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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Mihai Bostan

N° ????

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Thème NUM

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*Rapport
de recherche*



The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime

Mihai Bostan*

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Abstract: We study here the finite Larmor radius regime for the Vlasov-Poisson equations with strong external magnetic field. The proofs follow by compactness techniques. One of the key points is to replace the particle distribution by the center distribution of the Larmor circles. The limit of these densities satisfies a transport equation, whose velocity is given by the gyro-average of the electric field.

Key-words: Vlasov-Maxwell equations, Finite Larmor radius, Gyro-average

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Equations de Vlasov-Poisson avec champ magnétique extérieur fort. Régime du rayon de Larmor fini

Résumé : Nous abordons ici le régime asymptotique du rayon de Larmor fini pour les équations de Vlasov-Poisson avec champ magnétique extérieur fort. Un des points clé consiste à remplacer la densité de particules par la densité de centres de Larmor. La limite de ces densités vérifie une équation de transport dont la vitesse est donnée par la gyro-moyenne du champ électrique.

Mots-clés : Equations de Vlasov-Maxwell, Rayon de Larmor fini, Gyro-moyenne

1 Introduction

The dynamics of a population of charged particles interacting only through electro-magnetic fields created collectively is described by the Vlasov-Maxwell equations. The Cauchy problem for this model is now well understood [11], [15], [16], [17]. Nevertheless the existence of global smooth solution for the three dimensional Vlasov-Maxwell system is still a classical open problem. Conditional results for the global existence of strong solutions, depending on the behavior of the support of the particle densities, have been obtained by different approaches in [18], [9], [24]. It is also worth mentioning the recent results when considering data close to equilibrium [21], [29] or reduced models [10], [4].

The numerical resolution of the Vlasov-Maxwell equations is also a challenge problem since we are working in a phase space with three spatial dimensions and three momentum dimensions. Moreover new difficulties appear when studying asymptotic regimes due to the multi-scale character of the problem. Motivated by the magnetic confinement fusion (MCF) the study of strong magnetic field effect is now of crucial importance. Results for the Vlasov or Vlasov-Poisson equations with large external magnetic fields have been investigated recently [13], [19], [14], [7]. The guiding-center approximation for the Vlasov-Maxwell system with strong initial magnetic field was studied in [6] by the modulated energy method, see also [5], [8], [20] for other applications of this technique.

The asymptotic regime we wish to address here is the gyro-kinetic model with finite Larmor radius. We consider a population of non relativistic electrons whose density is denoted by f . We work in the two dimensional setting assuming that $f = f(t, x, p)$, $(t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$ and that the electro-magnetic field has the form $(E, B) = (E_1, E_2, 0, 0, 0, B_3)(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$. The particle density satisfies the Vlasov equation

$$\partial_t f + \frac{p}{m_e} \cdot \nabla_x f - e \left(E(t, x) + B_3(t, x) \frac{\perp p}{m_e} \right) \cdot \nabla_p f = 0, \quad (1)$$

where $-e < 0$ is the electron charge, $m_e > 0$ is the electron mass and the notation $\perp v$ stands for $(v_2, -v_1)$ for any $v = (v_1, v_2) \in \mathbb{R}^2$. The self-consistent electro-magnetic field (E, B) verifies the Maxwell equations

$$\partial_t E - c_0^2 \perp \nabla_x B_3 = \frac{e}{\varepsilon_0} \int_{\mathbb{R}^2} \frac{p}{m_e} f(t, x, p) dp, \quad (2)$$

$$\partial_t B_3 + \operatorname{div}_x \perp E = 0, \quad (3)$$

$$\operatorname{div}_x E = \frac{e}{\varepsilon_0} \left(n - \int_{\mathbb{R}^2} f(t, x, p) dp \right). \quad (4)$$

