Induction for Positive Almost Sure Termination - Extended version -

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ABSTRACT

In this paper, we propose an inductive approach to prove positive almost sure termination of probabilistic rewriting under the innermost strategy. We extend to the probabilistic case a technique we proposed for termination of usual rewriting under strategies. The induction principle consists in assuming that terms smaller than the starting terms for an induction ordering are positively almost surely terminating. The proof is developed in generating proof trees, modelizing rewriting trees, in alternatively applying abstraction steps, expressing the application of the induction hypothesis, and narrowing steps, simulating the possible rewriting steps after abstraction. This technique is fully automatable for rewrite systems on constants, very useful to modelize probabilistic protocols.

Categories and Subject Descriptors

F.3.1 [**LOGICS AND MEANINGS OF PROGRAMS**]: Specifying and Verifying and Reasoning about Programs—*Logics of programs, Mechanical verification, Specification techniques*; F.4.2 [**MATHEMATICAL LOGIC AND FORMAL LANGUAGES**]: Grammars and Other Rewriting Systems; F.4.3 [**MATHEMATICAL LOGIC AND FORMAL LANGUAGES**]: Formal Languages—*Algebraic language theory*; G.3 [**PROBABILITY AND STATISTICS**]; I.1.3 [**SYMBOLIC AND ALGEBRAIC MANIPULATION**]: Languages and Systems—*Evaluation strategies*; I.2.3 [**ARTIFICIAL INTELLIGENCE**]: Deduction and Theorem Proving—*Deduction, Inference engines, Mathematical induction*; D.3.1 [**PROGRAMMING LANGUAGES**]: Formal Definitions and Theory; D.2.4 [**SOFTWARE ENGINEERING**]: Software/Program Verification—*Correctness proofs, Formal methods, Validation*

General Terms

Algorithms, Languages, Verification

Keywords

Abstraction, Constraint, Narrowing, Probability, Termination

1. INTRODUCING THE PROBLEM

Probabilistic rewriting has recently been introduced to modelize systems, where probabilistic and undeterministic phenomena are combined [5]. A lot of models of systems, formalisms or techniques have already been enriched with probabilities, but most of them are restricted to finite state systems. Let us cite automata based models [7, 34], Petri Nets [2, 30], process algebra [18], model checking techniques [22]. Note also the existence of the PRISM [23], and the APMC [19] tools. Rewriting allows for expressing complex relations on infinite sets of states in a finite way, provided they are countable.

In the context of probabilistic rewriting, the problem of termination naturally arises and in [3], the notions of simple almost sure termination and positive almost sure (PAS in short) termination have been proposed, as well as a method based on interpretations on the reals to ensure the second property. The first termination notion expresses that the probability for a given rewriting derivation to terminate is 1; the second, stronger and more useful from a practical point of view, expresses that the mean length of the derivations from a term is finite.

Then, in [4], rewriting strategies have been considered, and sufficient criterions, still based on interpretations on the reals, have been given for PAS termination under strategies.

Here, we try to go one step further. In the previously cited paper, the considered strategies defined themselves with probabilities, expressing the ratio of the selection of a rule w.r.t to another. We tackle here the PAS termination problem for position strategies, defined by the position of the redexes in the terms to be rewritten, using an inductive approach we proposed for proving termination of non probabilistic rewriting under the innermost [10], the outermost [11] and local strategies [9]. In this paper, we adapt our inductive technique to the probabilistic case, investigate how it then works, and give a class of systems for which it is of interest.

We focus here on the innermost strategy, consisting in always rewriting at the lowest possible positions. This strategy is widely used in programming. It is often used as a built-in mechanism in the evaluation of rule-based or functional languages. In addition, for non-overlapping or locally confluent overlay systems [15], or systems satisfying critical

peak conditions [16], innermost termination is equivalent to standard termination (i.e. termination for standard rewriting, which consists in rewriting without any strategy). Note that as proved in [20], termination of rewriting is equivalent for the leftmost innermost and the innermost strategies.

A formalism has recently been proposed to extend the Constraint Handling Rule process with probabilistic capabilities applied to the rewrite rules themselves [12, 26, 27, 28, 29]. This is, to our knowledge, the only alternative attempt to formalize probabilistic transitions using rule based languages. Notice that these papers do not focus on techniques for proving termination of such systems.

There are other works about termination with probabilities, but in the context of concurrent programs [32, 31]. They deal with almost sure termination, whereas we deal with positive almost sure termination.

The basic idea of our approach is the following. We introduce the notion of innermost PAS (IPAS in short) termination for a term, and suppose, for every term t of a ground term algebra, that the terms smaller than t for an induction ordering are IPAS terminating. We then try to deduce that t is also IPAS terminating. The principle of our inductive method lies on a double mechanism allowing to generate proof trees, which represent, by a lifting mechanism, the rewriting trees of the ground terms: abstraction and narrowing.

The paper is structured as follows. In Section 2, the background is presented. Section 3 is devoted to definitions of probabilistic rewriting. In Section 4, the material for our inductive technique in the probabilistic case is defined. Section 5 gives the algorithm generating proof trees and the IPAS termination result for finite proof trees. Finally, Section 6 presents a generalization to a given class of infinite proof trees.

2. THE BACKGROUND

We assume that the reader is familiar with the basic definitions and notations of term rewriting given for instance in [1, 8, 33]. $\mathcal{T}(\mathcal{F},\mathcal{X})$ is the set of terms built from a given finite set $\mathcal{F}$ of function symbols f having arity $n \in \mathbb{N}$, and a set $\mathcal{X}$ of variables denoted $x, y \ldots$. $\mathcal{T}(\mathcal{F})$ is the set of ground terms (without variables). The terms reduced to a symbol of arity 0 are called *constants*. Positions in a term are represented as sequences of integers. The empty sequence ϵ denotes the top position. Let p and p' be two positions. The position p is said to be (a strict) prefix of p' (and p' suffix of p) if $p' = p\lambda$, where λ is a non-empty sequence of integers. For a position p of a term t, we note $t|_p$ the subterm of t at position p, and $t[s]_p$ the term obtained in replacing by s the subterm at position p in t.

A substitution is an assignment from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F},\mathcal{X})$, written $\sigma = (x = t, \ldots, y = u)$. It uniquely extends to an endomorphism of $\mathcal{T}(\mathcal{F},\mathcal{X})$. The result of applying σ to a term $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is written $\sigma(t)$ or σt. The domain of σ, denoted $Dom(\sigma)$ is the finite subset of $\mathcal{X}$ such that $\sigma x \neq x$. The range of σ, denoted $Ran(\sigma)$, is defined by $Ran(\sigma) = \bigcup_{x \in Dom(\sigma)} Var(\sigma x)$. An instantiation or ground substitution is an assignment from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F})$. Id denotes the identity substitution. The composition of substitutions σ_1 followed by σ_2 is denoted $\sigma_2\sigma_1$.

A set $\mathcal{R}$ of rewrite rules or rewrite system (RS in short) on $\mathcal{T}(\mathcal{F},\mathcal{X})$ is a set of pairs of terms of $\mathcal{T}(\mathcal{F},\mathcal{X})$, denoted $l \to r$, such that $Var(r) \subseteq Var(l)$. Given a rewrite system $\mathcal{R}$, a function symbol in $\mathcal{F}$ is called a *constructor* iff it does not occur in $\mathcal{R}$ at the top position of a left-hand side of rule, and is called a *defined function symbol* otherwise. The set of constructors of $\mathcal{F}$ for $\mathcal{R}$ is denoted $\mathcal{C}_R$, and the set of defined function symbols $\mathcal{D}_R$ ($\mathcal{R}$ is omitted when there is no ambiguity). In this paper, we only consider finite sets of function symbols and of rewrite rules.

The rewriting relation induced by $\mathcal{R}$ is denoted by $\to^{\mathcal{R}}$ ($\to$ if there is no ambiguity on $\mathcal{R}$), and defined by $s \to t$ iff there is a substitution σ and a position p in s such that $s|_p = \sigma l$ for some rule $l \to r$ of $\mathcal{R}$, and $t = s[\sigma r]_p$. This is written $s \to^{\mathcal{R}}_{p,l\to r,\sigma} t$ where p, $l \to r$, σ or $\mathcal{R}$ may be omitted; $s|_p$ is called a redex. The reflexive transitive closure of the rewriting relation induced by $\mathcal{R}$ is denoted by $\overset{*}{\to}^{\mathcal{R}}$. The innermost rewriting relation consists in always rewriting at the lowest possible positions.

Let $\mathcal{R}$ be a rewrite system on $\mathcal{T}(\mathcal{F},\mathcal{X})$. A term t is *narrowed* into t', at the non-variable position p, using the rewrite rule $l \to r$ of $\mathcal{R}$ and the substitution σ, when σ is a most general unifier of $t|_p$ and l, and $t' = \sigma(t[r]_p)$. This is denoted $t \leadsto^{\mathcal{R}}_{p,l\to r,\sigma} t'$ where p, $l \to r$, σ or $\mathcal{R}$ may be omitted. It is always assumed that there is no variable in common between the rule and the term, i.e. that $Var(l) \cap Var(t) = \emptyset$.

An ordering $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ is said to be noetherian iff there is no infinitely decreasing chain for this ordering. It is monotone iff for any pair of terms t, t' of $\mathcal{T}(\mathcal{F},\mathcal{X})$, for any context $f(\ldots\ \ldots)$, $t \succ t'$ implies $f(\ldots t \ldots) \succ f(\ldots t' \ldots)$. It has the subterm property iff for any t of $\mathcal{T}(\mathcal{F},\mathcal{X})$, $f(\ldots t \ldots) \succ t$.

For $\mathcal{F}$ and $\mathcal{X}$ finite, if $\succ$ is monotone and has the subterm property, then it is noetherian [21]. If, in addition, $\succ$ is stable under substitution (for any substitution σ, any pair of terms $t, t' \in \mathcal{T}(\mathcal{F},\mathcal{X})$, $t \succ t'$ implies $\sigma t \succ \sigma t'$), then it is called a simplification ordering. A RS $\mathcal{R}$ (innermost) terminates if and only if every (innermost) derivation of the rewriting relation induced by $\mathcal{R}$ is finite. For any term t of $\mathcal{T}(\mathcal{F})$, t (innermost) terminates if and only if every (innermost) rewriting derivation starting from t is finite.

3. PROBABILISTIC REWRITING

A *σ-algebra* on a set Ω is a set of subsets of Ω which contains the empty-set, and is stable by countable union and complementation. In particular, the set of subsets is a natural σ-algebra for any countable set. A *measurable space* (Ω, σ) is a set with a σ-algebra on it. A *probability* is a function P from a σ-algebra to $[0, 1]$, which is countably additive, and such that $P(\Omega) = 1$. A triplet (Ω, σ, P) is called *a probability space*. For more details, see [17].

A *stochastic sequence on a set A* is a family $(X_i)_{i\in\mathbb{N}}$, of random variables defined on some fixed probability space (Ω, σ, P) with values on A.

