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Analysis of Steiner subtrees of Random Trees for Traceroute Algorithms

Fabrice Guillemin, Philippe Robert

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ON THE DISCOVERY OF RANDOM TREES

FABRICE GUILLEMIN AND PHILIPPE ROBERT

ABSTRACT. We consider in this paper the problem of discovering, via a traceroute algorithm, the topology of a network, whose graph is spanned by an infinite branching process. A subset of nodes is selected according to some criterion. The minimal sub-tree containing the selected nodes is the set of those nodes discovered by the algorithm. For the selection of nodes, two criteria are considered: A node is randomly selected with a probability, which is either independent of the depth of the node (uniform model) or else in the depth biased model, is exponentially decaying with respect to its depth. The limiting behavior the size of the discovered subtree is investigated for both models.

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1. INTRODUCTION

In the past ten years, the Internet has known an extraordinary expansion and still experiences today a sustained growth. The counterpart of this success is that the different autonomous systems composing the global Internet have been developed by different operators independently. As a consequence, the physical topology of a component of the Internet is in general very difficult to describe. For network operators, however, the knowledge of the physical layout of a network is of prime interest for different issues related to quality of service, security, etc. To establish a representation of the whole or a part of the Internet, some topology exploration methods have to be devised. Various topology discovery experiments have been initiated by different organizations in order to infer the topology of the global Internet, notably the Skitter project by CAIDA [3], the DIMES project [11] and many other papers. The method generally proposed for analyzing the topology of a network is based on the traceroute facility offered by routers. The purpose of

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this paper is to investigate one of the main properties of the traceroute algorithm. Roughly speaking, a traceroute procedure consists of sending traceroute messages between hosts as follows:

TRACEROUTE ALGORITHM

If H and G are hosts participating in the topology discovery experiment, H sends to G a traceroute message so that all the hosts/routers on the path (H, G) are identified.

While a large number of experimental papers are available in the technical literature on the analysis of the topology of the Internet, a very few studies provide analytical insight into the efficiency of these topology discovery methods; see Vespignani *et al.* [5] for a discussion and Azzana *et al.* [2] for an analysis in the case of specific deterministic trees.

In this paper, a more realistic model is proposed to include some randomness in the degree of the nodes of the graph representing the topology of a network. One considers a network with a random tree architecture. To get more insight into the topology discovery process in the case of a large network, it is assumed that the underlying branching process does not terminate with probability 1; in particular the depth of the tree is infinite. The discovery process is as follows: a random number of nodes are selected among the nodes of the tree. After the selected nodes have performed the traceroute algorithm, the set of the nodes discovered is a sub-tree. The performance criterion used in this paper is the ratio of the size of this sub-tree and the size of the original tree where the nodes have been selected. Two stochastic models for selecting the nodes in the network are considered.

In the first model, the uniform model, nodes whose depth is less than $N > 0$ are randomly chosen with probability $1 - \exp(-\lambda)$ for some $\lambda > 0$ independently of the position of the node in the tree. The quantity analyzed here is the ratio of the mean size $\mathbb{E}(R_N)$ of the sub-tree discovered to the mean total size $\mathbb{E}(T_N)$ of the tree. The quantity $\rho_N(\lambda)$ denotes the fraction of the tree discovered. The asymptotic results of this paper first determine as N tends to infinity the limits $\rho(\lambda)$ of $\rho_N(\lambda)$. In a second step, the asymptotic behavior of $\rho(\lambda)$ for $\lambda \rightarrow 0$ is investigated. This last point gives an indication of the efficiency of the algorithm when only a few nodes are selected in the topology discovery experiment. It is shown in Theorem 1 that, for λ small, the exploration rate $\rho(\lambda)/\lambda$ is equivalent to $\log_m \lambda$ where m is the mean value of the offspring distribution of a node, so that at the first order the algorithm is very efficient. A second order analysis, Proposition 3 reveals however that the error is of the order of $\sqrt{\lambda} \log \lambda$ instead of $\sqrt{\lambda \log \lambda}$ indicating larger fluctuations than expected for a possible central limit theorem in the performances of the algorithm.