Here ε_0 is the vacuum permittivity, c_0 is the light speed in the vacuum and n is the concentration of the background ion distribution. We introduce the following characteristic scales

$$t = Tt', \quad x = \lambda_D x', \quad p = Pp',$$

where $\lambda_D = \left(\frac{\varepsilon_0 K_B T_{\text{th}}}{e^2 n}\right)^{1/2}$ is the Debye length, K_B is the Boltzmann constant and T_{th} is the temperature. Consider also the thermal momentum p_{th} and the inverse of the plasma frequency T_p given respectively by

$$\frac{p_{\text{th}}^2}{m_e} = K_B T_{\text{th}}, \quad T_p^2 = \frac{m_e \varepsilon_0}{e^2 n}.$$

Let us denote by f' , E' , B' the dimensionless unknowns given by

$$f = \frac{n}{P^2} f' \left(\frac{t}{T}, \frac{x}{\lambda_D}, \frac{p}{P} \right), \quad E = \frac{\lambda_D e n}{\varepsilon_0} E' \left(\frac{t}{T}, \frac{x}{\lambda_D} \right), \quad B = \frac{\lambda_D e n m_e}{\varepsilon^2 \varepsilon_0 P} B' \left(\frac{t}{T}, \frac{x}{\lambda_D} \right).$$

Observing that

$$\frac{\lambda_D e n m_e}{\varepsilon_0 P} = \frac{p_{\text{th}} m_e}{P e T_p},$$

we also have

$$B(t, x) = \frac{1}{\varepsilon^2} \frac{m_e p_{\text{th}}}{e T_p P}.$$

In this case the equations become having dropped the primes

$$\partial_t f + \frac{T}{T_p} \frac{P}{p_{\text{th}}} p \cdot \nabla_x f - \frac{T}{T_p} \frac{p_{\text{th}}}{P} \left(E(t, x) + \frac{\perp p}{\varepsilon^2} B_3(t, x) \right) \cdot \nabla_p f = 0,$$

$$\partial_t E - \frac{1}{\varepsilon} \frac{T}{T_p} \frac{p_{\text{th}}}{P} \left(\frac{m_e c_0}{p_{\text{th}}} \right)^2 \perp \nabla_x \left(\frac{B_3}{\varepsilon} \right) = \frac{T}{T_p} \frac{P}{p_{\text{th}}} j(t, x),$$

$$\partial_t \left(\frac{B_3}{\varepsilon} \right) + \varepsilon \frac{T}{T_p} \frac{P}{p_{\text{th}}} \text{div}_x^\perp E = 0,$$

$$\text{div}_x E = 1 - \rho(t, x),$$

where $\rho = \int_{\mathbb{R}^2} f dp$, $j = \int_{\mathbb{R}^2} p f dp$. We choose the reference units such that

$$\frac{T}{T_p} \frac{P}{p_{\text{th}}} = \frac{1}{\varepsilon^2}, \quad \frac{T}{T_p} \frac{p_{\text{th}}}{P} = 1,$$

which means $\frac{T}{T_p} = \frac{P}{p_{\text{th}}} = \frac{1}{\varepsilon}$. Note that in this case the unit for the magnetic field becomes $\frac{1}{\varepsilon} \frac{m_e}{e T_p}$ and the Larmor radius corresponding to this magnetic field and the reference velocity $\frac{P}{m_e}$ is given by $R = T_p \frac{p_{\text{th}}}{m_e} = \lambda_D$. Up to a multiplication constant $\left(\frac{m_e c_0}{p_{\text{th}}}\right)^2$ we obtain the following system, known as the finite Larmor radius regime see [12]

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon^2} \cdot \nabla_x f^\varepsilon - \left(E^\varepsilon(t, x) + B_3^\varepsilon(t, x) \frac{\perp p}{\varepsilon^2} \right) \cdot \nabla_p f^\varepsilon = 0, \quad (5)$$