DEFINITION 1 (PARS). *[3] Given some countable set S, we note $Dist(S)$ for the set of probability distributions on S: $\mu \in Dist(S)$ is a function $S \to [0, 1]$ that satisfies $\sum_{i\in S} \mu(i) = 1$.*

A probabilistic abstract reduction system (PARS) is a pair $\mathcal{A} = (A, \to)$ consisting of a countable set A and a relation $\to \subset A \times Dist(A)$. A state $a \in A$ with no μ such that $a \to \mu$ is said terminal.

A PARS is said deterministic if, for all a, there is at most one μ with $a \to \mu$. We denote $Dist(\mathcal{A})$ for the set of distributions μ with $a \to \mu$ for some a.

A *history* is a finite sequence $a_0a_1\cdots a_n$ of elements of the state space A. It is non-terminal if a_n is as well. A history expresses the evolution of a PARS.

DEFINITION 2 (DETERMINISTIC POLICY). *[3] A (deterministic) policy ϕ, that can also be called a (deterministic) strategy, is a function that maps non-terminal histories to distributions in such a way that $\phi(a_0a_1\cdots a_n) = \mu$ is always one (of the many possible) distribution μ with $a_n \to \mu$. A history is said realizable, if for all $i < n$, if μ_i denotes $\phi(a_0a_1\cdots a_i)$, one has $\mu_i(a_{i+1}) > 0$.*

The above definition assumes that strategies must be deterministic.

A *derivation of* $\mathcal{A}$ is then a stochastic sequence where the non-deterministic choices are given by some policy ϕ, and the probabilistic choices are governed by the corresponding distributions.

DEFINITION 3 (DERIVATIONS). *[3] A derivation π of $\mathcal{A}$ over policy ϕ is a stochastic sequence $\pi = (\pi_i)_{i\in\mathbb{N}}$ on the set $A \cup \{\bot\}$ (where $\bot$ is a new element: $\bot \notin A$) such that for all n,*

- $P(\pi_{n+1} = \bot|\pi_n = \bot) = 1$,
- $P(\pi_{n+1} = \bot|\pi_n = s) = 1$ *if $s \in A$ is terminal,*
- $P(\pi_{n+1} = \bot|\pi_n = s) = 0$ *if $s \in A$ is non-terminal,*
- *and for all $t \in A$:* $P(\pi_{n+1} = t|\pi_n = a_n, \pi_{n-1} = a_{n-1}, \ldots, \pi_0 = a_0) = \mu(t)$

whenever $a_0a_1\cdots a_n$ is a realizable non-terminal history and $\mu = \phi(a_0a_1\ \ldots a_n)$.

If a derivation is such that $\pi_n = \bot$ for some n, then $\pi_{n'} = \bot$ almost surely for all $n' \geq n$. Such a derivation is said to be *terminating*. If k is the greatest integer for which $\pi_k \neq \bot$, then π_k is called a normal form (of π_0). A non-terminating derivation is such that $\pi_n \in A$ ($\pi_n \neq \bot$) for all n.

The following definition is generalized from [4] to a class of policies Φ.

DEFINITION 4 (PAS TERMINATION). *A PARS $\mathcal{A} = (A, \to)$ will be said positively almost surely (PAS) terminating (under a class of strategies Φ) if for all policies ϕ ($\in \Phi$), for all states $a \in A$, the mean number of reduction steps before termination under policy ϕ starting from a, denoted by $T[a, \phi]$, is finite.*

DEFINITION 5 (PROBABILISTIC REWRITE SYSTEM). *[3] Given a set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$, a probabilistic rewrite rule is an element $l \to M$ of $\mathcal{T}(\mathcal{F}, \mathcal{X}) \times Dist(\mathcal{T}(\mathcal{F}, \mathcal{X}))$, such that for every $r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, if $M(r) > 0$, then $Var(r) \subseteq Var(l)$.*

A probabilistic rewrite system is a finite set $\mathcal{R}$ of probabilistic rewrite rules.

A probabilistic abstract reduction system $(\mathcal{T}(\mathcal{F}, \mathcal{X}), \to_{\mathcal{R}})$ over the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is associated to a probabilistic rewrite system where $\to_{\mathcal{R}}$ is defined as follows.

DEFINITION 6 (REDUCTION RELATION). *[3] The following PARS $(\mathcal{T}(\mathcal{F}, \mathcal{X}), \to)$ over terms is associated to a probabilistic rewrite system $\mathcal{R}$ as follows: $t \to_{\mathcal{R}} \mu$ iff there is a rule $l \to M = (r_1 : p_1, \ldots, r_k : p_k) \in \mathcal{R}$, some position p in t, some substitution σ, such that $t|_p = \sigma(l)$, and, for all t', $\mu(t') = \sum_{r_i, i\in[1..k]|t'=t[\sigma(r_i)]_p} M(r_i)$.*

For example, with the probabilistic rewrite rule $\{f(x, y) \to g(a) : 1/2|y : 1/2\}$ whose right hand side denotes the distribution with value $1/2$ on $g(a)$ and value $1/2$ on y, $f(b, c)$ rewrites to $g(a)$ with probability $1/2$, to c with probability $1/2$ and $f(b, g(a))$ rewrites to $g(a)$ with probability 1.

Innermost probabilistic rewriting consists in always applying the above definition at the lowest possible positions in the terms to be rewritten.

A probabilistic rewrite system $\mathcal{R}$ is innermost positively almost surely (IPAS) terminating if the associated PARS is PAS terminating under the class Φ_{Inn} of policies ϕ corresponding to innermost rewriting derivations.

A term t on which no rule of $\mathcal{R}$ applies is said to be in normal form for $\mathcal{R}$. If such a term t is on a(n) (innermost) rewriting derivation of a term u, then t is called (innermost) normal form of u, and is noted $u\downarrow$. To every rewriting derivation $t_0, t_1, \ldots, t_n = t_0\downarrow$ corresponds a derivation $\pi_0, \pi_1, \ldots, \pi_n, \pi_{n+1}, \ldots$ where $\pi_i = \bot$ for $i > n$.

Thus, in other words, a RS $\mathcal{R}$ is (innermost) PAS terminating if for every ϕ ($\in \Phi_{Inn}$) the mean length of derivations reaching a normal form is finite.

4. INDUCTIVELY PROVING POSITIVE ALMOST SURE TERMINATION

For proving that a probabilistic rewrite system on $\mathcal{T}(\mathcal{F})$ is IPAS terminating, we introduce a local notion of IPAS termination on terms, and prove this property for every term of $\mathcal{T}(\mathcal{F})$.

DEFINITION 7. *Let $\mathcal{R}$ be a probabilistic rewrite system on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. A term t of $\mathcal{T}(\mathcal{F})$ is said to be IPAS terminating if for every $\phi \in \Phi_{Inn}$, the mean number $T[t, \phi]$ of rewriting steps from t with $\mathcal{R}$ under the strategy ϕ before termination is finite.*

For proving that a term t of $\mathcal{T}(\mathcal{F})$ is IPAS terminating, we proceed by induction on $\mathcal{T}(\mathcal{F})$ with a noetherian ordering $\succ$, assuming the property for every t' such that $t \succ t'$. To warrant non emptiness of $\mathcal{T}(\mathcal{F})$, and a basis for the induction, we assume that $\mathcal{F}$ contains at least one constructor constant. The main intuition is to observe probabilistic rewriting derivations starting from a ground term $t \in \mathcal{T}(\mathcal{F})$ which is any instance of a pattern $g(x_1, \ldots, x_m) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, for some defined function symbol $g \in \mathcal{D}$, and variables $x_1, \ldots, x_m$. Proving the property of IPAS termination on ground terms amounts to proving that every ground instance of the patterns $g(x_1, \ldots, x_m)$ is IPAS terminating.

Rewriting derivations are simulated, using a lifting mechanism, by a proof tree developed from $g(x_1, \ldots, x_m)$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, for every $g \in \mathcal{D}$, by alternatively using two main concepts, namely narrowing and abstraction. More precisely, narrowing schematizes the rewriting possibilities of terms. Abstraction simulates the reduction of subterms in the derivations until these subterms become normal forms. It expresses the application of the induction hypothesis on these subterms: if they are IPAS terminating, with a mean number of reduction steps, they rewrite into a normal form.

The schematization of ground rewriting derivations is achieved through constraints. The nodes of the developed proof trees are composed of a current term of $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and a set of ground substitutions represented by a constraint progressively built along the successive abstraction and narrowing steps. Each node in an abstract tree schematizes a set of

ground terms: all ground instances of the current term, that are solutions of the constraint.

The constraint is in fact composed of two kinds of formulas: ordering constraints, set to warrant the validity of the inductive steps, and abstraction constraints combined to narrowing substitutions, which effectively define the relevant sets of ground terms.

For a term t of $\mathcal{T}(\mathcal{F},\mathcal{X})$ occurring in a proof tree issued from a reference term $t_{ref} = g(x_1,\ldots,x_m)$,

- first, the ground instances of some subterms $t|_j$ of t (characterized by the constraint associated to t) are supposed to be IPAS terminating, by the induction hypothesis, if $\theta t_{ref} \succ \theta t|_j$ for the induction ordering $\succ$ and for every θ solution of the constraint associated to t. They are replaced in t by *abstraction variables* X_j representing respectively any of their normal forms, implicitely corresponding to one of the normal forms they have when rewriting under any policy $\phi \in \Phi_{Inn}$. Reasoning by induction allows us to only suppose the existence of the normal forms *without explicitly computing them.* If the ground instances of the resulting term are IPAS terminating (either if the induction hypothesis can be applied to them, or if they can be proved IPAS terminating by other means, we will present later), then the ground instances of the initial term are IPAS terminating. Otherwise,
- the resulting term $u = t[X_j]_{\{i_1,\ldots,i_p\}}$ (where $i_1,\ldots,i_p$ are the abstraction positions in t) is narrowed in all possible ways into distributions μ, according to the possible instances of the X_j. This corresponds to rewriting ground instances of u (characterized by the constraint associated to u) according to all non-deterministic choices and all probabilistic choices. Thus, all policies $\phi \in \Phi_{Inn}$ are explicitely expressed by the narrowing mechanism.

Then IPAS termination of the ground instances of t is reduced to IPAS termination of the ground instances of the terms v of the distributions μ. Now, if $\theta t_{ref} \succ \theta v$ for every ground substitution θ that is a solution of the constraint associated to v, by the induction hypothesis, θv is supposed to be IPAS terminating. Otherwise, the process is iterated on v, until we get a term t' such that either $\theta t_{ref} \succ \theta t'$, or $\theta t'$ can be proved IPAS terminating.

This technique is inspired from the one we proposed for proving innermost termination of non probabilistic rewrite systems.

We now introduce some concepts to formalize and automate this mechanism.

4.1 Ordering constraints

The induction ordering is constrained along the proof by inequalities between terms that must be comparable, each time the induction hypothesis is used in the abstraction mechanism.

This ordering is not defined a priori, but just has to verify inequalities of the form $t > u_1,\ldots,u_m$, accumulated along the proof, and which are called *ordering constraints.* Thus, for establishing the inductive termination proof, it is sufficient to decide whether ordering constraints are satisfiable.

