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# *First hitting time of Double Integral Processes to curved boundaries*

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# First hitting time of Double Integral Processes to curved boundaries

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**Abstract:** The problem of finding the first hitting time of a Double Integral Process (DIP) such as the Integrated Wiener Process (IWP) has been a central and difficult endeavor in stochastic calculus and has applications in many fields of physics (first exit time of a particle in a noisy force field) or in biology and neuroscience (spike time of an integrate-and-fire neuron with exponentially decaying synaptic current). The only results available so far were an approximation of the stationary mean crossing time and the distribution of the first hitting time of the IWP to a constant boundary. In this paper, we generalize those results and find an analytical formula for the first hitting time of the IWP to piecewise cubic boundaries. We use this formula to approximate the law of the first hitting time of a general DIP to a smooth curved boundary, and we provide an estimation of the convergence of this method. This approximation formula is the first analytical description of the hitting time of a DIP to a curved boundary, and allows us to infer properties of this random variable and provides a way for computing accurately its law. The accuracy of the approximation is computed in the general case for the IWP and the calculation of crossing probability can be carried out through a Monte-Carlo method.

**Key-words:** First hitting time, first passage time, curved boundary, integrated Wiener process, Double integral process, second-order stochastic differential equation, numerical computation.

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# Premier temps d'atteinte d'un processus double-integral a une courbe

**Résumé :** Le problème de trouver le premier temps d'atteinte d'un processus double-integral (DIP) tel que le mouvement brownien intégré (IWP) est un probleme central et difficile dans le domaine du calcul stochastique et a des applications dans beaucoup de domaines physiques (premier temps de sortie d'une particule dans un champ de force bruité) ou en biologie et en neurosciences (temps de spike d'un neurone integre-et-tire avec des courants synaptic à décroissance exponentielle). Les seuls résultats disponibles jusqu'ici étaient la distribution du premier temps d'atteinte de l'IWP a une constante et une approximation de la moyenne de cette variable aléatoire. Dans cet article, nous généralisons ces résultats et trouvons une formule analytique pour la distribution du premier temps d'atteinte de l'IWP a une courbe cubique par morceaux. Nous utilisons cette formule pour approximer la loi de probabilité du premier temps d'atteinte d'un DIP général à une courbe lisse, et donnons une estimation de la convergence de cette méthode. Cette approximation est la première description analytique du premier temps d'atteinte d'un DIP à une courbe, et nous permet de trouver des propriétés de cette variable aléatoire et une méthode afin de la calculer efficacement. La précision de cette méthode est estimée dans le cas général pour l'IWP et le calcul numérique de cette loi peut-etre effectué en utilisant un algorithme de Monte-Carlo.

**Mots-clés :** premier temps d'atteinte, premier temps de passage, frontiere courbe, primitive du mouvement brownien, processus double-integral, equation differentielle stochatique du second order, calcul numérique

## Introduction

First passage time problems for one-dimensional diffusion processes through a time-dependent boundary have received a lot of attention over the last three decades. Unfortunately, the evaluation of the first passage time probability distribution function (pdf) through a constant or time dependent boundary is in general an arduous task which has still not received a satisfactory solution. Analytical results are scarce and fragmentary, even if closed form solutions exist for some very particular cases. Since no analytical method seem to solve the problem, one is led either to the study of the asymptotic behavior of this function and of its moments (see e.g. [22, 23]), or to the use of somewhat *ad-hoc* numerical procedures yielding approximate evaluations of the first passage time distributions. Such procedures can be classified as follows: (i) those that are based on probabilistic approaches (see e.g. [27, 5, 3, 28, 6, 20]), and (ii) purely numerical methods, such as the widely used Monte-Carlo method which applies without any restriction, but whose results are generally too coarse (for numerical methods, see e.g. [14, 10, 1, 8]).

In two and higher dimensions, the problem is even more complex and results can hardly be found. For the simplest Double Integral Process (DIP), the Integrated Wiener Process (IWP) defined in (1.10), McKean [19] Goldman [11], Lachal [15, 16, 17] found the probability distribution of the first hitting time to a constant boundary using stochastic calculus methods. Lefebvre used the Kolmogorov (Fokker-Planck) equation to find in some special cases closed-form solutions [18]. Generalizations of these formulas to other boundaries and other kinds of processes are simply not available. In the present paper, we propose a closed-form solution for the first hitting time of the IWP to a piecewise cubic function, and apply this formula to find an approximation of the first hitting time of a DIP to any smooth curved boundary. We also provide an estimation of the rate of convergence of this approximation.

In the first section, we introduce a motivation of this study, define the Double Integral Process and prove the main properties which will be useful for us in the rest of the paper. In the second section, we study the first hitting times of the IWP and provide a closed-form formula for the first hitting time of this process to a piecewise cubic function. In the third section, we introduce the approximation method of the first hitting time of the IWP to any smooth curved boundary, and find the rate of convergence of this method. Finally in the last section we provide an approximation formula for the first hitting time of a general DIP to a curved boundary. The fifth section describes quickly a numerical Monte-Carlo algorithm which can be used to compute the probability repartition function efficiently.

## 1 The Double Integral Process

In this section we introduce the Double Integral Process (DIP) and prove some useful properties. But before the mathematical study of the problem, we motivate this theoretical work by a specific problem arising in neuroscience: the distribution of the spikes for an integrate-and-fire neuron with exponentially decaying synaptic conductances.

## 1.1 Motivation

The definition of the DIP and the study of its first hitting times of curved boundaries has been motivated by numerous physical and biological problems. For instance a problem arising in neuroscience is to characterize the probability distribution of the spikes (action potentials) in presence of synaptic noise (see [9] for an introduction of the neuronal modelization of spiking neurons and [31] for review of the problem of spike time distribution).

A classical neuron model is the leaky integrate-and-fire model, where the membrane potential  $V(t)$  of a neural cell integrates external inputs and the noise at the synapses, and emits a spike when the membrane potential reaches a deterministic threshold function  $\theta(t)$  (which is constant in general). Hence in this model, the membrane potential is solution of the equation:

$$\tau_m \frac{dV(t)}{dt} = -(V(t) - V_{\text{rest}}) + I_e(t) + I_s(t) \quad (1.1)$$

In this equation  $\tau_m$  is the characteristic time of integration of the membrane potential,  $V_{\text{rest}}$  is the rest potential of the neuron,  $I_e$  represents deterministic external inputs and  $I_s$  the noisy synaptic inputs (see for instance [9, 31, 7]). The simplest model of synaptic noise is a standard Brownian motion, if we neglect the integration time of the synapse. Nevertheless, real post-synaptic currents have a very short rise time and a larger decay time.

If we take into account the decay time of the synapse  $\tau_s$ , then the synaptic current is solution of the stochastic differential equation:

$$\tau_s dI_s(t) = -I_s(t)dt + \sigma dW_t$$

We can integrate this system of stochastic differential equations as follows. The equation governing the membrane potential yields

$$V(t) = V_{\text{rest}}(1 - e^{-\frac{t}{\tau_m}}) + \frac{1}{\tau_m} \int_0^t e^{-\frac{s-t}{\tau_m}} I_e(s) ds + \frac{1}{\tau_m} \int_0^t e^{-\frac{s-t}{\tau_m}} I_s(s) ds,$$

and the synaptic current equation can be integrated as

$$I_s(t) = I_s(0)e^{-\frac{t}{\tau_s}} + \frac{\sigma}{\tau_s} \int_0^t e^{-\frac{s-t}{\tau_s}} dW_s,$$

where  $I_s(0)$  is a given random variable. We define  $\frac{1}{\alpha} = \frac{1}{\tau_m} - \frac{1}{\tau_s}$ . Replacing in the first equation  $I_s(t)$  by its value in the second equation we obtain

$$V(t) = V_{\text{rest}}(1 - e^{-\frac{t}{\tau_m}}) + \frac{1}{\tau_m} \int_0^t e^{-\frac{s-t}{\tau_m}} I_e(s) ds + \frac{I_s(0)}{1 - \frac{\tau_m}{\tau_s}} (e^{-\frac{t}{\tau_s}} - e^{-\frac{t}{\tau_m}}) + \frac{\sigma}{\tau_m \tau_s} e^{-\frac{t}{\tau_m}} \int_0^t e^{\frac{s}{\alpha}} \left( \int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) ds$$

The time of the spike emission is the first hitting time of  $V(t)$  to the threshold  $\theta(t)$ , so the first hitting time of the stochastic process

$$X_t := \int_0^t e^{\frac{s}{\alpha}} \left( \int_0^s e^{\frac{s'}{\tau_s}} dW_{s'} \right) ds \quad (1.2)$$

to the curved boundary

$$a(t) := \theta(t) - \left( V_{\text{rest}} \left( 1 - e^{-\frac{t}{\tau_m}} \right) + \frac{1}{\tau_m} \int_0^t e^{\frac{s-t}{\tau_m}} I_e(s) ds + \frac{I_s(0)}{1 - \frac{\tau_m}{\tau_s}} \left( e^{-\frac{t}{\tau_s}} - e^{-\frac{t}{\tau_m}} \right) \right).$$

This process  $X_t$  is a particular case of the class of DIP we consider in this article.

## 1.2 Definitions and main properties of DIPs

In this section, we define a class of stochastic processes containing the process (1.2), and prove some useful properties of these processes.

**Definition 1.1** (DIP). Let  $f \in \mathbb{L}^2(\mathbb{R})$  and  $g \in \mathbb{L}^1(\mathbb{R})$ . Let  $M_t$  be the martingale defined by  $M_t := \int_0^t f(s) dW_s$ .

The *double integral process (DIP)* associated to the functions  $f$  and  $g$  is defined for all  $t$  by:

$$X_t := \int_0^t g(s) M_s ds = \int_0^t g(s) \left( \int_0^s f(u) dW_u \right) ds \quad (1.3)$$

**Proposition 1.1.** The two-dimensionnal process  $(X_t, M_t)$  is a Gaussian Markov process.

*Proof.* First of all, note that if  $\mathcal{F}_t^X$  (resp.  $\mathcal{F}_t^M$ ) defines the canonical filtration associated to the process  $X$  (resp.  $M$ ) then it is clear that  $\forall t \geq 0, \mathcal{F}_t^X \subset \mathcal{F}_t^M$ .

