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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

# *Approximating Pareto Sets with Stochastic Search Algorithms*

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## Approximating Pareto Sets with Stochastic Search Algorithms

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**Abstract:** Recent studies show the growing interest in the design of suitable archiving strategies for their use with evolutionary multi-objective optimization algorithms. So far, nearly all works deal with Pareto *front* approximations, that is, with archiving strategies which provide a certain approximation quality in the limit which is measured in the image space of the underlying model. However, there are as well cases where it is of great importance for the decision maker to have the knowledge about the entire Pareto *set*. Investigations in this field are scarce, and so far no theoretical investigation has been done. We propose and investigate a novel archiving strategy for the approximation of a superset of the Pareto set, which includes the set of all  $\epsilon$ -efficient points of a given multi-objective optimization problem defined in continuous space. For this, we propose the set of interest, investigate its topology and state a convergence result for a generic stochastic search algorithm toward this set of interest. Finally, we present some numerical results indicating the practicability of the novel approach.

**Key-words:** multi-objective optimization, convergence,  $\epsilon$ -dominance, stochastic search algorithms.) performances

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**Résumé :** Pas de résumé

**Mots-clés :** Pas de motclef

## 1 Introduction

The knowledge about the entire Pareto set of a given multi-objective optimization problem can be of great importance for the decision maker, e.g. in product design: after having selected an appropriate solution in image space, it could be interesting for the product designer to know all the different solutions in decision space which lead to this or a similar objective value ([5]). There already exist quite a few archiving strategies to maintain the ‘essential’ solutions found by a stochastic search process (e.g., [1], [4], [3], [9], [8]), but, however, none of them can be taken for our purpose since they are either not suitable for continuous models for which we aim in this study or do not consider possible noninjectivities of the given model, which naturally arise e.g. when models with symmetries are being considered (for instance, knapsack problems which contain equal weights): all of the proposed archiving strategies store only nondominated solutions, and can thus run into trouble in this case ([8], or see Appendix). A start for Pareto set approximations with evolutionary strategies which can deal with this phenomena has been done in ([5]), where a multistart approach is proposed. Here we propose an alternative method which is based on  $\epsilon$ -dominance (see e.g. [2] and references therein). This allows that – under mild assumptions on the generation process – the limit archive provably covers the entire Pareto set as well as all  $\epsilon$ -efficient points. The storage of exactly those (dominated) points makes the archiver more robust with respect to the approximation quality, and can also help in practice to get a better understanding of the explored Pareto front (e.g., when not enough function calls can be spent to obtain a sufficiently exact front). Further, all these points are relevant for the decision maker in case small changes in the objective values cannot be distinguished (e.g., when uncertainties in the production process are given).

The remainder of this paper is organized as follows: in Section 2 we give the required background for the understanding of the sequel. In Section 3 we propose a set of interest which includes the set of  $\epsilon$ -efficient solutions, analyze its topology, and state a convergence result. We present numerical results on two examples in Section 4 and finally conclude in Section 5.

## 2 Background

In the following we consider continuous multi-objective optimization problems

$$\min_{x \in \mathbb{R}^n} \{F(x)\}, \quad (\text{MOP})$$

where the function  $F$  is defined as the vector of the objective functions

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad F(x) = (f_1(x), \dots, f_k(x)),$$

and where each  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable. Later we will restrict the search to a compact set  $Q \subset \mathbb{R}^n$ , the reader may think of an  $n$ -dimensional box.

- Definition 2.1** (a) Let  $v, w \in \mathbb{R}^k$ . Then the vector  $v$  is less than  $w$  ( $v <_p w$ ), if  $v_i < w_i$  for all  $i \in \{1, \dots, k\}$ . The relation  $\leq_p$  is defined analogously.
- (b) A point  $y \in \mathbb{R}^n$  is dominated by a point  $x \in \mathbb{R}^n$  ( $x \prec y$ ) with respect to (MOP) if  $F(x) \leq_p F(y)$  and  $F(x) \neq F(y)$ , else  $y$  is called nondominated by  $x$ .
- (c) A point  $x \in \mathbb{R}^n$  is called a Pareto point if there is no  $y \in \mathbb{R}^n$  which dominates  $x$ .
- (d) A point  $x \in \mathbb{R}^n$  is weakly Pareto optimal if there does not exist another point  $y \in \mathbb{R}^n$  such that  $F(y) <_p F(x)$ .

We now define a weaker concept of dominance, called  $\epsilon$ -dominance, which is the basis of the approximation concept used in this study.

**Definition 2.2** Let  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{R}_+^k$  and  $x, y \in \mathbb{R}^n$ .  $x$  is said to  $\epsilon$ -dominate  $y$  ( $x \prec_\epsilon y$ ) with respect to (MOP) if  $F(x) - \epsilon \leq_p F(y)$  and  $F(x) - \epsilon \neq F(y)$ .