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon} \perp \nabla_x \left(\frac{B_3^\varepsilon}{\varepsilon} \right) = \frac{j^\varepsilon(t, x)}{\varepsilon^2}, \quad (6)$$

$$\partial_t \left(\frac{B_3^\varepsilon}{\varepsilon} \right) + \frac{1}{\varepsilon} \operatorname{div}_x^\perp E^\varepsilon = 0, \quad (7)$$

$$\operatorname{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t, x). \quad (8)$$

We address here the case of a large external constant magnetic field $B_3^\varepsilon = B_{0,3} \neq 0$ *i.e.*, we investigate the Vlasov-Poisson equations

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon^2} \cdot \nabla_x f^\varepsilon - \left(E^\varepsilon(t, x) + B_{0,3} \frac{\perp p}{\varepsilon^2} \right) \cdot \nabla_p f^\varepsilon = 0, \quad (9)$$

$$\operatorname{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t, x), \quad \operatorname{div}_x^\perp E^\varepsilon = 0, \quad (10)$$

with the initial condition

$$f^\varepsilon(0, x, p) = f_0^\varepsilon(x, p). \quad (11)$$

In order to simplify our analysis we assume periodicity with respect to the space variable $x \in \mathbb{T}^2$, where $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ is equipped with the restriction of the Lebesgue measure of \mathbb{R}^2 on $[0, 1]^2$. We are searching for electric fields E^ε deriving from space periodic potentials Φ^ε satisfying

$$E^\varepsilon = \nabla_x \Phi^\varepsilon, \quad \Delta_x \Phi^\varepsilon = 1 - \rho^\varepsilon(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2.$$

The above problem has solution provided that the global neutrality condition is satisfied *i.e.*, $\int_{\mathbb{T}^2} \rho^\varepsilon(t, x) dx = 1$. Note also that the solution is unique up to a constant. We consider here zero average solutions $\int_{\mathbb{T}^2} \Phi^\varepsilon(t, x) dx = 0$. There are in the literature a lot of studies concerning the Cauchy problem for the Vlasov-Poisson system. The existence of weak solutions has been studied in [1], [23]. For smooth solutions the reader can refer to [28], [22], [2], [27], [25], [26].

The analysis of (9) is carried out by replacing the particle distribution functions f^ε by the center distribution functions g^ε given by

$$g^\varepsilon(t, y, p) = f^\varepsilon(t, x, p),$$

where $y = x - \frac{\perp p}{B_{0,3}}$ is the center of the Larmor circle. Performing the above change of unknown yields the equation

$$\partial_t g^\varepsilon - \frac{B_{0,3} \perp p}{\varepsilon^2} \cdot \nabla_p g^\varepsilon + \frac{1}{B_{0,3}} \perp E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \cdot \nabla_y g^\varepsilon - E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \cdot \nabla_p g^\varepsilon = 0.$$

Observing that

$$\frac{1}{B_{0,3}} \operatorname{div}_y \left\{ \perp E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\} = \operatorname{div}_p \left\{ E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\},$$

the above equation can be written in the conservative form

$$\partial_t g^\varepsilon - \frac{B_{0,3}}{\varepsilon^2} \operatorname{div}_p (g^\varepsilon \perp p) + \operatorname{div}_y \left(g^\varepsilon \frac{\perp E^\varepsilon(t, x)}{B_{0,3}} \right) - \operatorname{div}_p (g^\varepsilon E^\varepsilon(t, x)) = 0, \quad (12)$$

supplemented with the initial condition

$$g^\varepsilon(0, y, p) = f_0^\varepsilon \left(y + \frac{\perp p}{B_{0,3}}, p \right), \quad (y, p) \in \mathbb{T}^2 \times \mathbb{R}^2.$$

The asymptotic behavior of (12) follows by using the weak formulation. As ε goes to zero we expect that the limit center distribution g is invariant under the rotation of \mathbb{R}^2 , since formally we have $\perp p \cdot \nabla_p g = 0$. The limit model is obtained by multiplying with test functions ψ invariant under the rotations of \mathbb{R}^2 , since in this case the singular term $\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{B_{0,3}}{\varepsilon^2} g^\varepsilon \perp p \cdot \nabla_p \psi \, dp dy dt$ vanishes. The treatment of the electric Gauss law is performed in a natural way.