It is also clear that  $M$  is a martingale, and satisfies the Markov property. Let  $s \leq t$ . We have:

$$\begin{aligned} X_t &= \int_0^t g(u) M_u du \\ &= \int_0^s g(u) M_u du + \int_s^t g(u) M_u du \\ X_t &= X_s + \int_s^t g(u) (M_u - M_s) du + M_s \int_s^t g(u) du \end{aligned} \quad (1.4)$$

Conditionally to  $M_s$ , the process  $\int_s^t g(u) (M_u - M_s) du$  is independent of  $\mathcal{F}_s^M$  so the law of  $X_t$  knowing  $(X_s, M_s)$  is independent of the filtration  $\mathcal{F}_t^{(X,M)} = \mathcal{F}_t^{(M)}$  (which we denote in the sequel  $(\mathcal{F}_t)_{t \geq 0}$ ) and so does  $M$ , so eventually the pair  $(X, M)$  is Markov.



The pair is clearly a Gaussian process since its two components are. Indeed,  $M$  is Gaussian as the limit of the Riemann sums of brownian increments, which are Gaussian, and  $X$  is also the limit of Riemann sums of a Gaussian process, namely  $M$ , with the weights given by  $g$ .

□

**Remark 1.** In the proof of proposition 1.1, we proved also that conditionally to  $M_s$ , the increments  $(X_t - X_s, M_t - M_s)$  are independent of the  $\sigma$ -field  $\mathcal{F}_s$ .

**Proposition 1.2.** For each value of  $t \geq 0$ , the random variable  $Y_t := (X_t, M_t)$  is a two-dimensional Gaussian variable of parameters:

$$\begin{cases} \mathbb{E}[Y_t] &= (0, 0) \\ \mathbb{E}[Y_t^T \cdot Y_t] &= \begin{pmatrix} \rho_X(t) & C_{(X,M)}(t) \\ C_{(X,M)}(t) & \rho_M(t) \end{pmatrix} \end{cases} \quad (1.5)$$

where the functions  $\rho_X(t)$ ,  $C_{(X,M)}(t)$  and  $\rho_M(t)$  are defined by:

$$\begin{cases} \rho_M(t) &:= \int_0^t f(s)^2 ds \\ \rho_X(t) &:= 2 \int_0^t g(s) \left( \int_0^s g(u) \rho_M(u) du \right) ds \\ C_{(X,M)}(t) &:= \int_0^t g(s) \rho_M(s) ds \end{cases} \quad (1.6)$$

The transition measure of the Markov process  $(Y_t)_t$  has a Gaussian density w.r.t. Lebesgue's measure:

$$\mathcal{N} \left( \begin{pmatrix} x_s + m_s \int_s^t g(u) du \\ m_s \end{pmatrix}, \tilde{C}(s, t) \right) \quad (1.7)$$

where the correlation matrix  $\tilde{C}(s, t)$  reads:

$$\tilde{C}(s, t) = \begin{pmatrix} 2 \int_s^t g(u) \left( \int_s^u g(v) \int_s^v f(w)^2 dw dv \right) du & \int_s^t g(u) \left( \int_s^u f(v)^2 dv \right) du \\ \int_s^t g(u) \left( \int_s^u f(v)^2 dv \right) du & \int_s^t f(u)^2 du \end{pmatrix} \quad (1.8)$$

*Proof.* The calculations are essentially straightforward. To compute the transition density function, we use the equation (1.4) and write:

$$\begin{pmatrix} X_t \\ M_t \end{pmatrix} = \begin{pmatrix} X_s + M_s \int_s^t g(u) du \\ M_s \end{pmatrix} + \begin{pmatrix} \int_s^t g(u) (M_u - M_s) du \\ M_t - M_s \end{pmatrix} \quad (1.9)$$

The first term in the sum in the righthand side of (1.9) is  $\mathcal{F}_s$  measurable. Given  $X_s = x_s$  and  $M_s = m_s$ , it is equal to

$$\begin{pmatrix} x_s + m_s \int_s^t g(u) du \\ m_s \end{pmatrix}$$

The second term is independent of  $\mathcal{F}_s$  and is Gaussian.

Eventually, the process  $Y_t$  knowing  $Y_s = (x_s, m_s)$  has the same law as the Gaussian process:

$$\mathcal{N}\left(\begin{pmatrix} x_s + m_s \int_s^t g(u) du \\ m_s \end{pmatrix}, \tilde{C}(s, t)\right)$$

□

**Definition 1.2** (IWP). The *Integrated Wiener Process* is a special case of the DIP where the functions  $f$  and  $g$  are identically equal to 1 :

$$X_t := \int_0^t W_s ds \quad M_s = W_s \quad (1.10)$$

Its transition measure reads:

$$\begin{aligned} \mathbb{P}\left[X_{t+s} \in du, W_{t+s} \in dv \mid X_s = x, W_s = y\right] &\stackrel{\text{def}}{=} p_t(uv; x, y) du dv = \\ &\frac{\sqrt{3}}{\pi t^2} \exp\left[-\frac{6}{t^3}(u-x-ty)^2 + \frac{6}{t^2}(u-x-ty)(v-y) - \frac{2}{t}(v-y)^2\right] du dv \end{aligned} \quad (1.11)$$

**Lemma 1.3.** Let  $(X_t)_{t \geq 0}$  be a DIP defined by (1.3). Assume that  $f(s) \neq 0$  for all  $s \geq 0$  **on veut  $f \neq 0$  p.p. mais alors c'est  $\tilde{A} \textcircled{C}$  equivalent  $\tilde{A}$  une fonction non nulle partout.** **John: c'est a dire????**. The study of the hitting times of the DIP  $X$  is equivalent to the study of the simpler process:

$$\tilde{X}_t = \int_0^t \tilde{g}(s) W_s ds,$$

where  $\tilde{g}$  is defined in the proof.

*Proof.* Let  $(M_t)_t$  be the martingale defined by:

$$M_t = \int_0^t f(s) dW_s$$

Dubins-Schwarz' theorem<sup>1</sup> ensures us that there exists a Brownian motion  $(W_t)_t$  such that almost surely

$$M_t = W_{\langle M \rangle_t}$$

We note

$$\Phi(t) = \langle M \rangle_t = \int_0^t f^2(s) ds$$

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<sup>1</sup>Even though  $\langle M \rangle_\infty \neq \infty$  because of our hypothesis on  $f$ , see [13].

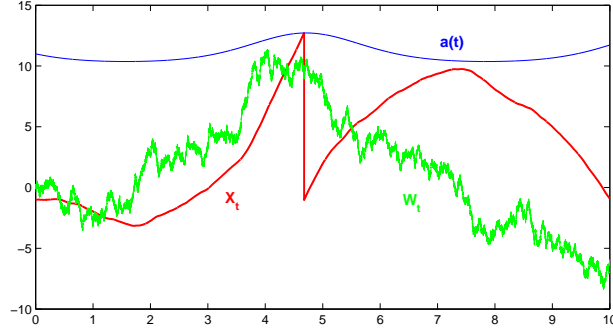


Figure 1: A sample path of the process  $U_t = (X_t, W_t)$  and the boundary curve  $a(t)$ . The IWP  $X_t$  is reseted at 0 when it crosses the boundary.

$\Phi$  is continuous and since we assumed that  $f(s) \neq 0$  for all  $s \geq 0$ , stricly increasing, so it is a bijection. Its derivative  $\Phi'(t)$  exists and is nonzero for all  $t \geq 0$ . We use the change of variable  $u = \Phi(s)$ . We have:

$$\begin{aligned} X_t &= \int_0^t g(s) M_s ds \\ &\stackrel{\mathcal{L}}{=} \int_0^t g(s) W_{\Phi(s)} ds \\ &= \int_0^{\Phi^{-1}(t)} \frac{g(\Phi^{-1}(u))}{\Phi'(\Phi^{-1}(u))} W_u du \end{aligned}$$

Hence the hitting time of a general DIP can be deduced from the hitting time of the process  $\tilde{X}_t = X_{\Phi(t)}$  which is of type  $\int_0^t \tilde{g}(s) W_s ds$ , where  $\tilde{g}(t) = \frac{g(t)}{\Phi'(t)}$   $\square$

## 2 Hitting time of the integrated Wiener process

We consider the special case  $(W_t)_{t \geq 0}$ , a standard Brownian motion. We are interested in the first hitting time to a curved boundary  $a(t)$  of the stochastic process:

$$X_t := \int_0^t W_s ds \tag{2.1}$$

This problem has been widely studied and has received no satisfactory solution so far. One of the main difficulties comes from the fact that the process is non Markov, so we have

to refer to the underlying Wiener process. Classical approaches based on Volterra equations or Durbin's method, work for the Brownian motion, but fail in providing a solution to this problem (see for instance [31] for a review). To achieve the program of characterizing those hitting times, we first recall existing results on the first hitting times to constant boundaries, generalize them to cubic and piecewise cubic boundaries, to end with the approximation formula for general boundaries.

## 2.1 First hitting time to constant boundaries

Lachal in [15] studies this problem in the case where the boundary is a constant. More precisely, in this section we study the process  $U_t = (X_t + x + ty, W_t + y)$  where  $X_t$  is the standard IWP. We denote by

$$\tau_a := \inf \{t > 0 ; X_t + x + ty = a\}$$

the first passage time at  $a$  of the first component of the bidimensional Markov process  $U_t$ . The work of Lachal [15] follows the work of McKean [19], where the joint law of the process  $(\tau_a, W_{\tau_a})$  is computed in the case  $x = a$ . The result is:

$$\begin{aligned} \mathbb{P}[\tau_a \in dt ; |W_{\tau_a}| \in dz | U_0 = (a, y)] &\stackrel{\text{def}}{=} \mathbb{P}_{(a, y)}(\tau_a \in dt ; |W_{\tau_a}| \in dz) \\ &= \frac{3z}{\pi\sqrt{2}t^2} e^{-(2/t)(y^2 - |y|z + z^2)} \left( \int_0^{4|y|z/t} e^{-3\theta/2} \frac{d\theta}{\sqrt{\pi\theta}} \right) \mathbb{1}_{[0, +\infty)}(z) dz dt \quad (2.2) \end{aligned}$$

We denote this density by  $m(t, y, z)$ .