In the second model, the depth biased model, the probability of selecting a node depends on its depth in the tree so that the mean number of selected nodes at depth n is α^n for some $\alpha > 0$. An instability phenomenon also holds in this context. It is shown in Theorem 2 that the ratio of the average of the size $R(\alpha)$ of the sub-tree discovered and $1/(1 - \alpha)$, the average number of selected nodes can be expressed as $\kappa(\alpha)/(1 - \alpha)$ where $\kappa(\alpha)$ is an oscillating function between two values as the number of selected nodes goes to infinity, i.e. $\alpha \nearrow 1$. Asymptotic oscillating expressions are quite common in the analysis of algorithms with an underlying tree structure, see Flajolet *et al.* [6], Mohamed and Robert [9] and Mahmoud [8] for

a general presentation. In general they can be expressed as $F(\phi(x))$ where F is some periodic function and ϕ is a slowly increasing function. Technically it comes from the fact that a Mellin transform has an infinite number of poles on a vertical axis (for an analytic approach, see [6]) or the fact that a discrete renewal theorem has to be used (for a probabilistic approach, see [9]). In the present case, the corresponding oscillating term cannot be represented in such a way; this unusual asymptotic behavior is obtained through an integral representation of the main quantity of interest. Related problems in a slightly different context have recently been investigated by Christophi and Mahmoud [4], where the asymptotic behavior of the distance between two random nodes of the tree is considered.

The paper is organized as follows: In Section 2, the models for the selection of the nodes of the tree are introduced. The uniform model is investigated in Section 4 and the depth biased model in Section 5. The main ingredients for the analysis of these models are Kesten-Stigum Theorem for Galton-Watson branching processes and a limit theorem proved in Section 3.

2. PROBLEM FORMULATION

Throughout this paper, we consider a Galton-Watson branching process, whose graph is a tree denoted by \mathcal{T} . A node of the tree gives birth to G nodes at the $(n + 1)$ th level, where the offspring G is some *integrable* random variable. (See Athreya and Ney [1] and Lyons and Peres [7] for an introduction to random trees.)

It is assumed that $\mathbb{P}(G=0)=0$ and $P(G \geq 2) > 0$, in particular the tree is supercritical, i.e. $m = \mathbb{E}(G) > 1$. For $n \geq 0$, the variable Z_n denotes the number of nodes at level n , in particular $Z_0 = 1$. For $1 \leq \ell \leq Z_n$, a node of the tree can be represented as a couple (n, ℓ) , where n is its generation and ℓ its rank within the generation. (For notational conventions, see Neveu [10] for example.) Let $\mathcal{T}_k^{n, \ell}$ denote the sub-tree of \mathcal{T} with depth less than or equal to k and with root at node (n, ℓ) . The size of $\mathcal{T}_k^{n, \ell}$ is denoted by $T_k^{n, \ell}$. When (n, ℓ) is the root node, i.e. $(n, \ell) = (0, 0)$, the upper index $(0, 0)$ is omitted. With the above notation, one gets easily that for all $N > 1$ and $n = 1, \dots, N$

$$(1) \quad T_N = \sum_{i=0}^{n-1} Z_i + \sum_{\ell=1}^{Z_n} T_{N-n}^{n, \ell}.$$

Let us consider a counting measure \mathcal{N} on the tree representing the distribution of the points selected in the tree: For a subset A of the nodes of the tree, $\mathcal{N}(A)$ denotes the total number of points in A . By selecting nodes, a sub-tree from \mathcal{T} is obtained through the traceroute algorithm; this sub-tree is referred to as sampled tree. See Figure 1.

Because of the assumption that the root node is always selected, a node (n, ℓ) of the tree \mathcal{T} at level n belongs to the sampled tree whenever $\mathcal{N}(\mathcal{T}_{N-n}^{n, \ell})$ is not 0. This implies that a node of the original tree belongs to the discovered tree if at least one of his descendants has been selected. This assumption is from a practical point of view not very restrictive since topology discovery experiments are spread all over the Internet.