**Theorem 2.3 ([6])** Let (MOP) be given and  $q : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $q(x) = \sum_{i=1}^k \hat{\alpha}_i \nabla f_i(x)$ , where  $\hat{\alpha}$  is a solution of

$$\min_{\alpha \in \mathbb{R}^k} \left\{ \left\| \sum_{i=1}^k \alpha_i \nabla f_i(x) \right\|_2^2; \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

Then either  $q(x) = 0$  or  $-q(x)$  is a descent direction for all objective functions  $f_1, \dots, f_k$  in  $x$ . Hence, each  $x$  with  $q(x) = 0$  fulfills the first-order necessary condition for Pareto optimality.

In case  $q(x) \neq 0$  it obviously follows that  $q(x)$  is an ascent direction for all objectives. Next, we need the following distances between different sets.

**Definition 2.4** Let  $u \in \mathbb{R}^n$  and  $A, B \subset \mathbb{R}^n$ . The semi-distance  $\text{dist}(\cdot, \cdot)$  and the Hausdorff distance  $d_H(\cdot, \cdot)$  are defined as follows:

- (a)  $\text{dist}(u, A) := \inf_{v \in A} \|u - v\|$
- (b)  $\text{dist}(B, A) := \sup_{u \in B} \text{dist}(u, A)$
- (c)  $d_H(A, B) := \max \{ \text{dist}(A, B), \text{dist}(B, A) \}$

Denote by  $\bar{A}$  the closure of a set  $A \in \mathbb{R}^n$ , by  $\overset{\circ}{A}$  its interior, and by  $\partial A = \bar{A} \setminus \overset{\circ}{A}$  the boundary of  $A$ .

Algorithm 1 gives a framework of a generic stochastic multi-objective optimization algorithm, which will be considered in this work. Here,  $Q \subset \mathbb{R}^n$  denotes the domain of the MOP,  $P_j$  the candidate set (or population) of the generation process at iteration step  $j$ , and  $A_j$  the corresponding archive.

**Algorithm 1** Generic Stochastic Search Algorithm

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1:  $P_0 \subset Q$  drawn at random
2:  $A_0 = \text{ArchiveUpdate}(P_0, \emptyset)$ 
3: for  $j = 0, 1, 2, \dots$  do
4:    $P_{j+1} = \text{Generate}(P_j)$ 
5:    $A_{j+1} = \text{ArchiveUpdate}(P_{j+1}, A_j)$ 
6: end for

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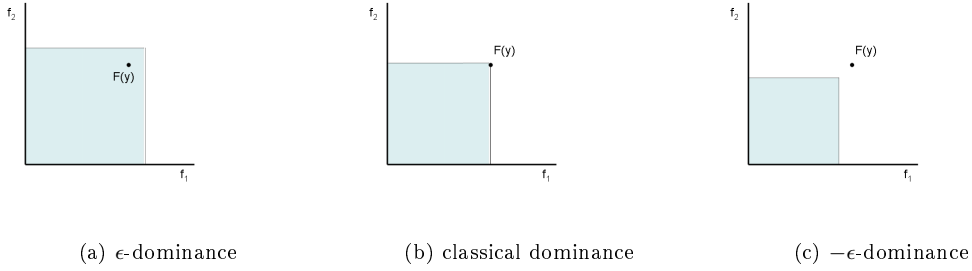


Figure 1: Regions of dominance for the different dominance concepts. A point  $x \in \mathbb{R}^n$  dominates the point  $y$  if its image is inside the shaded region in Figure (b). Figures (a) and (c) are analogue for  $\epsilon$ -dominance and  $-\epsilon$ -dominance.

### 3 The Archiving Strategy

In this section we define the set of interest, investigate the topology of this object, and finally state a convergence result.

**Definition 3.1** Let  $\epsilon \in \mathbb{R}_+^k$  and  $x, y \in \mathbb{R}^n$ .  $x$  is said to  $-\epsilon$ -dominate  $y$  ( $x \prec_{-\epsilon} y$ ) with respect to (MOP) if  $F(x) + \epsilon \leq_p F(y)$  and  $F(x) + \epsilon \neq F(y)$ .

This definition is of course analogous to the ‘classical’  $\epsilon$ -dominance relation but with a value  $\tilde{\epsilon} \in \mathbb{R}_-^k$ . However, we highlight it here since it will be used frequently in this work. While the  $\epsilon$ -dominance is a weaker concept of dominance, the  $-\epsilon$ -dominance concept is a stronger one. (see Figure 1).

**Definition 3.2** A point  $x \in Q$  is called  $-\epsilon$  weak Pareto point if there exists no point  $y \in Q$  such that  $F(x) + \epsilon <_p F(y)$ .