Later, Goldman in [11] computed the distribution of the random variable  $\tau_a$  in the case where  $x < a$  and  $y \leq 0$  and obtained the formula:

$$\begin{aligned} \mathbb{P}[\tau_a \in dt | U_0 = (x, y)] &= dt \left[ \sqrt{\frac{3}{8\pi t^3}} \left( \frac{3(a-x)}{t} - y \right) e^{-3(a-x-ty)^2/(2t^3)} \right. \\ &\quad \left. + \int_0^{+\infty} z dz \int_0^t \int_0^\infty \mathbb{P}[\tau_0 \in ds ; |W_{\tau_0}| \in d\mu | U_0 = (0, z)] q_{t-s}(x, y; a, z) \right] \quad (2.3) \end{aligned}$$

where  $q_t(x, y; u, v) = p_t(x, y; u, v) - p_t(x, y; u, -v)$ . We denote this density by  $g(a, t, x, y)$ .

Lastly, Lachal in [15] extended all these results and gave the joint distribution of the pair  $(\tau_a, W_{\tau_a})$  in all cases. The quite complex formula reads:

$$\begin{aligned} \mathbb{P}_{(x, y)}[\tau_a \in dt ; W_{\tau_a} \in dz] &= |z| \left[ p_t(x, y; a, z) - \right. \\ &\quad \left. \int_0^t \int_0^{+\infty} \mathbb{P}_{(0, -|z|)}\{\tau_0 \in ds ; W_{\tau_0} \in d\mu\} p_{t-s}(x, y; a, -\varepsilon\mu) \right] \mathbb{1}_A(z) dz dt \quad (2.4) \end{aligned}$$

where  $A = [0, \infty)$  if  $x < a$ ,  $A = (-\infty, 0]$  if  $x > a$ ,  $\varepsilon = \text{sign}(a - x)$  and  $\mathbb{P}_{(0, -|z|]}$  is given by McKean's formula (2.2). We denote this density by  $l_{x,y}^a(t, z)$ .

## 2.2 First Hitting time of cubic functions

The problem we adress now is the question of finding similar formulas for more general boundaries. **The problem of finding closed-form expressions for hitting times of different stochastic processes is a very difficult problem: ON SE REPETE UN PEU, NON?** For the Brownian motion itself, results are very scarces. A formula has been found for a linear boundary using Girsanov's theorem, for quadratic boundaries using the Laplace transform characterization (see [12]). Lastly, the method of images can give also closed form results for very particular cases (see [25] for a review). This difficulty in finding closed-form characterizations for hittings times of the Brownian motion incited people to search for approximation formulas of these hitting times. Monte-Carlo simulation is often used. Even if it can be used with no restriction, it is often considered too coarse and computationally inefficient. Furthermore, it is purely numeric and global, and we cannot get any analytical and qualitative information on the hitting time. For these reason, other semi-analytical methods of approximation have been developped to provide analytical approximations [5, 6, 4, 32, 26], sometimes together with error estimations [24, 2].

These problem is even more complex for the first hitting times of the integrated Wiener process.

In this section we apply Girsanov's theorem to find a closed-form expression of the first hitting time of the IWP to a cubic function. More precisely, we prove that under a certain probability, the process  $W_t + \alpha t + \frac{\beta}{2}t^2$  is a Wiener process. Under this probability, the process  $X_t + \frac{\alpha}{2}t^2 + \frac{\beta}{6}t^3$  has the law of an IWP. Hence the knowledge of the probability density function (pdf) of the first hitting time of the IWP to a constant will give us, using Girsanov's theorem, the pdf of the hitting time of the IWP to the cubic

$$t \mapsto \frac{\beta}{6}t^3 + \frac{\alpha}{2}t^2 + bt + a.$$

For the sake of generality we compute the new probability starting at a general time  $s$  at the point  $(x, y)$ . The index  $\{s, (x, y)\}$  denotes in the following the conditioning on the event  $(X_s = x, W_s = y)$ .

**Theorem 2.1.** *The process  $W_t + \alpha(t - s) + \frac{\beta}{2}(t - s)^2 + \mathbf{y}$ ,  $0 \leq s \leq t$  is a Wiener process starting at  $\mathbf{y}$  at time  $s$ , under the probability:*

$$\mathbb{P}_{s,(x,y)}^{\alpha,\beta} \Big|_{\mathcal{F}_{s,t}} = D_{s,(x,y)}^{\alpha,\beta}(t) \mathbb{P}_{s,(x,y)} \Big|_{\mathcal{F}_{s,t}} \quad (2.5)$$

where:

$$D_{s,(x,y)}^{\alpha,\beta}(t) = \exp \left( -\frac{1}{6}\beta^2(t^3 - s^3) - \frac{1}{2}\alpha\beta(t^2 - s^2) - \frac{1}{2}\alpha^2(t - s) - (\alpha + t\beta)W_t + (\alpha + s\beta)y + \beta(X_t - x) \right) \quad (2.6)$$

*Proof.* We consider the full process  $U_t^{\alpha,\beta} = (X_t^{\alpha,\beta}, W_t^{\alpha,\beta}) = (X_t + x + yt + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3, W_t + y + \alpha(t-s) + \frac{\beta}{2}(t-s)^2) = (X_t^{\alpha,\beta}, W_t^{\alpha,\beta})$ . This is a diffusion process satisfying the two-dimensional stochastic differential equation:

$$\begin{cases} dX_t^{\alpha,\beta} &= W_t^{\alpha,\beta} dt \\ dW_t^{\alpha,\beta} &= (\alpha + (t-s)\beta) dt + dW_t \end{cases} \quad (2.7)$$

to be solved for  $t \geq s \geq 0$ , with initial conditions  $U_s^{\alpha,\beta} = (x, y)$ .

We want  $W_t^{\alpha,\beta}$  to be a Brownian motion under a new probability. This is a straightforward application of Girsanov's theorem (or the particular case of Cameron-Martin's formula). We define

$$L_t^{\alpha,\beta} := - \int_s^t (\alpha + \beta h) dW_h, \quad 0 \leq s \leq t$$

This is a martingale for  $\mathcal{F}_{s,t}$  satisfying  $\frac{1}{2} \langle L^{\alpha,\beta}, L^{\alpha,\beta} \rangle_t = \int_s^t (\alpha + \beta h)^2 dh$ , therefore  $\mathbb{E}[\exp\{\frac{1}{2} \langle L^{\alpha,\beta}, L^{\alpha,\beta} \rangle_t\}] < \infty$ ,  $0 \leq s \leq t < \infty$ . We conclude from Novikov's criterion that  $W_t - \langle W, L^{\alpha,\beta} \rangle_t = W_t^{\alpha,\beta}$  is a Brownian motion under a new probability, noted  $\mathbb{P}_{s,(x,y)}^{\alpha,\beta}$ .

The Radon-Nicodym derivative  $D_{s,(x,y)}^{\alpha,\beta}(t)$  of this new probability with respect to the initial probability  $\mathbb{P}_{s,(x,y)}$  is given by Girsanov's theorem and is equal to

$$\exp \left( L_t^{\alpha,\beta} - \frac{1}{2} \langle L^{\alpha,\beta}, L^{\alpha,\beta} \rangle_t \right).$$

This can be written

$$\begin{aligned} D_{s,(x,y)}^{\alpha,\beta}(t) &= \frac{d\mathbb{P}^{\alpha,\beta}}{d\mathbb{P}} \Big|_{\mathcal{F}_{s,t}} \\ &= \exp \left( - \frac{1}{2} \int_s^t (\alpha + h\beta)^2 dh + \int_s^t (\alpha + h\beta) dW_h \right) \\ &= \exp \left( - \frac{1}{2} (\alpha^2(t-s) + \alpha\beta(t^2 - s^2) + \frac{1}{3}\beta^2(t^3 - s^3)) \right. \\ &\quad \left. - \alpha(W_t - y) - \beta(tW_t - sy) + \beta(X_t - x) \right) \end{aligned}$$

which is equivalent to (2.6). □

We note

$$\begin{aligned} d^{\alpha,\beta}(s, x, y; t, u, v) &= \exp \left( - \frac{1}{6}\beta^2(t^3 - s^3) - \frac{1}{2}\alpha\beta(t^2 - s^2) - \frac{1}{2}\alpha^2(t - s) \right. \\ &\quad \left. - (\alpha + t\beta)v + (\alpha + s\beta)y + \beta(u - x) \right) \end{aligned} \quad (2.8)$$

**the probability density function of the new probability wrt the initial one**

**Theorem 2.2.** Let  $\tau_C$  be the first hitting time of the standard IWP to the cubic curve  $C$  of

$$C(t-s) = a + b(t-s) + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3, \quad t \geq s$$

Under the reference probability  $\mathbb{P}$ , the law of the random variable  $(\tau_C, W_{\tau_C})$  satisfies the equation:

$$\begin{aligned} \mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) &= d^{\alpha,\beta}(s, x, y + b; t, C(t-s), z) \\ &\times \mathbb{P}_{s,(x,y+b)}(\tau_a \in dt, W_{\tau_a} + b + \alpha(\tau_a - s) + \frac{\beta}{2}(\tau_a - s)^2 \in dz) \end{aligned} \quad (2.9)$$

The second term of the righthand side is given by Lachal's formula (2.4) if  $a \neq x$  or by McKean's formula (2.2) if  $a = x$ .