To complete the description of the problem, it remains to specify how the nodes of the original tree are selected. In the following, we shall consider two selection criteria:

3. AN ASYMPTOTIC EXPANSION

To prove asymptotic expansions in the following sections, the following proposition will be used repeatedly. Its proof is based on integral representations and Fubini's Theorem instead of complex analysis techniques as it is usually the case in this setting. See Robert [12] for a presentation of these methods.

Proposition 1. *Let V be a positive integrable random variable, $m > 1$ and h be non-negative bounded differentiable function on $[0, +\infty)$ such that $h(0) = 0$. Assume in addition that the function h' is integrable and $h'(0) \neq 0$. The function $\Psi(h)(x)$ defined by*

$$(4) \quad \Psi(h)(x) = \sum_{n=0}^{+\infty} \frac{1}{m^n} \mathbb{E}(h(xVm^n)), \quad x \geq 0,$$

is such that

$$\lim_{x \rightarrow 0} \frac{\Psi(h)(x)}{x \log_m(1/x)} = \mathbb{E}(V)h'(0).$$

Proof. The function h being bounded and $m > 1$ the function $\Psi(h)$ is well defined. Since h is non-negative and $|h'|$ integrable with respect to Lebesgue measure on \mathbb{R}_+ , Fubini's Theorem applied twice shows that $\Psi(h)$ can be expressed as

$$\begin{aligned} \Psi(h)(x) &= \sum_{n=0}^{+\infty} \frac{1}{m^n} \mathbb{E}(h(xVm^n)) = \mathbb{E}\left(\sum_{n=0}^{+\infty} \frac{1}{m^n} h(xVm^n)\right) \\ &= \mathbb{E}\left(\sum_{n=0}^{+\infty} \frac{1}{m^n} \int_0^{+\infty} h'(u) \mathbb{1}_{\{u \leq xVm^n\}} du\right) \\ &= \mathbb{E}\left(\int_0^{+\infty} h'(u) \sum_{n=0}^{+\infty} \frac{1}{m^n} \mathbb{1}_{\{u \leq xVm^n\}} du\right). \end{aligned}$$

Hence,

$$(5) \quad (m-1)\Psi(h)(x) = m\mathbb{E}\left(\int_0^{xV} h'(u) du\right) + \mathbb{E}\left(\int_{xV}^{+\infty} \frac{1}{m^{\lfloor \log_m(u/xV) \rfloor}} h'(u) du\right),$$

where $\lfloor y \rfloor$ is the integer part of $y \in \mathbb{R}$.

For $u \geq V$, one has

$$xm^{\lfloor \log_m(u/xV) \rfloor} \geq xm^{\lfloor \log_m(1/x) \rfloor} \geq xm^{\log_m(1/x)-1} = m^{-1},$$

and hence,

$$\mathbb{E}\left(\int_V^{+\infty} \frac{1}{xm^{\lfloor \log_m(u/xV) \rfloor}} |h'(u)| du\right) \leq m^{-1} \int_0^{+\infty} |h'(u)| du.$$

It follows that we have the equivalence as $x \rightarrow 0$

$$\begin{aligned} \frac{\Psi(h)(x)}{x} &\sim \mathbb{E}\left(\int_{xV}^V \frac{1}{xm^{\lfloor \log_m(u/xV) \rfloor}} h'(u) du\right) \\ &= \mathbb{E}\left(V \int_{xV}^V m^{\{\log_m(u/xV)\}} \frac{h'(u)}{u} du\right), \end{aligned}$$

where $\{y\} = y - [y]$ is the fractional value of $y \in \mathbb{R}$. The above equation can be rewritten as

$$\begin{aligned} \frac{\Psi(h)(x)}{x} &= \mathbb{E} \left(V \int_{xV}^V m^{\{\log_m(u/xV)\}} \frac{h'(u) - h'(0)}{u} du \right) \\ &\quad + h'(0) \mathbb{E} \left(V \int_{xV}^V \frac{m^{\{\log_m(u/xV)\}}}{u} du \right) \end{aligned}$$

