Normalization in Supernatural deduction and in Deduction modulo

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Abstract. Deduction modulo and Supernatural deduction are two extentions of predicate logic with computation rules. Whereas the application of computation rules in deduction modulo is transparent, these rules are used to build non-logical deduction rules in Supernatural deduction. In both cases, adding computation rules may jeopardize proof normalization, but various conditions have been given in both cases, so that normalization is preserved. We prove in this paper that normalization in Supernatural deduction and in Deduction modulo are equivalent, *i.e.* the set of computation rules for which one system strongly normalizes is the same as the set of computation rules for which the other is.

1 Introduction

Deduction modulo is an extension of predicate logic where a theory is defined by a set of axiom Γ and a set of rewrite rules $\mathcal{R}$. The deduction rules are modified to take the rewrite rules into account. For instance, the modus ponens is not stated as usual

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

but as

$$\frac{\Gamma \vdash C \quad \Gamma \vdash A}{\Gamma \vdash B}\; C \equiv (A \Rightarrow B)$$

where $\equiv$ is the congruence generated by the rewrite rules. For instance, in arithmetic, if we replace the axioms of addition and multiplication by the obvious rewrite rules, then the formula $1000 + 1000 = 2000$ is congruent to the formula $2000 = 2000$ and it has a much shorter proof than in the original formulation of arithmetic. By removing computation steps from proofs, Deduction modulo allows to reduce of the size of proofs.

A theory formed with rewrite rules only, *i.e.* no axioms, is called *purely computational*. When we restrict to rules such as $0 + y \rightarrow y$ that rewrite terms to terms and thus do not affect the complexity of formulae, it can be proved that all purely computational theories have a one-element model. Thus, arithmetic, for instance, cannot be expressed as a purely computational theory. This has motivated the introduction of rules directly rewriting formulae to formulae such as the rule $R : (X \subseteq Y) \rightarrow \forall z\ (z \in X \Rightarrow z \in Y)$. The left hand side of such rules has to be restricted to atomic formulae. When the right hand side is atomic also rewriting does not affect the logical complexity of formulae, and such rules can therefore be classified in the same category as term rules, but those whose right hand side is compound introduce logical power. For instance, theories such as first-order arithmetic, simple type theory, higher-order arithmetic and set theory can be expressed as purely computational theories.

If they introduce logical power, the rules rewriting atomic formulae to non atomic ones cannot be claimed to reduce the size of proofs in a significant way. Indeed, each connector or quantifier introduced by such a rule has to be used by a deduction rule. Indeed, as we shall see, there is no point in rewriting $X \subseteq Y$ to $\forall z\ (z \in X \Rightarrow z \in Y)$ if we do not use this quantifier in a introduction rule or an elimination rule. More precisely, we can decide to restrict the use of these rewrite rule to patterns where it is used together with a introduction rule or an elimination rule of the conjunction in patterns of the form

$$\forall\text{-INTRO}\ \frac{\Gamma \vdash (z \in X \Rightarrow z \in Y)}{\Gamma \vdash X \subseteq Y}\ z \notin \mathcal{FV}(\Gamma) \qquad \forall\text{-ELIM}\ \frac{\Gamma \vdash X \subseteq Y}{\Gamma \vdash t \in X \Rightarrow t \in Y}$$

Thus, restricting in Deduction modulo the use of the congruence to these patterns leads to introduce specializations of the introduction and elimination rules of the universal quantifier, that we can call R-introduction and R-elimination rules.

Such an extension of predicate logic has been introduced in [1] under the name *Supernatural deduction*. Supernatural deduction goes much further. Indeed, when, as in this example, the right-hand side of the rule contains several connectors and quantifiers, we can further apply the introduction and elimination rules of these connectors and quantifiers leading to the rules

$$(\subseteq I)\ \frac{\Gamma, z \in X \vdash z \in Y}{\Gamma \vdash X \subseteq Y}\ z \notin \mathcal{FV}(\Gamma) \qquad (\subseteq E)\ \frac{\Gamma \vdash X \subseteq Y \qquad \Gamma \vdash t \in X}{\Gamma \vdash t \in Y}$$

where all connectors and quantifiers have disappeared.