*Proof.* Let  $\Gamma \subset \mathbb{R}$  be a measurable set and  $t \geq s \geq 0$ . We have by the change of probability formula:

$$\begin{aligned} \mathbb{P}_{s,(x,y)}^{\alpha,\beta}(\tau_a \leq t, W_{\tau_a} \in \Gamma) &= \mathbb{E}_{s,(x,y)} \left[ \mathbb{1}_{s \leq \tau_a \leq t, W_{\tau_a} \in \Gamma} D_{s,(x,y)}^{\alpha,\beta}(t) \right] \\ &= \mathbb{E}_{s,(x,y)} \left[ \mathbb{1}_{s \leq \tau_a \leq t, W_{\tau_a} \in \Gamma} \mathbb{E}_{s,(x,y)} \left( D_{s,(x,y)}^{\alpha,\beta}(t) \middle| \mathcal{F}_{s \vee \tau_a, t \wedge \tau_a} \right) \right] \\ &= \mathbb{E}_{s,(x,y)} \left[ \mathbb{1}_{s \leq \tau_a \leq t, W_{\tau_a} \in \Gamma} D_{s,(x,y)}^{\alpha,\beta}(t \wedge \tau_a) \right] \\ &= \mathbb{E}_{s,(x,y)} \left[ \mathbb{1}_{s \leq \tau_a \leq t, W_{\tau_a} \in \Gamma} D_{s,(x,y)}^{\alpha,\beta}(\tau_a) \right] \\ &= \int_{\Gamma} \int_s^t D_{s,(x,y)}^{\alpha,\beta}(t') \mathbb{P}_{s,(x,y)}(\tau_a \in dt' ; W_{\tau_a} \in dz) \end{aligned}$$

In going from the second to the third equation we use the fact that, according to Girsanov's theorem,  $D_{s,(x,y)}^{\alpha,\beta}(t)$  is a martingale.

If  $x \neq a$ , the last probability is given by Lachal's formula (2.4) and gives the density  $c_{s,(x,y)}^{\alpha,\beta}(t, z)$  of the hitting time of  $(\tau_a, W_{\tau_a})$  under the new probability  $\mathbb{P}_{s,(x,y)}^{\alpha,\beta}$ :

$$\begin{aligned} c_{s,(x,y)}^{\alpha,\beta}(t, z) &= D_{s,(x,y)}^{\alpha,\beta}(t) |z| \left[ p_t(x, y; a, z) - \right. \\ &\quad \left. \int_0^t \int_0^{+\infty} \mathbb{P}_{(0,-|z|)}\{\tau_0 \in ds'; W_{\tau_0} \in d\mu\} p_{t-s'}(x, y; a, -\varepsilon\mu) \right] \mathbb{1}_A(z) \end{aligned}$$

where  $A = [0, \infty)$  if  $x < a$  and  $A = (-\infty, 0]$  if  $x > a$ ,  $\varepsilon = \text{sign}(a - x)$  and  $\mathbb{P}_{(0,-|z|)}$  is given by McKean's formula (2.2).

Under the probability  $\mathbb{P}_{s,(x,y)}^{\alpha,\beta}$ , the process  $(X_t^{\alpha,\beta}, W_t^{\alpha,\beta})_{t \geq s}$  is a standard IWP, Wiener process pair with initial condition  $(X_s^{\alpha,\beta} = x, W_s^{\alpha,\beta} = y)$ .

Now we recall that:

$$\begin{cases} W_t &= W_t^{\alpha,\beta} - \alpha(t-s) - \frac{\beta}{2}(t-s)^2 \\ X_t &= X_t^{\alpha,\beta} - \frac{\alpha}{2}(t-s)^2 - \frac{\beta}{6}(t-s)^3 \end{cases}$$

Let us use these relations to find a new interpretation of  $\tau_a$ .

$$\begin{aligned}
\tau_a &= \inf \left\{ t > s, X_t + (t-s)y + x = a \mid X_s = 0, W_s = 0 \right\} \\
&= \inf \left\{ t > s, X_t^{\alpha,\beta} - \frac{\alpha}{2}(t-s)^2 - \frac{\beta}{6}(t-s)^3 + (t-s)y + x = a \mid \right. \\
&\quad \left. X_s^{\alpha,\beta} = 0, W_s^{\alpha,\beta} = 0 \right\} \\
&= \inf \left\{ t > s, X_t^{\alpha,\beta} + (t-s)(b+y) + x = a + b(t-s) \right. \\
&\quad \left. + \frac{\alpha}{2}(t-s)^2 + \frac{\beta}{6}(t-s)^3 \mid X_s^{\alpha,\beta} = 0, W_s^{\alpha,\beta} = 0 \right\}
\end{aligned}$$

The corresponding location of the underlying Wiener process is  $W_t^{\alpha,\beta} + b$ , which is equal to  $W_t + b + \alpha(t-s) + \frac{\beta}{2}(t-s)^2$ . Under the probability  $\mathbb{P}_{s,(x,y)}^{\alpha,\beta}$ , the process  $X_t^{\alpha,\beta} + (t-s)(b+y) + x$  is a standard IWP, noted  $\tilde{X}_t$ , starting at  $x$  at  $t = s$  obtained from the standard Brownian motion  $W_t^{\alpha,\beta}$ , noted  $\tilde{W}_t$ , starting at  $b+y$  at time  $t = s$ . Hence the last equality can be rewritten

$$\tau_a = \inf \{ t > s, \tilde{X}_t = C(t-s) \mid \tilde{X}_s = x, \tilde{W}_s = b+y \},$$

Therefore  $\tau_a$  under  $\mathbb{P}_{s,(x,y+b)}^{\alpha,\beta}$  has the same law as  $\tau_C$  under  $\mathbb{P}_{s,(x,y)}$ .

So eventually the law of  $\tau_C, W_{\tau_C}$  reads:

$$\begin{aligned}
\mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) &= d^{\alpha,\beta}(s, x, y+b; t, C(t-s), z) \\
&\quad \times \mathbb{P}_{s,(x,y+b)}(\tau_a \in dt, W_{\tau_a} + b + \alpha(\tau_a - s) + \frac{\beta}{2}(\tau_a - s)^2 \in dz)
\end{aligned}$$

**which is exactly (2.9).**

If  $x \neq a$ , this formula reads:

$$\begin{aligned}
\mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) &= d^{\alpha,\beta}(s, x, y+b; t, C(t-s), z) \\
&\quad \times l_{x,y+b}^a(t-s, z-b-\alpha(t-s) - \frac{\beta}{2}(t-s)^2) dt dz \quad (2.10)
\end{aligned}$$

where  $l_{x,y}^\alpha$  is Lachal's density (2.4).

If  $x = a$  the same calculus using McKean's formula (2.2) gives us the formula of the hitting time probability density using the same method:

$$\begin{aligned}
\mathbb{P}_{s,(x,y)}(\tau_C \in dt, W_{\tau_C} \in dz) &= d^{\alpha,\beta}(s, x, y+b; t, C(t-s), z) \\
&\quad \times m(t-s, y+b, z-b-\alpha(t-s) - \frac{\beta}{2}(t-s)^2) dt dz \quad (2.11)
\end{aligned}$$

□



### 3 Approximation of hitting times of the IWP to a general boundary

In this section we provide the formula of the first hitting time of the IWP to a piecewise cubic function and use it to compute an approximation formula of the law of the first hitting time of the IWP to a smooth curved boundary. We also provide the convergence rate of this approximation.

#### 3.1 First hitting time to a piecewise cubic function

In this section we consider the hitting time of an IWP to a continuous piecewise cubic function. We denote by  $(U_t)_{t \geq 0}$  the bidimensional process  $(X_t, W_t)_{t \geq 0}$  and assume that the starting point is fixed:

$$X_0 = x_0, W_0 = y_0$$

We recall that the process  $(U_t)_t$  is strongly Markovian with transition density (1.11).

Let  $0 = t_0 < t_1 < t_2 < \dots < t_{n-1}$  the points where the cubic coefficients of the boundary change, and  $\{(a_i, b_i, \alpha_i, \beta_i), i = 1 \dots n\}$  the coefficients of the cubic in the interval  $[t_i, t_{i+1})$  (with the convention  $t_n = \infty$ ).

$$C(t) = \sum_{i=0}^n \left( a_i + b_i(t - t_i) + \frac{\alpha_i}{2}(t - t_i)^2 + \frac{\beta_i}{6}(t - t_i)^3 \right) \mathbb{1}_{[t_i, t_{i+1})}(t) \quad (3.1)$$

The only requirement we have on the curve  $C$  is the continuity assumption of this function, which reads in terms of the parameters of the curve:

$$\forall i \in \{1, \dots, n-1\} \quad a_{i+1} = a_i + b_i(t_{i+1} - t_i) + \frac{\alpha_i}{2}(t_{i+1} - t_i)^2 + \frac{\beta_i}{6}(t_{i+1} - t_i)^3$$

We denote by  $\tau_C^s$  the first hitting time of the process  $(X_t)_{t \geq s}$  to the curve  $C$ :

$$\tau_C^s := \inf\{t > s; X_t = C(t)\}$$

Let  $t \in [0, T]$  fixed and note  $p$  the index of the bin containing  $t$  (i.e.  $t \in [t_p, t_{p+1})$ , see Fig.2).