Since the first term of the right hand side of the above equation is bounded as x goes to 0, only the second term has to be considered. For $x < 1$, we have

$$\begin{aligned} \int_{xV}^V \frac{m^{\{\log_m(u/xV)\}}}{u} du &= \int_1^{1/x} \frac{m^{\{\log_m(u)\}}}{u} du \\ &= \sum_{k \geq 0: m^k \leq 1/x} \int_{m^k}^{m^{k+1}} \frac{m^{\{\log_m(u)\}}}{u} du + O(1) = (m-1)[- \log_m(x)] + O(1) \end{aligned}$$

and the result follows. \square

By using the terminology of Flajolet *et al.* [6], for non-negative sequences (λ_n) and (μ_n) , a series like

$$(6) \quad G(x) = \sum_{n \geq 0} \lambda_n g(\mu_n x),$$

for some function g is defined as an *harmonic sum*. Because of the integration of the random variable V and given that one wants the weakest assumptions on this random variable, Series (4) could be seen as a generalization of harmonic sums. The fact that the sequences (λ_n) and (μ_n) are specific in Expression (4) is not a real restriction, see Robert [12].

Flajolet *et al.* [6] derives the asymptotic expansion of $G(x)$ when x goes to 0 or $+\infty$ by using Mellin transform techniques. For $s \in \mathbb{C}$, if $h^*(s)$ is the Mellin transform of h , i.e. for s in some vertical strip of \mathbb{C} ,

$$h^*(s) = \int_0^{+\infty} h(x) x^{s-1} dx,$$

it is easy to check that the Mellin transform of $\Psi(h)$ is given by

$$\Psi(h)^*(s) = \frac{1}{m^{s+1} - 1} \mathbb{E}(V^{-s}) h^*(s).$$

Following the methods of Flajolet *et al.* [6], to get the asymptotic behavior of $\Psi(h)(x)$ as x goes to infinity, one has to identify the first singularity of $\Psi(h)^*$ on the right of the maximal vertical strip where it is defined. In particular, some conditions on the finiteness of some fractional moments of the random variable V have to be assumed (as well as growth conditions on h^*). From this point of view, our approach is minimal since only the finiteness of $\mathbb{E}(V)$ is assumed. It turns out that it is important as it will be seen in the following sections, since in practice little is known on the fractional moments of the corresponding variable V .

4. THE EXPLORATION RATE IN THE UNIFORM MODEL

In this section, nodes are selected at random with uniform probability among the nodes of the tree with depth less than N . The variable R_N is the size of the underlying sub-tree (or sampled tree) containing the selected nodes. The asymptotic behavior of $\rho(N) = \mathbb{E}(R_N)/\mathbb{E}(T_N)$, the fraction of discovered nodes, when N tends to infinity is investigated. In the second part of this section, the ratio $\text{var}(R_N)/\mathbb{E}(T_N)$ is analyzed.

4.1. First Order Asymptotics. In the uniform case, the limiting behavior of the ratio ρ_N when N tends to infinity is given by the following result.

Theorem 1. *The ratio of the average size R_N of the sampled tree to the total average size of the tree $\mathbb{E}(T_N)$ satisfies the relation*

$$\rho(\lambda) \stackrel{\text{def.}}{=} \lim_{N \rightarrow +\infty} \rho_N = \sum_{n=0}^{+\infty} \frac{m-1}{m^{n+1}} \left(1 - \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) \right).$$

If additionally the condition $\mathbb{E}(G \log G) < +\infty$ holds then

$$(7) \quad \lim_{\lambda \rightarrow 0} \frac{\rho(\lambda)}{\lambda \log_m(1/\lambda)} = 1.$$

Relation (7) shows that the rate of increase of the discovery process is infinite near the origin. This implies that with only a few selected nodes a significant proportion of the network is discovered.

