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***The Vlasov-Maxwell system with strong initial
magnetic field. Part II : Finite Larmor radius regime***

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The Vlasov-Maxwell system with strong initial magnetic field. Part II : Finite Larmor radius regime

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Abstract: We study here the finite Larmor radius regime for the Vlasov-Maxwell equations with strong initial magnetic field. The proofs follow by the modulated energy technique. We establish convergence results for the electro-magnetic field and we justify rigorously the limit for the circle center distribution.

Key-words: Vlasov-Maxwell equations, Finite Larmor radius, Modulated energy

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Equations de Vlasov-Maxwell avec champ magnétique initial fort. Partie II : régime du rayon de Larmor fini

Résumé : Nous abordons ici le régime asymptotique du rayon de Larmor fini pour les équations de Vlasov-Maxwell avec champ magnétique fort. En utilisant la technique de l'énergie modulée on montre des résultats de convergence pour le champ électromagnétique et on justifie également la limite de la distribution des centres.

Mots-clés : Equations de Vlasov-Maxwell, Rayon de Larmor fini, Energie modulée

1 Introduction

The dynamics of a population of charged particles interacting only through electro-magnetic fields created collectively is described by the Vlasov-Maxwell equations. The Cauchy problem for this model is now well understood [8], [12], [13], [14]. Nevertheless the existence of global smooth solution for the three dimensional Vlasov-Maxwell system is still a classical open problem. Conditional results for the global existence of strong solutions, depending on the behavior of the support of the particle densities, have been obtained by different approaches in [15], [6], [19]. It is also worth mentioning the recent results when considering data close to equilibrium [18], [20] or reduced models [7], [1].

The numerical resolution of the Vlasov-Maxwell equations is also a challenge problem since we are working in a phase space with 3 spatial dimensions and 3 momentum dimensions. Moreover new difficulties appear when studying asymptotic regimes due to the multi-scale character of the problem. Motivated by the magnetic confinement fusion (MCF) the study of strong magnetic field effect is now of crucial importance. Results for the Vlasov or Vlasov-Poisson equations with large external magnetic fields have been investigated recently [10], [16], [11], [4]. The guiding-center approximation for the Vlasov-Maxwell system with strong initial magnetic field was studied in [3] by the modulated energy method, see also [2], [5], [17] for other applications of this technique.

The asymptotic regime we wish to address here is the gyro-kinetic model with finite Larmor radius. We consider a population of non relativistic electrons whose density is denoted by f . We work in the two dimensional setting assuming that $f = f(t, x, p)$, $(t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$ and that the electro-magnetic field has the form $(E, B) = (E_1, E_2, 0, 0, 0, B_3)(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$. The particle density satisfies the Vlasov equation

$$\partial_t f + \frac{p}{m_e} \cdot \nabla_x f - e \left(E(t, x) + B_3(t, x) \frac{\perp p}{m_e} \right) \cdot \nabla_p f = 0, \quad (1)$$

where $-e < 0$ is the electron charge, $m_e > 0$ is the electron mass and the notation $\perp v$ stands for $(v_2, -v_1)$ for any $v = (v_1, v_2) \in \mathbb{R}^2$. The self-consistent electro-magnetic field (E, B) verifies the Maxwell equations

$$\partial_t E - c_0^2 \perp \nabla_x B_3 = \frac{e}{\varepsilon_0} \int_{\mathbb{R}^2} \frac{p}{m_e} f(t, x, p) dp, \quad (2)$$

$$\partial_t B_3 + \operatorname{div}_x \perp E = 0, \quad (3)$$

$$\operatorname{div}_x E = \frac{e}{\varepsilon_0} \left(n - \int_{\mathbb{R}^2} f(t, x, p) dp \right). \quad (4)$$

Here ε_0 is the vacuum permittivity, c_0 is the light speed in the vacuum and n is the concentration of the background ion distribution. We introduce the following characteristic scales

$$t = Tt', \quad x = \lambda_D x', \quad p = Pp',$$

where $\lambda_D = \left(\frac{\varepsilon_0 K_B T_{\text{th}}}{e^2 n}\right)^{1/2}$ is the Debye length, K_B is the Boltzmann constant and T_{th} is the temperature. Consider also the thermal momentum p_{th} and the inverse of the plasma frequency T_p given respectively by

$$\frac{p_{\text{th}}^2}{m_e} = K_B T_{\text{th}}, \quad T_p^2 = \frac{m_e \varepsilon_0}{e^2 n}.$$

Let us denote by f' , E' , B' the dimensionless unknowns given by

$$f = \frac{n}{P^2} f' \left(\frac{t}{T}, \frac{x}{\lambda_D}, \frac{p}{P} \right), \quad E = \frac{\lambda_D e n}{\varepsilon_0} E' \left(\frac{t}{T}, \frac{x}{\lambda_D} \right), \quad B = \frac{\lambda_D e n m_e}{\varepsilon^2 \varepsilon_0 P} B' \left(\frac{t}{T}, \frac{x}{\lambda_D} \right).$$

Observing that

$$\frac{\lambda_D e n m_e}{\varepsilon_0 P} = \frac{p_{\text{th}} m_e}{P e T_p},$$

we also have

$$B(t, x) = \frac{1}{\varepsilon^2} \frac{m_e p_{\text{th}}}{e T_p P}.$$

In this case the equations become having dropped the primes

$$\partial_t f + \frac{T}{T_p} \frac{P}{p_{\text{th}}} p \cdot \nabla_x f - \frac{T}{T_p} \frac{p_{\text{th}}}{P} \left(E(t, x) + \frac{\perp p}{\varepsilon^2} B_3(t, x) \right) \cdot \nabla_p f = 0,$$

$$\partial_t E - \frac{1}{\varepsilon} \frac{T}{T_p} \frac{p_{\text{th}}}{P} \left(\frac{m_e c_0}{p_{\text{th}}} \right)^2 \perp \nabla_x \left(\frac{B_3}{\varepsilon} \right) = \frac{T}{T_p} \frac{P}{p_{\text{th}}} j(t, x),$$

$$\partial_t \left(\frac{B_3}{\varepsilon} \right) + \varepsilon \frac{T}{T_p} \frac{P}{p_{\text{th}}} \text{div}_x^\perp E = 0,$$

$$\text{div}_x E = 1 - \rho(t, x),$$

where $\rho = \int_{\mathbb{R}^2} f \, dp$, $j = \int_{\mathbb{R}^2} p f \, dp$. We choose the reference units such that

$$\frac{T}{T_p} \frac{P}{p_{\text{th}}} = \frac{1}{\varepsilon^2}, \quad \frac{T}{T_p} \frac{p_{\text{th}}}{P} = 1,$$

which means $\frac{T}{T_p} = \frac{P}{p_{\text{th}}} = \frac{1}{\varepsilon}$. Note that in this case the unit for the magnetic field becomes $\frac{1}{\varepsilon} \frac{m_e}{e T_p}$ and the Larmor radius corresponding to this magnetic field and the reference velocity $\frac{P}{m_e}$ is given by $R = T_p \frac{p_{\text{th}}}{m_e} = \lambda_D$. Up to a multiplication constant $\left(\frac{m_e c_0}{p_{\text{th}}}\right)^2$ we obtain the following system, known as the finite Larmor radius regime see [9]

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon^2} \cdot \nabla_x f^\varepsilon - \left(E^\varepsilon(t, x) + B_3^\varepsilon(t, x) \frac{\perp p}{\varepsilon^2} \right) \cdot \nabla_p f^\varepsilon = 0, \quad (5)$$

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon} \perp \nabla_x \left(\frac{B_3^\varepsilon}{\varepsilon} \right) = \frac{j^\varepsilon(t, x)}{\varepsilon^2}, \quad (6)$$

$$\partial_t \left(\frac{B_3^\varepsilon}{\varepsilon} \right) + \frac{1}{\varepsilon} \operatorname{div}_x^\perp E^\varepsilon = 0, \quad (7)$$

$$\operatorname{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t, x). \quad (8)$$