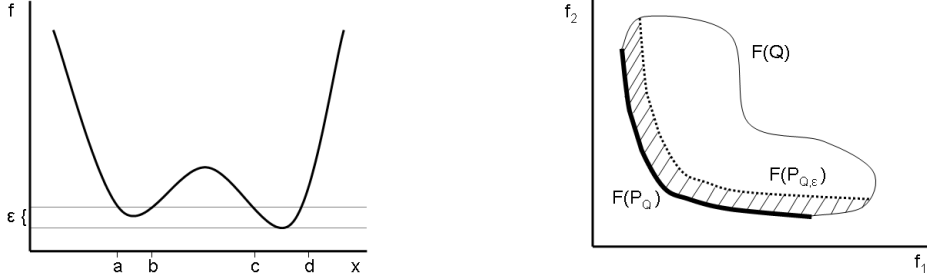


Figure 2: Two different examples for sets  $P_{Q,\epsilon}$ . Left for  $k = 1$  and in parameter space, and right an example for  $k = 2$  in image space.

**Definition 3.3** Denote by  $P_{Q,\epsilon}$  the set of points in  $Q \subset \mathbb{R}^n$  which are not  $-\epsilon$ -dominated by any other point in  $Q$ , i.e.

$$P_{Q,\epsilon} := \{x \in Q \mid \nexists y \in Q : y \prec_{-\epsilon} x\}. \quad (1)$$

**Example 3.4** (a) Figure 2 shows two examples for sets  $P_{Q,\epsilon}$ , one for the single-objective case (left), and one for  $k = 2$  (right). In the first case we have  $P_{Q,\epsilon} = [a, b] \cup [c, d]$ .

(b) Consider the MOP

$$\begin{aligned} F : \mathbb{R} &\rightarrow \mathbb{R}^2 \\ F(x_1, x_2) &= ((x-1)^2, (x+1)^2) \end{aligned} \quad (2)$$

For  $\epsilon = (1, 1)$  and  $Q$  sufficiently large, say  $Q = [-3, 3]$ , we obtain  $P_Q = [-1, 1]$ , where  $P_Q$  denotes the Pareto set of  $F|_Q$ , and  $P_{Q,\epsilon} = (-2, 2)$ . Note that the boundary of  $P_{Q,\epsilon}$ , i.e.

$$\partial P_{Q,\epsilon} = \overline{P_{Q,\epsilon}} \setminus \overset{\circ}{P_{Q,\epsilon}} = [-2, 2] \setminus (2, 2) = \{-2, 2\},$$

is given by  $-\epsilon$  weak Pareto points which are not included in  $P_{Q,\epsilon}$  (see also Lemma 1): for  $x_1 = -2$  and  $x_2 = 2$  it is  $F(x_1) = (9, 1)$  and  $F(x_2) = (1, 9)$ . Since there exists no  $x \in Q$  with  $f_i(x) < 0, i = 1, 2$ , there is also no point  $x \in Q$  where all objectives are less than at  $x_1$  or  $x_2$ . Further, since  $F(-1) = (4, 0)$  and  $F(1) = (0, 4)$  there exist points which  $-\epsilon$ -dominate these points, and they are thus not included in  $P_{Q,\epsilon}$ .

**Lemma 3.5** (a) Let  $Q \subset \mathbb{R}^n$  be compact. Under the following assumptions

(A1) Let there be no weak Pareto point in  $Q \setminus P_Q$ , where  $P_Q$  denotes the set of Pareto points of  $F|_Q$ .

(A2) Let there be no  $-\epsilon$  weak Pareto point in  $Q \setminus \overline{P_{Q,\epsilon}}$ ,

(A3) Define  $\mathcal{B} := \{x \in Q \mid \exists y \in P_Q : F(y) + \epsilon = F(x)\}$ . Let  $\mathcal{B} \subset \overset{\circ}{Q}$  and  $q(x) \neq 0$  for all  $x \in \mathcal{B}$ , where  $q$  is as defined in Theorem 2.3,

it holds:

$$\begin{aligned} \overline{P_{Q,\epsilon}} &= \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon <_p F(x)\} \\ \overset{\circ}{P_{Q,\epsilon}} &= \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon \leq_p F(x)\} \\ \partial P_{Q,\epsilon} &= \{x \in Q \mid \exists y_1 \in P_Q : F(y_1) + \epsilon \leq_p F(x) \text{ and } \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\} \end{aligned} \quad (3)$$

(b) Let in addition to the assumptions made above be  $q(x) \neq 0 \forall x \in \partial P_{Q,\epsilon}$ . Then

$$\overline{\overset{\circ}{P_{Q,\epsilon}}} = \overline{P_{Q,\epsilon}} \quad (4)$$