Definition 8 (ordering constraint). An ordering constraint *is a pair of terms of* $\mathcal{T}(\mathcal{F},\mathcal{X})$ *noted* $(t > t')$. *It is said to be* satisfiable *if there is an ordering* $\succ$, *such that for every instantiation* θ *whose domain contains* $\mathcal{V}ar(t) \cup \mathcal{V}ar(t')$, *we have* $\theta t \succ \theta t'$. *We say that* $\succ$ *satisfies* $(t > t')$.

A conjunction C *of ordering constraints is satisfiable if there is an ordering satisfying all conjuncts. The empty conjunction, always satisfied, is denoted by* $\top$.

Satisfiability of a constraint conjunction C of this form is undecidable. But a sufficient condition for an ordering $\succ_{\mathcal{P}}$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ to satisfy C is that $t \succ_{\mathcal{P}} t'$ for every constraint $t > t'$ of C, and $\succ_{\mathcal{P}}$ is stable under substitution.

Simplification orderings fulfill such a condition. So, in practice, it is sufficient to find a simplification ordering $\succ_{\mathcal{P}}$ such that $t \succ_{\mathcal{P}} t'$ for every constraint $t > t'$ of C.

The ordering $\succ_{\mathcal{P}}$, defined on $\mathcal{T}(\mathcal{F},\mathcal{X})$, can then be seen as an extension of the induction ordering $\succ$ on $\mathcal{T}(\mathcal{F})$. For convenience sake, $\succ_{\mathcal{P}}$ will also be written $\succ$.

Solving ordering constraints in finding simplification orderings is a well-known problem. The simplest way and an automatable way to proceed is to test simple existing orderings like the subterm ordering, the Recursive Path Ordering, or the Lexicographic Path Ordering. This is often sufficient for the constraints considered here: thanks to the power of induction, they are often simpler than for termination methods directly using ordering for orienting rewrite rules.

If these simple orderings are not powerful enough, automatic solvers like C*i*me [1] can provide adequate polynomial orderings.

4.2 Abstraction

To abstract a term t at positions $i_1,\ldots,i_p$, where the $t|_j$ are supposed to have a normal form $t|_j\!\downarrow$, we replace the $t|_j$ by abstraction variables X_j representing respectively any of their possible normal forms for any policy $\phi \in \Phi_{Inn}$. Let us define these special variables more formally.

Definition 9. *Let* $\mathcal{N}$ *be a set of variables disjoint from* $\mathcal{X}$. *Symbols of* $\mathcal{N}$ *are called* abstraction variables. *Substitutions and instantiations are extended to* $\mathcal{T}(\mathcal{F},\mathcal{X}\cup\mathcal{N})$ *in the following way: for any substitution* σ *(resp. instantiation* θ*) such that* $Dom(\sigma)$ *(resp.* $Dom(\theta)$*) contains a variable* $X \in \mathcal{N}$, σX *(resp.* θX*) is in innermost normal form.*

Definition 10 (term abstraction). *The term* $t[t|_j]_{j\in\{i_1,\ldots,i_p\}}$ *is said to be* abstracted *into the term* u *(called* abstraction *of* t*) at positions* $\{i_1,\ldots,i_p\}$ *iff* $u = t[X_j]_{j\in\{i_1,\ldots,i_p\}}$, *where the* $X_j, j \in \{i_1,\ldots,i_p\}$ *are fresh distinct abstraction variables.*

Termination on $\mathcal{T}(\mathcal{F})$ is in fact proved by reasoning on terms with abstraction variables, i.e. on terms of $\mathcal{T}(\mathcal{F},\mathcal{X}\cup\mathcal{N})$. Ordering constraints are extended to pairs of terms of $\mathcal{T}(\mathcal{F},\mathcal{X}\cup\mathcal{N})$. When subterms $t|_j$ are abstracted by X_j, we state constraints on abstraction variables, called *abstraction constraints* to express that their instances can only be normal forms of the corresponding instances of $t|_j$. Initially, they are of the form $t\!\downarrow = X$ where $t \in \mathcal{T}(\mathcal{F},\mathcal{X}\cup\mathcal{N})$, and $X \in \mathcal{N}$, but we will see later how they are combined with the substitutions used for the narrowing process.

4.3 Narrowing

After abstracting the current term t into $t[X_j]_{j\in\{i_1,\ldots,i_p\}}$, we test whether the possible ground instances of $t[X_j]_{j\in\{i_1,}$

[1] Available at http://cime.lri.fr/

$_{...,i_p\}}$ are reducible, according to the possible values of the instances of the X_j. This is achieved by innermost narrowing $t[X_j]_{j\in\{i_1,...,i_p\}}$.

To schematize innermost rewriting on ground terms, we need to refine the usual notion of narrowing. In fact, with the usual innermost narrowing relation, if a position p in a term t is a narrowing position, no suffix position of p can be a narrowing position as well. However, if we consider ground instances of t, we can have rewriting positions p for some instances, and p' for other instances, such that p' is a suffix of p. So, when using the narrowing relation to schematize innermost rewriting of ground instances of t, the narrowing positions p to consider depend on a set of ground instances of t, which is defined by excluding the ground instances of t that would be narrowable at some suffix position of p. For instance, with the RS $R = \{g(a) \to a, f(g(x)) \to b\}$, the innermost narrowing positions of the term $f(g(X))$ are 1 with the narrowing substitution $\sigma = (X = a)$, and ϵ with any σ such that $\sigma X \neq a$.

Let σ be a substitution on $\mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{N})$. In the following, we identify σ with the equality formula $\bigwedge_i (x_i = t_i)$, with $x_i \in \mathcal{X} \cup \mathcal{N}$, $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{N})$. Similarly, we call *negation* $\overline{\sigma}$ of the substitution σ the formula $\bigvee_i (x_i \neq t_i)$.

Definition 11. *A substitution σ is said* to satisfy *a constraint $\bigwedge_j \bigvee_{i_j} (x_{i_j} \neq t_{i_j})$, iff for every ground instantiation θ, $\bigwedge_j \bigvee_{i_j} (\theta\sigma x_{i_j} \neq \theta\sigma t_{i_j})$. A* constrained substitution *σ is a formula $\sigma_0 \wedge \bigwedge_j \bigvee_{i_j} (x_{i_j} \neq t_{i_j})$, where σ_0 is a substitution, and $\bigwedge_j \bigvee_{i_j} (x_{i_j} \neq t_{i_j})$ the constraint to be satisfied by σ_0.*

Definition 12 (inner. proba. narrowing). *A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{N})$* innermost narrows *into a distribution μ at the non-variable position p, using the rule $l \to M = (r_1 : p_1, \ldots, r_k : p_k) \in \mathcal{R}$ with the constrained substitution $\sigma = \sigma_0 \wedge \bigwedge_{h\in[1..m]} \overline{\sigma_h}$, which is written $t \leadsto^{Inn}_{p,l\to M,\sigma} \mu = (v_1 : p'_1, \ldots, v_q : p'_q)$ iff*

- $\sigma_0(l) = \sigma_0(t|_p)$
- *for all $v_j, j \in [1..q], v_j = \sigma_0(t[r_i]_p)$ for some $i \in [1..k]$*
- $\mu(v_j) = p'_j = \sum_{r_i, i\in[1..k]|v_j=\sigma_0(t[r_i]_p)} M(r_i)$

where σ_0 is the most general unifier of t and l at position p, and $\sigma_h, h \in [1..m]$ are all the most general unifiers of $\sigma_0 t$ and a left-hand side of rule of $\mathcal{R}$, at suffix positions of p.

Notice that we are interested in the narrowing substitution applied to the current term t, but not in its definition on the variables of the left-hand side of the rule. So, the narrowing substitutions we consider are restricted to the variables of the narrowed term t.

4.4 Cumulating constraints

Abstraction constraints have to be combined with the narrowing substitutions to characterize the ground terms schematized by the current term t in the proof tree. Indeed, a narrowing branch on the current term u with narrowing substitution σ represents a rewriting branch for any ground instance of σu.

In addition, σ has to satisfy the constraints on variables of u, already set in A. So, σ, considered as the narrowing constraint attached to the narrowing branch, is added to A. This leads to the introduction of abstraction constraint formulas.

Definition 13. *An* abstraction constrained formula *(ACF in short) is a formula $\bigwedge_i (t_i\downarrow = t'_i) \wedge \bigwedge_j (x_j = u_j)$, where $x_j \in \mathcal{X} \cup \mathcal{N}$, $t_i, t'_i, u_j, \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{N})$.*

Definition 14. *An abstraction constrained formula $A = \bigwedge_i (t_i\downarrow = t'_i) \wedge \bigwedge_j (x_j = u_j)$ is* satisfiable *iff there is at least one instantiation θ such that $\bigwedge_i (\theta t_i\downarrow = \theta t'_i) \wedge \bigwedge_j (\theta x_j = \theta u_j)$. The instantiation θ is then said to satisfy the ACF A and is called solution of A.*

For a better readability on examples, we can propagate σ into A (by applying σ to A), thus getting instantiated abstraction constraints of the form $t_i\downarrow = t'_i$ from initial abstraction constraints of the form $t_i\downarrow = X_i$.

An ACF A is attached to each term u in the proof trees; the ground substitutions solutions of A define the instances of the current term u, for which we are observing IPAS termination. When A has no solution, the current node of the proof tree represents no ground term. Such nodes are then irrelevant for the proof. Detecting and suppressing them during a narrowing step allows us to control the narrowing mechanism, well known to easily diverge. So, we have the choice between generating only the relevant nodes of the proof tree, by testing the satisfiability of A at each step, or stopping the proof on a branch on an irrelevant node, by testing the unsatisfiability of A.

Checking the satisfiability of A is in general undecidable, but it is often easy in practice to exhibit an instantiation satisfying it. Automatable sufficient conditions are also under study. The unsatisfiability of A is also undecidable in general, but simple automatable sufficient conditions can be used [14], as to test whether A contains equalities $t\downarrow = u$, where u is reducible. In the following, we present the procedure exactly simulating the rewriting trees, i.e. dealing with the satisfiability of A.

5. THE ALGORITHM

We are now ready to describe the inference rules defining our mechanism. They transform a set T of 3-tuples (U, A, C) where $U = \{t\}$ or $\emptyset$, t is the current term whose ground instances have to be proved IPAS terminating, A is an abstraction constraint formula, C is a conjunction of ordering constraints.

- The first rule abstracts the current term t at given positions $i_1, \ldots, i_p$ into $t[X_j]_{j\in\{i_1,\ldots,i_p\}}$. The constraint $\bigwedge_{j\in\{i_1,\ldots,i_p\}} t_{ref} > t|_j$ is set in C. We do not need to associate any probability to the resulting term. The abstraction constraint $\bigwedge_{j\in\{i_1,\ldots,i_p\}} t|_j\downarrow = X_j$ is added to the ACF A. We call this rule **Abstract**.