Performing now the change of unknown

$$f^\varepsilon(t, x, p) = \frac{1}{\varepsilon^4} F^\varepsilon(t, x, u), \quad p = \varepsilon^2 u,$$

we have the equalities

$$\rho^\varepsilon = \int_{\mathbb{R}^2} f^\varepsilon dp = \int_{\mathbb{R}^2} F^\varepsilon du, \quad j^\varepsilon = \int_{\mathbb{R}^2} p f^\varepsilon dp = \varepsilon^2 \int_{\mathbb{R}^2} u F^\varepsilon du =: \varepsilon^2 J^\varepsilon.$$

Thus the previous equations can be written

$$\partial_t F^\varepsilon + u \cdot \nabla_x F^\varepsilon - \frac{1}{\varepsilon^2} (E^\varepsilon(t, x) + B_3^\varepsilon(t, x)^\perp u) \cdot \nabla_u F^\varepsilon = 0, \quad (9)$$

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon} \perp \nabla_x \left(\frac{B_3^\varepsilon}{\varepsilon} \right) = J^\varepsilon(t, x), \quad (10)$$

$$\partial_t \left(\frac{B_3^\varepsilon}{\varepsilon} \right) + \frac{1}{\varepsilon} \operatorname{div}_x^\perp E^\varepsilon = 0, \quad (11)$$

$$\operatorname{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t, x). \quad (12)$$

We also prescribe initial conditions

$$F^\varepsilon(0, x, u) = F_0^\varepsilon(x, u), \quad (E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)(0, x) = (E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)(x).$$

In order to simplify our analysis we assume periodicity with respect to the space variable $x \in \mathbb{T}^2$, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is equipped with the restriction of the Lebesgue measure of \mathbb{R}^2 on $[0, 1]^2$. The asymptotic behavior of the previous system when ε goes to zero was investigated in [3] (see Section 4). We assume that the initial conditions satisfy the hypotheses

H1) $F_0^\varepsilon \geq 0$, $\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F_0^\varepsilon(x, u) du dx = 1$;

H2) $\lim_{\varepsilon \searrow 0} \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|u|^2}{2} F_0^\varepsilon(x, u) du dx = 0$;

H3) $(E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon) \in L^2(\mathbb{T}^2)^3$ and $\operatorname{div}_x E_0^\varepsilon = 1 - \rho_0^\varepsilon$ where $\rho_0^\varepsilon = \int_{\mathbb{R}^2} F_0^\varepsilon du$;

H4) there are $E_0 = (E_{0,1}, E_{0,2}) \in L^2(\mathbb{T}^2)^2$ verifying $\partial_{x_1} E_{0,2} - \partial_{x_2} E_{0,1} = 0$ and a constant magnetic field $(0, 0, B_{0,3})$ with $B_{0,3} \neq 0$ such that

$$\lim_{\varepsilon \searrow 0} \left\{ \frac{1}{2} \int_{\mathbb{T}^2} |E_0^\varepsilon(x) - E_0(x)|^2 dx + \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{B_{0,3}^\varepsilon(x) - B_{0,3}}{\varepsilon} \right)^2 dx \right\} = 0;$$

H5) $\lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^2} \frac{B_{0,3}^\varepsilon(x) - B_{0,3}}{\varepsilon^2} dx = 0$.

Notice that H2 is equivalent to

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|p|^2}{2} f_0^\varepsilon(x, p) dp dx = 0.$$

Under the above conditions it was shown in [3] that

$$\lim_{\varepsilon \searrow 0} \left\{ \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|p|^2}{2} f^\varepsilon dp dx + \frac{1}{2} \int_{\mathbb{T}^2} |E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 dx \right\} = 0,$$

uniformly with respect to $t \in [0, T]$ and

$$\lim_{\varepsilon \searrow 0} \rho^\varepsilon = \rho, \quad \lim_{\varepsilon \searrow 0} \frac{j^\varepsilon}{\varepsilon} = J, \quad \lim_{\varepsilon \searrow 0} \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon^2} = A_3 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2),$$

provided that (ρ, J, E, A_3) is a smooth solution of the limit system

$$J = \rho \frac{\perp E}{B_{0,3}}, \quad \partial_t E - \perp \nabla_x A_3 = J, \quad \operatorname{div}_x \perp E = 0, \quad \operatorname{div}_x E = 1 - \rho, \quad (13)$$

with the initial condition $E(0, \cdot) = E_0$. It remains to study the limit as ε goes to zero of the particle distributions $(f^\varepsilon)_{\varepsilon > 0}$ given by (5). Actually this analysis can be carried out if we replace the particle distribution functions f^ε by the center distribution functions g^ε given by

$$g^\varepsilon(t, y, p) = f^\varepsilon(t, x, p),$$

where $y = x - \frac{\perp p}{B_{0,3}}$ is the center of the Larmor circle. Performing the above change of unknown yields the equation

$$\begin{aligned} \partial_t g^\varepsilon - \frac{B_{0,3}}{\varepsilon^2} \perp p \cdot \nabla_p g^\varepsilon - \frac{1}{B_{0,3}} A_3^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) p \cdot \nabla_y g^\varepsilon + \frac{1}{B_{0,3}} \perp E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \cdot \nabla_y g^\varepsilon \\ - E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \cdot \nabla_p g^\varepsilon - A_3^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \perp p \cdot \nabla_p g^\varepsilon = 0, \end{aligned}$$

where $A_3^\varepsilon = \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon^2}$. Observing that

$$\nabla_p \left\{ A_3^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\} + \frac{1}{B_{0,3}} \perp \nabla_y \left\{ A_3^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\} = 0,$$

and

$$\frac{1}{B_{0,3}} \operatorname{div}_y \left\{ \perp E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\} = \operatorname{div}_p \left\{ E^\varepsilon(t, y + \frac{\perp p}{B_{0,3}}) \right\},$$

the above equation can be written in conservative form

$$\partial_t g^\varepsilon - \frac{B_{0,3}}{\varepsilon^2} \operatorname{div}_p (g^\varepsilon \perp p) - \operatorname{div}_y \left(\frac{g^\varepsilon}{B_{0,3}} (A_3^\varepsilon p - \perp E^\varepsilon) \right) - \operatorname{div}_p (g^\varepsilon (E^\varepsilon + A_3^\varepsilon \perp p)) = 0, \quad (14)$$

supplemented with the initial condition

$$g^\varepsilon(0, y, p) = f_0^\varepsilon \left(y + \frac{\perp p}{B_{0,3}}, p \right), \quad (y, p) \in \mathbb{T}^2 \times \mathbb{R}^2.$$

In order to identify the limit of $(g^\varepsilon)_{\varepsilon>0}$ as ε goes to zero we intend to pass to the limit in the weak formulation of (14). Besides the treatment of the second term of (14), which becomes singular as ε goes to zero, the analysis of the center distribution limit requires a strong convergence result for $(A_3^\varepsilon)_{\varepsilon>0}$. Recall that for the moment we have proved the convergence in distribution sense of $(A_3^\varepsilon)_{\varepsilon>0}$ towards the unknown A_3 appearing in the limit system (13). We will see that under appropriate hypotheses this convergence holds also strongly in $L^2(\mathbb{T}^2)$, uniformly on compact subsets of \mathbb{R}_+ . But this requires a higher order approximation analysis for the system (9), (10), (11), (12).

The paper is organized as follows: in Section 2 we determine the first order correction terms for (13). We identify these terms formally by Hilbert method; we expand in powers of ε the solution of (9), (10), (11), (12). Then we justify this higher approximation by using a first order corrected version of the modulated energy introduced in [3]. Section 3 is devoted to the limit procedure for the center distributions. We obtain a transport equation for the limit density whose velocity field depends on the gyro-averages of the macroscopic quantities E, A_3 .