The paper is organized as follows. In Section 2 we establish uniform estimates with respect to the small parameter $\varepsilon > 0$. These estimates come basically from the conservation of the total energy, combined with Sobolev and interpolation inequalities. In Section 3 we investigate the limit of the center distributions. We use compactness methods.

2 Uniform estimates

We assume that the initial conditions $(f_0^\varepsilon)_{\varepsilon > 0}$ satisfy the hypotheses

H1) $f_0^\varepsilon \geq 0$, $\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f_0^\varepsilon(x, p) \, dp \, dx = 1$, $\forall \varepsilon > 0$;

H2) $\sup_{\varepsilon > 0} \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |p|^2 f_0^\varepsilon(x, p) \, dp \, dx < +\infty$;

H3) there is a bounded non increasing function $F_0 \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; r dr)$ such that $f_0^\varepsilon(x, p) \leq F_0(|p|)$, $\forall (x, p) \in \mathbb{T}^2 \times \mathbb{R}^2$, $\varepsilon > 0$.

We consider weak solutions $(f^\varepsilon, E^\varepsilon)_{\varepsilon > 0}$ of the Vlasov-Poisson problem (9), (10), (11). Integrating with respect to the momentum the Vlasov equation yields the continuity equation

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon^2} \operatorname{div}_x j^\varepsilon = 0, \quad (13)$$

where $\rho^\varepsilon = \int_{\mathbb{R}^2} f^\varepsilon \, dp$, $j^\varepsilon = \int_{\mathbb{R}^2} p f^\varepsilon \, dp$. We deduce easily by H1 that the global neutrality condition holds true for any $t \in \mathbb{R}_+$

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f^\varepsilon(t, x, p) \, dp \, dx = 1, \quad \forall \varepsilon > 0.$$

By standard arguments we deduce that the total energy is non increasing on \mathbb{R}_+

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |p|^2 f^\varepsilon(t, x, p) \, dp \, dx + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^2} |E^\varepsilon(t, x)|^2 \, dx \right\} \leq 0. \quad (14)$$

Usual computations involving interpolation inequalities provide an estimate for the L^2 norm of the charge densities $(\rho^\varepsilon)_{\varepsilon>0}$.

$$\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T}^2)} \leq (8\pi \|f_0^\varepsilon\|_{L^\infty})^{1/2} \left(\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|p|^2}{2} f^\varepsilon(t, x, p) dp dx \right)^{1/2}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

In particular H2, H3 imply that

$$\sup_{\varepsilon>0} \|E_0^\varepsilon\|_{H^1(\mathbb{T}^2)^2} < +\infty,$$

and by (14) we deduce that

$$\sup_{0<\varepsilon<1, t \in \mathbb{R}_+} \left\{ \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |p|^2 f^\varepsilon(t, x, p) dp dx + \int_{\mathbb{T}^2} |\rho^\varepsilon(t, x)|^2 dx \right\} < +\infty, \quad (15)$$

and

$$\sup_{0<\varepsilon<1, t \in \mathbb{R}_+} \|E^\varepsilon(t)\|_{H^1(\mathbb{T}^2)^2} < +\infty. \quad (16)$$