  The abstraction positions are chosen so that the abstraction mechanism captures the greatest possible number of rewriting steps: then we abstract all of the greatest possible subterms of $t = f(t_1, \ldots, t_m)$. More concretely, we try to abstract $t_1, \ldots, t_m$ and, for each $t_i = g(t'_1, \ldots, t'_n)$ that cannot be abstracted, we try to abstract $t'_1, \ldots, t'_n$, and so on. In the worst case, we are driven to abstract leaves of the term, which are either variables, or constants.

  Note also that it is not useful to abstract non narrowable subterms of $\mathcal{T}(\mathcal{F}, \mathcal{N})$. Indeed, by Definition 9, every ground instance of such subterms is in normal form.

- The second rule narrows the resulting term u, if it is not a term of $\mathcal{T}(\mathcal{C},\mathcal{N})$, in all possible ways in one step, with all possible rewrite rules of the rewrite system $\mathcal{R}$, and all possible substitutions, into distributions $\mu_1,\ldots,\mu_n$, according to Definition 12. This step is a branching step, creating $q_1+\ldots+q_n=q'$ states, where $q_i, i\in[1..n]$ is the number of terms (with probability >0) in the distribution μ_i. The substitution σ is integrated to A. This is the **Narrow** rule.

  For example, if $\mathcal{R}$ is $\{f(x)\to g(x):1/2|h(x):1/2, f(a)\to a:1/10|b:9/10\}$ then the state $\{(f(X),A,C)\}$ generates the states $(\{g(X):1/2\},A\wedge\sigma_1,C)$, $(\{h(X):1/2\},A\wedge\sigma_1,C)$, $(\{a:1/10\},A\wedge\sigma_2,C)$, $(\{b:9/10\},A\wedge\sigma_2,C)$ with the respective associated narrowing substitutions $\sigma_1=Id$, $\sigma_1=Id$, $\sigma_2=(X=a)$, $\sigma_2=(X=a)$.

- We finally have a **Stop** rule halting the proof process on the current branch of the proof tree, when the ground instances of the current term can be stated as IPAS terminating. This happens when the whole current term u can be abstracted, i.e. when the induction hypothesis is applied to it, or when $u\in\mathcal{T}(\mathcal{F},\mathcal{N})$ and is not narrowable.

Let us note that the inductive reasoning can be completed as follows. When the induction hypothesis cannot be applied to a term u, it may be possible to prove IPAS termination of every ground instance of u in another way. Let $IPAST(u)$ be a predicate that is true iff every ground instance of u is IPAS terminating. In the previous first and third steps of the inductive reasoning, we then associate the alternative predicate $IPAST(u)$ to the condition $t>u$. It is true in particular when $u\in\mathcal{T}(\mathcal{F},\mathcal{N})$ and is not narrowable, as said above. Otherwise, we can use the notion of usable rule, as in [14].

The rules are given in Table 1. They use a reference term $t_{ref}=g(x_1,\ldots,x_m)$, where $x_1,\ldots,x_m\in\mathcal{X}$ and $g\in\mathcal{D}$ (if g is a constant, then $t_{ref}=g$). Note that, when a rule applies to a state, the current term has an associated probability if it has been generated by **Narrow**, and does not have any if it has been generated by **Abstract**. Hence the notation $\{t(:p)\}$ in Table 1.

We generate the proof trees of $\mathcal{R}$ by applying, for each defined symbol $g\in\mathcal{D}$, the inference rules using the reference term $t_{ref}=g(x_1,\ldots,x_m)$ on the initial set of 3-tuples $\{(\{t_{ref}=g(x_1,\ldots,x_m)\},\top,\top)\}$, with a specific strategy S, repeating the following steps: first, apply **Abstract**, and then try **Stop**. Then try all possible applications of **Narrow**. Then, try **Stop** again.

Let us clarify that if A is satisfiable, the transformed forms of A by **Abstract** and **Stop** are also satisfiable. Moreover, the first application of **Abstract** generates $A=(\bigwedge_i x_i{\downarrow}=X_i)$, always satisfied by the constructor constant supposed to exist in $\mathcal{F}$. Thus, with strategy S, it is useless to prove the satisfiability of A in the **Abstract** and **Stop** rules.

The process may not terminate if there is an infinite number of applications of **Abstract** and **Narrow** on the same branch of a proof tree. Nothing can be said in that case about termination. The process stops if no inference rule applies anymore. Then, when all branches of the proof trees end with an application of **Stop**, IPAS termination is established.

Given a proof tree, to every policy $\phi\in\Phi_{Inn}$ is associated a deterministic subtree of the proof tree, called ϕ-deterministic subtree of the proof tree, expressing only probabilistic choices. In practice, it is obtained by only considering, at every branching node, the branches corresponding to a same probabilistic narrowing step, for a given position and a given rule.

A finite proof tree or one of its subtrees is said to be *successful* if its leaves are states of the form $(\emptyset,A,C)$. We write $SUCCESS(g,\succ)$ if the application of S on $(\{g(x_1,\ldots,x_m)\},\top,\top)$ gives a successful proof tree, whose sets C of ordering constraints are satisfied by the same ordering $\succ$.

PROPOSITION 1. *Let $\mathcal{R}$ be a probabilistic rewrite system on $\mathcal{T}(\mathcal{F},\mathcal{X})$ having at least one constructor constant. If there is a noetherian ordering $\succ$ such that for each symbol $g\in\mathcal{D}$, we have $SUCCESS(g,\succ)$, then every term of $\mathcal{T}(\mathcal{F})$ is IPAS terminating.*

In the proof of Proposition 1, the information given by probabilities is not used. This means that for RS's whose proof trees are finite, our method works as in the non probabilistic case. This corroborates -and gives a formal proof of- the fact that if we remove the probabilities in a given RS, and replace the probabilistic choice by an undeterministic choice, innermost termination of the resulting RS implies IPAS termination of the initial probabilistic system. So the probabilistic extension of our inductive approach is of real interest for systems whose IPAS termination is due to a probabilistic argument on infinite rewriting chains. We investigate this case in the next section.

Example 1. The following RS

$$\begin{array}{ll} f(0,1,x) & \to f(x,x,x):1 \\ g(x,y) & \to x:1/10 \mid y:9/10 \end{array}$$

whose non probabilistic transformation:

$$\begin{array}{ll} f(0,1,x) & \to f(x,x,x) \\ g(x,y) & \to x \\ g(x,y) & \to y \end{array}$$

is well known to be innermost terminating, illustrates the above purpose.

Let us develop nevertheless the IPAS termination proof on the probabilistic RS to show how our technique works. The defined symbols of $\mathcal{F}$ are here f and g. Applying the rules on $f(x_1,x_2,x_3)$, we get:

Table 1: Inference rules for IPAS-temination

Abstract: $$\frac{\{t(:p)\},\ \ A,\ \ C}{\{u\},\ A \wedge \bigwedge_{j\in\{i_1,\ldots,i_p\}} t|_j\downarrow = X_j,\ C \wedge \bigwedge_{j\in\{i_1,\ldots,i_p\}} H_C(t|_j)}$$

where t is abstracted into u at positions $i_1,\ldots,i_p \neq \epsilon$
if $C \wedge H_C(t|_{i_1}) \ldots \wedge H_C(t|_{i_p})$ is satisfiable

Narrow: $$\frac{\{t(:p)\},\ \ A,\ \ C}{\{v_i : p_i\},\ \ A\wedge\sigma,\ \ C}$$

where $i \in [1..q]$
if $t \leadsto^{Inn}_{\sigma} \mu = (v_1 : p_1 \ldots v_q : p_q)$ and $A \wedge \sigma$ is satisfiable

Stop: $$\frac{\{t(:p)\},\ \ A,\ \ C}{\emptyset,\ \ A\wedge H_A(t),\ \ C\wedge H_C(t)}$$

if $(C \wedge H_C(t))$ is satisfiable.

and $H_A(t) = \begin{cases} \top & t \text{ is in } \mathcal{T}(\mathcal{F},\mathcal{N}) \text{ and is not narrowable} \\ t\downarrow = X & \text{otherwise.} \end{cases}$ $\quad H_C(t) = \begin{cases} \top & \text{if } IPAST(t) \\ t_{ref} > t & \text{otherwise.} \end{cases}$

$$\begin{array}{c} t_{ref} = f(x_1,x_2,x_3) \\ A = \top \\ C = \top \\ \downarrow \\ \textbf{Abstract} \\ \downarrow \\ f(X_1,X_2,X_3) \\ A = (x_1\downarrow = X_1 \wedge x_2\downarrow = X_2 \wedge x_3\downarrow = X_3) \\ C = (f(x_1,x_2,x_3) > x_1,x_2,x_3) \\ \downarrow{\scriptstyle \sigma=(X_1=0\wedge X_2=1)} \ \textbf{Narrow} \\ f(X_3,X_3,X_3) : 1 \\ A = (x_1\downarrow = 0 \wedge x_2\downarrow = 1 \wedge x_3\downarrow = X_3) \\ C = (f(x_1,x_2,x_3) > x_1,x_2,x_3) \\ \downarrow \\ \textbf{Stop} \\ \downarrow \\ \emptyset \\ A = (x_1\downarrow = X_1 \wedge x_2\downarrow = X_2 \wedge x_3\downarrow = X_3) \\ C = (f(x_1,x_2,x_3) > x_1,x_2,x_3) \end{array}$$

Abstract applies since $f(x_1,x_2,x_3) > x_1,x_2,x_3$ is satisfiable by any simplification ordering.

Narrow applies because $A \wedge \sigma = (x_1\downarrow = 0 \wedge x_2\downarrow = 1 \wedge x_3\downarrow = X_3)$, where $\sigma = (X_1 = 0 \wedge X_2 = 1)$, is satisfiable by any ground instantiation θ such that $\theta x_1 = 0$, $\theta x_2 = 1$ and $\theta x_3 = \theta X_3 = 0$.

Then **Stop** applies because $f(X_3,X_3,X_3)$ is a non narrowable term whose all variables are abstraction variables, and hence we have $IPAST$ $(f(X_3,X_3X_3))$.

Considering now $g(x_1,x_2)$, we get the proof tree in Table 2.

Abstract applies since $g(x_1,x_2) > x_1,x_2$ is satisfiable by any simplification ordering.

Narrow applies because $A\wedge\sigma = (x_1\downarrow = X_1 \wedge x_2\downarrow = X_2)$, where $\sigma = Id$, is satisfiable by any ground instantiation θ such that $\theta x_1 = \theta X_1 = 0$ and $\theta x_2 = \theta X_2 = 0$.

Then **Stop** applies on both branches because X_1 and X_2 are abstraction variables, hence we trivially have $IPAST(X_1)$ and $IPAST(X_2)$.