*Proof:* Define

$$W := \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon <_p F(x)\}.$$

We show the equality  $\overline{P_{Q,\epsilon}} = W$  by mutual inclusion.  $W \subset \overline{P_{Q,\epsilon}}$  follows directly by assumption (A2). To see the other inclusion assume that there exists an  $x \in \overline{P_{Q,\epsilon}} \setminus W$ . Since  $x \notin W$  there exists an  $y \in Q$  such that  $F(y) + \epsilon <_p F(x)$ . Further, since  $F$  is continuous there exists further a neighborhood  $U$  of  $x$  such that

$$F(y) + \epsilon <_p F(u), \quad \forall u \in U.$$

Thus,  $y$  is  $-\epsilon$ -dominating all  $u \in U$  (i.e.,  $U \cap P_{Q,\epsilon} = \emptyset$ ), a contradiction to the assumption that  $x \in \overline{P_{Q,\epsilon}}$ . Thus, we have

$$\overline{P_{Q,\epsilon}} := \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon <_p F(x)\}.$$

as claimed. Next we show that the interior of  $P_{Q,\epsilon}$  is given by

$$I := \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon \leq_p F(x)\}, \quad (5)$$

which we do again by mutual inclusion. To see that  $\overset{\circ}{P_{Q,\epsilon}} \subset I$  assume that there exists an  $x \in \overset{\circ}{P_{Q,\epsilon}} \setminus I$ . Since  $x \notin I$  we have

$$\exists y_1 \in Q : F(y_1) + \epsilon \leq_p F(x). \quad (6)$$

Since  $x \in \overset{\circ}{P_{Q,\epsilon}}$  there exists no  $y \in Q$  which  $-\epsilon$ -dominates  $x$ , and hence, equality holds in equation (6). Further, by assumption (A1) it follows that  $y_1$  must be in  $P_Q$ . Thus, we can reformulate (6) by

$$\exists y_1 \in P_Q : F(y_1) + \epsilon = F(x) \quad (7)$$

Since  $x \in P_{Q,\epsilon}^\circ$  there exists a neighborhood  $\tilde{U}$  of  $x$  such that  $\tilde{U} \subset P_{Q,\epsilon}^\circ$ . Further, since  $q(x) \neq 0$  by assumption (A1), there exists a point  $\tilde{x} \in \tilde{U}$  such that  $F(\tilde{x}) >_p F(x)$ . Combining this and (7) we obtain

$$F(y_1) + \epsilon = F(x) <_p F(\tilde{x}), \quad (8)$$

and thus  $y_1 \prec_{-\epsilon} \tilde{x} \in \tilde{U} \subset P_{Q,\epsilon}^\circ$ , which is a contradiction. It remains to show that  $I \subset P_{Q,\epsilon}^\circ$ : assume there exists an  $x \in I \setminus P_{Q,\epsilon}^\circ$ . Since  $x \notin P_{Q,\epsilon}^\circ$  there exists a sequence  $x_i \in Q \setminus P_{Q,\epsilon}^\circ$ ,  $i \in \mathbb{N}$ , such that  $\lim_{i \rightarrow \infty} x_i = x$ . That is, there exists a sequence  $y_i \in Q$  such that  $y_i \prec_{-\epsilon} x_i$  for all  $i \in \mathbb{N}$ . Since all the  $y_i$  are inside  $Q$ , which is a bounded set, there exists a subsequence  $y_{i_j}$ ,  $j \in \mathbb{N}$ , and an  $y \in Q$  such that  $\lim_{j \rightarrow \infty} y_{i_j} = y$  (Bolzano-Weierstrass). Since  $F(y_{i_j}) + \epsilon \leq_p F(x_{i_j})$ ,  $\forall j \in \mathbb{N}$ , it follows for the limit points that also  $F(y) + \epsilon \leq_p F(x)$ , which is a contradiction to  $x \in I$ . Thus, we have

$$P_{Q,\epsilon}^\circ = \{x \in Q \mid \nexists y \in Q : F(y) + \epsilon \leq_p F(x)\}. \quad (9)$$

Having stated the analytical expressions for the closure and the interior of  $P_{Q,\epsilon}$ , we obtain for the boundary

$$\begin{aligned} \partial P_{Q,\epsilon} &= \overline{P_{Q,\epsilon}} \setminus P_{Q,\epsilon}^\circ \\ &= \{x \in Q \mid \exists y_1 \in Q : F(y_1) + \epsilon \leq_p F(x) \text{ and } \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\} \end{aligned} \quad (10)$$

Since by (A1) the point  $y_1$  in (10) must be in  $P_Q$ , we obtain

$$\partial P_{Q,\epsilon} = \{x \in Q \mid \exists y_1 \in P_Q : F(y_1) + \epsilon \leq_p F(x) \text{ and } \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\} \quad (11)$$