Proof. The probability that node ℓ at level n does not belong to the sampled tree is given by

$$\mathbb{P}(\mathcal{N}(\mathcal{T}_n^{N-n, \ell}) \neq 0) = 1 - \exp(-\lambda T_n^{N-n, \ell}).$$

By summing-up these relations, one gets that the expected value of R_N , the average number of nodes in the sampled tree is given by

$$\begin{aligned} \mathbb{E}(R_N) &= \sum_{n=0}^N \mathbb{E}(Z_{N-n}) (1 - \mathbb{E}(\exp(-\lambda T_n))) \\ &= \sum_{n=0}^N m^{N-n} \left(1 - \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) \right). \end{aligned}$$

The limit when $N \rightarrow \infty$ of the ratio ρ_N is then given by

$$\rho(\lambda) \stackrel{\text{def.}}{=} \lim_{N \rightarrow +\infty} \rho_N = \sum_{n=0}^{+\infty} \frac{m-1}{m^{n+1}} \left(1 - \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) \right)$$

since $\mathbb{E}(T_N) \sim m^{N+1}/(m-1)$ for large N . This proves the first equality stated in Theorem 1.

We now study the behavior of $\rho(\lambda)$ when λ goes to 0. Since $\mathbb{E}(G \log G) < +\infty$, Kesten-Stigum's Theorem ensures the existence of a random variable W such that $\mathbb{P}(W > 0) = 1$ (because of the assumption on the distribution of G) and $\mathbb{E}(W) = 1$ (See Lyons and Peres [7]) and that, almost surely,

$$\lim_{n \rightarrow +\infty} \frac{Z_n}{m^n} = W.$$

Let us define

$$f(\lambda) \stackrel{\text{def.}}{=} \sum_{n=0}^{+\infty} \frac{m-1}{m^{n+1}} \left(1 - \mathbb{E} \left(\exp \left(-\lambda W \frac{m^{n+1} - m}{m-1} \right) \right) \right).$$

Then,

$$(8) \quad \frac{|\rho(\lambda) - f(\lambda)|}{\lambda} \leq \sum_{n=0}^{+\infty} \frac{m-1}{\lambda m^{n+1}} \left| \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) - \mathbb{E} \left(\exp \left(-\lambda W \frac{m^{n+1} - m}{m-1} \right) \right) \right|.$$

Since W is integrable, Lebesgue's Theorem gives that

$$(9) \quad \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) - \exp \left(-\lambda W \frac{m^{n+1} - m}{m-1} \right) \right) = \mathbb{E} \left(\sum_{i=1}^n Z_i - W \frac{m^{n+1} - m}{m-1} \right) = 0.$$

We have

$$\begin{aligned} \frac{1}{m^{n+1}\lambda} \left| \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) - \mathbb{E} \left(\exp \left(-\lambda W \frac{m^{n+1} - m}{m-1} \right) \right) \right| \\ \leq \frac{1}{m^{n+1}} \sum_{i=1}^n \mathbb{E} |Z_i - m^i W|. \end{aligned}$$

From Athreya and Ney [1, Theorem 1, page 54], for $n \geq 1$, there exists a sequence (W^i) of i.i.d. random variables with the same distribution as W such that

$$Z_n - m^n W = \sum_{i=1}^{Z_n} (1 - W^i).$$

By using Cauchy-Schwartz Inequality, we obtain

$$(10) \quad \begin{aligned} \mathbb{E} (|Z_n - m^n W|) &\leq \sqrt{\mathbb{E} ((Z_n - m^n W)^2)} \\ &= \text{Var}(1 - W) \sqrt{\mathbb{E}(Z_n)} \\ &= \text{Var}(1 - W) m^{n/2}. \end{aligned}$$

From the above inequality, we deduce that

$$\begin{aligned} \frac{1}{m^{n+1}\lambda} \left| \mathbb{E} \left(\exp \left(-\lambda \sum_{i=1}^n Z_i \right) \right) - \mathbb{E} \left(\exp \left(-\lambda W \frac{m^{n+1} - m}{m-1} \right) \right) \right| \\ \leq \frac{\text{Var}(1 - W)}{\sqrt{m-1}} \frac{1}{m^{(n+1)/2}}. \end{aligned}$$