Actually following the ideas in [3] it is possible to obtain more uniform bounds with respect to the parameter ε . By using regularization procedure it is sufficient to treat the case of smooth solutions $(f^\varepsilon, E^\varepsilon)$. For any $R > 0$ we can write

$$\rho^\varepsilon(t, x) = \rho_1^\varepsilon(t, x) + \rho_2^\varepsilon(t, x),$$

with $\rho_1^\varepsilon := \int_{\mathbb{R}^2} f^\varepsilon \mathbf{1}_{\{|p| \leq 2R\}} dp$, $\rho_2^\varepsilon := \int_{\mathbb{R}^2} f^\varepsilon \mathbf{1}_{\{|p| > 2R\}} dp$. In order to estimate the charge density ρ_1^ε we combine the Hölder inequality and the bound for the kinetic energy. For any $\eta > 0$ we denote by $(2 + \eta)'$ the conjugate exponent of $2 + \eta$ i.e., $\frac{1}{2+\eta} + \frac{1}{(2+\eta)'} = 1$. We have

$$\rho_1^\varepsilon(t, x) \leq \left(\int_{|p| \leq 2R} |p|^2 f^\varepsilon dp \right)^{\frac{1}{2+\eta}} \left(\int_{|p| \leq 2R} |p|^{-2\frac{(2+\eta)'}{2+\eta}} f^\varepsilon dp \right)^{\frac{1}{(2+\eta)'}}.$$

We deduce that

$$\|\rho_1^\varepsilon\|_{L^{2+\eta}(\mathbb{T}^2)} \leq C_1 R^{\frac{2\eta}{2+\eta}}, \quad t \in \mathbb{R}_+, \quad 0 < \varepsilon < 1, \quad (17)$$

for some constant C_1 not depending on R or t . For estimating the charge density ρ_2^ε observe that

$$f^\varepsilon(t, x, p) = f_0^\varepsilon(X^\varepsilon(0; t, x, p), P^\varepsilon(0; t, x, p)),$$

where $(X^\varepsilon, P^\varepsilon)$ solve the characteristic system

$$\frac{dX^\varepsilon}{ds} = \frac{P^\varepsilon(s)}{\varepsilon^2}, \quad \frac{dP^\varepsilon}{ds} = - \left(E^\varepsilon(s, X^\varepsilon(s)) + B_{0,3} \frac{\perp P^\varepsilon(s)}{\varepsilon^2} \right),$$

and the conditions $X^\varepsilon(t) = x, P^\varepsilon(t) = p$. Multiplying the second characteristic equation by $P^\varepsilon(s)$ we obtain easily that $\left| \frac{d}{ds} |P^\varepsilon(s)| \right| \leq \|E^\varepsilon(s)\|_{L^\infty(\mathbb{T}^2)^2}$ implying that

$$|P^\varepsilon(0; t, x, p)| \geq |p| - t \|E^\varepsilon\|_{L^\infty(]0, t[\times \mathbb{T}^2)^2}.$$

We fix now $t > 0$ and let us consider $R = R(t) = t \|E^\varepsilon\|_{L^\infty(]0,t[\times \mathbb{T}^2)^2}$. The hypothesis H3 yields for any $s \in [0, t]$

$$\begin{aligned} \rho_2^\varepsilon(s, x) &= \int_{|p| > 2R} f_0^\varepsilon(X^\varepsilon(0; s, x, p), P^\varepsilon(0; s, x, p)) dp \\ &\leq \int_{|p| > 2R} F_0(|p| - R) dp \\ &\leq 4\pi \int_0^{+\infty} r F_0(r) dr =: C_2. \end{aligned} \quad (18)$$

Finally combining (17), (18) and Sobolev inequalities we deduce

$$\|E^\varepsilon(s)\|_{L^\infty(\mathbb{T}^2)^2} \leq C_3 \|\rho^\varepsilon(s)\|_{L^{2+\eta}(\mathbb{T}^2)} \leq C(t, \eta) \left(1 + \|E^\varepsilon\|_{L^\infty(]0,t[\times \mathbb{T}^2)^2}^{\frac{2\eta}{2+\eta}} \right), \quad \forall s \in [0, t],$$

and by taking $0 < \eta < 2$ we obtain that the following estimate holds

$$\sup_{0 < \varepsilon < 1} \|E^\varepsilon\|_{L^\infty(]0,t[\times \mathbb{T}^2)^2} < +\infty, \quad t \in \mathbb{R}_+,$$