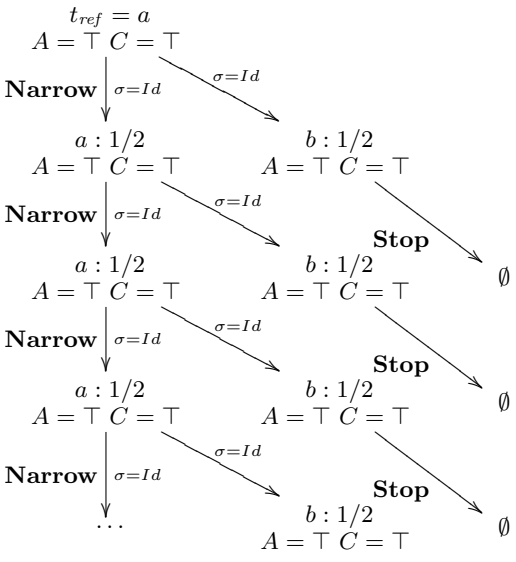
6. ONE STEP FURTHER: CONSIDERING INFINITE PROOF TREES

Consider the following RS $\mathcal{R}$, which is IPAS terminating, but not terminating.

Example 2. $\{a \to a : 1/2 \mid b : 1/2\}$.

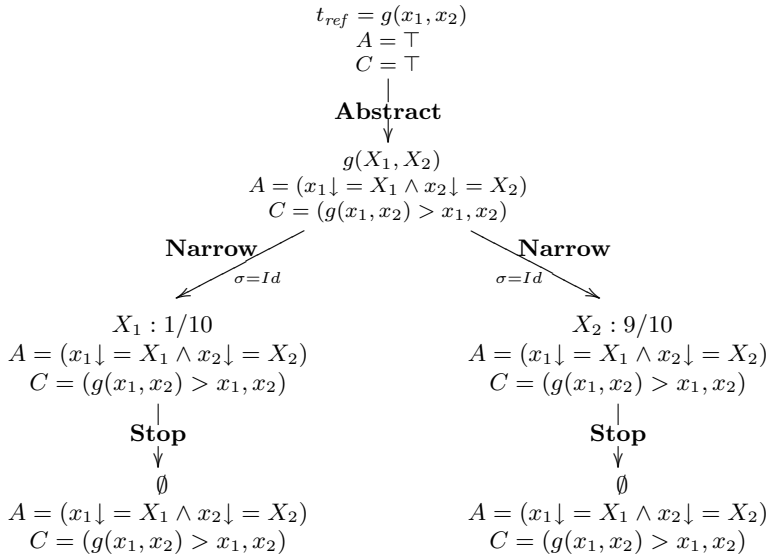
Here, innermost termination is equivalent to termination since we only have constants.

The only defined symbol of $\mathcal{R}$ is a. So the previous algorithm generates the unique following proof tree:



The first branch of this proof tree is infinite. Thanks to the lifting mechanism (obvious here since terms are constants), its represents the infinite rewriting branch of the derivation tree starting from a. All other possible branches

Table 2: Proof tree of the symbol g in Example 1



are finite. If we now consider the narrowing steps with the probabilities defined by the rule used, we observe that the infinite branch has the probability $1/2*1/2*1/2... = 0$. But for every possible ground term represented by t_{ref} (here, the only constant a), there is at least one finite branch. Then, by definition of IPAS termination, a is IPAS terminating. Let us now generalize and formalize this reasoning.

DEFINITION 15. *A proof tree, whose root state is noted s_0, is said infinitely successful if for every $\phi \in \Phi_{Inn}$, the ϕ-deterministic subtree of the proof tree either is successful or fulfills the following conditions:*

- *there is one branch starting from s_0 with two states s_m and s_n such that $s_n = s_m$,*
- *the states $s_i = (\{t_i : p_i\}, A_i, C_i)$ on this branch between s_m and s_n are such that $A_i = A_m$ and $C_i = C_m$,*
- *every state on this branch from s_0 until s_{n-1} has only brother states that are roots of successful subtrees.*

Note that this definition subsumes the previous definition of successful proof tree given in Section 5. Note also that it implies that the sequence $s_m, \ldots, s_n$ defines a cycle. Indeed, strategy S applies the inferences rules in the same way on two equal states. Moreover, the cycle is unique because of the third condition of the definition.

We write $I{-}SUCCESS(g, \succ)$ if the application of S on $(\{t_{ref} = g(x_1, \ldots, x_m)\}, \top, \top)$ gives an infinitely successful proof tree, whose sets C of ordering constraints are satisfied by the same ordering $\succ$.

THEOREM 1. *Let $\mathcal{R}$ be a probabilistic rewrite system on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ having at least one constructor constant. If there is a noetherian ordering $\succ$ such that*

- *for each symbol $g \in \mathcal{D}$, we have $I{-}SUCCESS(g, \succ)$,*
- *for the cycle $(s_i = (\{t_i : p_i\}, A_m, C_m), i \in [m..n]$ with $s_n = s_m)$, if it exists, of every ϕ-deterministic proof subtree of the proof trees, there is i such that $p_i < 1$,*

then every term of $\mathcal{T}(\mathcal{F})$ is IPAS terminating.

Consider now the branch from s_0 to s_n in Definition 15. We observe that if A and C do not change between s_m and s_n, then the **Abstract** rule has not been applied between the two states. Only the **Narrow** rule has been applied and with narrowing substitutions equal to Id (up to a variable renaming) on the given branch.

The following proposition defines a class of RS's fulfilling the above conditions on **Abstract** and **Narrow**.

PROPOSITION 2. *Let $\mathcal{R}$ be a RS. If the possible cycles in the ϕ-deterministic proof subtrees of the proof trees of $\mathcal{R}$ are such that:*

- *the first term of the cycle is of the form $f(x_1, \ldots, x_m)$ where the x_i are either variables or constructor constants, and f can be a constant,*
- *the successive rewrite rules of $\mathcal{R}$ used in the k* **Narrow** *steps of the cycle are of the form*

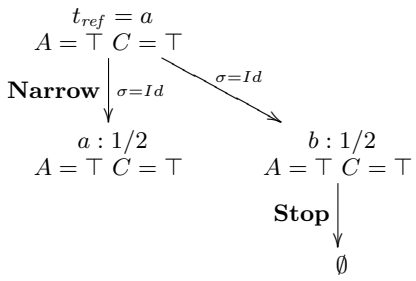
  $$f_j(x_1^j, \ldots, x_{m_j}^j) \to M_j = |_{i_j} t_{i_j} : p_{i_j} \; j \in [1..k]$$

  where $x_1^j, \ldots, x_{m_j}^j$ are also either variables or constructor constants, and the f_j can be constants,
- $f_1(x_1^1, \ldots, x_{m_1}^1) = f(x_1, \ldots, x_m)$
- *for $j \in [1..k-1]$, the term t_{i_j}, for some i_j, generated by the rule $f_j(x_1^j, \ldots, x_{m_j}^j) \to M_j = |_{i_j} t_{i_j} : p_{i_j}$ on the branch of the cycle is equal to $f_{j+1}(x_1^{j+1}, \ldots, x_{m_{j+1}}^{j+1})$ (if $k = 1$, this condition is void),*
- *the term t_{i_k}, for some i_k, generated by the rule $f_k(x_1^k, \ldots, x_{m_k}^k) \to M_k = |_{i_k} t_{i_k} : p_{i_k}$ on the branch of the cycle is equal to $f(x_1, \ldots, x_m)$.*

then, the only inference rule applied in the steps of the cycles is **Narrow***, and with narrowing substitutions equal to Id.*

An important subclass of this class is the class $\mathcal{A}$ of RS's on constants, like the RS of the previous example, whose (I)PAS termination can now be proved.

Thanks to Theorem 1, the proof tree just has to be developed as follows. The branch having a cycle is stopped as soon as the cycle is detected, i.e. when a same state arises twice on the branch.



Another important subclass of this class is the class $\mathcal{B}$ of RS's of the form

$$\{f_j(x_1^j, \ldots, x_{m_j}^j) \to M_j = |_{ij} t_{ij} : p_{ij},\ j \in [1..k]\}$$

where

- $x_1^j, \ldots, x_{m_j}^j$ are either variables or constructor constants, and the f_j can be constants,
- for each $j \in [1..k]$, at most one t_{i_j} is a left-hand side of rule $f_l(x_1^l, \ldots, x_{m_l}^l)$ for some $l \in [1..k]$, and the other t_{i_j} of M_j are not narrowable.

For this class, it can even be proved that all proof trees are infinitely successful (with any simplification ordering).

PROPOSITION 3. *Let $\mathcal{R} \in \mathcal{B}$. Then every proof tree of $\mathcal{R}$ is infinitely successful.*

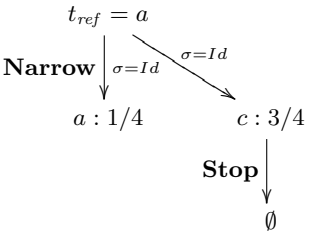
If in addition, for every rule, at least one of the t_{i_j} of M_j is not narrowable, the second condition of Theorem 1 is fulfilled, hence the following result.

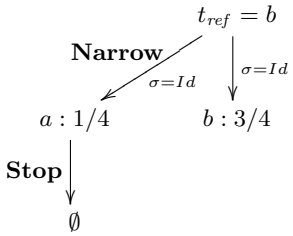
COROLLARY 1. *Let $\mathcal{R} \in \mathcal{B}$. If every rule of $\mathcal{R}$ hat at least a non narrowable term in the distribution of its right-hand side, then $\mathcal{R}$ is IPAS terminating.*

The previous example can also be proved (I)PAS terminating directly using Corollary 1.

This is not the case for the following RS, in the class $\mathcal{A}$ but not in $\mathcal{B}$, that requires to develop the proof trees.

Example 3. The RS $\{a \to a : 1/4 | c : 3/4, b \to a : 1/4 | b : 3/4\}$ is (I)PAS terminating. The proof trees are:

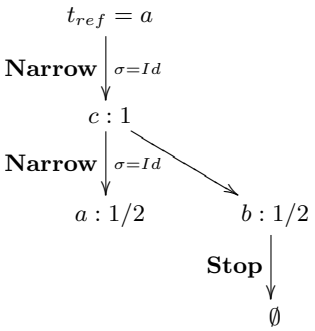


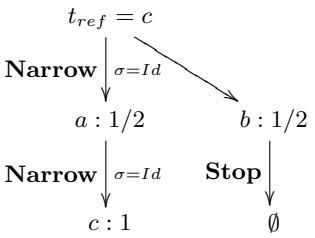


In the second proof tree, **Stop** applies on a because a can be supposed to be (I)PAST by setting $b > a$ for any noetherian ordering on constant terms.

Note that on such an example, where the inductive principle is crucial, the real interpretation technique of [3, 4], is very hard to apply. Because this technique involves arguments that are local to one rule, and are not modular w.r.t rules, this is also the case for examples where the cycle is generated by more than one rule like $\{a \to c : 1, c \to a : 1/2 | b : 1/2\}$, and that we easily handle.

Example 4. The two proof trees for $\{a \to c : 1, c \to a : 1/2 | b : 1/2\}$ are:





Example 5. Consider the RS $\{f(0, 1, x) \to f(0, 1, x) : 1/2 | f(x, x, x) : 1/2, g(x, y) \to x : 1/10 | y : 9/10\}$.

The proof tree of g is the same as in Example 1. The proof tree of f is given in Table 3.