It may be argued that these rules are much closer to usual mathematical practice than either predicate logic with axioms, Deduction modulo or predicate logic extended with rules *à la* Prawitz, where atomic formulae are *folded* and *unfolded* through specialized inference rules:

$$\frac{\Gamma \vdash \forall z\ (z \in X \Rightarrow z \in Y)}{\Gamma \vdash X \subseteq Y} \qquad \frac{\Gamma \vdash X \subseteq Y}{\Gamma \vdash \forall z\ (z \in X \Rightarrow z \in Y)}$$

Both in deduction modulo and in Supernatural deduction , there are some rewrite systems for which there exists proof that do not normalize. But [1] introduces a proof-term language or Supernatural deduction based on the rewriting calculus ([2]) and [3] proves its strong normalization, under some hypotheses on the rewrite system. On the other hand, several sufficient conditions for strong normalization in Deduction modulo have been given in [4].

This raises the question on the relation between proof normalization in Deduction modulo and in Supernatural deduction. Are the conditions the same on the rewrite system for proofs in Deduction modulo to strongly normalize and for proofs and in Supernatural deduction to normalize ?

In this paper, we give a partial positive answer to this problem. We show that for a orthogonal rewrite system in the fragment of predicate logic formed with the connector $\Rightarrow$ and quantifier $\forall$, Deduction modulo strongly normalizes if and only if Supernatural deduction strongly normalizes. The result could easily be extrended to the connectors $\top$ and $\wedge$ but the extension to the connectors and quantifier $\perp$, $\vee$ and $\exists$ and to non orthogonal rewrite systems is more challenging. Indeed, Supernatural deduction rules are already trickier to define when such quantifiers are involved or when the system is not orthogonal. A possible solution seems to move from natural deduction to sequent calculus [5].

2 Deduction modulo

Consider an orthogonal rewrite system $\mathcal{R}$ rewriting atomic formulae to formlae and $\equiv$ the congruence defined by $\mathcal{R}$. The rules of Deduction modulo are the following.

$$(Ax)\ \frac{}{\Gamma, A \vdash B}\ A \equiv B$$

$$(\Rightarrow I_{\equiv})\ \frac{\Gamma, A \vdash B}{\Gamma \vdash C}\ C \equiv A \Rightarrow B \qquad (\Rightarrow E_{\equiv})\ \frac{\Gamma \vdash C \qquad \Gamma \vdash A}{\Gamma \vdash B}\ C \equiv A \Rightarrow B$$

$$(\forall I_{\equiv})\ \frac{\Gamma \vdash B}{\Gamma \vdash A}\ A \equiv \forall x.B \text{ AND } x \notin \mathcal{FV}(\Gamma) \qquad (\forall E_{\equiv})\ \frac{\Gamma \vdash A}{\Gamma \vdash B}\ A \equiv \forall x.C \text{ AND } B \equiv C[x := t]$$

The proof-term language is that of predicate logic

$$\mathcal{T} ::= \alpha \mid \lambda\alpha : A.\mathcal{T} \mid \mathcal{T}\,\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}\,t$$

but these rules are typed with different typing rules

$$(Ax)\ \frac{}{\Gamma, \alpha : A \vdash \alpha : B}\ A \equiv B$$

$$(\Rightarrow I_{\equiv})\ \frac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash (\lambda\alpha : A.\pi) : C}\ C \equiv A \Rightarrow B \qquad (\Rightarrow E_{\equiv})\ \frac{\Gamma \vdash \pi_1 : C \qquad \Gamma \vdash \pi_2 : A}{\Gamma \vdash (\pi_1\ \pi_2) : B}\ C \equiv A \Rightarrow B$$

$$(\forall I_{\equiv})\ \frac{\Gamma \vdash \pi : B}{\Gamma \vdash (\lambda x.\pi) : A}\ A \equiv \forall x.B \text{ AND } x \notin \mathcal{FV}(\Gamma) \qquad (\forall E_{\equiv})\ \frac{\Gamma \vdash \pi : A}{\Gamma \vdash (\pi\ t) : B}\ A \equiv \forall x.C \text{ AND } B \equiv C[x := t])$$