We know that  $\tau_C^s$  is a stopping time under the filtration associated to  $U$ , which is strongly Markovian. The event  $\{U_{t_1} = u_1, \tau_C^0 \geq t_1, U_0\}$  is in  $\mathcal{F}^{U_{t_1}}$  therefore  $\mathbb{P}(\tau_C^0 \geq t | U_{t_1} = u_1, \tau_C^0 \geq t_1, U_0) = \mathbb{P}(\tau_C^{t_1} \geq t | U_{t_1} = u_1)$ . It follows that

$$\begin{aligned} \mathbb{P}(\tau_C^0 \geq t | U_0) &= \int^{(2)} \mathbb{P}(\tau_C^0 \geq t | U_{t_1} = u_1, \tau_C^0 \geq t_1, U_0) \mathbb{P}(U_{t_1} = du_1, \tau_C^0 \geq t_1 | U_0) \\ &= \int^{(2)} \mathbb{P}(\tau_C^{t_1} \geq t | U_{t_1} = u_1) \mathbb{P}(U_{t_1} = du_1, \tau_C^0 \geq t_1 | U_0) \end{aligned}$$

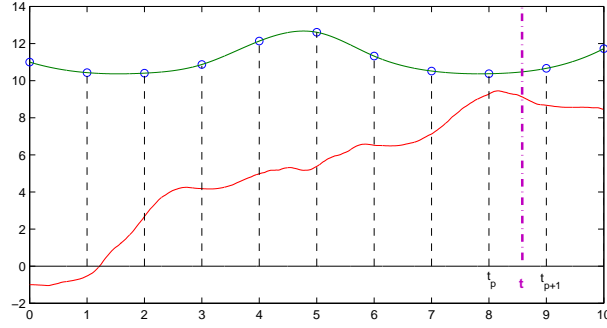


Figure 2: Principle of the demonstration: probability that the first hitting time is greater than  $t \in [t_p, t_{p+1})$

The first term in this integral is identical to the lefthand of the equation. By an immediate recursion we get:

$$\begin{aligned}
 \mathbb{P} \left( \tau_C^0 \geq t \middle| U_0 \right) &= \int^{(4)} \mathbb{P} \left( \tau_C^{t_2} \geq t \middle| U_{t_2} = u_2 \right) \\
 &\quad \times \mathbb{P} \left( U_{t_2} = du_2, \tau_C^{t_1} \geq t_2 \middle| U_{t_1} = u_1 \right) \\
 &\quad \times \mathbb{P} \left( U_{t_1} = du_1, \tau_C^0 \geq t_1 \middle| U_0 \right) \\
 &\quad \dots \\
 &= \int^{(2p)} \mathbb{P} \left( \tau_C^{t_p} \geq t \middle| U_{t_p} = u_p \right) \\
 &\quad \times \mathbb{P} \left( U_{t_p} = du_p, \tau_C^{t_{p-1}} \geq t_p \middle| U_{t_{p-1}} = u_{p-1} \right) \\
 &\quad \times \mathbb{P} \left( U_{t_{p-1}} = du_{p-1}, \tau_C^{t_{p-2}} \geq t_{p-1} \middle| U_{t_{p-2}} = u_{p-2} \right) \\
 &\quad \times \dots \\
 &\quad \times \mathbb{P} \left( U_{t_1} = du_1, \tau_C^0 \geq t_1 \middle| U_0 \right)
 \end{aligned} \tag{3.2}$$

where  $\int^{(N)}$  denotes the integral on  $\mathbb{R}^N$ . Note that the integration variables  $(u_i)_{\{i=1, \dots, p\}}$  are two-dimensional.

The terms of the product (3.2) are of the same kind. Their expression is given by the result of section 2.2.

We need to compute is the conditional probability  $q_k(du_k, u_{k-1})$  defined by:

$$q_k(du_k, u_{k-1}) = \mathbb{P}(U_{t_k} = du_k, \tau_C^{t_{k-1}} \geq t_k | U_{t_{k-1}} = u_{k-1})$$

Since

$$\{U_{t_k} = du_k, \tau_C^{t_{k-1}} \geq t_k\} = \{U_{t_k} = du_k\} \div \{U_{t_k} = du_k, \tau_C^{t_{k-1}} < t_k\},$$

where  $\div$  is the set difference,

$$\begin{aligned} q_k(du_k, u_{k-1}) &= \mathbb{P}(U_{t_k} = du_k | U_{t_{k-1}}) - \mathbb{P}(U_{t_k} = du_k, \tau_C^{t_{k-1}} \leq t_k | U_{t_{k-1}}) \\ &= \mathbb{P}(U_{t_k} = du_k | U_{t_{k-1}}) - \int_{t_{k-1}}^{t_k} \mathbb{P}(U_{t_k} = du_k, \tau_C^{t_{k-1}} \in ds | U_{t_{k-1}}) \\ &= \mathbb{P}(U_{t_k} = du_k | U_{t_{k-1}}) - \\ &\quad \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \mathbb{P}(U_{t_k} = du_k | \tau_C^{t_{k-1}} = s, W_s = y, U_{t_{k-1}}) \mathbb{P}(\tau_C^{t_{k-1}} \in ds, W_s = dy | U_{t_{k-1}}) \\ &= (p_{t_k - t_{k-1}}(u_k; u_{k-1}) \\ &\quad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s}(u_k; C(s), y) \mathbb{P}(\tau_C^{t_{k-1}} \in ds, W_s = dy | U_{t_{k-1}})) du_k \end{aligned}$$

where  $p_t(x, y; u, v)$  is the transition density function (1.11) of the process  $U$ . The curve  $C$  on the interval  $[t_{k-1}, t_k]$  is a fixed cubic function; The hitting time of the IWP starting at  $u_{k-1}$  to  $C$  has a known density computed in section 2.2 and the term we are interested in can be deduced from the expression we derived previously.

Hence we can state the following theorem:

**Theorem 3.1.** *The law of the first hitting time of a piecewise cubic function to a curved boundary is given by the formula:*

$$\begin{aligned} \mathbb{P}(\tau_C^0 \geq t | U_0) &= \int^{(2p)} \mathbb{P}(\tau_C^{t_p} \geq t | U_{t_p} = u_p) \prod_{k=1}^p (p_{t_k - t_{k-1}}(u_k; u_{k-1}) \\ &\quad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s}(u_k; C(s), y) \mathbb{P}(\tau_C^{t_{k-1}} \in ds, W_s = dy | U_{t_{k-1}})) du_k \end{aligned} \quad (3.3)$$

## 3.2 Approximation of the first hitting time to a general boundary

In this section we derive an approximation of the first hitting time before a given time  $T$  of the IWP to a general smooth boundary using the results of the previous section.

Let  $f : \mathbb{R} \mapsto \mathbb{R}$  be a continuously differentiable function. Let also  $T > 0$  and

$$0 = t_0 < t_1 < \dots < t_n = T$$

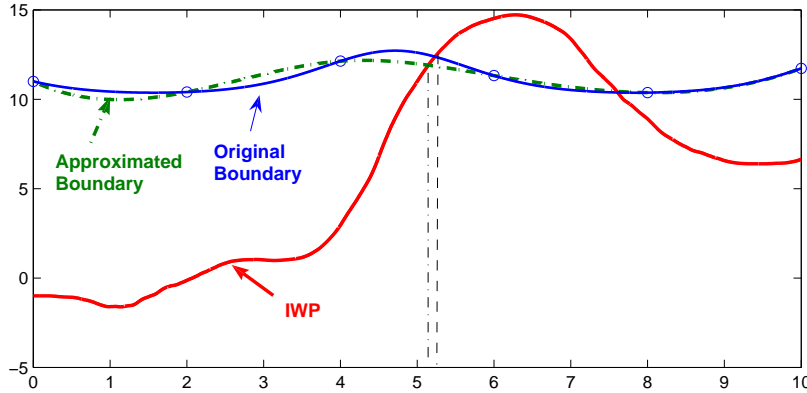


Figure 3: Principle of the approximation method: approximating the boundary by a regular piecewise cubic function

be a partition  $\pi$  of the interval  $[0, T]$ . We denote the  $\delta(\pi)$  the mesh step defined by:

$$\delta(\pi) = \max\{t_{i+1} - t_i, i = 0 \dots n\}$$

The principle of the present method is to approximate the first hitting time of the IWP to the boundary  $f$  by the first hitting time of the IWP to a smooth piecewise cubic function  $C_\pi$  (see fig. 3). The constraints we choose to apply to the curve is to pass at the control points  $\{(t_i, f(t_i)), i = 1 \dots n\}$  and to be at least continuously differentiable. There are many ways for defining our interpolating function.

1. The cubic Hermite approximation is particularly interesting because it has a continuous derivative, passes at the points of the initial curve and has the same derivative at the control point as the initial curve, i.e.:

$$\begin{cases} C_\pi(t_i) = f(t_i) \\ C'_\pi(t_i) = f'(t_i) \\ C_\pi|_{[t_i, t_{i+1}]} \text{ is a cubic polynomial} \end{cases}$$

This approximation does not oscillate between two data points so it preserves the shape of  $f$  when  $\delta(\pi)$  is small enough. When the original function  $f$  is has continuous derivatives up to order four on the interval  $[0, T]$ , then

$$\sup_{t \in [0, T]} |f(t) - C_\pi(t)| \leq \frac{1}{384} \sup_{t \in [0, T]} |f^{(iv)}(t)| \delta(\pi)^4 \quad (3.4)$$

where  $f^{(iv)}(t)$  denotes the fourth order derivative of the function  $f$ .

It is defined on  $[0, T]$  by the expression:

$$C_\pi(t) = \sum_{i=0}^{n-1} \left( (2t^3 - 3t^2 + 1)f(t_i) + (t^3 - 2t^2 + t)f'(t_i) \right. \\ \left. + (-2t^3 + 3t^2)f(t_{i+1}) + (t^3 - t^2)f'(t_{i+1}) \right) \mathbb{1}_{[t_i, t_{i+1})}(t) \quad (3.5)$$

2. The cubic spline interpolation is a  $C^2$  interpolation of the function  $f$ , passing at the control points  $\{(t_i, f(t_i)), i = 1 \dots n\}$ . The most natural boundary conditions for our problem is the *Hermite boundary condition*:  $C_\pi''(0) = f''(0)$  and  $C_\pi''(T) = f''(T)$ . This polynomial approximation is also of order four, i.e.  $\exists K(f) > 0$  a constant depending on the function  $f$  such that:

$$\sup_{t \in [0, T]} |f(t) - C_\pi(t)| \leq K(f)\delta(\pi)^4 \quad (3.6)$$

For the sake of generality and simplicity we denote:

$$C_\pi(t) = \sum_{i=0}^{n-1} \left( a_i + b_i(t - t_i) + \frac{\alpha_i}{2}(t - t_i)^2 + \frac{\beta_i}{6}(t - t_i)^3 \right) \mathbb{1}_{[t_i, t_{i+1})}(t) \quad (3.7)$$

**Theorem 3.2.** *The first hitting time of the IWP to a Lipschitz continuous boundary has a density with respect to Lebesgue's measure.*

*Proof.* We assume that  $f$  is Lipschitz continuous. Let  $L$  denote the Lipschitz continuity constant of  $f$ , we have:

$$f(s + h) - f(s) \geq -Lh \quad \forall 0 < s < s + h < t$$

For a fixed  $t \in [0, T]$ , we introduce the boundary  $f_t(s) := f(t) + L(t - s)$ . Let now

$$\begin{aligned} \tau_t &:= \inf\{s > 0; X_s > f_t(s) | X_0 = x_0, W_0 = y_0\} \\ &= \inf\{s > 0; X_s + Ls > f(t) + Lt | X_0 = x_0, W_0 = y_0\} \\ &= \inf\{s > 0; X_s > f(t) + Lt | X_0 = x_0, W_0 = y_0 + L\} \end{aligned}$$

Obviously, we have:

$$\mathbb{P}(\tau_f \in (t, t + h)) \leq \mathbb{P}(\tau_t \in (t, t + h)) \quad (3.8)$$

and hence the stopping time  $\tau_f$  has a density  $p(t)$  with respect to Lebesgue's measure. From Lachal's formula we have moreover:

$$p(t) \leq \int_{\mathbb{R}} l_{x_0, y_0 + L}^{f(t) + Lt}(t, z) dz$$

□

Now we provide an efficient method to approximate the law of this first hitting time.