Relation (9) and Lebesgue's Theorem then imply that

$$\lim_{\lambda \rightarrow 0} \frac{\rho(\lambda) - f(\lambda)}{\lambda} = 0.$$

Hence, up to an expression which is of the order of $o(\lambda)$, the behavior at 0 of $\rho(\lambda)$ is equivalent to the behavior of $f(\lambda)$ as λ gets small. By using Proposition 1, we have by taking $h(u) = 1 - e^{-u}$ and $V = Wm/(m-1)$,

$$\sum_{n=0}^{+\infty} \frac{m-1}{m^{n+1}} (1 - \mathbb{E}(\exp(-xVm^n))) = \Psi(h)(x) \sim x \log_m(1/x)$$

as $x \rightarrow 0$. To conclude the proof, we note that

$$\lim_{x \rightarrow 0} \frac{\Psi(h)(x) - f(x)}{x \log_m x} = 0$$

and the result follows. \square

4.2. Second Order Properties. The results obtained in the previous section show that the size of the sampled tree is of the same order of magnitude as the original tree if the intensity of the number of selected nodes attached to each node is fixed. When this intensity is very small, the speed of the discovery process is even very fast. In this section, we evaluate the second moment of the random variable R_N in order to estimate the dispersion of the size of the sampled tree around the mean value.

In the following, we use the following notation: If (n, ℓ) and (n', ℓ') are two nodes of the tree, the relation $(n', \ell') < (n, \ell)$ is said to hold whenever the nodes are distinct and if $\mathcal{T}_{N-n'}^{n', \ell'} \subset \mathcal{T}_{N-n}^{n, \ell}$, i.e. when (n', ℓ') is a node of the sub-tree $\mathcal{T}_{N-n}^{n, \ell}$ but not its root.

Proposition 2 (Asymptotic behavior of the variance). *When the size N of the original tree goes to infinity, the variance of the size of the sampled tree is such that*

$$(11) \quad \rho_2(\lambda) \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \frac{\text{Var}(R_N)}{\mathbb{E}(T_N)} = \frac{m-1}{m} \sum_{n=1}^{+\infty} \frac{1}{m^n} \left[\mathbb{E}(e^{-\lambda T_n}) (1 - \mathbb{E}(e^{-\lambda T_n})) \right. \\ \left. + 2 \sum_{k=0}^{n-1} \mathbb{E} \left(e^{-\lambda T_{n-k-1}} Z_{n-k} \mathbb{E} \left(e^{-\lambda T_k} \right)^{Z_{n-k}-1} \mathbb{E} \left(e^{-\lambda T_k} (1 - e^{-\lambda T_k}) \right) \right) \right].$$

Proof. Using Representation (2) for the size of the sampled tree, one obtains the relation

$$R_N - \mathbb{E}(R_N) = \sum_{n=0}^N \sum_{\ell=1}^{Z_{N-n}} \Delta_n^\ell$$

with $\Delta_n^\ell = \mathbb{1}_{\{\mathcal{N}(T_n^{N-n, \ell}) \neq \emptyset\}} - \mathbb{P}(\mathcal{N}(T_n^{N-n, \ell}) \neq \emptyset)$. By using the independence in the selection of nodes in the tree and the fact that the random variables Δ_n^ℓ are centered, we have the identity

$$\text{Var}(R_N) - \sum_{n=0}^N \mathbb{E}(Z_{N-n}) \text{Var}(\Delta_n) = 2\mathbb{E} \left(\sum_{\substack{(n, \ell), (n', \ell') \in \mathcal{T} \\ (n', \ell') < (n, \ell)}} \Delta_n^\ell \Delta_{n'}^{\ell'} \right).$$

Conditioning on the state of the tree, when $(n', \ell') < (n, \ell)$, one has the identity

$$\mathbb{E} \left(\Delta_n^\ell \Delta_{n'}^{\ell'} \mid \mathcal{T} \right) = \exp(-\lambda T_n) (1 - \exp(-\lambda T_{n'})).$$