It remains to show the second claim. It is  $\overline{P_{Q,\epsilon}} = P_{Q,\epsilon}^\circ \cup \partial P_{Q,\epsilon}$ . Assume that  $\overline{P_{Q,\epsilon}} \neq \overline{P_{Q,\epsilon}^\circ}$ , i.e., that there exists an  $x \in \partial P_{Q,\epsilon}$  and a neighborhood  $U$  of  $x$  such that  $U \cap P_{Q,\epsilon}^\circ = \emptyset$ . Since  $x \in \partial P_{Q,\epsilon}$  there exists a point  $y \in P_Q$  such that  $F(y) + \epsilon \leq_p F(x)$ . By assumption it is  $q(x) \neq 0$ , and thus there exists an  $\bar{x} \in U$  such that  $F(\bar{x}) <_p F(x)$ . Since  $\bar{x} \notin P_{Q,\epsilon}^\circ$  there exists an  $\bar{y} \in Q$  such that  $F(\bar{y}) + \epsilon \leq_p F(\bar{x}) <_p F(x)$ , which contradicts the assumption that  $x \in \partial P_{Q,\epsilon}$ . Thus, we have that the closure of the interior of  $P_{Q,\epsilon}$  is equal to its closure as claimed.

**Remarks 3.6** (a) Note that in general  $P_{Q,\epsilon}$  is neither an open nor a closed set, and that  $\overline{P_{Q,\epsilon}}$  gets ‘completed’ by  $-\epsilon$  weak Pareto points (see also Example 1).

(b) Since for  $x$  and  $y_1$  in equation (11) it must hold that there exists an index  $j \in \{1, \dots, k\}$  such that

$$f_j(y_1) + \epsilon_j = f_j(x).$$

Thus, the boundary of  $P_{Q,\epsilon}$  can be characterized by the set of  $-\epsilon$  weak Pareto points which are bounded in objective space from  $P_Q$  by  $\epsilon$ .

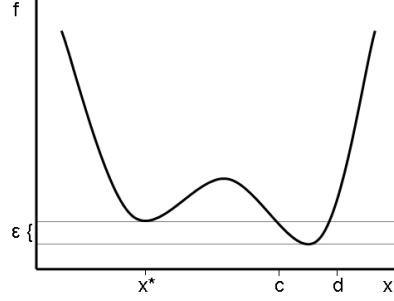


Figure 3: Example of a set  $P_{Q,\epsilon}$  where the closure of its interior is not equal to its closure.

The next example shows that the closure of the interior of  $P_{Q,\epsilon}$  does in general not have to be equal to its closure, which causes trouble to approximate  $\partial P_{Q,\epsilon}$  using stochastic search algorithms. However, the following Lemma shows that this is – despite for theoretical investigations – not problematic since  $\overset{\circ}{P}_{Q,\epsilon}$ , which can be approximated in any case, already contains all the interesting parts.

**Example 3.7** Figure 3 shows an example which is a modification of the MOP in Example 3.4 (a). We have  $P_{Q,\epsilon} = \{x^*\} \cup [c, d]$  and hence

$$\overline{\overset{\circ}{P}_{Q,\epsilon}} = [c, d] \neq \overline{P_{Q,\epsilon}}.$$

Note that here we have  $f'(x^*) = 0$ , and thus that (A3) is violated. The problem with the approximation of the entire set  $P_{Q,\epsilon}$  in this case is the following: assume that  $\text{argmin} f$  is already a member of the archive, then every candidate solution near  $x^*$  will be rejected by all further archives. Thus, the entire set  $P_{Q,\epsilon}$  can only be approximated if  $x^*$  is a member of a population  $P_i, i \in \mathbb{N}$ , and the probability for this event is zero. Such problems do not occur for points in  $\overset{\circ}{P}_{Q,\epsilon}$  (see proof of Theorem 3.9).

**Lemma 3.8** The following inclusions hold:

$$(a) P_Q \subset \overset{\circ}{P}_{Q,\epsilon}$$

$$(b) P_{Q,\epsilon}^c := \{x \in Q \mid \exists p \in P_Q : x \prec_\epsilon p\} \subset \overset{\circ}{P}_{Q,\epsilon}$$

*Proof:* It is of course sufficient to prove (b) – we have added the claim in (a) only due to the significance of  $P_Q$ . For this, assume there exists an  $x \in P_{Q,\epsilon}^c \setminus \overset{\circ}{P}_{Q,\epsilon}$ . Since  $x \in P_{Q,\epsilon}^c$  there

exists a Pareto optimal point  $p \in P_Q$  with  $p \prec_\epsilon p$ . Further, since  $x \notin P_{Q,\epsilon}^\circ$  there exists an  $y \in Q$  such that  $F(y) + \epsilon \leq_p F(x)$ . Combining both we obtain

$$\begin{aligned} F(y) \leq_p F(x) - \epsilon \leq F(p), \quad \text{and} \\ \exists j \in \{1, \dots, k\} : f_j(y) \leq f_j(x) - \epsilon < f_j(p) \quad (\Rightarrow F(y) \neq F(p)), \end{aligned} \quad (12)$$

which means that  $y \prec p$ , a contradiction to  $p \in P_Q$ , and we are done.