The reduction rules on these proof-terms are the two kinds of β-reduction

$$((\lambda\alpha : A\ \pi_1)\ \pi_2) \rhd \pi_1[\alpha := \pi_2]$$

$$((\lambda x\ \pi)\ t) \rhd \pi[x := t]$$

3 Supernatural deduction

3.1 Super-rules computation

For a rewrite rule $P \to \varphi$, we may add Prawitz' *folding* and *unfolding* rules [6,7]:

$$\frac{\varphi}{P} \qquad \textit{and} \qquad \frac{P}{\varphi}$$

The basic idea of Supernatural deduction is to go one step further and incorporate in these rules the introductions and eliminations for all the connectives of φ. More formally, we define the super-rules as follows.

Definition 1 (Computation of the introduction super-rules). *Consider a rewrite rule $R : P \to \varphi$. Consider a sequence of variables $l = x_1, x_2, \ldots$ that do not occur in the rule. We associate to R an introduction rule of the form*

$$\frac{premiseI(\Gamma,\varphi,l)}{\Gamma \vdash P}\ cond(\Gamma,\varphi,l)$$

Where the sequent $premiseI(\Gamma,\varphi,l)$ and the condition $cond(\Gamma,\varphi,l)$ are defined by induction on the stucture of φ as follows

- *if φ is atomic, then $premiseI(\Gamma,\varphi,l) = (\Gamma \vdash \varphi)$ and $cond(\Gamma,\varphi,l) = \varnothing$,*
- *if $\varphi = \varphi_1 \Rightarrow \varphi_2$ then $premiseI(\Gamma,\varphi,l) = premiseI((\Gamma,\varphi_1),\varphi_2,l)$ and $cond(\Gamma,\varphi,l) = cond((\Gamma,\varphi_1),\varphi_2,l)$*
- *if $\varphi = \forall x\ \varphi_1$ then $premiseI(\Gamma,\varphi,y.l) = premiseI(\Gamma,\varphi_1[x := y],l)$ and $cond(\Gamma,\varphi,y.l) = cond(\Gamma,\varphi_1[x := y],l) \cup \{y \notin \mathcal{FV}(\Gamma)\}$*

Definition 2 (Computation of the elimination super-rules). *Consider a rewrite rule $R : P \to \varphi$. Consider a sequence of names $l = t_1, t_2, \ldots$. We associate to R an elimination rule of the form*

$$\frac{\Gamma \vdash P \qquad premisesE(\Gamma,\varphi,l)}{conclusion(\Gamma,\varphi,l)}$$

Where the multiset of sequents $premisesE(\Gamma,\varphi,l)$ and the sequent $conclusion(\Gamma,\varphi)$ are defined by induction on the stucture of φ as follows

- *if φ is atomic then $premisesE(\Gamma,\varphi,l) = \varnothing$ and $conclusion(\Gamma,\varphi) = (\Gamma \vdash \varphi)$*
- *if $\varphi = \varphi_1 \Rightarrow \varphi_2$ then $premisesE(\Gamma,\varphi,l) = \{\Gamma \vdash \varphi_1\} \cup premisesE(\Gamma,\varphi_2,l)$ and $conclusion(\Gamma,\varphi,l) = conclusion(\Gamma,\varphi_2,l)$*

- *if* $\varphi = \forall x\ \varphi_1$ *then let* $premisesE(\Gamma, \varphi, t.l) = premisesE(\Gamma, \varphi_1[x := t], l)$ *and* $conclusion(\Gamma, \varphi, t.l) = conclusion(\Gamma, \varphi_1[x := t], l)$

Example 1 (Super-rules for inclusion definition). Given the rewrite rule $\subseteq: X \subseteq Y \to \forall z.(z \in X \Rightarrow z \in Y)$, the associated super deduction rules are:

$$(\subseteq I)\ \frac{\Gamma, z \in X \vdash z \in Y}{\Gamma \vdash X \subseteq Y}\ z \notin \mathcal{FV}(\Gamma) \qquad (\subseteq E)\ \frac{\Gamma \vdash X \subseteq Y \qquad \Gamma \vdash t \in X}{\Gamma \vdash t \in Y}$$