**Theorem 3.3.** *The first hitting time of the IWP to the curve  $(C_\pi(t))$  before  $T$  converges in law to the first hitting time of the IWP to the curve  $f$  before  $T$ .*

*Furthermore, if  $f$  is four times continuously differentiable, the convergence of this approximation is of order 4. More precisely, if  $P(t, g)$  denotes the probability:*

$$P(t, g) = \mathbb{P}\left(X_t \geq g(t) \text{ for some } t \in [0, T]\right),$$

*for a real function  $g$ , there exists a constant  $\tilde{K}(f)$  depending on the function  $f$  such that we have:*

$$\boxed{|P(t, C_\pi) - P(t, f)| \leq \tilde{K}(f) \delta(\pi)^4} \quad (3.9)$$

To prove this theorem, we will use the following lemma giving the density of the random variable  $\sup_{s \in [0, t]} X_s$ .

**Lemma 3.4.** Let  $t > 0$  a fixed real and  $S_t$  be the random variable defined by:

$$S_t = \sup_{s \in [0, t]} X_s$$

Then the law of this random variable is characterized by:

$$\begin{aligned} \mathbb{P}_{0, x, y}(S_t \in da) = & -da \int_{-\infty}^{\infty} \int_0^t |z| \left\{ \frac{\partial p_s(x, y; a, z)}{\partial a} \right. \\ & \left. - \int_0^s \int_0^{+\infty} \mathbb{P}_{(0, -|z|)}\{\tau_0 \in du; W_{\tau_0} \in d\mu\} \frac{\partial p_{s-u}(x, y; a, -\varepsilon\mu)}{\partial a} \right\} \mathbb{1}_A(z) dz dt \end{aligned} \quad (3.10)$$

*Proof of lemma 3.4.* We have:

$$\begin{aligned} \mathbb{P}_{0, x, y}(S_t \geq a) &= \mathbb{P}_{0, x, y}(\tau_a \leq t) \\ &= \int_0^t \int_{-\infty}^{\infty} l_{x, y}^a(t, z) dt dz \end{aligned}$$

where  $l_{x, y}^a(t, z)$  is Lachal's density. From the expression of this density and its dependency in  $a$ , we can see that the random variable  $S_t$  has a density with respect to Lebesgue's measure, and this measure has the expression (3.10) (this formula is obtained using Lebesgue's theorem of derivation under the sum sign). □

*Proof of theorem 3.3.* This result comes directly from the existence of a smooth density of the random variable  $S_t$ , the uniform convergence of the processes  $C_\pi$  to  $f$  and the smoothness of the function  $f$ . Let  $U_0$  be the initial conditions:  $(X_0 = x, W_0 = y)$ , such that  $x < f(0)$ .

We have:

$$P(t, f + \varepsilon) \leq P(t, f) \leq P(t, f - \varepsilon)$$

Assume that  $f_\varepsilon$  is a uniform approximation of  $f$  such that

$$\sup_{s \in [0, t]} |f_\varepsilon(s) - f(s)| \leq \varepsilon.$$

We have:

$$\begin{aligned} |P(t, f_\varepsilon) - P(t, f)| &\leq P(t, f - \varepsilon) - P(t, f + \varepsilon) \\ &= P(t, f - \varepsilon) - P(t, f) + P(t, f) - P(t, f + \varepsilon) \\ &= \mathbb{P}\left(-\varepsilon \leq \sup_{s \in [0, t]} (X_s - f(s)) \leq 0\right) + \mathbb{P}\left(0 \leq \sup_{s \in [0, t]} (X_s - f(s)) \leq \varepsilon\right) \\ &\stackrel{\text{def}}{=} \Delta_{-\varepsilon}(f) + \Delta_\varepsilon(f) \end{aligned} \tag{3.11}$$

We have:

$$\begin{aligned} \Delta_\varepsilon(f) &= \mathbb{P}\left(0 \leq \sup_{s \in [0, t]} (X_s - f(s)) \leq \varepsilon \middle| U_0\right) \\ &= \int_0^t \int_{\mathbb{R}} \mathbb{P}\left(\tau_f \in ds, W_{\tau_f} \in dz \middle| U_0\right) \mathbb{P}\left(\sup_{v \in [s, t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s), W_s = z\right) \\ &= \int_0^t \int_{f'(s)}^\infty \mathbb{P}\left(\tau_f \in ds, W_{\tau_f} \in dz \middle| U_0\right) \mathbb{P}\left(\sup_{v \in [s, t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s), W_s = z\right) \end{aligned}$$

Finally, we know that  $W_{\tau_f} > f'(\tau_f)$ , i.e.,  $z > f'(s)$ . At this point we can conclude on the convergence of the approximation as an application of Lebesgue's theorem. Indeed, let  $s \in [0, t]$  and  $z > f'(s)$ . Then the process  $(X_t - f(t))_{t \geq s}$  is a differentiable process starting from 0 with a strictly positive derivative, so

$$\mathbb{P}\left(\sup_{v \in [s, t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s), W_s = z\right) \xrightarrow{\varepsilon \rightarrow 0} 0$$

Furthermore,  $\mathbb{P}\left(\sup_{v \in [s, t]} (X_v - f(v)) \leq \varepsilon \middle| X_s = f(s), W_s = z\right) \leq 1$  which is integrable under the measure  $\mathbb{P}\left(\tau_f \in ds, W_{\tau_f} \in dz \middle| U_0\right)$  so Lebesgue's theorem applies and we have the expected result:

$$\Delta_\varepsilon(f) \xrightarrow{\varepsilon \rightarrow 0} 0$$

The same argument applies for the term  $\Delta_{-\varepsilon}(f)$  of equation (3.11). Indeed, we can bound this probability for  $\varepsilon \leq \frac{f(0)-x}{2}$  by:

$$\begin{aligned} \Delta_{-\varepsilon}(f) &= \mathbb{P}\left(-\varepsilon \leq \sup_{s \in [0, t]} X_s - f(s) \leq 0 \mid X_0 = x, W_0 = y\right) \\ &= \mathbb{P}\left(0 \leq \sup_{s \in [0, t]} X_s - f(s) \leq \varepsilon \mid X_0 = x + \varepsilon, W_0 = y\right) \\ &\leq \mathbb{P}\left(0 \leq \sup_{s \in [0, t]} X_s - f(s) \leq \varepsilon \mid X_0 = \frac{x + f(0)}{2}, W_0 = y\right) \\ &\xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned} \tag{3.12}$$

Nevertheless, the expected result is stronger : we want to compute the order of convergence of the approximation. We now assume that the boundary  $f$  is twice continuously differentiable.

Lemma 3.4 ensures us that the random variable  $\sup_{s \in [0, t]} X_s$  has a smooth density with respect to Lebesgue's measure. From Girsanov's theorem, under the probability  $\mathbb{Q}$  defined by:

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left[ \int_0^t f''(s) dW_s - \frac{1}{2} f''(s)^2 ds \right]$$

the process  $\sup_{v \in [s, t]} (X_v - f(v))$  has the law of the sup over  $[0, t]$  of an IWP, which is given by (3.10), so under the probability measure  $\mathbb{P}$ , has a smooth density for  $z > f'(s)$ .

So eventually, using again the same bound as (3.12), we have the expected result: there exist a constant  $\tilde{K}(f)$  depending on the function  $f$  such that

$$|P(t, f_\varepsilon) - P(t, f)| \leq \tilde{K}(f)\varepsilon$$

Now if  $f$  is four times continuously differentiable,  $C_\pi$  is a cubic spline inpolation of  $f$  and we have (see for instance [30]) the convergence estimation (3.4) or (3.6).

Finally we have:

$$\begin{aligned} |P(t, C_\pi) - P(t, f)| &\leq 2\tilde{K}(f)\|C_\pi - f\| \\ &\leq 2\tilde{K}(f)\delta(\pi)^4 \end{aligned}$$

□



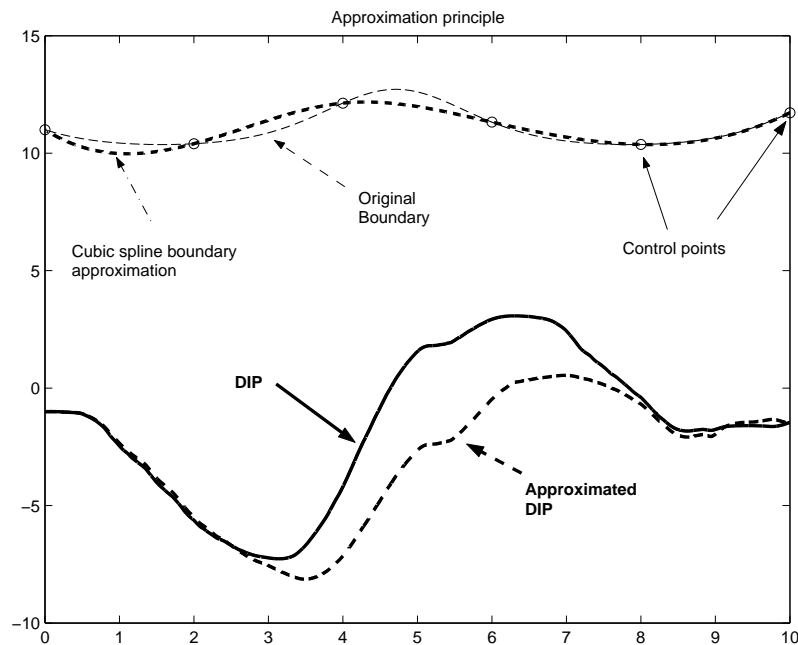


Figure 4: Approximation principle for the DIP: The boundary is approximated by a smooth piecewise cubic boundary while the process is approximated by a continuous piecewise IWP process

## 4 Approximation of hitting times of a general DIP to a general boundary

In this section we derive an approximation formula for the probability density function of the first hitting time of a general double integral process to a general smooth boundary. Here again the idea is to use the formulas we obtained in the section 2 to build approximations on a partition of a given time interval  $[0, T]$  (see Fig.4).