3 Center distribution limit

In this section we investigate the limit of the center distribution functions $(g^\varepsilon)_{\varepsilon > 0}$ as ε goes to zero. By the hypothesis H3 we deduce that $\sup_{\varepsilon > 0} \|g^\varepsilon\|_{L^\infty} \leq \|F_0\|_{L^\infty}$ and therefore there is a sequence $(\varepsilon_k)_k$ converging towards zero such that $(g^{\varepsilon_k})_k$ converges to some function g with respect to the weak \star topology of $L^\infty(\mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2)$ and $(g^{\varepsilon_k}(0))_k$ converges to some function g_0 with respect to the weak \star topology of $L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$. We assume also that $\lim_{k \rightarrow +\infty} E^{\varepsilon_k} = E$ strongly in $L^2(]0, T[\times \mathbb{T}^2)^2$, $\forall T \in \mathbb{R}_+$. Since $\operatorname{div}_x^\perp E^{\varepsilon_k} = 0$ for any k we have $\operatorname{div}_x^\perp E = 0$. Multiplying (12) by a test function $\theta(t, y)\varphi(p)$ where $\theta \in C_c^1(]0, T[\times \mathbb{T}^2)$, $\varphi \in C_c^1(\mathbb{R}^2)$ yields after integration by parts

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \varphi(p) g^\varepsilon(t, y, p) dp dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \varphi(p) g^\varepsilon(0, y, p) dp dy \\ & + \frac{B_{0,3}}{\varepsilon^2} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y)^\perp p \cdot \nabla_p \varphi g^\varepsilon dp dy dt \\ & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot^\perp E^\varepsilon \left(t, y + \frac{^\perp p}{B_{0,3}} \right) \varphi g^\varepsilon dp dy dt \\ & + \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \nabla_p \varphi \cdot E^\varepsilon \left(t, y + \frac{^\perp p}{B_{0,3}} \right) g^\varepsilon dp dy dt = 0. \end{aligned} \quad (19)$$

Obviously we have the convergences

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \varphi g^{\varepsilon_k} dp dy dt = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \varphi g dp dy dt,$$

and

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \varphi(p) g^{\varepsilon_k}(0, y, p) dp dy = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \varphi(p) g_0 dp dy.$$

We claim that for any test function $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ we have

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} E^{\varepsilon_k} \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi g^{\varepsilon_k} dp dy dt = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} E \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi g dp dy dt. \quad (20)$$

Indeed we have

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} E^{\varepsilon_k} \psi g^{\varepsilon_k} dp dy dt &= \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k} - E) \psi g^{\varepsilon_k} dp dy dt \\ &+ \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} E \psi g^{\varepsilon_k} dp dy dt \\ &= I_1^k + I_2^k. \end{aligned}$$

We can write

$$\begin{aligned} |I_1^k| &= \left| \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k} - E)(t, x) \psi \left(t, x - \frac{\perp p}{B_{0,3}}, p \right) f^{\varepsilon_k} dp dx dt \right| \\ &\leq \|\psi\|_{L^\infty} \int_0^T \int_{\mathbb{T}^2} |(E^{\varepsilon_k} - E)(t, x)| \rho^{\varepsilon_k}(t, x) dx dt \\ &\leq \|\psi\|_{L^\infty} \|E^{\varepsilon_k} - E\|_{L^2([0, T] \times \mathbb{T}^2)^2} \|\rho^{\varepsilon_k}\|_{L^2([0, T] \times \mathbb{T}^2)}, \end{aligned}$$

and therefore by (15) we deduce that $\lim_{k \rightarrow +\infty} I_1^k = 0$. Obviously we have the convergence

$$\lim_{k \rightarrow +\infty} I_2^k = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} E \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi(t, y, p) g dp dy dt,$$

and thus (20) holds. Taking $\varepsilon = \varepsilon_k$ in (19) and passing to the limit for $k \rightarrow +\infty$ we deduce easily that