Having analyzed the topology of  $P_{Q,\epsilon}$  we are now in the position to state the following result. The archiving strategy is simply the one which keeps all obtained points which are not  $-\epsilon$ -dominated by any other test point.

**Theorem 3.9** *Let an MOP  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be given, where  $F$  is continuous, let  $Q \subset \mathbb{R}^n$  be a compact set and  $\epsilon \in \mathbb{R}_+^k$ . Further let*

$$\forall x \in Q \text{ and } \forall \delta > 0 : P(\exists l \in \mathbb{N} : P_l \cap B_\delta(x) \cap Q \neq \emptyset) = 1 \quad (13)$$

Then, under the assumptions made in Lemma 3.5, an application of Algorithm 1, where

$$\text{ArchiveUpdate}_{P_{Q,\epsilon}}(P, A) := \{x \in P \cup A : y \not\prec_{-\epsilon} x \ \forall y \in P \cup A\}, \quad (14)$$

is used to update the archive, leads to a sequence of archives  $A_l, l \in \mathbb{N}$ , with

$$\lim_{l \rightarrow \infty} d_H(P_{Q,\epsilon}, A_l) = 0, \quad \text{with probability one.} \quad (15)$$

*Proof:* Since  $\text{dist}(A, B) = \text{dist}(\overline{A}, \overline{B})$  for all sets  $A, B \subset \mathbb{R}^n$  and since  $\overline{P_{Q,\epsilon}^\circ} = \overline{P_{Q,\epsilon}}$  (see Lemma 3.5), it is sufficient to show that the Hausdorff distance between  $A_l$  and  $\overline{P_{Q,\epsilon}}$  vanishes in the limit with probability one.

First we show that  $\text{dist}(A_l, P_{Q,\epsilon}^\circ) \rightarrow 0$  with probability one for  $l \rightarrow \infty$ . It is

$$\text{dist}(A_l, P_{Q,\epsilon}^\circ) = \max_{a \in A_l} \inf_{p \in P_{Q,\epsilon}^\circ} \|a - p\|.$$

We have to show that every  $x \in Q \setminus \overline{P_{Q,\epsilon}}$  will be discarded (if added before) from the archive after finitely many steps, and that this point will never be added further on.

Let  $x \in Q \setminus \overline{P_{Q,\epsilon}}$ . Since  $x$  is by assumption (A2) not a  $-\epsilon$ -weak Pareto point, there exists a point  $p \in P_{Q,\epsilon}$  such that  $F(p) + \epsilon <_p F(x)$ . Further, since  $F$  is continuous there exists a neighborhood  $U$  of  $x$  such that

$$F(p) + \epsilon <_p F(u), \quad \forall u \in U. \quad (16)$$

By (13) it follows that there exists with probability one a number  $l_0 \in \mathbb{N}$  such that there exists a point  $x_{l_0} \in P_{l_0} \cap U \cap Q$ . Thus, by construction of  $\text{ArchiveUpdate}_{P_{Q,\epsilon}}$ , the point

$x$  will be discarded from the archive if it is a member of  $A_{l_0}$ , and never be added to the archive further on.

Now we consider the limit behavior of  $\text{dist}(P_{P,\epsilon}, A_l)$ . It is

$$\text{dist}(P_{Q,\epsilon}, A_l) = \sup_{p \in P_{Q,\epsilon}} \min_{a \in A_l} \|p - a\|.$$

Let  $\bar{p} \in P_{Q,\epsilon}$ . For  $i \in \mathbb{N}$  there exists by (13) a number  $l_i$  and a point  $p_i \in P_{l_i} \cap B_{1/i}(\bar{p}) \cap Q$ , where  $B_\delta(p)$  denotes the open ball with center  $p$  and radius  $\delta \in \mathbb{R}_+$ . Since  $\lim_{i \rightarrow \infty} p_i = \bar{p}$  and since  $\bar{p} \in P_{Q,\epsilon}$  there exists an  $i_0 \in \mathbb{N}$  such that  $p_i \in P_{Q,\epsilon}$  for all  $i \geq i_0$ . By construction of  $\text{ArchiveUpdate}P_{Q,\epsilon}$ , all the points  $p_i, i \geq i_0$ , will be added to the archive (and never discarded further on). Thus, we have  $\text{dist}(\bar{p}, A_l) \rightarrow 0$  for  $l \rightarrow \infty$  as desired, which completes the proof.