Example 6. The following example involves constrained substitutions:
$\{f(g(x)) \to g(a) : 1/2 | c : 1/2, g(a) \to g(a) : 1/2 | c : 1/2\}$.

The proof tree of f is given in Table 4.

The proof tree of g is similar.

7. CONCLUSION

In this paper, we have studied the termination problem of probabilistic rewrite systems. We have adapted the inductive technique, which we had proposed for termination of rewriting under strategies, to the probabilistic case. It

Table 3: Proof tree of the symbol f in Example 5

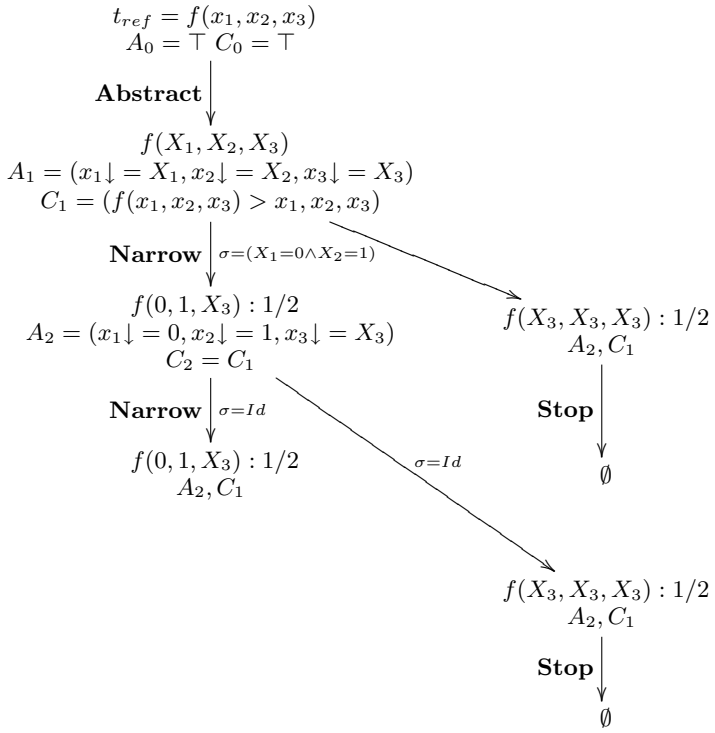
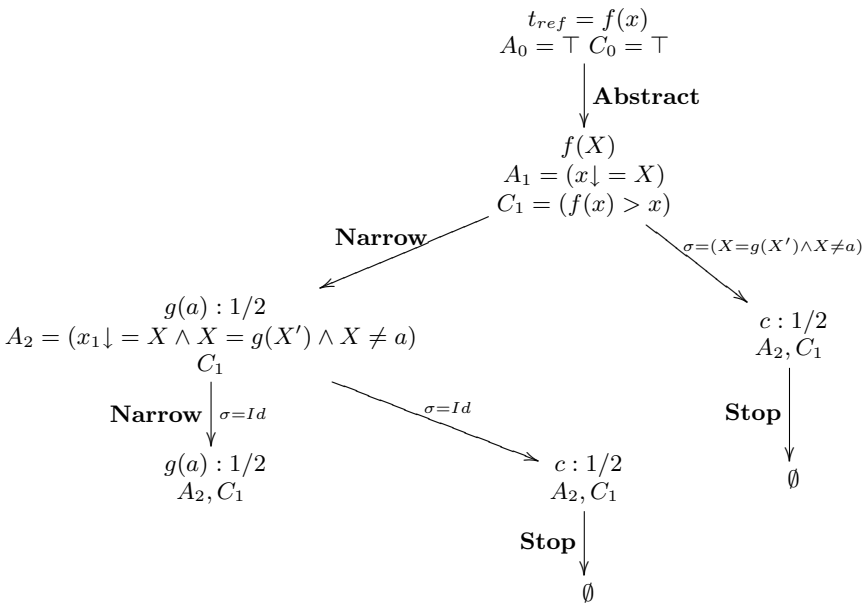


Table 4: Proof tree of the symbol f in Example 6



consists in generating proof trees modelizing rewriting trees on ground terms, by alternatively applying abstracting and narrowing steps. As a non-probabilistic RS can be seen as a probabilistic RS whose right-hand sides only have distributions with a unique term of probability 1, the theorem given here subsumes the results given in [10, 14] for termination under the innermost strategy.

We have also given a class of RS's for which this generalization is of interest. An interesting subclass of this class is composed by the RS's on constants, like the first three examples of Section 6. Indeed, constants can modelize states of automata used for expressing protocols, and it often happens that probabilistic protocols regularly fall in the same state when they evolve. It can then be crucial to prove that such cycling situations have a null probability of occurring. Our technique allows it to happen.

In a more general way, our application area can seem limited, because of the restricted form of the rules in cycles we tackle at the moment, but most randomized algorithms [25] or telecommunication protocols (e.g. CSMA-CA protocol [6]) based on probabilistic arguments rely on very simple arguments involving very simple probabilistic rewrite rules. The reasoning for these rules, however, is often difficult to do [25]. This paper provides a way to do inductive reasoning for probabilistic systems. As far as we know, there have not been many investigations on this subject.

Moreover, the completeness results of [3, 4], based on real interpretations, are nice from a theoretical point of view, but not constructive, and no algorithmic help exists yet, to exhibit ad-hoc interpretations.

To the contrary, our method is operational. Detecting a cycle as specified in Definition 15 is automatable. As said before, for our approach, there are sufficient conditions for testing the unsatisfiability of A, and C is often easy to satisfy. In the interesting case of RS's on constants, $A = C = \top$, and the method is completely automatable.

Finally, note the important fact that, if $\mathcal{R}$ is deterministic, innermost derivations are equivalent to standard derivations. So, our proof technique also establishes PAS termination of $\mathcal{R}$ for the standard strategy.

We now plan to generalize our theorem using infinite proof trees on a larger class of systems, and to investigate other techniques to ensure PAS termination.

8. ACKNOWLEDGMENTS

We would like to thank Olivier Bournez for fruitful discussions on termination of probabilistic rewriting and on this paper, and for giving us motivating examples.

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APPENDIX

Appendix

This appendix contains the proof of the propositions and of the theorem.

A. THE LIFTING LEMMA

For the proof of Proposition 1, we need the following lifting lemma, which is a generalization to the probabilistic case of the innermost instance of the lifting lemma given in [14].

LEMMA 1 (PROBA. INNERMOST LIFTING LEMMA). *Let $\mathcal{R}$ be a probabilistic rewrite system. Let $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, α a normalized ground substitution, and $\mathcal{Y} \subseteq \mathcal{X}$ a set of variables such that $Var(s) \cup Dom(\alpha) \subseteq \mathcal{Y}$. If $\alpha s \rightarrow^{Inn}_{p,l\rightarrow M} \nu = (t'_1 : p''_1, \ldots, t'_n : p''_n)$ with $M = (r_1 : p_1, \ldots, r_k : p_k)$, then there is a term $s' \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and substitutions $\beta, \sigma = \sigma_0 \wedge \bigwedge_{h\in[1..m]} \overline{\sigma_h}$ such that:*

1. $s \rightsquigarrow^{Inn}_{p,l\rightarrow M,\sigma} \mu = (s'_1 : p'_1, \ldots, s'_q : p'_q)$,
2. $q = n$ and for $i \in [1..n], p'_i = p''_i$
3. for $i \in [1..n], \beta s'_i = t'_i$,
4. $\beta\sigma_0 = \alpha[\mathcal{Y} \cup Var(l)]$
5. β satisfies $\bigwedge_{h\in[1..m]} \overline{\sigma_h}$.

where σ_0 is the most general unifier of $s|_p$ and l, and $\sigma_h, h \in [1..m]$ are all the most general unifiers of $\sigma_0 s|_{p'}$ and a left-hand side l' of a rule of $\mathcal{R}$, for all suffix positions p' of p in s.

Recall the innermost instance of the lifting lemma given in [14], which is itself an adaptation to rewriting under strategies of the well-known lifting lemma of Middeldorp and Hamoen [24].

LEMMA 2 (INNERMOST LIFTING LEMMA). *Let $\mathcal{R}$ be a rewrite system. Let $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, α a normalized ground substitution, and $\mathcal{Y} \subseteq \mathcal{X}$ a set of variables such that $Var(s) \cup Dom(\alpha) \subseteq \mathcal{Y}$. If $\alpha s \rightarrow^{Inn}_{p,l\rightarrow r} t'$, then there is a term $s' \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and substitutions $\beta, \sigma = \sigma_0 \wedge \bigwedge_{h\in[1..m]} \overline{\sigma_h}$ such that:*

1. $s \rightsquigarrow^{Inn}_{p,l\rightarrow r,\sigma} s'$,
2. $\beta s' = t'$,
3. $\beta\sigma_0 = \alpha[\mathcal{Y} \cup Var(l)]$
4. β satisfies $\bigwedge_{h\in[1..m]} \overline{\sigma_h}$.

where σ_0 is the most general unifier of $s|_p$ and l and $\sigma_h, h \in [1..m]$ are all the most general unifiers of $\sigma_0 s|_{p'}$ and a left-hand side l' of a rule of $\mathcal{R}$, for all positions p' which are suffix positions of p in s.

PROOF. of Lemma 1

Let $\mathcal{R}$ be a probabilistic rewrite system. Let $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, α a normalized ground substitution and $\mathcal{Y} \subseteq \mathcal{X}$ a set of variables such that $Var(s) \cup Dom(\alpha) \subseteq \mathcal{Y}$.

If $\alpha s \rightarrow^{Inn}_{p,l\rightarrow M} \nu = (t'_1 : p''_1, \ldots, t'_n : p''_n)$ with $M = (r_1 : p_1, \ldots, r_k : p_k)$, then, by definition of probabilistic rewriting, $\alpha s \rightarrow^{Inn}_{p,l\rightarrow M} t'_1 : p_1, \ldots, t'_k : p_k$, where $t'_i = t'_j$ for some possible i and j, and for $e \in [1..n]$, $p''_e = \sum_{r_i, i\in[1..k]|t'_e = t[\sigma(r_i)]_p} p_i$.

By Lemma 2, there are the terms $s'_1, \ldots, s'_k \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and substitutions $\beta, \sigma = \sigma_0 \wedge \bigwedge_{h\in[1..m]} \overline{\sigma_h}$ such that:

- $s \rightsquigarrow^{Inn}_{p,l\rightarrow M,\sigma} s'_1 : p_1, \ldots, s'_k : p_k$,
- for $i \in [1..k], \beta s'_i = t'_i$,
- $\beta\sigma_0 = \alpha[\mathcal{Y} \cup Var(l)]$
- β satisfies $\bigwedge_{h\in[1..m]} \overline{\sigma_h}$.

where σ_0 is the most general unifier of $s|_p$ and l and $\sigma_h, h \in [1..m]$ are all the most general unifiers of $\sigma_0 s|_{p'}$ and a left-hand side l' of a rule of $\mathcal{R}$, for all suffix positions p' of p in s.