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) (\perp p \cdot \nabla_p \varphi) g dp dy dt = 0,$$

saying that g is invariant under the rotations of \mathbb{R}^2 . Passing in polar coordinates we deduce that for any function $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ we have

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \psi(t, y, p) g(t, y, p) dp dy dt = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \tilde{\psi}(t, y, r) \tilde{g}(t, y, r) dr dy dt, \quad (21)$$

where

$$\tilde{\psi}(t, y, r) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, y, r \cos \alpha, r \sin \alpha) d\alpha, \quad (t, y, r) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}_+,$$

and

$$\tilde{g}(t, y, r) = 2\pi r g(t, y, r, 0), \quad (t, y, r) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}_+.$$

We multiply now (12) by a test function $\theta(t, y)\psi(|p|)$ where $\theta \in C_c^1([0, T] \times \mathbb{T}^2)$, $\psi \in C_c^1(\mathbb{R}_+)$, $\psi'(0) = 0$. Notice that the function $p \rightarrow \psi(|p|)$ belongs to $C_c^1(\mathbb{R}^2)$, $\nabla_p \psi(|\cdot|)|_{p=0} = (0, 0)$ and thus we obtain

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \psi(|p|) g^\varepsilon dp dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \psi(|p|) g^\varepsilon(0, y, p) dp dy \\ & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot \perp E^\varepsilon \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi(|p|) g^\varepsilon dp dy dt \\ & + \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E^\varepsilon \left(t, y + \frac{\perp p}{B_{0,3}} \right) g^\varepsilon dp dy dt = 0. \end{aligned}$$

Taking $\varepsilon = \varepsilon_k$ and letting $k \rightarrow +\infty$ we obtain as before

$$\begin{aligned} & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \psi(|p|) g dp dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \psi(|p|) g_0 dp dy \\ & - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot \perp E \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi(|p|) g dp dy dt \\ & + \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E \left(t, y + \frac{\perp p}{B_{0,3}} \right) g dp dy dt = 0. \end{aligned} \quad (22)$$

Using now (21) we have

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \psi(|p|) g dp dy dt = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \partial_t \theta \psi(r) \tilde{g} dr dy dt, \quad (23)$$

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \psi(|p|) g_0 dp dy = \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \theta(0, y) \psi(r) \tilde{g}_0 dr dy, \quad (24)$$

where $\tilde{g}_0(y, r) = r \int_0^{2\pi} g_0(y, r \cos \alpha, r \sin \alpha) d\alpha$. Similarly we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_y \theta \cdot \perp E \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi(|p|) g dp dy dt &= \frac{1}{2\pi} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \psi(r) \nabla_y \theta \\ &\cdot \int_0^{2\pi} \perp E \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha \tilde{g} dr dy dt. \end{aligned} \quad (25)$$

Finally observe that the last term in (22) vanishes. Indeed we have

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E \left(t, y + \frac{\perp p}{B_{0,3}} \right) g dp dy dt &= \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \theta(t, y) \psi'(r) \\ &\times \frac{1}{2\pi} \int_0^{2\pi} (\cos \alpha E_1 + \sin \alpha E_2) \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha \tilde{g} dr dy dt. \end{aligned}$$

Since $\operatorname{div}_x^\perp E = 0$ there is Φ such that $E = \nabla_x \Phi$ and thus

$$\frac{d}{d\alpha} \left\{ \Phi \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) \right\} = \left(\frac{r \cos \alpha}{B_{0,3}} E_1 + \frac{r \sin \alpha}{B_{0,3}} E_2 \right),$$

implying that

$$\int_0^{2\pi} (\cos \alpha E_1 + \sin \alpha E_2) \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha = 0, \quad \forall r \in \mathbb{R}_+.$$