3.2 A proof-term language for Supernatural deduction

The proof-term language is that of proof-terms for predicate logic, enhanced with a pattern language and the corresponding abstraction:

$$\mathcal{T} ::= \alpha \mid \lambda\alpha : A.\mathcal{T} \mid \mathcal{T}\,\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}\,t \mid \lambda R(\overline{m}).\mathcal{T} \mid \mathcal{T}\,R(\overline{t})$$

The variables $x, y, \ldots$ are variables of the theory while $\alpha, \beta, \ldots$ are proof variables. The two last constructs allow to interprete the super-rules. In the pattern $R(\overline{m})$ the constructor R is applied to a sequence of variables that may be ever term or proof variables and in the term $R(\overline{t})$ it is applied to a sequence of terms that may be either terms of the theory or proof-terms.

We can now define the typing rules that correspond to the super-rules above.

Definition 3. *The* arity *of a formula* φ *is a sequence of* $\forall$ *and* $\Rightarrow$ *symbols defined by induction on* φ *as follows*

- *if* φ *is atomic* $arity(\varphi) = [\ ]$,
- *if* $\varphi = \varphi_1 \Rightarrow \varphi_1$ *then* $arity(\varphi) = (\Rightarrow .(arity(\varphi_2)))$,
- *if* $\varphi = \forall x\ \varphi_1$ *then* $arity(\varphi) = (\forall.(arity(\varphi_1)))$.

Let φ *be a formula, a sequence for* φ *is a sequence of distinct variables such that the* n*-th variable of the sequence is a proof variable if the* n*-th element of the arity of* φ *is* $\Rightarrow$ *and a term variable otherwise.*

Definition 4 (Computation of the abstraction rules). *Consider a rewrite rule* $R : P \to \varphi$. *Consider a sequence* l *for* φ *of variables that do not occur in the rule. We associate to* R *an abstraction rule of the form*

$$\frac{premiseI(\Gamma, \varphi, l)}{\Gamma \vdash (\lambda R(l).\pi) : P}\ cond(\Gamma,\varphi,l)$$

Where the sequent $premiseI(\Gamma, \varphi, l)$ *and the condition* $cond(\Gamma, \varphi, l)$ *are defined by induction on the stucture of* φ *as follows*

- *if* φ *is atomic, then* $premiseI(\Gamma, \varphi, l) = (\Gamma \vdash \pi : \varphi)$ *and* $cond(\Gamma, \varphi, l) = \varnothing$,
- *if* $\varphi = \varphi_1 \Rightarrow \varphi_2$ *then* $premiseI(\Gamma, \varphi, \alpha.l) = premiseI((\Gamma, \alpha : \varphi_1), \varphi_2, l)$ *and* $cond(\Gamma, \varphi, \alpha.l) = cond((\Gamma, \varphi_1), \varphi_2, l)$
- *if* $\varphi = \forall x\ \varphi_1$ *then* $premiseI(\Gamma, \varphi, y.l) = premiseI(\Gamma, \varphi_1[x := y], l)$ *and* $cond(\Gamma, \varphi, y.l) = cond(\Gamma, \varphi_1[x := y], l) \cup \{y \notin \mathcal{FV}(\Gamma)\}$

Definition 5 (Computation of the application rules). *Consider a rewrite rule $R : P \to \varphi$. Consider a sequence l for φ of names. We associate to R the application rule of the form*

$$\frac{\Gamma \vdash \pi : P \qquad premisesE(\Gamma, \varphi, l)}{conclusion(\Gamma, (\pi\ R(l)), \varphi, l)}$$

Where the multiset of sequents $premisesE(\Gamma, \varphi, l)$ and the sequent $conclusion(\Gamma, \pi', \varphi, l)$ are defined by induction on the stucture of φ as follows