2 Second order approximation

In this section we discuss the second order approximation for (9), (10), (11), (12). As usual we start with a formal analysis. Let us search for

$$\begin{aligned} F^\varepsilon &= F + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + \dots, \\ E^\varepsilon &= E + \varepsilon E^{(1)} + \varepsilon^2 E^{(2)} + \dots, \\ B_3^\varepsilon &= B_{0,3} + \varepsilon^2 A_3 + \varepsilon^3 A_3^{(1)} + \dots \end{aligned}$$

Notice that since $\lim_{\varepsilon \searrow 0} \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} = 0$ in $L^2(]0, T[; L^2(\mathbb{T}^2))$ for any $T \in \mathbb{R}_+$ there is no first order term in the expansion of B_3^ε . We denote by $(\rho, J), (\rho^{(k)}, J^{(k)})_{k \geq 1}$ the charge and current densities of $F, (F^{(k)})_{k \geq 1}$. Since $\partial_t \rho^\varepsilon + \operatorname{div}_x J^\varepsilon = 0$ we have $\partial_t \rho + \operatorname{div}_x J = 0$ and $\partial_t \rho^{(k)} + \operatorname{div}_x J^{(k)} = 0$ for any $k \geq 1$. Plugging these ansatz into the Vlasov equation (9) yields

$$-(E + B_{0,3} \perp u) \cdot \nabla_u F = 0, \quad (15)$$

$$-(E + B_{0,3}{}^\perp u) \cdot \nabla_u F^{(1)} - E^{(1)} \cdot \nabla_u F = 0, \quad (16)$$

$$\partial_t F + u \cdot \nabla_x F - (E + B_{0,3}{}^\perp u) \cdot \nabla_u F^{(2)} - E^{(1)} \cdot \nabla_u F^{(1)} - (E^{(2)} + A_3{}^\perp u) \cdot \nabla_u F = 0. \quad (17)$$

Multiplying (15), (16) by u and integrating with respect to u yields

$$\rho E + B_{0,3}{}^\perp J = 0, \quad \rho^{(1)} E + B_{0,3}{}^\perp J^{(1)} + \rho E^{(1)} = 0,$$

which is equivalent to

$$J = \rho \frac{{}^\perp E}{B_{0,3}}, \quad J^{(1)} = \frac{\rho^{(1)}{}^\perp E + \rho {}^\perp E^{(1)}}{B_{0,3}}. \quad (18)$$

Multiplying now (17) by u we obtain after integration with respect to u

$$\partial_t J + \operatorname{div}_x \int_{\mathbb{R}^2} (u \otimes u) F \, du + \rho^{(2)} E + B_{0,3}{}^\perp J^{(2)} + \rho^{(1)} E^{(1)} + \rho E^{(2)} + A_3{}^\perp J = 0,$$

which is equivalent to

$$\partial_t {}^\perp J + \operatorname{div}_x \int_{\mathbb{R}^2} ({}^\perp u \otimes u) F \, du + \rho^{(2)}{}^\perp E - B_{0,3} J^{(2)} + \rho^{(1)}{}^\perp E^{(1)} + \rho {}^\perp E^{(2)} - A_3 J = 0. \quad (19)$$

Multiplying now (15) by u_1^2 and $u_1 u_2$ implies

$$\int_{\mathbb{R}^2} u_1 u_2 F \, du = -\frac{E_1 J_1}{B_{0,3}} = -\rho \frac{E_1 E_2}{(B_{0,3})^2}, \quad (20)$$

and

$$\int_{\mathbb{R}^2} (u_2^2 - u_1^2) F \, du = -\frac{E_1 J_2 + E_2 J_1}{B_{0,3}} = -\rho \frac{(E_2)^2 - (E_1)^2}{(B_{0,3})^2}. \quad (21)$$

We deduce that

$$(\nabla_x \otimes \nabla_x) : \int_{\mathbb{R}^2} ({}^\perp u \otimes u) F \, du = -\frac{1}{(B_{0,3})^2} (\nabla_x \otimes \nabla_x) : (\rho {}^\perp E \otimes E), \quad (22)$$

and finally by taking the divergence in (19) we find

$$\begin{aligned} \partial_t \operatorname{div}_x {}^\perp J - \frac{1}{(B_{0,3})^2} (\nabla_x \otimes \nabla_x) : (\rho {}^\perp E \otimes E) + \operatorname{div}_x (\rho^{(2)}{}^\perp E) - B_{0,3} \operatorname{div}_x J^{(2)} \\ + \operatorname{div}_x (\rho^{(1)}{}^\perp E^{(1)}) + \operatorname{div}_x (\rho {}^\perp E^{(2)}) - \operatorname{div}_x (A_3 J) = 0. \end{aligned} \quad (23)$$

Plugging now the asymptotic expansions into the Maxwell equations (10), (11), (12) and combining with the previous equations yields the systems

$$J = \rho \frac{{}^\perp E}{B_{0,3}}, \quad \partial_t E - {}^\perp \nabla_x A_3 = J, \quad \operatorname{div}_x {}^\perp E = 0, \quad \operatorname{div}_x E = 1 - \rho, \quad (24)$$

$$J^{(1)} = \frac{\rho^{(1)\perp} E + \rho^\perp E^{(1)}}{B_{0,3}}, \quad \partial_t E^{(1)} - \perp \nabla_x A_3^{(1)} = J^{(1)}, \quad \operatorname{div}_x^\perp E^{(1)} = 0, \quad \operatorname{div}_x E^{(1)} = -\rho^{(1)}, \quad (25)$$

$$\begin{cases} \partial_t A_3 + \operatorname{div}_x^\perp E^{(2)} = 0, & \operatorname{div}_x E^{(2)} = -\rho^{(2)}, \\ B_{0,3} \partial_t \rho^{(2)} + \perp E \cdot \nabla_x \rho^{(2)} + \perp E^{(2)} \cdot \nabla_x \rho = -\partial_t \operatorname{div}_x^\perp J + \frac{1}{(B_{0,3})^2} (\nabla_x \otimes \nabla_x) : (\rho^\perp E \otimes E) \\ \quad - \perp E^{(1)} \cdot \nabla_x \rho^{(1)} + \operatorname{div}_x (A_3 J) + \rho \partial_t A_3. \end{cases} \quad (26)$$

Obviously the system (24) is exactly the limit system (13). In order to solve the second system it is convenient to eliminate $J^{(1)}$ by taking the divergence of the time evolution equation for $E^{(1)}$ (or by using the continuity equation $\partial_t \rho^{(1)} + \operatorname{div}_x J^{(1)} = 0$). We obtain

$$B_{0,3} \partial_t \rho^{(1)} + \perp E \cdot \nabla_x \rho^{(1)} + \perp E^{(1)} \cdot \nabla_x \rho = 0, \quad \operatorname{div}_x^\perp E^{(1)} = 0, \quad \operatorname{div}_x E^{(1)} = -\rho^{(1)}.$$

The last equation of the third system was obtained by eliminating $J^{(2)}$ in (23) using the continuity equation $\partial_t \rho^{(2)} + \operatorname{div}_x J^{(2)} = 0$. The equations $\operatorname{div}_x^\perp E = \operatorname{div}_x^\perp E^{(1)} = 0$ and $\operatorname{div}_x^\perp E^{(2)} = -\partial_t A_3$ have been used as well. We assume that all these systems have smooth solutions (essentially we need that these solutions belong to $W^{2,\infty}([0, T] \times \mathbb{T}^2)$ for any $T \in \mathbb{R}_+$). We define the modulated energy

$$\begin{aligned} \mathcal{H}_4^\varepsilon(t) &= \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{1}{2} \left| u - \frac{\perp(E + \varepsilon E^{(1)})}{B_{0,3}} \right|^2 F^\varepsilon(t, x, u) \, du \, dx \\ &\quad + \frac{1}{2} \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2(t, x) \, dx + \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2(t, x) \, dx. \end{aligned}$$