Combining now (22 – 25) yields

$$- \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} (\partial_t(\theta\psi) + \mathcal{V}(t, y, r) \cdot \nabla_y(\theta\psi)) \tilde{g} dr dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}_+} \theta(0, y) \psi(r) \tilde{g}_0 dr dy = 0, \quad (26)$$

where

$$\mathcal{V}(t, y, r) = \frac{1}{2\pi B_{0,3}} \int_0^{2\pi} {}^\perp E \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha.$$

Taking into account that $\operatorname{div}_y^\perp E = 0$ we deduce that $\operatorname{div}_y \mathcal{V} = 0$ and therefore (26) can be written formally

$$\begin{cases} \partial_t \tilde{g} + \mathcal{V}(t, y, r) \cdot \nabla_y \tilde{g} = 0, \\ \tilde{g}(0, y, r) = r \int_0^{2\pi} g_0(y, r \cos \alpha, r \sin \alpha) d\alpha. \end{cases} \quad (27)$$

It remains to pass to the limit in the electric Gauss law. For any function $\varphi \in C^1([0, T] \times \mathbb{T}^2)$ we have

$$\begin{aligned} - \int_0^T \int_{\mathbb{T}^2} E^{\varepsilon_k} \cdot \nabla_x \varphi dx dt - \int_0^T \int_{\mathbb{T}^2} \varphi dx dt &= - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \varphi f^{\varepsilon_k} dp dx dt \\ &= - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \varphi \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) g^{\varepsilon_k} dp dy dt, \end{aligned}$$

and after letting $k \rightarrow +\infty$ we obtain

$$\begin{aligned} &- \int_0^T \int_{\mathbb{T}^2} E \cdot \nabla_x \varphi dx dt - \int_0^T \int_{\mathbb{T}^2} \varphi dx dt = - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \varphi \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) g dp dy dt \\ &= - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \varphi(t, x) g \left(t, x - \frac{{}^\perp p}{B_{0,3}}, p \right) dp dx dt \\ &= - \int_0^T \int_{\mathbb{T}^2} \varphi(t, x) \int_{\mathbb{R}_+} \frac{1}{2\pi} \int_0^{2\pi} \tilde{g} \left(t, x_1 - \frac{r \sin \alpha}{B_{0,3}}, x_2 + \frac{r \cos \alpha}{B_{0,3}}, r \right) d\alpha dr dx dt. \end{aligned}$$

Therefore the finite Larmor radius regime is characterized by the system

$$\begin{cases} \partial_t \tilde{g} + \mathcal{V}(t, y, r) \cdot \nabla_y \tilde{g} = 0, & \tilde{g}(0, y, r) = r \int_0^{2\pi} g_0(y, r \cos \alpha, r \sin \alpha) d\alpha, \\ \mathcal{V}(t, y, r) = \frac{1}{2\pi B_{0,3}} \int_0^{2\pi} {}^\perp E \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha, \\ \operatorname{div}_x E = 1 - \int_{\mathbb{R}_+} \frac{1}{2\pi} \int_0^{2\pi} \tilde{g} \left(t, x_1 - \frac{r \sin \alpha}{B_{0,3}}, x_2 + \frac{r \cos \alpha}{B_{0,3}}, r \right) d\alpha dr, & \operatorname{div}_x {}^\perp E = 0. \end{cases}$$

Note that when the Larmor radius goes to zero, meaning that the particle distribution and the center distribution coincide (*i.e.*, $x = y$) the above model converges to the model

$$\partial_t \rho + \mathcal{V}(t, x) \cdot \nabla_x \rho = 0, \quad \mathcal{V} = \frac{{}^\perp E}{B_{0,3}}, \quad \operatorname{div}_x E = 1 - \rho, \quad \operatorname{div}_x {}^\perp E = 0,$$

which is the guiding-center approximation regime see [6].

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Contents

1	Introduction	3
2	Uniform estimates	6
3	Center distribution limit	8



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