- *if φ is atomic then $premisesE(\Gamma, \varphi, l) = \varnothing$ and $conclusion(\Gamma, \pi', \varphi, l) = (\Gamma \vdash \pi' : \varphi)$*
- *if $\varphi = \varphi_1 \Rightarrow \varphi_2$ then $premisesE(\Gamma, \varphi, \tau.l) = \{\Gamma \vdash \tau : \varphi_1\} \cup premisesE(\Gamma, \varphi_2, l)$ and $conclusion(\Gamma, \pi', \varphi, \tau.l) = conclusion(\Gamma, \pi', \varphi_2, l)$*
- *if $\varphi = \forall x\ \varphi_1$ we let $premisesE(\Gamma, \varphi, t.l) = premisesE(\Gamma, \varphi_1[x := t], l)$ and $conclusion(\Gamma, \pi', \varphi, t.l) = conclusion(\Gamma, \pi', \varphi_1[x := t], l)$*

Example 2 (Proof-terms for the inclusion). Our definition of $\subseteq$ uses a witness and charges an assumption into the context. Thus, the associated proof-terms are those given by the following typing rules:

$$(\subseteq I)\ \frac{\Gamma, \alpha{:}(x \in X) \vdash A : (x \in Y)}{\Gamma \vdash \lambda_{\subseteq}(x, \alpha).A : (X \subseteq Y)} \qquad (\subseteq E)\ \frac{\Gamma \vdash A : (X \subseteq Y) \qquad \Gamma \vdash B : (t \in X)}{\Gamma \vdash A_{\subseteq}(t, B) : (t \in Y)}$$

Definition 6 (Generalized cut elimination). *The elimination of a generalized cut is represented by a reduction which transmits the witnesses and the lemmas to the proof.*

$$\lambda R(\overline{m}).\pi\ R(\bar{t}) \rhd_\rho \pi[\overline{m := t}]$$

When seeing Supernatural deduction proof-terms as very simple ρ-terms of the rewriting calculus, the generalized cut elimination is then a ρ-reduction step. Hence the notation.

4 If Supernatural deduction is normalizes then Deduction modulo does

To prove that the strong normalization of proofs in Supernatural deduction implies that of the proofs in Deduction modulo, we introduce an intermediate system called *one-step Deduction modulo* and prove that if all proofs in Supernatural deduction normalize then so do proofs in one-step Deduction modulo and then that if all proofs in one-step Deduction modulo normalize then all proofs in Deduction modulo do.

4.1 One-step Deduction modulo

We introduce here a restriction of Deduction modulo in one step Deduction modulo, conversion is oriented and limited to one-step conversion of atomic formulae.

$$(Ax)\ \frac{}{\Gamma, A \vdash A}$$

$$(\Rightarrow I)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad (\Rightarrow E)\ \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

$$(\Rightarrow I_1)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash P}\ P \to A \Rightarrow B \qquad (\Rightarrow E_1)\ \frac{\Gamma \vdash P \qquad \Gamma \vdash A}{\Gamma \vdash B}\ P \to A \Rightarrow B$$

$$(\forall I)\ \frac{\Gamma \vdash_1 A}{\Gamma \vdash_1 \forall x.A}\ x \notin \mathcal{FV}(\Gamma) \qquad (\forall E)\ \frac{\Gamma \vdash_1 \forall x.A}{\Gamma \vdash_1 A[x := t]}$$

$$(\forall I_1)\ \frac{\Gamma \vdash_1 A}{\Gamma \vdash_1 P}\ P \to \forall x.A \text{ AND } x \notin \mathcal{FV}(\Gamma) \qquad (\forall E_1)\ \frac{\Gamma \vdash_1 P}{\Gamma \vdash_1 A[x := t]}\ P \to \forall x.A$$

Notice that when reducing a proof in this restriction, we obtain a proof that is also in this restriction.

For the proof-terms of one-step deducttion modulo, we use the same notation than for full Deduction modulo.

4.2 Normalization of Supernatural deduction implies normalization of one-step Deduction modulo

Definition 7. *To each rule $R : P \to \varphi$, we associate two proof-terms in Supernatural deduction of type $\varphi \Rightarrow P$ and $P \Rightarrow \varphi$*

$$\chi^l_R = \lambda f\ \lambda R(m_1, \ldots, m_n).(f\ m_1 \ldots m_n)$$

and

$$\chi^r_R = \lambda p\ \lambda m_1 \ldots \lambda m_n.(p\ R(m_1, \ldots, m_n))$$

where $m_1, \ldots, m_n$ is a sequence for φ.