We assume that $\sup_{\varepsilon > 0} \varepsilon^{-4} \mathcal{H}_4^\varepsilon(0) < +\infty$, which is equivalent to

$$\begin{aligned} \sup_{\varepsilon > 0} \varepsilon^{-4} \left\{ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\varepsilon^2}{2} \left| u - \frac{\perp E_0}{B_{0,3}} \right|^2 F_0^\varepsilon \, du \, dx + \frac{1}{2} \int_{\mathbb{T}^2} |E_0^\varepsilon - E_0 - \varepsilon E_0^{(1)}|^2 \, dx \right. \\ \left. + \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_{0,3} \right)^2 \, dx \right\} < +\infty. \end{aligned}$$

We write the Vlasov equation (9) in the form

$$\begin{aligned} \varepsilon^2 (\partial_t F^\varepsilon + \operatorname{div}_x (F^\varepsilon u)) - \operatorname{div}_u \left(F^\varepsilon (E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u) \right) \\ - \operatorname{div}_u \left(F^\varepsilon (E^\varepsilon - E - \varepsilon E^{(1)} + (B_{0,3}^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u) \right) = 0. \end{aligned}$$

We multiply the above equation by $h_4^\varepsilon(t, x, u) = \frac{1}{2} \left| u - (D + \varepsilon D^{(1)})(t, x) \right|^2$, where $D = \frac{\perp E}{B_{0,3}}$, $D^{(1)} = \frac{\perp E^{(1)}}{B_{0,3}}$, and we perform integration by parts. The computations are standard but long.

Therefore we split them into three lemmas. The notation C stands for generic constants depending only on the $W^{2,\infty}([0, T] \times \mathbb{T}^2)$ norms of the solutions to the limit systems and T . These constants are allowed to change from line to line.

Lemma 2.1 *For any $0 < \varepsilon < 1$, $T \in \mathbb{R}_+$, $t \in [0, T]$ we have*

$$\begin{aligned} & \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \varepsilon^2 (\partial_t F^\varepsilon + \operatorname{div}_x(F^\varepsilon u)) h_4^\varepsilon \, du \, dx = -Q_3(t) + \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon \, du \, dx \\ & - \frac{\varepsilon^2}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} \left\{ \varepsilon^2 \int_{\mathbb{R}^2} u F^\varepsilon \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right\} \cdot^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) \, dx, \end{aligned}$$

where $D^\varepsilon = D + \varepsilon D^{(1)}$ and $|Q_3(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. We have

$$\begin{aligned} Q_3 &+ \frac{\varepsilon^2}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} \left\{ \varepsilon^2 \int_{\mathbb{R}^2} u F^\varepsilon \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right\} \cdot^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) \, dx \\ &= \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon (\partial_t h_4^\varepsilon + u \cdot \nabla_x h_4^\varepsilon) \, du \, dx \\ &= -\varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon (u - D^\varepsilon) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) \, du \, dx \\ &\quad - \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon (u - D^\varepsilon) \cdot (\nabla_x D^\varepsilon) (u - D^\varepsilon) \, du \, dx \\ &=: Q_4(t) + Q_5(t). \end{aligned} \tag{27}$$

It is easily seen that

$$|Q_5(t)| \leq C\varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon \, du \, dx. \tag{28}$$

Observe that we have

$$Q_4(t) = -\varepsilon^2 \int_{\mathbb{T}^2} (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) \, dx.$$

We check easily that

$$J^\varepsilon - \rho^\varepsilon D^\varepsilon = J^\varepsilon - J - \varepsilon J^{(1)} - (\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}) D^\varepsilon - \varepsilon^2 \rho^{(1)} D^{(1)}. \tag{29}$$

Using now the equation $\operatorname{div}_x(E^\varepsilon - E - \varepsilon E^{(1)}) = -(\rho^\varepsilon - \rho - \varepsilon \rho^{(1)})$ we deduce that

$$\begin{aligned} \varepsilon^2 \left| \int_{\mathbb{T}^2} (\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}) D^\varepsilon \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) \, dx \right| &\leq C\varepsilon^2 \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}| \, dx \\ &\leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 \, dx. \end{aligned} \tag{30}$$

Obviously we have

$$\varepsilon^2 \left| \int_{\mathbb{T}^2} \varepsilon^2 \rho^{(1)} D^{(1)} \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx \right| \leq C \varepsilon^4. \quad (31)$$

It remains to analyze the term $\varepsilon^2 \int_{\mathbb{T}^2} (J^\varepsilon - J - \varepsilon J^{(1)}) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx$. Using the momentum conservation

$$\varepsilon^2 \left(\partial_t \int_{\mathbb{R}^2} F^\varepsilon u du + \operatorname{div}_x \int_{\mathbb{R}^2} F^\varepsilon (u \otimes u) du \right) + \rho^\varepsilon E^\varepsilon + {}^\perp J^\varepsilon B_3^\varepsilon = 0,$$

we obtain

$$B_{0,3}{}^\perp (J^\varepsilon - J - \varepsilon J^{(1)}) = -T_6 - T_7 + T_8, \quad (32)$$

where

$$T_6 = \varepsilon^2 \left(\partial_t \int_{\mathbb{R}^2} F^\varepsilon u du + \operatorname{div}_x \int_{\mathbb{R}^2} F^\varepsilon (u \otimes u) du \right), \quad T_7 = E^\varepsilon - E - \varepsilon E^{(1)},$$

$$T_8 = \operatorname{div}_x E^\varepsilon E^\varepsilon - \operatorname{div}_x E E^\varepsilon - \varepsilon (\operatorname{div}_x E E^{(1)} + \operatorname{div}_x E^{(1)} E) - (B_3^\varepsilon - B_{0,3}){}^\perp J^\varepsilon.$$

We have

$$\varepsilon^2 B_{0,3} \int_{\mathbb{T}^2} (J^\varepsilon - J - \varepsilon J^{(1)}) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx = -Q_6(t) - Q_7(t) + Q_8(t),$$

where $Q_j = \varepsilon^2 \int_{\mathbb{T}^2} T_j \cdot {}^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx$, $j \in \{6, 7, 8\}$. It is easily seen that

$$\left| Q_6(t) - \varepsilon^4 \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon u \cdot {}^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) du dx \right| \leq C \varepsilon^4 + C \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon du dx, \quad (33)$$

and

$$|Q_7(t)| \leq C \varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx. \quad (34)$$

For any $w \in C^1(\mathbb{T}^2)^2$ the notation $\mathcal{F}(w)$ stands for $\mathcal{F}(w) = \operatorname{div}_x w w - \operatorname{div}_x {}^\perp w {}^\perp w$. We have $T_8 = T_9 + T_{10}$ with

$$\begin{aligned} T_9 &= \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) + \operatorname{div}_x (E^\varepsilon - E - \varepsilon E^{(1)}) (E + \varepsilon E^{(1)}) \\ &\quad + \operatorname{div}_x (E + \varepsilon E^{(1)}) (E^\varepsilon - E - \varepsilon E^{(1)}) + \varepsilon^2 \operatorname{div}_x E^{(1)} E^{(1)}, \end{aligned}$$

and

$$T_{10} = \operatorname{div}_x {}^\perp E^\varepsilon {}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) - (B_3^\varepsilon - B_{0,3}){}^\perp J^\varepsilon.$$

Since for any $w \in C^1(\mathbb{T}^2)^2$ we have $\mathcal{F}(w) = \operatorname{div}_x (w \otimes w) - \frac{1}{2} \nabla_x |w|^2$ we check easily that

$$\begin{aligned} |Q_9(t)| &= \varepsilon^2 \left| \int_{\mathbb{T}^2} T_9 \cdot {}^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx \right| \\ &\leq C \varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx. \end{aligned} \quad (35)$$

For estimating the term T_{10} we use the equations

$$\partial_t(E^\varepsilon - E - \varepsilon E^{(1)}) - \frac{1}{\varepsilon} \perp \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) = J^\varepsilon - J - \varepsilon J^{(1)} - \varepsilon \perp \nabla_x A_3^{(1)},$$

and

$$\partial_t \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right) + \frac{1}{\varepsilon} \operatorname{div}_x^\perp E^\varepsilon = 0.$$

