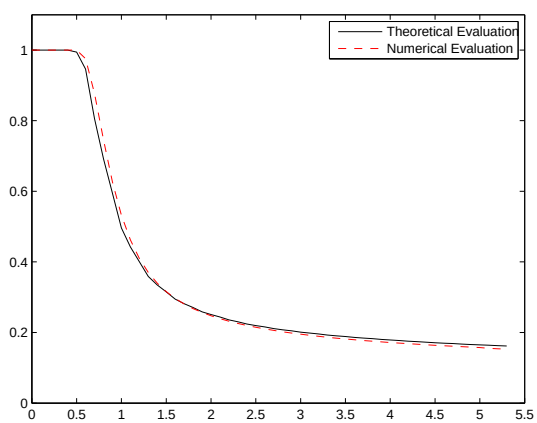


(b) Probability density function of the first hitting time of the IWP to the cubic curve: $t \mapsto 1-\frac{1}{2}t+t^3$ with the intial condition $X_0 = 0$, W total mass is ≈ 0.2578 in this case.

Figure 6: Probability density function of the first hitting time of the IWP to different cubic boundaries for the same initial conditions.



(a) Inverse cumulative distribution of the first hitting time of a linear curve of a DIP defined by the sinusoidal functions $f(x) = sin(x)$ and $g(x) = cos(x)$.



(b) Inverse cumulative distribution function of the IWP to a constant boundary 1 obtained by our method compared with the theoretical evaluation.

Figure 7: Inverse cumulative distribution function of the first hitting time of DIPs with different functions to different boundaries.

time	$\mathbb{P}(\tau_\pi \geq t \vert U_0)$
0.1	1.008819
0.2	1.003199
0.3	1.000000
0.4	0.999871
0.5	0.999576
0.6	0.996399
0.7	0.984581

Table 1: Table of the probabilities of crossing the sinusoidal function $t \mapsto 1 - sin(\frac{2t}{\pi})$ after time t by an IWP starting at the origin.

Probably other algorithms would be more efficient, but the corresponding computational issues are outside the scope of the present paper.

Conclusion

In this paper, we have provided a method of approximation of the probability distribution of the first hitting time of a Double Integral Process (DIP) to a curved boudary. To our knowledge this is the first result for this problem.

We first obtain a closed-form expression of the probability distribution of the first hitting time of the Integrated Wiener Process (IWP) to a continuous piecewise cubic boundary.

By approximating a general smooth boundary with a piecewise cubic function we use this expression to compute an approximation of the probability distribution of the first hitting time of the IWP to any smooth curved boundary, and prove that it converges (very fast in many cases) towards the probability distribution of the first hitting time of the IWP to the original curved boundary.

We then extend the method to solve the problem of approximating the probability distribution of the first hitting of a DIP to a smooth curved boundary.

Lastly we sketch a numerical procedure based on Monte-Carlo simulation to compute the probability distribution efficiently.

These results have potential applications in many fields of physics and biology.

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