Proposition 1. *The term $\chi^r\ (\chi^l\ x)$ reduces in two steps to $\lambda m_1 \ldots \lambda m_n.(x\ m_1 \ldots m_n)$.*

We introduce the notation $\lambda_{P \to^1 \varphi} x.\pi$ to differentiate $(\Rightarrow I_1)$ from $(\Rightarrow I)$ abstraction in one-step Deduction modulo proof-terms. We similarly denote $(\Rightarrow E)$ applications with a labelled underscore: $(\pi_1\ \__{P \to^1 \varphi}\ \pi_2)$.

Definition 8 (Translation). *To each proof π of a formula φ in one-step Deduction modulo, we associate a proof of φ in Supernatural deduction by induction on the structure of π as follows, where $m_1, \ldots, m_n$ is a sequence for φ.*

- *if* α *variable then* $[\![\alpha]\!] = \lambda m_1 \dots \lambda m_n \ \ (\alpha\ m_1 \dots m_n)$
- $[\![\lambda x.\pi']\!] = \lambda m_1 \dots \lambda m_n\ ((\lambda x.[\![\pi']\!])\ m_1 \dots m_n)$
- $[\![\lambda\alpha : A.\pi']\!] = \lambda m_1 \dots \lambda m_n\ ((\lambda\alpha : A.[\![\pi']\!])\ m_1 \dots m_n)$
- $[\![(\pi_1\ \pi_2)]\!] = \lambda m_1 \dots \lambda m_n\ ([\![\pi_1]\!]\ [\![\pi_2]\!]\ m_1 \dots m_n)$
- $[\![(\pi'\ t)]\!] = \lambda m_1 \dots \lambda m_n\ ([\![\pi']\!]\ t\ m_1 \dots m_n)$
- $[\![\lambda_{P \to^1 A \Rightarrow B} \alpha{:}A.\pi']\!] = \chi^l\ \lambda\alpha : A.[\![\pi']\!]$
- $[\![\lambda_{P \to^1 \forall x.A} x.\pi']\!] = \chi^l\ \lambda x.[\![\pi']\!]$
- $[\![(\pi_1\ _{-P \to^1 A \Rightarrow B}\ \pi_2)]\!] = \lambda m_1 \dots m_n\ (\chi^r\ [\![\pi_1]\!]\ [\![\pi_2]\!]\ m_1 \dots m_n)$
- $[\![(\pi'\ _{-P \to^1 \forall x.A}\ t)]\!] = \lambda m_1 \dots m_n\ (\chi^r\ [\![\pi']\!]\ t\ m_1 \dots m_n)$

Proposition 2. *If* π *is well-typed in one-step Deduction modulo, then* $[\![\pi]\!]$ *is well-typed in Supernatural deduction and has the same type.*

Proof. By induction on the structure of π. Let us detail the case of the application. $(\pi_1\ _{-P \to^1 A \Rightarrow B}\ \pi_2)$ where π_1 has type P, π_2 has type A and $(\pi_1\ \pi_2)$ type B. Notice that since the term is well-typed in one-step Deduction modulo and not just in Deduction modulo, the type of π_1 is P and not any type convertible to P. Thus, by induction hypothesis, $[\![\pi_2]\!]$ has type A and $[\![\pi_1]\!]$ has type P and since the term χ^r has type $P \Rightarrow (A \Rightarrow B)$, the term $[\![(\pi_1\ \pi_2)]\!]$ has type B.

Proposition 3. *Let* π *and* π' *be two proofs in one-step Deduction modulo such that* $\pi \rhd_\beta \pi'$, *then* $[\![\pi]\!] \rhd^+_{\beta\rho} [\![\pi']\!]$.

Thus, if Supernatural deduction normalizes for $\mathcal{R}$, then one-step Deduction modulo normalizes for $\mathcal{R}$.

4.3 Normalization of one-step Deduction modulo implies normalization of Deduction modulo

We define a translation from Deduction modulo to one-step Deduction modulo.