In accordance with lemma 1.3, we simply study the hitting times of a process  $Y$  defined by:

$$Y_t = \int_0^t g(s) W_s ds \quad (4.1)$$

where  $g(\cdot)$  is a continuously differentiable function and  $W$  a standard Brownian motion. Let  $\pi$  be a partition of the interval  $[0, T]$ :

$$(\pi) := \{0 =: t_0 < t_1 < t_2 < \dots < t_N := T\}$$

We denote by  $g^\pi$  the piecewise constant approximation of  $g$ :

$$g^\pi(t) = \sum_{i=0}^{N-1} g(t_i) \mathbb{1}_{[t_i, t_{i+1})}(t) \quad (4.2)$$

and  $Y^\pi$  the associated DIP:

$$Y_t^\pi = \int_0^t g^\pi(s) W_s ds. \quad (4.3)$$

**Proposition 4.1.** The process  $Y_t^\pi$  converges almost surely to the process  $Y_t$ . Furthermore, there exists a real positive process  $Z_t$  such that:

$$\sup_{0 \leq s \leq t} |Y_s^\pi - Y_s| \leq \delta(\pi) Z_t \quad (4.4)$$

*Proof.*

$$\begin{aligned} \sup_{0 \leq s \leq t} |Y_s^\pi - Y_s| &= \sup_{0 \leq s \leq t} \left| \int_0^s (g^\pi(u) - g(u)) W_u du \right| \\ &\leq \sup_{0 \leq s \leq t} \int_0^s |g^\pi(u) - g(u)| |W_u| du \\ &\leq \left( \sup_{0 \leq s \leq t} |g^\pi(u) - g(u)| \right) \sup_{0 \leq s \leq t} \int_0^s |W_u| du \end{aligned}$$

We assumed that  $g$  was continuously differentiable on  $\mathbb{R}^+$ , so it is uniformly Lipschitz on  $[0, T]$ . We note

$$\|g'\|_{\infty, T} = \sup_{0 \leq s \leq T} |g'(s)|$$

the uniform Lipschitz constant. We eventually have:

$$\sup_{0 \leq s \leq t} |Y_s^\pi - Y_s| \leq \delta(\pi) \|g'\|_{\infty, T} \sup_{0 \leq s \leq t} \int_0^s |W_u| du$$

The process  $Z_t := \sup_{0 \leq s \leq t} \int_0^s |W_u| du$  is almost surely finite, thus the process  $Y^\pi$  converges almost surely to  $Y$  when  $\delta(\pi) \rightarrow 0$  and we have the majoration (4.4).  $\square$

Let now  $f$  be a smooth function (at least two times continuously differentiable) and  $C^\pi$  the approximating function (3.7).

**Theorem 4.2.** *The first crossing time  $\tau^\pi$  of the process  $Y^\pi$  and the process  $C^\pi$  tends in law to the first hitting time of the process  $Y$  to the curve  $f$ , denoted by  $\tau_f$ .*

*Proof.* A sufficient condition for this convergence is the convergence in law of the process  $\sup_{s \in [0, t]} (Y_s^\pi - C_s^\pi)$  to the process  $\sup_{s \in [0, t]} (Y_s - f(s))$ , and this convergence comes directly from the calculations above. Here we only assume that  $f$  is continuously differentiable.

In this case, the cubic spline interpolation  $C_t^\pi$  converges to  $f$  linearly, i.e. there exists a constant  $K(f) > 0$  depending on  $f$  such that:

$$\sup_{t \in [0, T]} |f(t) - C_t^\pi| \leq K(f) \delta(\pi) \quad (4.5)$$

We have:

$$\begin{aligned} \sup_{s \in [0, t]} (Y_s^\pi - C_s^\pi) &\leq \sup_{s \in [0, t]} (Y_t^\pi - Y_t) + \sup_{s \in [0, t]} (f(t) - C_t^\pi) + \sup_{s \in [0, t]} (Y_t - f(t)) \\ &\leq \delta(\pi) Z_t + K(f) \delta(\pi) + \sup_{s \in [0, t]} (Y_t - f(t)) \end{aligned}$$

Writing the same estimation on  $\sup_{s \in [0, t]} Y_s - f(s)$  yields:

$$\left| \sup_{s \in [0, t]} (Y_s^\pi - C_s^\pi) - \sup_{s \in [0, t]} (Y_t - f(t)) \right| \leq \delta(\pi) (Z_t + K(f))$$

Hence we have the expected result:

$$\begin{aligned} \mathbb{P}(\tau^\pi \leq t) &= \mathbb{P}\left(\sup_{s \in [0, t]} (Y_s^\pi - C_s^\pi) > 0\right) \\ &\xrightarrow{\delta(\pi) \rightarrow 0} \mathbb{P}\left(\sup_{s \in [0, t]} (Y_s - f(s)) > 0\right) = \mathbb{P}(\tau_f \leq t) \end{aligned}$$

□

We now compute the formula of the probability density function of the first crossing time of  $Y^\pi$  to the curve  $C^\pi$ , as we did in the section 3.

We consider  $\pi$  a fixed partition of the interval  $[0, T]$ . We note  $\tau_\pi^s$  the first hitting time after time  $s$  of the process  $Y^\pi$  to the curve  $C^\pi$  and  $U^\pi$  the two-dimensional process  $(Y_t^\pi, W_t)$ . The same proof as in section 3 gives us a formula analog to (3.2):

$$\begin{aligned} \mathbb{P}(\tau_\pi^0 \geq t | U_0^\pi) &= \int^{(2p)} \mathbb{P}(\tau_\pi^{t_p} \geq t | U_{t_p}^\pi = u_p) \\ &\quad \times \mathbb{P}(U_{t_p}^\pi = du_p, \tau_\pi^{t_{p-1}} \geq t_p | U_{t_{p-1}}^\pi = u_{p-1}) \\ &\quad \times \mathbb{P}(U_{t_{p-1}}^\pi = du_{p-1}, \tau_\pi^{t_{p-2}} \geq t_{p-1} | U_{t_{p-2}}^\pi = u_{p-2}) \\ &\quad \times \dots \\ &\quad \times \mathbb{P}(U_{t_1}^\pi = du_1 | U_0^\pi) \end{aligned} \quad (4.6)$$

Here again the terms in the recursion formula are of the same kind, and the only quantity we need to calculate is the conditionnal probability  $q_k(du_k, u_{k-1})$  defined for a given  $k$  by the formula:

$$q_k(du_k, u_{k-1}) = \mathbb{P}\left(U_{t_k}^\pi = du_k, \tau_\pi^{t_k-1} \geq t_k | U_{t_{k-1}}^\pi = u_{k-1}\right)$$

Note that in the interval  $[t_i, t_{i+1})$ , the process  $Y_t^\pi$  reads:

$$Y_t^\pi = Y_{t_i}^\pi + g(t_i)(X_t - X_{t_i}) \quad (4.7)$$

and  $(X_t - X_{t_i})_{t \geq t_i}$  is simply an IWP starting from 0.

$$\begin{aligned} q_k(du_k, u_{k-1}) &= \mathbb{P}\left(U_{t_k}^\pi = du_k | U_{t_{k-1}}^\pi\right) \\ &\quad - \mathbb{P}\left(U_{t_k}^\pi = du_k, \tau_\pi^{t_k-1} \leq t_{p-1} | U_{t_{k-1}}^\pi\right) \\ &= \mathbb{P}\left(U_{t_k}^\pi = du_k | U_{t_{k-1}}^\pi\right) \\ &\quad - \int_{t_{k-1}}^{t_k} \mathbb{P}\left(U_{t_k}^\pi = du_k, \tau_\pi^{t_k-1} \in ds | U_{t_{k-1}}^\pi\right) \\ &= \mathbb{P}\left(U_{t_k}^\pi = du_k | U_{t_{k-1}}^\pi\right) \\ &\quad - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \mathbb{P}\left(U_{t_k}^\pi = du_k | \tau_\pi^{t_k-1} = s, W_s = y, U_{t_{k-1}}^\pi\right) \\ &\quad \times \mathbb{P}\left(\tau_\pi^{t_k-1} \in ds, W_s = dy | U_{t_{k-1}}^\pi\right) \\ &= \tilde{p}(t_k, du_k; t_{k-1}, u_{k-1}) - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \tilde{p}(t_k, du_k; s, (C^\pi(s), y)) \\ &\quad \times \mathbb{P}\left(\tau_\pi^{t_k-1} \in ds, W_s = dy | U_{t_{k-1}}^\pi\right) \end{aligned} \quad (4.8)$$

where  $\tilde{p}(t, x, y; s, u, v)$  is the transition function of the process  $U^\pi$  (for  $t \geq s$ ). This function is deduced from the function (1.11) and the relation (4.7) for  $s$  and  $t$  in the same bin  $[t_i, t_{i+1})$  and reads:

$$\begin{aligned} \tilde{p}(t, x, y; s, u, v) &= \mathbb{P}\left(Y_t^\pi = x, W_t = y \middle| Y_s^\pi = u, W_s = v\right) \\ &= \mathbb{P}\left(Y_s^\pi + g(t_i)(X_t - X_s) = x, W_t = y \middle| Y_s^\pi = u, W_s = v\right) \\ &= \mathbb{P}\left(X_t - X_s = \frac{x - u}{g(t_i)}, W_t = y \middle| Y_s^\pi = u, W_s = v\right) \\ &= \mathbb{P}\left(X_t - X_s = \frac{x - u}{g(t_i)}, W_t - W_s = y - v \middle| Y_s^\pi = u, W_s = v\right) \end{aligned} \quad (4.9)$$