We obtain

$$\begin{aligned} T_{10} + (B_3^\varepsilon - B_{0,3})^\perp (J + \varepsilon J^{(1)} + \varepsilon \perp \nabla_x A_3^{(1)}) &= -\partial_t \{ (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \} \\ &- \frac{1}{2} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 \\ &- \varepsilon A_3 \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right). \end{aligned}$$

Therefore we deduce that the term $Q_{10} := \varepsilon^2 \int_{\mathbb{T}^2} T_{10} \cdot \perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx$ satisfies

$$\begin{aligned} |Q_{10}(t) + \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot \perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon) dx| &\leq C\varepsilon^4 \\ + C \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx. & \end{aligned} \quad (36)$$

Combining the partial computations (27 – 36) we deduce that $|Q_3(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$. \square

Remark 2.1 *Using the computations of the previous proof yields also the inequalities*

$$\|\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}\|_{L^\infty([0, T]; H^{-1}(\mathbb{T}^2))} \leq C \|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty([0, T]; L^2(\mathbb{T}^2))},$$

and

$$\|J^\varepsilon - J - \varepsilon J^{(1)}\|_{W^{-1,1}([0, T] \times \mathbb{T}^2)} \leq C\varepsilon^2 + C \|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty([0, T]; L^2(\mathbb{T}^2))} + C \|\mathcal{H}_4^\varepsilon\|_{L^\infty([0, T])}.$$

For further computations we retain also the following estimate. Consider a smooth function $\varphi \in W^{1,\infty}([0, T] \times \mathbb{T}^2)$. Then there is a constant C , depending also on $\|\varphi\|_{W^{1,\infty}([0, T] \times \mathbb{T}^2)}$ such that for any $t \in [0, T]$ we have

$$\begin{aligned} \left| \varepsilon^2 \int_{\mathbb{T}^2} \perp (J^\varepsilon - J - \varepsilon J^{(1)}) \cdot \varphi dx + \frac{\varepsilon^2}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} (\varepsilon^2 \int_{\mathbb{R}^2} u F^\varepsilon du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)})) \cdot \varphi dx \right| \\ \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t). \end{aligned} \quad (37)$$

The above inequality follows by similar computations as those in the proof of Lemma 2.1: start with the equality (32) and then transform the terms T_6, T_7, T_9, T_{10} performing eventually integration by parts.

Lemma 2.2 For any $\varepsilon > 0, T \in \mathbb{R}_+, t \in [0, T]$ we have

$$\begin{aligned} & - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \operatorname{div}_u \left(F^\varepsilon (E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u) \right) h_4^\varepsilon \, du \, dx + Q_{13}(t) \\ & = \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} \frac{A_3 D^\varepsilon}{B_{0,3}} \cdot \left(\varepsilon^2 \int_{\mathbb{R}^2} F^\varepsilon u \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx, \end{aligned}$$

where $|Q_{13}(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. Observe that for any $u \in \mathbb{R}^2$ we have

$$(E + \varepsilon E^{(1)} + B_{0,3}^\perp u) \cdot (u - D^\varepsilon) = 0.$$

Thus integrating by parts and using (29) we obtain

$$\begin{aligned} & - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \operatorname{div}_u (F^\varepsilon (E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u)) h_4^\varepsilon \, du \, dx \\ & = -\varepsilon^2 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot {}^\perp (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \, dx \\ & = -Q_{11}(t) + Q_{12}(t), \end{aligned}$$

where

$$Q_{11}(t) = \varepsilon^2 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot {}^\perp (J^\varepsilon - J - \varepsilon J^{(1)}) \, dx, \quad Q_{12}(t) = \varepsilon^4 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot {}^\perp (\rho^{(1)} D^{(1)}) \, dx.$$

Obviously we have $|Q_{12}(t)| \leq C\varepsilon^4$ and by (37) we deduce that

$$\left| Q_{11} + \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} \frac{A_3 D^\varepsilon}{B_{0,3}} \cdot \left(\varepsilon^2 \int_{\mathbb{R}^2} F^\varepsilon u \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx \right| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t),$$

which implies our conclusion. \square

Lemma 2.3 For any $0 < \varepsilon < 1, T \in \mathbb{R}_+, t \in [0, T]$ we have

$$\begin{aligned} & - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \operatorname{div}_u \left((E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u) F^\varepsilon \right) h_4^\varepsilon \, du \, dx \\ & = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left(|E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 \right) dx \\ & - \frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot \left(D^\varepsilon + \varepsilon^2 \frac{{}^\perp E^{(2)}}{B_{0,3}} \right) dx \\ & - \frac{\varepsilon^4}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} {}^\perp E^{(2)} \cdot \left(\int_{\mathbb{R}^2} F^\varepsilon u \, du + A_3 {}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx - Q_{22}(t), \end{aligned}$$

where $|Q_{22}(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. Integrating by parts with respect to u and using (29) yields

$$\begin{aligned}
& - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \operatorname{div}_u \left((E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u) F^\varepsilon \right) h_4^\varepsilon \, du \, dx \\
& = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left(E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u \right) \cdot (u - D^\varepsilon) F^\varepsilon \, du \, dx \\
& = \int_{\mathbb{T}^2} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \, dx - \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3) D^\varepsilon \cdot {}^\perp J^\varepsilon \, dx \\
& = Q_{14}(t) - Q_{15}(t) + Q_{16}(t), \tag{38}
\end{aligned}$$

where

$$Q_{14} = \int_{\mathbb{T}^2} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (J^\varepsilon - J - \varepsilon J^{(1)}) \, dx, \quad Q_{15}(t) = \varepsilon^2 \int_{\mathbb{T}^2} \rho^{(1)} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot D^{(1)} \, dx,$$

$$Q_{16} = - \int_{\mathbb{T}^2} D^\varepsilon \cdot \left((\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}) (E^\varepsilon - E - \varepsilon E^{(1)}) + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \right) \, dx.$$

It is easily seen that

$$|Q_{15}(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 \, dx. \tag{39}$$

We use now the equations

$$\partial_t (E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}) - \frac{{}^\perp \nabla_x}{\varepsilon} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right) = J^\varepsilon - J - \varepsilon J^{(1)} - \varepsilon^2 \partial_t E^{(2)}, \tag{40}$$

$$\partial_t \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right) + \frac{1}{\varepsilon} \operatorname{div}_x {}^\perp (E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}) = -\varepsilon^2 \partial_t A_3^{(1)}, \tag{41}$$

and we deduce that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left(|E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 \right) \, dx \\
& = \int_{\mathbb{T}^2} (J^\varepsilon - J - \varepsilon J^{(1)} - \varepsilon^2 \partial_t E^{(2)}) \cdot (E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}) \, dx \\
& - \varepsilon^2 \int_{\mathbb{T}^2} \partial_t A_3^{(1)} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right) \, dx =: Q_{17}(t). \tag{42}
\end{aligned}$$

By using (37) we obtain

$$\begin{aligned}
Q_{17}(t) & = Q_{14}(t) + \frac{\varepsilon^4}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} {}^\perp E^{(2)} \int_{\mathbb{R}^2} F^\varepsilon u \, du \, dx \\
& + \frac{\varepsilon^2}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot {}^\perp E^{(2)} \, dx + Q_{18}(t), \tag{43}
\end{aligned}$$

where $|Q_{18}(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$. Using now (29) yields

$$\begin{aligned}
Q_{16}(t) &= \int_{\mathbb{T}^2} D^\varepsilon(\operatorname{div}_x(E^\varepsilon - E - \varepsilon E^{(1)})(E^\varepsilon - E - \varepsilon E^{(1)}) \\
&\quad - (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp(J^\varepsilon - J - \varepsilon J^{(1)})) dx \\
&\quad + \varepsilon^2 \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3) D^\varepsilon \cdot \perp(\rho^{(1)} D^{(1)}) dx \\
&= Q_{19}(t) + Q_{20}(t).
\end{aligned} \tag{44}$$