Definition 9 (Conversion step). *Let* A *and* A' *be two formulae such that* $A = A'$, $A \to^1 A'$ *or* $A' \to^1 A$. *The* conversion step $\rho_{A,A'}$ *is a proof of the judgment* $\vdash_1 A \Rightarrow A'$ *defined by induction on the structure of* A *as follows.*

- *If both formulae are atomic then they are equal, we take* $\rho_{A,A'} = \lambda\alpha{:}A\ \alpha$.
- *If* A *is atomic and* $A' = B' \Rightarrow C'$ *then* $A \to^1 A'$. *We let* $\rho_{A,A'} = \lambda\alpha{:}A\ \lambda\beta{:}B'\ (\alpha\ \beta)$.
- *If* A *is atomic and* $A' = \forall x\ B'$ *then* $A \to^1 A'$. *We let* $\rho_{A,A'} = \lambda\alpha{:}A\ \lambda x\ (\alpha\ x)$.
- *if* A' *is atomic and* $A = B \Rightarrow C$ *then* $A' \to^1 A$. *We let* $\rho_{A,A'} = \lambda\alpha{:}B \Rightarrow C\ \lambda\beta{:}B\ (\alpha\ \beta)$.
- *if* A' *is atomic and* $A = \forall x\ B$ *then* $A' \to^1 A$. *We let* $\rho_{A,A'} = \lambda\alpha{:}\forall x.B\ \lambda x\ (\alpha\ x)$.
- *If* $A = B \Rightarrow C$ *and* $A' = B' \Rightarrow C'$ *then* $B = B'$, $B \to^1 B'$ *or* $B' \to^1 B$ *and* $C = C'$, $C \to^1 C'$ *or* $C \to^1 C'$. *We let* $\rho_{A,A'} = \lambda\alpha{:}B \Rightarrow C\ \lambda\beta{:}B'\ (\rho_{C,C'}\ (\alpha\ (\rho_{B',B}\ \beta)))$
- *If* $A = \forall x\ B$ *and* $A' = \forall x\ B'$ *then* $B = B'$ *or* $B \to^1 B'$ *or* $B' \to^1 B$. *We let* $\rho_{A,A'} = \lambda\alpha{:}\forall x{:}B\ \lambda x\ (\rho_{B,B'}\ (\alpha\ x))$.

Notice that all these terms are well-typed in one-step Deduction modulo.

Definition 10 (Enrichment). *Let π and π' be two proofs in Deduction modulo. The proof π' is said to be an* enrichment *of π if it is obtained by inserting conversion steps in π.*

More formally the enrichment relation is the smallest reflexive relation compatible with term structure such that $\rho\pi$ is an enrichment of π where ρ is a conversion step.

Proposition 4. *Let π be a proof of $\Gamma \vdash A$ in Deduction modulo. There exists a proof π^+ in one-step Deduction modulo that is an enrichment of π.*

Proof. By induction on π.

Proposition 5. *Let A and A' be two formulae such that $A = A'$, $A \to^1 A'$ or $A' \to^1 A$ and π be a proof of $\Gamma \vdash_1 A$ that is an introduction, then the proof $(\rho_{A,A'}\ \pi)$ of $\Gamma \vdash_1 A'$ reduces to an introduction that is an enrichment of π.*

Proof. We consider seven cases according to the form of A and A'.