We have seen in remark 1 that conditionnaly to  $W_s$  the increment of the two-dimensional process are independant of  $\mathcal{F}_s$ , so we have:

$$\begin{aligned}\tilde{p}(t, x, y; s, u, v) &= \mathbb{P}\left(X_{t-s} = \frac{x-u}{g(t_i)}, W_{t-s} = y-v \mid X_0 = 0, W_0 = v\right) \\ &= p_{t-s}\left(\frac{x-u}{g(t_i)}, y-v; 0, 0\right)\end{aligned}\quad (4.10)$$

Hence the general term (4.8) of the expansion reads:

$$\begin{aligned}dx_k dy_k \Big\{ p_{t_k-t_{k-1}}\left(\frac{x_k-x_{k-1}}{g(t_{k-1})}, y_k-y_{k-1}; 0, 0\right) - \\ \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k-s}\left(\frac{x_k-C^\pi(s)}{g(t_{k-1})}, y_k-y_s; 0, 0\right) \mathbb{P}\left(\tau^\pi \in ds, W_s = dy_s \mid U_{t_{k-1}}^\pi\right) \Big\}\end{aligned}\quad (4.11)$$

The last thing to compute is  $\mathbb{P}\left(\tau^\pi \in ds, W_s = dy \mid U_{t_{k-1}}^\pi\right)$  the law of the hitting time  $\tau^\pi$  in a given bin  $[t_k, t_{k-1})$ , which appears in the expression (4.8). This law is can be deduced from the law of the first hitting time of the IWP using the formula (4.7).

Indeed, to compute this probability, we use the fact that on the event  $\{U_{t_{k-1}}^\pi, t \geq t_{k-1}\}$  we have:

$$\begin{aligned}\tau^\pi &= \inf\{s > t_{k-1}, Y_s^\pi = C^\pi(s) \mid U_{t_{k-1}}^\pi\} \\ &= \inf\{s > t_{k-1}, Y_{t_i}^\pi + g(t_i)(X_t - X_{t_i}) = C^\pi(s) \mid U_{t_{k-1}}^\pi\} \\ &= \inf\{s > t_{k-1}, X_t - X_{t_i} = \frac{C^\pi(s) - Y_{t_i}^\pi}{g(t_i)} \mid U_{t_{k-1}}^\pi\} \\ &= \inf\{s > 0, \tilde{X}_t = \frac{C^\pi(s) - Y_{t_i}^\pi}{g(t_i)} \mid \tilde{X}_0 = 0, \tilde{W}_0 = W_{t_{k-1}}\}\end{aligned}$$

where  $(\tilde{X}_t, \tilde{W}_t)_t$  is a standard IWP and  $W_{t_{k-1}} := U_{t_{k-1}, 2}^\pi$  is the second component of  $U_{t_{k-1}}^\pi$ . Hence we have:

$$\mathbb{P}\left(\tau^\pi \in ds, W_s \in dy \mid U_{t_{k-1}}^\pi\right) = \mathbb{P}_{s, (0, W_{t_{k-1}})}\left(\tau_{(C-Y_{t_{k-1}}^\pi)/g(t_{k-1})} \in ds, W_s \in dy\right)\quad (4.12)$$

and the last expression is given by (2.10) and (2.11).

Eventually, the following theorem is proved:

**Theorem 4.3.** *Let  $g$  be a Lipchitz continuous real function,  $T > 0$  and  $\pi$  a partition of the interval  $[0, T]$*

$$0 =: t_0 < t_1 < \dots < t_p := T$$

Let  $f$  be a continuously differentiable function. The first hitting time  $\tau^\pi$  of the approximated process  $Y^\pi$  defined by (4.3) to a cubic spline approximation of  $f$  on the partition  $\pi$ , denoted  $f^\pi$ , satisfies the equation:

$$\mathbb{P}(\tau^\pi \geq T | U_0) = \int^{(2p)} \prod_{k=1}^p \left\{ p_{t_k - t_{k-1}} \left( \frac{x_k - x_{k-1}}{g(t_{k-1})}, y_k - y_{k-1}; 0, 0 \right) - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} p_{t_k - s} \left( \frac{x_k - f^\pi(s)}{g(t_{k-1})}, y_k - y_s; 0, 0 \right) \mathbb{P}_{s, (0, y_s)}(\tau_{(C - x_{k-1})/g(t_{k-1})} \in ds, W_s \in dy_s) \right\} dx_k dy_k \quad (4.13)$$

where  $\mathbb{P}(\tau_C \in ds, W_s \in dy_s)$  is given by equations (2.11) or (2.10).

## 5 Numerical Evaluation

In this section we propose an algorithm to implement the approximation formula we derived in the previous section. The expressions we found for the first hitting time involve integrals on  $\mathbb{R}^{2p}$  when there are  $p$  points in the mesh, which have no closed-form expression. Numerical computation of these integrals can be quite intricate and unefficient, so a numerical approximation is needed and another approximation is done besides (3.9) and theorem 4.2. The principle of the numerical approximation we propose here is to express this integral as an expectation over a certain probability measure and use a Monte-Carlo algorithm to compute this expectation. The accuracy of this approximation can be assessed through standard procedures for Monte Carlo simulations [21, 29].

**Corollary 5.1** (of theorem 3.1). Let  $X$  be a standard IWP and  $f$  a smooth boundary function. The law of the first hitting time  $\tau$  of  $X$  to  $f$  satisfies for  $t \in [t_{p-1}, t_p]$  :

$$\mathbb{P}(\tau_C \geq t | U_0) = \mathbb{E} \left[ h_p(X_{t_1}, W_{t_1}, \dots, X_{t_{p-1}}, W_{t_{p-1}}, X_t, W_t) | U_0 \right] \quad (5.1)$$

where the function  $h$  is defined by:

$$h_p(x_1, y_1, \dots, x_p, y_p) := \prod_{k=1}^p \left( 1 - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \frac{p_{t_k - s}(x_k, y_k, C(s), y)}{p_{t_k - t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} \times \mathbb{P}(\tau_C \in ds, W_{\tau_C} = dy | X_{t_{k-1}} = x_{k-1}, W_{t_{k-1}} = y_{k-1}) \right) \quad (5.2)$$

The same method can be applied for the DIP for a given function  $g$ .

**Corollary 5.2** (of theorem 4.3). Let  $g$  be a Lipchitz continuous real function,  $T > 0$  and  $\pi$  a partition of the interval  $[0, T]$

$$0 =: t_0 < t_1 < \dots < t_p := T$$

Let  $f$  be a continuously differentiable function. The first hitting time  $\tau^\pi$  of the approximated process  $Y^\pi$  defined by (4.3) to a cubic spline approximation of  $f$  on the partition  $\pi$ , denoted  $f^\pi$ , can be computed as the expectation:

$$\mathbb{P}(\tau^\pi \geq T | U_0) = \mathbb{E} \left[ h_p^{g,\pi}(X_{t_1}, W_{t_1}, \dots, X_{t_p}, W_{t_p}) | U_0 \right] \quad (5.3)$$

where the function  $h_p^{g,\pi}$  is defined by:

$$h_p^{g,\pi}(x_1, \dots, x_p) := \prod_{k=1}^p \left\{ \frac{p_{t_k-t_{k-1}}\left(\frac{x_k-x_{k-1}}{g(t_{k-1})}, y_k-y_{k-1}; 0, 0\right)}{p_{t_k-t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} - \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}} \frac{p_{t_k-s}\left(\frac{x_k-f^\pi(s)}{g(t_{k-1})}, y_k-y_s; 0, 0\right)}{p_{t_k-t_{k-1}}(x_k, y_k, x_{k-1}, y_{k-1})} \mathbb{P}_{s,(0,y_s)}(\tau_{(C-x_{k-1})/g(t_{k-1})} \in ds, W_s \in dy_s) \right\} \quad (5.4)$$

where  $\mathbb{P}(\tau_C \in ds, W_s \in dy_s)$  is given by equations (2.11) or (2.10).

Hence the problem is now reduced to the computation of the expectation of a certain function of the Gaussian random vector  $(X_0, W_0, X_{t_1}, W_{t_1}, \dots, X_{t_p}, W_{t_p})$ . This vector is Gaussian of mean 0 and covariance matrix defined by blocks by the expression:

$$K(t_1, \dots, t_p) = \left( \begin{array}{cc} \frac{1}{3}|t_j-t_i|^3 & \frac{1}{2}|t_j-t_i|^2 \\ \frac{1}{2}|t_j-t_i|^2 & |t_j-t_i| \end{array} \right)_{(i,j) \in \{0 \dots p\}} \quad (5.5)$$

The Monte-Carlo algorithm we use to compute the expected probability is the following:

- (i). Compute the square root  $K(t_1, \dots, t_p)^{1/2}$  of the covariance matrix (5.5) (using for instance a Cholesky decomposition)
- (ii). Generate an i.i.d. sample  $u = (u_1, u_2, \dots, u_{2p})'$  from the normal standard distribution  $\mathcal{N}(0, 1)$ .
- (iii). Compute the transformation  $x = K(t_1, \dots, t_p)^{1/2} \cdot u$
- (iv). Calculate  $h_p(x)$  or  $h_p^{g,\pi}(x)$
- (v). Repeat steps (ii)-(iv)  $N$  times and calculate the frequency

$$P_N = \frac{1}{N} \sum_{\text{over the realisations}} h_p(x) (\text{resp. } h_p^{g,\pi}(x))$$

The probability  $\mathbb{P}(\tau \geq T)$  is then estimated by  $P_N$ . The standard error of this estimator is given by :

$$\mathcal{E}(N) = \sqrt{\frac{\sum [h(x) - P_N]^2}{N(N-1)}} \quad (5.6)$$

## Conclusion

In this paper, we provide a new approximation formula for the first hitting time of a Double Integral Process to curved boundaries. This result is the first one in this field.

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