Taking into account that

$$\perp(J^\varepsilon - J - \varepsilon J^{(1)}) = \partial_t^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) + \frac{\nabla_x}{\varepsilon} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) - \varepsilon \nabla_x A_3^{(1)},$$

we deduce that

$$\begin{aligned}
&\operatorname{div}_x(E^\varepsilon - E - \varepsilon E^{(1)})(E^\varepsilon - E - \varepsilon E^{(1)}) - (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp(J^\varepsilon - J - \varepsilon J^{(1)}) \\
&= \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) + \operatorname{div}_x^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) \cdot \perp(E^\varepsilon - E - \varepsilon E^{(1)}) \\
&\quad - (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3) \left(\partial_t^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) + \frac{1}{\varepsilon} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) - \varepsilon \nabla_x A_3^{(1)} \right) \\
&= \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) - \frac{1}{2} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 + \varepsilon^2 \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) \nabla_x A_3^{(1)} \\
&\quad - \partial_t \left\{ (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) \right\} - \varepsilon^2 \partial_t A_3^\perp(E^\varepsilon - E - \varepsilon E^{(1)}),
\end{aligned}$$

and therefore we obtain

$$Q_{19}(t) = -\frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) \cdot D^\varepsilon dx + Q_{21}(t), \tag{45}$$

where

$$|Q_{21}(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 + \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx.$$

Observe also that we have

$$|Q_{20}(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx,$$

and finally combining (38), (39), (43 – 45) yields our conclusion. \square

The previous lemmas allow us to justify that $\left(E + \varepsilon E^{(1)}, \frac{B_{0,3}}{\varepsilon} + \varepsilon A_3\right)$ are second order approximations for $\left(E^\varepsilon, \frac{B_3^\varepsilon}{\varepsilon}\right)$.

Proposition 2.1 *For any $T \in \mathbb{R}_+$ there is a constant C such that for $\varepsilon > 0$ small enough, $t \in [0, T]$ we have $\mathcal{H}_4^\varepsilon(t) \leq C(\varepsilon^4 + \mathcal{H}_4^\varepsilon(0))$. Moreover if the following hypothesis holds*

$$H6) \sup_{\varepsilon > 0} \varepsilon^{-4} \left\{ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\varepsilon^2}{2} \left| u - \frac{\perp E_0}{B_{0,3}} \right|^2 F_0^\varepsilon du dx + \frac{1}{2} \int_{\mathbb{T}^2} |E_0^\varepsilon - E_0 - \varepsilon E_0^{(1)}|^2 \right. \\ \left. + \left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_{0,3} \right)^2 dx \right\} < +\infty,$$

then $\sup_{\varepsilon > 0, t \in [0, T]} \varepsilon^{-4} \mathcal{H}_4^\varepsilon(t) < +\infty$ for any $T \in \mathbb{R}_+$. In particular

$$\sup_{\varepsilon > 0} \varepsilon^{-2} \|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty([0, T]; L^2(\mathbb{T}^2)^2)} + \sup_{\varepsilon > 0} \varepsilon^{-2} \left\| \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right\|_{L^\infty([0, T]; L^2(\mathbb{T}^2))} < +\infty,$$

and

$$\sup_{\varepsilon > 0} \varepsilon^{-2} \|\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}\|_{L^\infty([0, T]; H^{-1}(\mathbb{T}^2))} + \sup_{\varepsilon > 0} \varepsilon^{-2} \|J^\varepsilon - J - \varepsilon J^{(1)}\|_{W^{-1,1}([0, T] \times \mathbb{T}^2)^2} < +\infty.$$

Proof. By Lemmas 2.1, 2.2, 2.3 we deduce that

$$\frac{d}{dt} \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon du dx + \frac{d}{dt} \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 dx \\ + \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 dx - \frac{d}{dt} \mathcal{R}_4^\varepsilon \\ \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t),$$

where

$$B_{0,3} \mathcal{R}_4^\varepsilon(t) = \int_{\mathbb{T}^2} \left(\varepsilon^4 \int_{\mathbb{R}^2} F^\varepsilon u du + \varepsilon^2 (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) \\ \cdot \perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon + A_3^\perp D^\varepsilon) dx \\ + \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (B_{0,3} D^\varepsilon + \varepsilon^2 \perp E^{(2)}) dx \\ + \varepsilon^4 \int_{\mathbb{T}^2} \perp E^{(2)} \left(\int_{\mathbb{R}^2} F^\varepsilon u du + A_3^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx.$$

We check easily that $|\mathcal{R}_4^\varepsilon(t)| \leq C\varepsilon^4 + C\varepsilon \mathcal{H}_4^\varepsilon(t)$. Observe also that

$$\left| \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 dx - \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx \right| \leq 3\varepsilon^4 \int_{\mathbb{T}^2} |E^{(2)}|^2 dx \\ + \frac{1}{2} \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx,$$

implying that

$$\begin{aligned} -C\varepsilon^4 + \frac{1}{2} \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx &\leq \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 dx \\ &\leq C\varepsilon^4 + \frac{3}{2} \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx. \end{aligned}$$

Similarly we have

$$\begin{aligned} -C\varepsilon^4 + \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx &\leq \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 dx \\ &\leq C\varepsilon^4 + \frac{3}{2} \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx. \end{aligned}$$

Finally we deduce that

$$\mathcal{H}_4^\varepsilon(t) \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(0) + \int_0^t \mathcal{H}_4^\varepsilon(s) ds, \quad t \in [0, T].$$

Our conclusion follows immediately by Gronwall lemma and Remark 2.1. \square

3 Center distribution limit

In this section we investigate the limit of the center distribution functions $(g^\varepsilon)_{\varepsilon>0}$ as ε goes to zero. By Proposition 2.1 we know that under suitable hypotheses the following convergences hold

$$\lim_{\varepsilon \searrow 0} (E^\varepsilon, A_3^\varepsilon) = (E, A_3) \text{ strongly in } L^\infty(]0, T[; L^2(\mathbb{T}^2)^3), \quad \forall T \in \mathbb{R}_+,$$

where (E, A_3) solves (24) together with $\int_{\mathbb{T}^2} A_3(\cdot, x) dx = 0$. We denote by $\mathcal{M}_+^1(X)$ the set of bounded Radon non negative measures on X . Using the hypothesis H1 and the space periodicity we deduce that

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} g^\varepsilon(t, y, p) dp dy = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f^\varepsilon(t, x, p) dp dx = 1, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Therefore there is a sequence $(\varepsilon_k)_k$ converging towards zero such that $(g^{\varepsilon_k})_k$ converges vaguely to $dg \in \mathcal{M}_+^1([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ for any $T \in \mathbb{R}_+$ and $(g^{\varepsilon_k}(0))_k$ converges vaguely to $dg_0 \in \mathcal{M}_+^1(\mathbb{T}^2 \times \mathbb{R}^2)$. After extraction, if necessary, we can also assume that $(f_0^{\varepsilon_k})_k$ converges vaguely to $df_0 \in \mathcal{M}_+^1(\mathbb{T}^2 \times \mathbb{R}^2)$. Notice that we have

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} \varphi(y, p) dg_0 = \int_{\mathbb{T}^2 \times \mathbb{R}^2} \varphi\left(x - \frac{\perp p}{B_{0,3}}, p\right) df_0, \quad \forall \varphi \in C_c^0(\mathbb{T}^2 \times \mathbb{R}^2). \quad (46)$$