By symmetry, the above computations yield the following expression for the variance

$$\begin{aligned} U_N &\stackrel{\text{def.}}{=} \text{Var}(R_N) - \sum_{n=0}^N \mathbb{E}(Z_{N-n}) \text{Var}(\Delta_n) \\ &= 2\mathbb{E} \left(\sum_{n=1}^N \mathbb{E}(Z_{N-n}) \exp(-\lambda T_n) \sum_{\substack{(N-n',l') \in \mathcal{T}_N \\ (N-n',l') < (N-n,1)}} \left(1 - \exp(-\lambda T_{n'}^{N-n',l'})\right) \right). \end{aligned}$$

For two nodes of the tree such that $(N-n',l') < (N-n,1)$, Equation (1) gives the relation

$$T_n \stackrel{\text{dist.}}{=} \sum_{k=0}^{n-n'-1} \tilde{Z}_k + \sum_{\ell'=1}^{\tilde{Z}_{n-n'}} T_{n'}^{N-n',\ell'},$$

where $(\tilde{Z}_k, k \geq 0)$ denotes another independent Galton-Watson process independent of $(Z_n, n \geq 0)$ with the same offspring distribution. By using this relation, we have

$$\begin{aligned} &\mathbb{E} \left(\exp(-\lambda T_n) \sum_{\substack{(N-n',l') \in \mathcal{T}_N \\ (N-n',l') < (N-n,1)}} \left(1 - \exp(-\lambda T_{n'}^{N-n',\ell'})\right) \right) \\ &= \mathbb{E} \left(\sum_{n'=0}^{n-1} \sum_{\ell'=1}^{Z_{n-n'}} \exp(-\lambda T_n) \left(1 - \exp(-\lambda T_{n'}^{N-n',\ell'})\right) \right) \\ &= \mathbb{E} \left(\sum_{n'=0}^{n-1} \exp \left(-\lambda \sum_{k=0}^{n-n'-1} Z_k \right) \mathbb{E}(V_{n-n'}) \right), \end{aligned}$$

where

$$V_{n-n'} = \sum_{\ell'=1}^{Z_{n-n'}} \exp \left(-\lambda \sum_{\ell''=1}^{Z_{n-n'}} T_{n'}^{N-n',\ell''} \right) \left(1 - \exp(-\lambda T_{n'}^{N-n',\ell'})\right).$$

By using the independence of the different trees $\mathcal{T}_{n'}^{N-n',\ell'}$ for $\ell = 1, \dots, Z_{n-n'}$, we have

$$\begin{aligned} &\mathbb{E}(V_{n-n'}) \\ &= \mathbb{E} \left(Z_{n-n'} (\mathbb{E}(\exp(-\lambda T_{n'})))^{Z_{n-n'}-1} \mathbb{E}(\exp(-\lambda T_{n'})(1 - \exp(-\lambda T_{n'}))) \right). \end{aligned}$$

It follows that by using the above expression for U_N , one obtains

$$\begin{aligned} U_N &= 2 \sum_{n=1}^N \mathbb{E}(Z_{N-n}) \sum_{n'=0}^{n-1} \mathbb{E} \left(\exp \left(-\lambda \sum_{k=0}^{n-n'-1} Z_k \right) \right. \\ &\quad \left. Z_{n-n'} \mathbb{E} \left(\exp(-\lambda T_{n'}) \right)^{Z_{n-n'}-1} \mathbb{E} \left(\exp(-\lambda T_{n'})(1 - \exp(-\lambda T_{n'})) \right) \right). \end{aligned}$$

Dividing by $\mathbb{E}(T_N)$, we have

$$\frac{U_N}{\mathbb{E}(T_N)} = \frac{2(m-1)}{m} \sum_{n=1}^N \frac{1}{m^n} \sum_{n'=0}^{n-1} \mathbb{E} \left(\exp \left(-\lambda \sum_{k=0}^{n-n'-1} Z_k \right) Z_{n-n'} \mathbb{E} \left(\exp(-\lambda T_{n'}) \right)^{Z_{n-n'}-1} \mathbb{E} \left(\exp(-\lambda T_{n'}) (1 - \exp(-\lambda T_{n'})) \right) \right).$$

By letting N go to infinity, we finally obtain Relation (11). \square

As for the first moment of R_N , we turn now to the analysis of the behavior of $\rho_2(\lambda)$ in the neighborhood of 0.