**Remarks 3.10** (a) *In order to obtain a ‘complete’ convergence result we have postulated some (mild) assumptions in order to guarantee that  $\overline{P_{Q,\epsilon}} = \overline{P_{Q,\epsilon}}$ , which is in fact an important topological property needed for the proof. However, if we drop the assumptions we can still expect that the interior of  $P_{Q,\epsilon}$  – the ‘interesting’ part (see Lemma 3.8) – will be approximated in the limit. To be more precise, regardless of assumptions (A1)–(A3) it holds in the above theorem that*

$$\lim_{l \rightarrow \infty} \overline{\text{dist}(P_{Q,\epsilon}, A_l)} = 0, \quad \text{with probability one.}$$

(b) *Note the analogy of the approach proposed above to one approach to approximate the Pareto front (compare to [7]): in case all the nondominated solutions which are found so far are kept in the archive, i.e., when*

$$\text{ArchiveUpdateND}(A, P) := \{x \in A \cup P : y \not\prec x \forall y \in A \cup P\}$$

*is used, one can show that – under similar assumptions as in Theorem 3.9 – an application of Algorithm 1 leads to a sequence of archives  $\{A_i\}_{i \in \mathbb{N}}$ , such that*

$$\lim_{l \rightarrow \infty} d_H(F(P_Q), F(A_l)) = 0 \quad \text{with probability one,}$$

*where  $P_Q$  denotes the Pareto set of  $F|_Q$ , i.e.*

$$P_Q = \{x \in Q \mid \nexists y \in Q : y \prec x\}.$$

## 4 Numerical Results

Here we demonstrate the practicability of the novel archiver on two examples. For this, we compare  $\text{ArchiveUpdate}P_{Q,\epsilon}$  against the ‘classical’ archiving strategy which stores all

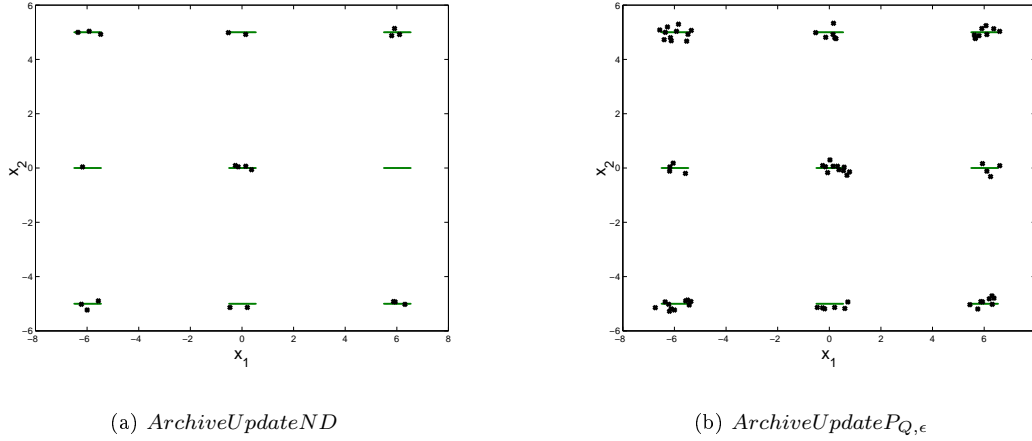


Figure 4: Numerical result for MOP (18) in parameter space.

nondominated solutions obtained during the search procedure (*ArchiveUpdateND*). To obtain a fair comparison of the two archivers we have decided to take a random search operator for the generation process (the same sequence of points for all settings).

#### 4.1 Example 1

First consider the following MOP proposed in [5]:

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$F(x_1, x_2) = \begin{pmatrix} (x_1 - t_1(c + 2a) + a)^2 + (x_2 - t_2b)^2 \\ (x_1 - t_1(c + 2a) - a)^2 + (x_2 - t_2b)^2 \end{pmatrix}, \quad (17)$$

where

$$t_1 = \text{sgn}(x_1) \min \left( \left\lceil \frac{|x_1| - a - c/2}{2a + c} \right\rceil, 1 \right), t_2 = \text{sgn}(x_2) \min \left( \left\lceil \frac{|x_2| - b/2}{b} \right\rceil, 1 \right).$$

The Pareto set consists of nine connected components of length  $a$  with identical images. We have chosen the values  $a = 0.5$ ,  $b = c = 5$ ,  $\epsilon = (0.1, 0.1)$ , the domain  $Q = [-20, 20]^2$ , and  $N = 10,000$  randomly chosen points within  $Q$ . Figure 4 displays two typical results in parameter space. It is obvious that the approximation quality obtained by *ArchiveUpdateP<sub>Q,ε</sub>* is better than the one obtained by *ArchiveUpdateND* in the Hausdorff sense. Such an advantage does not hold – as expected – for the comparison in the image space (Figure 5).