We use now interpolation inequalities. We have for any $\varepsilon > 0$, $t \in [0, T]$

$$\begin{aligned} \rho^\varepsilon(t, x) &= \int_{\mathbb{R}^2} F^\varepsilon(t, x, u) \mathbf{1}_{\{|u-D(t,x)| \leq R\}} du + \int_{\mathbb{R}^2} F^\varepsilon(t, x, u) \mathbf{1}_{\{|u-D(t,x)| > R\}} du \\ &\leq \|F^\varepsilon(t)\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} \pi R^2 + \frac{1}{R^2} \int_{\mathbb{R}^2} F^\varepsilon(t, x, u) |u - D(t, x)|^2 du. \end{aligned}$$

Taking the optimal value for R one gets

$$\rho^\varepsilon(t, x) \leq 2\pi^{1/2} \|F^\varepsilon(t)\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)}^{1/2} \left(\int_{\mathbb{R}^2} F^\varepsilon(t, x, u) |u - D(t, x)|^2 du \right)^{1/2},$$

and finally since by Proposition 2.1 we know that

$$\left\| \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon(\cdot, x, u) |u - D(\cdot, x)|^2 du dx \right\|_{L^\infty([0, T])} \leq C\varepsilon^2,$$

we obtain for any $\varepsilon > 0$, $t \in [0, T]$

$$\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T}^2)} \leq C\varepsilon^3 \|f_0^\varepsilon\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)}^{1/2}. \quad (47)$$

Multiplying (14) by a test function $\theta(t, y)\varphi(p)$ where $\theta \in C_c^1([0, T] \times \mathbb{T}^2)$, $\varphi \in C_c^1(\mathbb{R}^2)$ yields after integration by parts

$$\begin{aligned} &- \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \varphi(p) g^\varepsilon(t, y, p) dp dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \varphi(p) g^\varepsilon(0, y, p) dp dy \\ &+ \frac{B_{0,3}}{\varepsilon^2} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) {}^\perp p \cdot \nabla_p \varphi g^\varepsilon dp dy dt \\ &+ \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot (pA_3^\varepsilon - {}^\perp E^\varepsilon) \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) \varphi g^\varepsilon dp dy dt \\ &+ \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \nabla_p \varphi \cdot (E^\varepsilon + {}^\perp pA_3^\varepsilon) \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) g^\varepsilon dp dy dt = 0. \end{aligned} \quad (48)$$

Obviously we have the convergences

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \varphi g^{\varepsilon^k} dp dy dt = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \partial_t \theta \varphi dg,$$

and

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \varphi(p) g^{\varepsilon^k}(0, y, p) dp dy = \int_{\mathbb{T}^2 \times \mathbb{R}^2} \theta(0, y) \varphi(p) dg_0.$$

In order to pass to the limit in the other terms we assume the following growth hypothesis for the L^∞ norms of the initial densities

H7) $\lim_{\varepsilon \searrow 0} \varepsilon^8 \|f_0^\varepsilon\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} = 0$.

Notice that the above condition is very weak, due to the large power of ε in front of $\|f_0^\varepsilon\|_{L^\infty}$.

We claim that for any test function $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ we have

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k}, A_3^{\varepsilon_k}) \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi g^{\varepsilon_k} dp dy dt = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} (E, A_3) \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi dg. \quad (49)$$

Indeed we have

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k}, A_3^{\varepsilon_k}) \psi g^{\varepsilon_k} dp dy dt &= \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k} - E, A_3^{\varepsilon_k} - A_3) \psi g^{\varepsilon_k} dp dy dt \\ &+ \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E, A_3) \psi g^{\varepsilon_k} dp dy dt \\ &= I_1^k + I_2^k. \end{aligned}$$

By Proposition 2.1 we know that

$$\|(E^{\varepsilon_k} - E, A_3^{\varepsilon_k} - A_3)\|_{L^\infty([0, T]; L^2(\mathbb{T}^2)^3)} \leq C\varepsilon_k, \quad \forall k.$$

Combining with (47) we obtain

$$\begin{aligned} |I_1^k| &= \left| \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (E^{\varepsilon_k} - E, A_3^{\varepsilon_k} - A_3)(t, x) \psi \left(t, x - \frac{\perp p}{B_{0,3}}, p \right) f^{\varepsilon_k} dp dx dt \right| \\ &\leq C \int_0^T \int_{\mathbb{T}^2} |(E^{\varepsilon_k} - E, A_3^{\varepsilon_k} - A_3)(t, x)| \rho^{\varepsilon_k}(t, x) dx dt \\ &\leq C\varepsilon_k^4 \|f_0^{\varepsilon_k}\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)}^{1/2}, \end{aligned}$$

and therefore $\lim_{k \rightarrow +\infty} I_1^k = 0$. Obviously we have the convergence

$$\lim_{k \rightarrow +\infty} I_2^k = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} (E, A_3) \left(t, y + \frac{\perp p}{B_{0,3}} \right) \psi(t, y, p) dg,$$

and thus (49) holds. Taking $\varepsilon = \varepsilon_k$ in (48) and passing to the limit for $k \rightarrow +\infty$ we deduce easily that

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \theta(t, y) (\perp p \cdot \nabla_p \varphi) dg = 0,$$

saying that the measure g is invariant under the rotations of \mathbb{R}^2

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \psi(t, y, \mathcal{R}(\alpha)p) dg = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \psi(t, y, p) dg, \quad (50)$$

for any $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ and any $\alpha \in \mathbb{R}$. Here $\mathcal{R}(\alpha)$ stands for the rotation of angle α . For any function $\varphi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+)$, $\varphi \geq 0$ we have

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \varphi(t, y, |p|) dg \leq \|\varphi\|_{C^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+)} \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} dg,$$

and therefore there is a measure $d\tilde{g} \in \mathcal{M}_+^1([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+)$ such that

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \varphi(t, y, |p|) dg = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \varphi(t, y, r) d\tilde{g}, \quad \forall \varphi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+).$$

For any function $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ we denote by $\tilde{\psi} \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+)$ the function given by

$$\tilde{\psi}(t, y, r) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, y, r \cos \alpha, r \sin \alpha) d\alpha.$$

Applying (50) with $\psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$, $\alpha \in [0, 2\pi[$ and taking the angle average yields

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \psi(t, y, p) dg = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \frac{1}{2\pi} \int_0^{2\pi} \psi(t, y, \mathcal{R}(\alpha)p) d\alpha dg.$$

Consider now $(t, y, p) \in [0, T] \times \mathbb{T}^2 \times \mathbb{R}^2$. There is $\alpha_0 \in [0, 2\pi[$ such that $p = \mathcal{R}(\alpha_0)p_0$, where $p_0 = (|p|, 0)$. Therefore

$$\frac{1}{2\pi} \int_0^{2\pi} \psi(t, y, \mathcal{R}(\alpha)p) d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, y, \mathcal{R}(\alpha)p_0) d\alpha = \tilde{\psi}(t, y, |p|),$$

and we deduce that

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \psi(t, y, p) dg = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \tilde{\psi}(t, y, r) d\tilde{g}, \quad \forall \psi \in C_c^0([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2). \quad (51)$$

Remark 3.1 *If the measure $g \in \mathcal{M}_+^1([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ is absolutely continuous with respect to the Lebesgue measure and it is invariant under the rotations of \mathbb{R}^2*

$$dg = g(t, y, p) dt dy dp, \quad g \in L^1([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2), \quad g(t, y, p_1, p_2) = g(t, y, |p|, 0),$$

then the measure $\tilde{g} \in \mathcal{M}_+^1([0, T] \times \mathbb{T}^2 \times \mathbb{R}_+)$ is absolutely continuous and is given by

$$d\tilde{g} = 2\pi r g(t, y, r, 0) dt dy dr.$$

We multiply now (14) by a test function $\theta(t, y)\psi(|p|)$ where $\theta \in C_c^1([0, T] \times \mathbb{T}^2)$, $\psi \in C_c^1(\mathbb{R}_+)$, $\psi'(0) = 0$. Notice that the function $p \rightarrow \psi(|p|)$ belongs to $C_c^1(\mathbb{R}^2)$, $\nabla_p \psi(|\cdot|)|_{p=0} = (0, 0)$

and thus we obtain

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \partial_t \theta \psi(|p|) g^\varepsilon dp dy dt - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(0, y) \psi(|p|) g^\varepsilon(0, y, p) dp dy \\
& + \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot (p A_3^\varepsilon - {}^\perp E^\varepsilon) \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) \psi(|p|) g^\varepsilon dp dy dt \\
& + \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E^\varepsilon \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) g^\varepsilon dp dy dt = 0.
\end{aligned}$$