If we total the probabilities of the equal s'_i, according to Definition 12, we obtain $s \rightsquigarrow^{Inn}_{p,l\rightarrow M,\sigma} \mu = (s'_1 : p'_1, \ldots, s'_q : p'_q)$.

It remains to be proved that $q = n$ and for $i \in [1..n], p'_i = p''_i$. For any i and t'_i, and given σ, the term s'_i and the substitution β such that $\beta s'_i = t'_i$ are unique. Thus, if we have $t'_i = t'_j$, then $s'_i = s'_j$. Inversely, as $\beta s'_i = t'_i$, if $s'_i = s'_j$, then $t'_i = t'_j$. So $q = n$ and for $i \in [1..n], p'_i = p''_i$.

□

B. THE IPAS TERMINATION PROPOSITION

As a reminder, $SUCCESS(g,\succ)$ means that the application of S on $(\{g(x_1, \ldots, x_m)\}, \top, \top)$ gives a finite proof tree, whose sets C of ordering constraints are satisfied by the same ordering $\succ$, and whose leaves are states of the form $(\emptyset, A, C)$.

Proposition 1. *Let $\mathcal{R}$ be a probabilistic rewrite system on $\mathcal{T}(\mathcal{F},\mathcal{X})$ having at least one constructor constant. If there is a noetherian ordering $\succ$ such that for each symbol $g \in \mathcal{D}$, we have $SUCCESS(g,\succ)$, then every term of $\mathcal{T}(\mathcal{F})$ is IPAS terminating.*

PROOF. We use an abstraction lemma, a narrowing lemma, and a stopping lemma, which are given after this main proof.

We prove by induction on $\mathcal{T}(\mathcal{F})$ that any ground instance $\theta f(x_1, \ldots, x_m)$ of any term $f(x_1, \ldots, x_m) \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is IPAS terminating, i.e. that for all policies ϕ, $T[\theta f(x_1, \ldots, x_m), \phi]$ is finite.

The induction ordering is constrained along the proof. At the beginning, it has to be at least noetherian and to have the constructor subterm property. Such an ordering always exists on $\mathcal{T}(\mathcal{F})$ (for instance the embedding relation). Let us denote it $\succ$.

If f is a defined symbol, let us denote it g and prove that $g(\theta x_1, \ldots, \theta x_m)$ is IPAS terminating for any θ satisfying $A = \top$ if we have $SUCCESS{-}S\ (h,\succ)$ for every defined symbol h. Let us denote $g(x_1, \ldots, x_m)$ by t_{ref} in the sequel of the proof.

To each state s of the proof tree of g, characterized by a current term t and the set of constraints A, we associate the set of ground terms $G = \{\alpha t \mid \alpha \text{ satisfies } A\}$, that is the set of ground instances represented by s.

The **Abstract** inference rule (resp. **Narrow**) transforms $(\{t\}, A, C)$ into $(\{t'\}, A', C')$ to which is associated $G' = \{\beta t' \mid \beta \text{ satisfies } A'\}$ (resp. into $(\{t'_i\}, A'_i), i \in [1..q]$ to which are associated $G' = \{\beta_i t'_i \mid \beta_i \text{ satisfies } A'_i\}$).

By abstraction (resp. narrowing) Lemma, when applying **Abstract** (resp. **Narrow**), for each reducible αt in G, there is a $\beta t'$ (resp. there are $\beta_i t'_i$) in G' and such that IPAS termination of $\beta t'$ (resp. of the $\beta_i t'_i$) implies IPAS termination of αt.

When the **Stop** inference rule applies on $(\{t\}, A, C)$: by stopping lemma, every term of $G = \{\alpha t \mid \alpha \text{ satisfies } A\}$ is IPAS terminating. Therefore, IPAS termination is ensured for all terms in all sets G in the proof tree.

As the process is initialized with $\{t_{ref}\}$ and a set A of abstraction constraints satisfiable by any ground substitution, we get that $g(\theta x_1, \ldots, \theta x_m)$ is IPAS terminating, for any $t_{ref} = g(x_1, \ldots, x_m)$, and any ground instance θ.

If f is a constructor, either it is a constant, which is irreducible, and then IPAS terminating, or we consider the pattern $f(x_1, \ldots, x_m)$. The proof then works like in the case of defined symbols, but with just an application of **Abstract** and **Stop**. Indeed, $f(x_1, \ldots, x_m)$ always abstracts into $f(X_1, \ldots, X_m)$. Then **Stop** applies because $f(X_1, \ldots, X_m)$ is not narrowable and all its variables are in $\mathcal{N}$.

□

LEMMA 3 (ABSTRACTION LEMMA). *Let $(\{t\}, A, C)$ be a state of any proof tree, giving the state $(\{t' = t[X_j]_{j\in\{i_1,\ldots,i_p\}}\}, A', C')$ by application of* **Abstract**.

For any ground substitution α satisfying A, if αt is reducible, there is β such that IPAS termination of $\beta t'$ implies IPAS termination of αt. Moreover, β satisfies A'.

PROOF. We prove that $\alpha t \xrightarrow{*}_S \beta t'$, where $\beta = \alpha \cup \bigcup_{j\in\{i_1,\ldots,i_p\}} X_j = \alpha t|_j\downarrow$.

First, the abstraction positions in t are chosen so that the $\alpha t|_j$ can be supposed IPAS terminating. Indeed, each term $t|_j$ is such that:

- either $IPAST(S, t|_j)$ is true, and then by definition of the predicate $IPAST$, $\alpha t|_j$ is IPAS terminating;
- or $t_{ref} > t|_j$ is satisfiable by $\succ$, and then, by induction hypothesis, $\alpha t|_j$ is IPAS terminating.

So, $T[\alpha t|_j, \phi]$ is finite for every policy ϕ, i.e. $\alpha t|_j$ is reducible to a normal form $\alpha t|_j\downarrow$ with a finite mean length of derivation. Then, for every policy ϕ, whatever the positions $i_1, \ldots, i_p$ in the term t, we have $\alpha t \xrightarrow{*}_{Innermost} \alpha t[\alpha t|_{i_1}\downarrow]_{i_1} \ldots [\alpha t|_{i_p}\downarrow]_{i_p} = \beta t'$, and $T[\alpha t, \phi] = T[\alpha t|_{i_1}, \phi] + \ldots + T[\alpha t|_{i_p}, \phi] + T[\beta t', \phi]$. Thus, as $T[\beta t', \phi]$ is finite, then $T[\alpha t, \phi]$ is as well.

Finally, , β satisfies $A' = A \wedge t|_{i_1}\downarrow = X_{i_1} \ldots \wedge t|_{i_p}\downarrow = X_{i_p}$, provided the X_i are neither in A, nor in $Dom(\alpha)$, which is true since the X_i are fresh variables, neither appearing in A, nor in $Dom(\alpha)$.

□

LEMMA 4 (NARROWING LEMMA). *Let $(\{t\}, A, C)$ be a state of any proof tree, giving the states $(\{v_i : p_i\}, A'_i, C'_i), i \in [1..l]$, by application of* **Narrow**. *For any ground substitution α satisfying A, if αt is reducible, then, for each $i \in [1..l]$, there is β_i such that IPAS termination of the $\beta_i v_i, i \in [1..l]$, implies IPAS termination of αt. Moreover, β_i satisfies A'_i for each $i \in [1..l]$.*

PROOF. For any rewriting step $\alpha t \rightarrow^{Inn}_{p,l\to M} \nu = (t'_1 : p''_1, \ldots, t'_n : p''_n)$, corresponding to any policy $\phi \in \Phi_{Inn}$, by Lifting Lemma, there is a term $s' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and substitutions $\beta, \sigma = \sigma_0 \wedge \bigwedge_{h\in[1..m]} \overline{\sigma_h}$ such that:

1. $s \rightsquigarrow^{Inn}_{p,l\to M,\sigma} \mu = (s'_1 : p'_1, \ldots, s'_q : p'_q)$,
2. $q = n$ and for $i \in [1..n], p'_i = p''_i$
3. for $i \in [1..n], \beta s'_i = t'_i$,
4. $\beta\sigma_0 = \alpha[\mathcal{Y} \cup Var(l)]$
5. β satisfies $\bigwedge_{h\in[1..m]} \overline{\sigma_h}$.

where σ_0 is the most general unifier of $s|_p$ and l and $\sigma_h, h \in [1..m]$ are all the most general unifiers of $\sigma_0 s|_{p'}$ and a left-hand side l' of a rule of $\mathcal{R}$, for all suffix positions p' of p in s.

All possible narrowing steps, corresponding to all possible policies ϕ, are effectively produced in the proof tree by the **Narrow** rule, applied in all possible ways on t. Then, the narrowing step $t \rightsquigarrow^{Inn}_{p,l\to M,\sigma} (s'_1 : p'_1 \ldots s'_q : p'_q)$ is produced. We prove that IPAS termination of the $\beta_i s'_i, i \in [1..q]$ implies IPAS termination of αt.

We have $T[\alpha t, \phi] = p'_1(1 + T[\beta_1 s'_1, \phi]) + \ldots + p'_l(1 + T[\beta_1 s' q, \phi])$. As the $T[\beta_i s'_i, \phi], i \in [1..q]$ are finite and q is finite, $T[\alpha t, \phi]$ is as well.

Let us prove that β satisfies $A' = A \wedge \sigma_0 \wedge \bigwedge_{j\in[1..k]} \overline{\sigma_j}$.

By Lifting Lemma, we have $\alpha = \beta\sigma_0$ on $\mathcal{Y}$. As we can take $\mathcal{Y} \supseteq Var(A)$, we have $\alpha = \beta\sigma_0$ on $Var(A)$.

More precisely, on $Ran(\sigma_0)$, β is such that $\beta\sigma_0 = \alpha$ and on $Var(A) \setminus Ran(\sigma_0)$, $\beta = \alpha$. As $Ran(\sigma_0)$ only contains fresh variables, we have $Var(A) \cap Ran(\sigma_0) = \emptyset$, so $Var(A) \setminus Ran(\sigma_0) = Var(A)$. So $\beta = \alpha$ on $Var(A)$ and then, β satisfies A.

Moreover, as $\beta\sigma_0 = \alpha$ on $Dom(\sigma_0)$, β satisfies σ_0.

So β satisfies $A \wedge \sigma_0$. Finally, with the point 4. of the lifting lemma, we conclude that β satisfies $A' = A \wedge \sigma_0 \wedge \bigwedge_{j\in[1..k]} \overline{\sigma_j}$.

□

LEMMA 5 (STOPPING LEMMA). *Let $(\{t\}, A, C)$ be a state of any proof tree, with A satisfiable, and giving the state $(\emptyset, A', C')$ by application of an inference rule. Then for every ground substitution α satisfying A, αt is IPAS terminating.*

PROOF. The only rule giving the state $(\emptyset, A', C')$ is **Stop**. When **Stop** is applied, then

- either $IPAST(S, t)$ and then αt is IPAS terminating for every ground substitution α,
- or $(t_{ref} > t)$ is satisfiable. Then, for every ground substitution α satisfying A, $\alpha t_{ref} \succ \alpha t$. By induction hypothesis, αt is IPAS terminating.