- if A and A' are atomic then $\rho_{A,A'} = \lambda\alpha{:}A\ \alpha$ and the result is trivial.
- If A is atomic and $A' = B' \Rightarrow C'$ then $\rho_{A,A'} = \lambda\alpha{:}A\ \lambda\beta{:}B'\ (\alpha\ \beta)$ and $\pi = \lambda\beta{:}B'\ \pi'$ and $\rho\pi \rhd \lambda\beta{:}B'\pi' = \pi$ that is an enrichment of π.
- If A is atomic and $A' = \forall x\ B'$ then $\rho_{A,A'} = \lambda\alpha{:}A\ \lambda x\ (\alpha\ x)$ and $\pi = \lambda x\ \pi'$ and $\rho\pi \rhd \lambda x\ \pi' = \pi$ that is an enrichment of π.
- if A' is atomic and $A = B \Rightarrow C$ then $\rho_{A,A'} = \lambda\alpha{:}B \Rightarrow C\ \lambda\beta{:}B\ (\alpha\ \beta)$ and $\pi = \lambda\beta{:}B\ \pi'$ and $\rho\pi \rhd \lambda\beta\pi' = \pi$ that is an enrichment of π.
- if A' is atomic and $A = \forall x\ B$ then $\rho_{A,A'} = \lambda\alpha{:}\forall x.B\ \lambda x\ (\alpha\ x)$ and $\pi = \lambda x\ \pi'$ and $\rho\pi \rhd \lambda x\ \pi' = \pi$ that is an enrichment of π.
- If $A = B \Rightarrow C$ and $A' = B' \Rightarrow C'$ then $\rho_{A,A'} = \lambda\alpha{:}B \Rightarrow C\ \lambda\beta{:}B'\ (\rho_{C,C'}\ (\alpha\ (\rho_{B',B}\ \beta)))$ and $\pi = \lambda\gamma{:}B\pi'$ and $\rho\pi \rhd \lambda\beta{:}B'(\rho_{C,C'}\ \pi'[\gamma := (\rho_{B',B}\ \beta)])$ that is an enrichment of π.
- If $A = \forall x\ B$ and $A' = \forall x\ B'$ then $\rho_{A,A'} = \lambda\alpha{:}\forall x{:}B\ \lambda x\ (\rho_{B,B'}\ (\alpha\ x))$ and $\pi = \lambda x\ \pi'$ and $\rho\pi \rhd \lambda x\ (\rho_{B,B'}\pi')$ that is an enrichment of π.

Proposition 6. *Let π and π' two proofs in Deduction modulo such that $\pi \rhd \pi'$, let π^+ an enrichment of π then there exists an enrichment π'^+ of π' such that $\pi^+ \rhd^+ \pi'^+$ in one-step Deduction modulo.*

Proof. The reduced redex in π has the form $\lambda\alpha\pi_1\ \pi_2$ and π^+ contains a subterm of the form $(\rho\ \lambda\alpha\pi_1^+)\ \pi_2^+$. By Proposition 5 this term reduces to $(\lambda\alpha'\pi_1')\ \pi_2^+$ where π_1' is another enrichment of π_1 and then to $\pi_1'[\alpha' := \pi_2^+]$ which is a subterm of an enrichment of π'.

Proposition 7. *If one-step Deduction modulo normalizes then Deduction modulo does.*

Proof. Consider a reduction sequence $\pi_1, \pi_2, \ldots$ in Deduction modulo. By Proposition 6 there exists a reduction sequence $\pi_1^+, \pi_2^+, \ldots$ in one-step Deduction modulo. Hence the sequence is finite.

5 If Deduction modulo normalizes then Supernatural Deduction does

Here we translate Supernatural deduction proof terms into Deduction modulo proof terms in a way such that a Supernatural deduction reduction maps to at least one Deduction modulo reduction.

We leave λ-terms as usual and call τ the translation which translate:

- an abstraction $\lambda R(\overline{m}).\pi$ as the λ-proof-term corresponding to the same introductions in Deduction modulo.
- an application $\pi\, R(\bar{t})$ as the λ-proof-term corresponding to the same eliminations in Deduction modulo.

Then we have:

Lemma 1 *If $\pi \rhd_\rho \pi'$ then $\tau(\pi) \rhd_\beta^+ \tau(\pi')$.*

Proof. Immediate.

Thus, if natural Deduction modulo $\mathcal{R}$ normalizes, then so does Supernatural Deduction modulo $\mathcal{R}$.

Conclusion

Intergating computation rules to deduction can take several forms that materalize in several formalisms, such as Supernatural deduction, Deduction modulo, the extension of natural deduction with folding and unfolding rules, variants of all these systems in sequent calculus, ... The plurality of these systems raises the question of the stability of properties of rewrite rules with respect to the choice of a formalism or another. This paper contributes to show that the choice of a formalism or another seems to be a rather superficial choice, as properties such as strong normalization are quite stable when one switches from one system to another.

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