Taking $\varepsilon = \varepsilon_k$ and letting $k \rightarrow +\infty$ we obtain as before

$$\begin{aligned}
& - \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \partial_t \theta \psi(|p|) dg - \int_{\mathbb{T}^2 \times \mathbb{R}^2} \theta(0, y) \psi(|p|) dg_0 \\
& + \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \frac{\nabla_y \theta}{B_{0,3}} \cdot (p A_3 - {}^\perp E) \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) \psi(|p|) dg \\
& + \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) dg = 0. \tag{52}
\end{aligned}$$

Using now (51) we have

$$\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \partial_t \theta \psi(|p|) dg = \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \partial_t \theta \psi(r) d\tilde{g}, \tag{53}$$

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} \theta(0, y) \psi(|p|) dg_0 = \int_{\mathbb{T}^2 \times \mathbb{R}_+} \theta(0, y) \psi(r) d\tilde{g}_0, \tag{54}$$

$$\begin{aligned}
\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \frac{\nabla_y \theta \cdot p}{B_{0,3}} A_3 \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) \psi(|p|) dg &= \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \frac{r \psi(r)}{2\pi B_{0,3}} \nabla_y \theta \\
&\cdot \int_0^{2\pi} (\cos \alpha, \sin \alpha) A_3 \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha d\tilde{g}. \tag{55}
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \nabla_y \theta \cdot {}^\perp E \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) \psi(|p|) dg &= \frac{1}{2\pi} \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \psi(r) \nabla_y \theta \\
&\cdot \int_0^{2\pi} {}^\perp E \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha d\tilde{g}. \tag{56}
\end{aligned}$$

Finally observe that the last term in (52) vanishes. Indeed we have

$$\begin{aligned}
\int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2} \theta(t, y) \psi'(|p|) \frac{p}{|p|} \cdot E \left(t, y + \frac{{}^\perp p}{B_{0,3}} \right) dg &= \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}_+} \theta(t, y) \psi'(r) \\
&\times \frac{1}{2\pi} \int_0^{2\pi} (\cos \alpha E_1 + \sin \alpha E_2) \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha d\tilde{g}.
\end{aligned}$$

Since $\operatorname{div}_x^\perp E = 0$ there is Φ such that $E = \nabla_x \Phi$ and thus

$$\frac{d}{d\alpha} \left\{ \Phi \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) \right\} = \left(\frac{r \cos \alpha}{B_{0,3}} E_1 + \frac{r \sin \alpha}{B_{0,3}} E_2 \right),$$

implying that

$$\int_0^{2\pi} (\cos \alpha E_1 + \sin \alpha E_2) \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha = 0, \quad \forall r \in \mathbb{R}_+.$$

Combining now (52 – 56) yields

$$- \int_{[0,T] \times \mathbb{T}^2 \times \mathbb{R}_+} (\partial_t(\theta\psi) + \mathcal{V}(t, y, r) \cdot \nabla_y(\theta\psi)) d\tilde{g} - \int_{\mathbb{T}^2 \times \mathbb{R}_+} \theta(0, y)\psi(r) d\tilde{g}_0 = 0, \quad (57)$$

where

$$\mathcal{V}(t, y, r) = \frac{1}{2\pi B_{0,3}} \int_0^{2\pi} (\nabla_x^\perp E - r(\cos \alpha, \sin \alpha)A_3) \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha.$$

Notice that

$$\frac{d}{d\alpha} \left\{ A_3 \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) \right\} = \frac{r \cos \alpha}{B_{0,3}} \partial_{y_1} A_3 + \frac{r \sin \alpha}{B_{0,3}} \partial_{y_2} A_3,$$

and therefore

$$\operatorname{div}_y \int_0^{2\pi} r(\cos \alpha, \sin \alpha)A_3 \left(t, y_1 + \frac{r \sin \alpha}{B_{0,3}}, y_2 - \frac{r \cos \alpha}{B_{0,3}} \right) d\alpha = 0.$$

Taking into account that $\operatorname{div}_y^\perp E = 0$ we deduce that $\operatorname{div}_y \mathcal{V} = 0$ and therefore (57) can be written formally

$$\begin{cases} \partial_t \tilde{g} + \mathcal{V}(t, y, r) \cdot \nabla_y \tilde{g} = 0, \\ \tilde{g}(0) = \tilde{g}_0. \end{cases} \quad (58)$$

We introduce now the characteristics $Y(s; t, y, r)$ associated to the velocity field \mathcal{V}

$$\frac{dY}{ds} = \mathcal{V}(s, Y(s; t, y, r), r), \quad Y(s = t; t, y, r) = y.$$

Therefore the measure solution of (58) is given by

$$\int_{\mathbb{T}^2 \times \mathbb{R}_+} \varphi(y, r) d\tilde{g}(t) = \int_{\mathbb{T}^2 \times \mathbb{R}_+} \varphi(Y(t; 0, y, r), r) d\tilde{g}_0.$$

for any $\varphi \in C_c^0(\mathbb{T}^2 \times \mathbb{R}_+)$. Using now (51), (46) we deduce that the limit measure dg is defined by

$$\begin{aligned}
\int_{\mathbb{T}^2 \times \mathbb{R}^2} \psi(y, p) dg(t) &= \int_{\mathbb{T}^2 \times \mathbb{R}_+} \tilde{\psi}(y, r) d\tilde{g}(t) \\
&= \int_{\mathbb{T}^2 \times \mathbb{R}_+} \tilde{\psi}(Y(t; 0, y, r), r) d\tilde{g}_0 \\
&= \int_{\mathbb{T}^2 \times \mathbb{R}^2} \tilde{\psi}(Y(t; 0, y, |p|), |p|) dg_0 \\
&= \int_{\mathbb{T}^2 \times \mathbb{R}^2} \tilde{\psi}(Y(t; 0, x - \frac{\perp p}{B_{0,3}}, |p|), |p|) df_0 \\
&= \int_{\mathbb{T}^2 \times \mathbb{R}^2} \frac{1}{2\pi} \int_0^{2\pi} \psi(Y(t; 0, x - \frac{\perp p}{B_{0,3}}, |p| \cos \alpha, |p| \sin \alpha) d\alpha df_0,
\end{aligned} \tag{59}$$

for any $\psi \in C_c^0(\mathbb{T}^2 \times \mathbb{R}^2)$ and any $t \in \mathbb{R}_+$. Finally we have obtained a complete description for the finite Larmor radius regime. Observing that H6 is stronger than H2, H4, H5 we formulate our result as follows.

Theorem 3.1 *Assume that the initial conditions $(f_0^\varepsilon, E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)_{\varepsilon>0}$ are smooth and satisfy the hypotheses H1, H3, H6, H7. We denote by $(f^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$ the solutions of (5), (6), (7), (8) and we suppose that the limit systems (24), (25), (26) have smooth solutions. Then for any $T \in \mathbb{R}_+$ we have*

$$\begin{aligned}
\sup_{\varepsilon>0, t \in [0, T]} \frac{1}{2\varepsilon^2} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left| \frac{p}{\varepsilon^2} - \frac{\perp(E + \varepsilon E^{(1)})}{B_{0,3}} \right|^2 f^\varepsilon(t, x, p) dp dx &< +\infty, \\
\sup_{\varepsilon>0, t \in [0, T]} \frac{1}{2\varepsilon^4} \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx &< +\infty,
\end{aligned}$$

and the center distribution functions $(g^{\varepsilon k})_k$ converge vaguely towards the solution defined by (59).

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