□

C. THE IPAS TERMINATION THEOREM

Theorem 1. *Let $\mathcal{R}$ be a probabilistic rewrite system on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ having at least one constructor constant. If there is a noetherian ordering $\succ$ such that*

- *for each symbol $g \in \mathcal{D}$, we have $I{-}SUCCESS(g, \succ)$,*
- *for each infinitely successful proof tree having a cycle $(s_i = (\{t_i : p_i\}, A_m, C_m), i \in [m..n]$ with $s_n = s_m)$, there is i such that $p_i < 1$,*

then every term of $\mathcal{T}(\mathcal{F})$ is IPAS terminating.

PROOF. We prove that for every ground term s of $\mathcal{T}(\mathcal{F}, \mathcal{X})$, for every policy $\phi \in \Phi_{inn}$, $T[s, \phi]$ is finite.

If the top symbol of s is a defined symbol g, consider its proof tree, generated from the state $(\{t_{ref} = g(x_1, \ldots, x_m)\}, \top, \top)$. Consider the deterministic proof subtree of this proof tree, associated to ϕ.

If the proof subtree is successful, by a reasoning similar to the one of the proof of Proposition 1, we state that $T[\theta t_{ref}, \phi]$ is finite for every ground substitution θ.

If the proof subtree has a cycle, let us compute $T[\theta t_{ref}, \phi]$. We first observe how to express the mean length of a term of the cycling branch with respect to its successor(s) in the proof subtree.

- If from a state $(\{t\}, A, C)$ we generate $(\{t'\}, A', C')$ with an **Abstract** step, as said in the proof of Abstraction Lemma, for any ground substitution α, we have $T[\alpha t, \phi] = \sum_{j\in\{i_1,\ldots,i_p\}} T[\beta t|_j, \phi] + T[\beta t', \phi]$ where t' is the abstracted term of t at positions $\{i_1, \ldots, i_p\}$, β is linked to α as specified in the Abstraction lemma, and $\sum_{j\in\{i_1,\ldots,i_p\}} T[\beta t|_j, \phi]$ is finite. We note $T[\alpha t, \phi] = U_l + T[\beta t', \phi]$ if the **Abstract** step is the l-th of the branch.

- If from a state $(\{t\}, A, C)$ we generate the states $(\{v_j : p_j\}, A'_j, C'_j), j \in [1..q]$ with a **Narrow** step, as said in the proof of Narrowing Lemma, for any ground substitution α, we have $T[\alpha t, \phi] = \sum_{j\in[1..q]} p_j(1+T[\beta_j v_j, \phi])$, the β_j being linked to α as specified in the Narrowing lemma. If we isolate the state which is on the infinite branch, and whose rank is i, among the states generated by the **Narrow** step, the expression becomes $\sum_{j\in[1..q]} p_j + \sum_{j\in\{1..q\}\setminus\{i\}} p_j T[\beta_j v_j, \phi] + p_i T[\beta_i v_i, \phi]$, and we note it $N_l + p_i T[\beta_i v_i, \phi]$ if the **Narrow** step is the l-th of the branch. By hypothesis, all brother states of $(\{v_i\}, A'_i, C'_i)$ are roots of successful proof trees, and then, with a reasoning similar to the one of the proof of Proposition 1, we state that the $T[\beta_j v_j, \phi]$ are finite. for every ground substitution θ. Then the real number N_l is finite.

Now, consider the branch with the cycle, and let $t_0 = t_{ref}, t_1, \ldots, t_m$ be the terms of the branch before the cycle, and $t_{m+1}, \ldots, t_n = t_m$ the terms of the cycle.

We have $T[\theta t_{ref}, \phi]$
$= U_1 + T[\alpha_1 t_1, \phi]$ (with an **Abstract** step)
$= U_1 + N_1 + p_1 T[\alpha_2 t_2, \phi]$ (with a **Narrow** step)
$= U_1 + N_1 + p_1(U_2 + T[\alpha_3 t_3, \phi])$ (with an **Abstract** step)
$= U_1 + N_1 + p_1 U_2 + p_1(N_2 + p_2 T[\alpha_4 t_4, \phi])$ (with a **Narrow** step)
...
which is equal to $(U_1 + N_1) + p_1(U_2 + N_2) + \ldots + p_{k-1}(U_k + N_k) + p_k(U_{k+1} + T[\alpha_m t_m, \phi])$ for some k if the step generating t_m is an **Abstract** step, and equal to $(U_1 + N_1) + p_1(U_2 + N_2) + \ldots + p_{k-1}(U_k + N_k) + p_k T[\alpha_m t_m, \phi]$ for some k if the step generating t_m is a **Narrow** step.

Note that in the considered branch of the proof tree, some **Abstract** steps may not exist, so for some i we can have $U_i = 0$.

As the U_i and the N_i are finite, and the p_i are probabilities, then $T[\theta t_{ref}, \phi]$ is finite if and only if $T[\alpha_m t_m, \phi]$ is. We finally prove that $T[\alpha_m t_m, \phi]$ is finite.

From the term t_m, there are only **Narrow** steps along the branch of the proof subtree, and with narrowing substitutions equal to Id. So $T[\alpha_m t_m, \phi]$
$= N_{k+1} + p_{k+1} T[\alpha_{m+1} t_{m+1}, \phi]$
$= N_{k+1} + p_{k+1} T[\alpha_m t_{m+1}, \phi]$ (since the narrowing substitution is Id)
$= N_{k+1} + p_{k+1}(N_{k+2} + p_{k+2} T[\alpha_m t_{m+2}, \phi])$
$= N_{k+1} + p_{k+1} N_{k+2} + p_{k+1} p_{k+2}(N_{k+3} + p_{k+3} T[\alpha_m t_{m+3}, \phi])$
$= N_{k+1} + p_{k+1} N_{k+2} + p_{k+1} p_{k+2} N_{k+3} + p_{k+1} p_{k+2} p_{k+3}\ T[\alpha_m t_{m+3}, \phi]$
...
$= N_{k+1} + \sum_{j\in[1..n-m-1]} (\prod_{i\in[1..j]} p_{k+i}) N_{k+j+1} + (\prod_{i\in[1..n-m]} p_{k+i}) T[\alpha_m t_n, \phi]$.

So $T[\alpha_m t_m, \phi]$ is of the form $U + V.T[\alpha_m t_n, \phi]$, where U is finite and $V = \prod_{i\in[1..n-m]} p_{k+i}$ with $p_{k+i} \in [0, 1]$.

Since $t_n = t_m$, we have $T[\alpha_m t_m, \phi] = U + V.T[\alpha_m t_m, \phi]$ and then

$$T[\alpha_m t_m, \phi] = U \div (1 - V).$$

By hypothesis of Theorem 1, there is a $p_i, i \in [k + 1..k + n - m]$, such that $p_i < 1$. Thus $1 - V > 0$ and $T[\alpha_m t_m, \phi]$ is finite.

Thus $T[\theta t_{ref}, \phi]$ is finite for every t_{ref}, every ground substitution θ, and every ϕ.

The case where the top symbol of s is a constructor is treated like in the proof of Proposition 1.

□

D. RESULTS FOR INFINITE PROOF TREES

Proposition 2. *Let $\mathcal{R}$ be a RS. If the possible cycles in the ϕ-deterministic proof subtrees of the proof trees of $\mathcal{R}$ are such that:*

- *the first term of the cycle is of the form $f(x_1, \ldots, x_m)$ where the x_i are either variables or constructor constants, and f can be a constant,*
- *the successive rewrite rules of $\mathcal{R}$ used in the k **Narrow** steps of the cycle are of the form*

  $$f_j(x^j_1, \ldots, x^j_{m_j}) \to M_j = |_{i_j} t_{i_j} : p_{i_j}\ j \in [1..k]$$

  where $x^j_1, \ldots, x^j_{m_j}$ are also either variables or constructor constants, and the f_j can be constants,
- *$f_1(x^1_1, \ldots, x^1_{m_1}) = f(x_1, \ldots, x_m)$*
- *for $j \in [1..k-1]$, the term t_{i_j}, for some i_j, generated by the rule $f_j(x^j_1, \ldots, x^j_{m_j}) \to M_j = |_{i_j} t_{i_j} : p_{i_j}$ on the branch of the cycle is equal to $f_{j+1}(x^{j+1}_1, \ldots, x^{j+1}_{m_{j+1}})$ (if $k = 1$, this condition is void),*
- *the term t_{i_k}, for some i_k, generated by the rule $f_k(x^k_1, \ldots, x^k_{m_k}) \to M_k = |_{i_k} t_{i_k} : p_{i_k}$ on the branch of the cycle is equal to $f(x_1, \ldots, x_m)$.*

then, the only inference rule applied in the steps of the cycles is **Narrow**, *and with narrowing substitutions equal to Id.*

PROOF. Let $\mathcal{R}$ be a rewrite system, and a proof tree of $\mathcal{R}$, with root state s_0. Consider the branch from s_0 to s_n in the lemma above. Recall that only the **Narrow** rule has been applied between s_m and s_n, with narrowing substitutions equal to Id on the given branch, and that the states $s_i = (\{t_i\}, A_i, C_i)$ on the branch between s_m and s_n are such that $A_i = A_m$ and $C_i = C_m$.

Either the proof tree is successful and the result is immediate, or there is a branch as specified in the lemma. In that case, as inference rules are applied in the same way on equal states,

- the sequence of states from s_m to s_n is generated again from s_n, so we have a cycle
- the brother states of $s_n, \ldots, s_{n+(n-m)}$ are are roots of successful subtrees since they are the same states than the brother states of the $s_m, \ldots, s_n$, which are roots of successful subtrees.

Moreover, as every state on this branch from s_0 until s_m has only brother states that are roots of successful subtrees, the proof tree verifies Definition 15. □

Proposition 3. *Let $\mathcal{R} \in \mathcal{B}$. Then every proof tree of $\mathcal{R}$ is infinitely successful.*

Proof. Let be a proof tree of $\mathcal{R}$. Every ϕ-deterministic proof subtree of proof tree begins with an application of **Abstract** on a pattern $g(x_1, \ldots, x_m)$ to give $g(X_1, \ldots, X_m)$. Then, we have an application of **Narrow**, whose narrowing substitution can be different from Id on variables the $X_1, \ldots, X_m$, but equal to Id on the variables of the left-hand side of the rule used. Therefore, the terms generated by the **Narrow** step are exactly the terms of the distribution of the right-hand side of the rule used.

If all these terms are not narrowable, a **Stop** step applies on them and the ϕ-deterministic proof subtree id successful.

If not, there exists a unique term t, which is a left-hand side of rule. The other terms, if they exist, are not narrowable and generate **Stop** steps as above. On t, **Narrow** applies again, and from this step with Id as narrowing substitution: we then can reason in the same way than for the first application of **Narrow**.

After a finite number of applications of **Narrow**, either we only have non narrowable terms, and the ϕ-deterministic proof subtree is succesful, or we get a left-hand side of rule, already produced on the branch: we then have a cycle as specified in Definition 15.

□