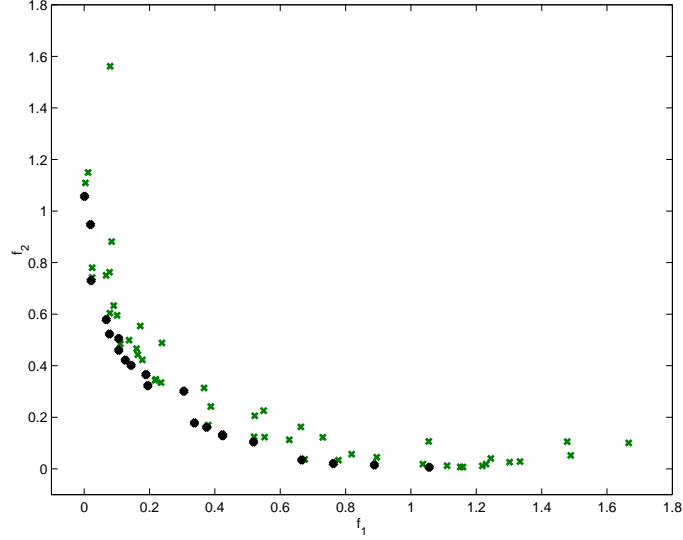


Figure 5: Comparison of the result of both archivers in objective space.

## 4.2 Example 2

Finally, we consider the MOP suggested by Tanaka ([10]):

$$\begin{aligned} F : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ F(x_1, x_2) &= (x_1, x_2) \end{aligned} \tag{18}$$

where

$$\begin{aligned} C_1(x) &= x_1^2 + x_2^2 - 1 - 0.1 \cos(16 \arctan(x_1/x_2)) \geq 0 \\ C_2(x) &= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5 \end{aligned}$$

Figure 6 shows two comparisons for  $N = 10,000$  and  $N = 100,000$  points within  $Q = [0, \pi]^2$  as domain<sup>1</sup>. Since  $F$  is injective the additional nondominated solutions kept in  $ArchiveUpdateP_{Q,\epsilon}$  can in this case not help to detect a further connected component. However, in both figures these additional points seemingly help to get a better understanding of the explored Pareto front.

<sup>1</sup>To fit into our framework, we consider in fact the (compact) domain  $Q' := [0, \pi]^2 \cap \{x \in \mathbb{R}^n : C_1(x) \geq 0 \text{ and } C_2(x) \leq 0.5\}$ .



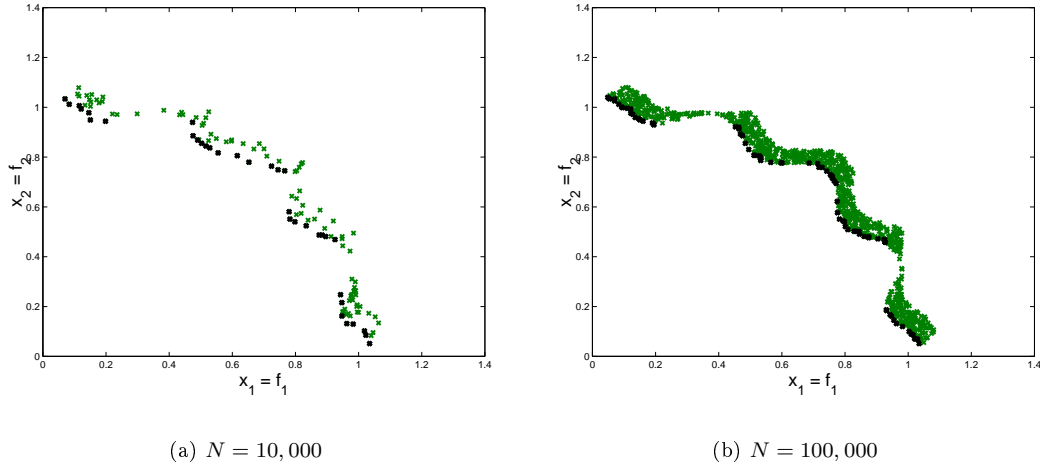


Figure 6: Numerical result for MOP (18) using  $\epsilon = (0.1, 0.1)$ .

## 5 Conclusion and Future Work

We have proposed and investigated a novel archiving strategy which allows – under mild assumptions on the generation process – to approximate the set  $P_{Q,\epsilon}$  which contains all  $\epsilon$ -efficient solutions within a compact domain  $Q$  and thus also the entire Pareto set of a given MOP within  $Q$ . We have proven the convergence of a generic stochastic search algorithm toward this set in the probabilistic sense, and have given two examples indicating the usefulness of the approach.

Since the set of  $\epsilon$ -dominated solutions forms an  $n$ -dimensional object, a suitable finite size representation of  $P_{Q,\epsilon}$  and the related archiving strategy are of major interest for further investigations.

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