Proposition 3 (Asymptotic Behavior of $\lambda \rightarrow \rho_2(\lambda)$ at 0). *Provided that the random variable G has a finite second moment, the function $\rho_2(\lambda)$ defined by Equation (11) is such that*

$$(12) \quad \lim_{\lambda \searrow 0} \frac{\rho_2(\lambda)}{\lambda (\log_m \lambda)^2} = 1.$$

Proof. Define

$$f_a(\lambda) \stackrel{\text{def.}}{=} \sum_{n=1}^{+\infty} \frac{m-1}{m^n} (\mathbb{E}(e^{-\lambda T_n}) (1 - \mathbb{E}(e^{-\lambda T_n})))$$

and

$$f_b(\lambda) \stackrel{\text{def.}}{=} \sum_{n=1}^{+\infty} \frac{m-1}{m^n} \sum_{k=0}^{n-1} \mathbb{E} \left(e^{-\lambda T_{n-k-1}} Z_{n-k} \mathbb{E}(e^{-\lambda T_k})^{Z_{n-k}-1} \right) \mathbb{E}(e^{-\lambda T_k} (1 - e^{-\lambda T_k})).$$

Equation (11) gives that $mM_2(\lambda) = 2f_b(\lambda) + f_a(\lambda)$.

Asymptotic behavior of f_a . With similar arguments as in the proof of Theorem 1 the asymptotic behavior of $f_a(\lambda)$ when λ goes to 0 is equivalent to the asymptotic behavior of

$$\sum_{n=1}^{+\infty} \frac{m-1}{m^n} \left(\mathbb{E} \left(\exp \left(-\lambda \frac{W m^{n+1}}{m-1} \right) \right) \left(1 - \mathbb{E} \left(\exp \left(-\lambda \frac{W m^{n+1}}{m-1} \right) \right) \right) \right).$$

If W_1 and W_2 are two independent random variables with the same distribution as W , the above series can be rewritten as

$$\begin{aligned} & \sum_{n=1}^{+\infty} \frac{m-1}{m^n} \mathbb{E} \left(\exp \left(-\lambda \frac{W_1 m^{n+1}}{m-1} \right) \left(1 - \mathbb{E} \left(\exp \left(-\lambda \frac{W_2 m^{n+1}}{m-1} \right) \right) \right) \right) \\ &= (m-1) \sum_{n=1}^{+\infty} \left(\frac{1}{m^n} \mathbb{E}(h(\lambda(W_1 + W_2)m^n/(m-1))) - \mathbb{E}(h(\lambda W_1 m^n/(m-1))) \right), \end{aligned}$$

with $h(u) = 1 - e^{-u}$. Consequently,

$$(13) \quad \lim_{\lambda \rightarrow 0} \frac{f_a(\lambda)}{-\lambda \log_m \lambda} = m$$

by Proposition 1.

Asymptotic behavior of f_b . Let us fix some $\varepsilon > 0$ and assume that $\lambda < \varepsilon$. The function $f_b(\lambda)$ can be rewritten as

$$(14) \quad f_b(\lambda) = \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \frac{m-1}{m^n} S(n; \lambda) + \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{m-1}{m^n} S(n; \lambda)$$

where

$$S(n; \lambda) = \sum_{k=0}^{n-1} \mathbb{E} \left(e^{-\lambda T_{n-k-1}} Z_{n-k} \mathbb{E} (e^{-\lambda T_k})^{Z_{n-k}-1} \right) \mathbb{E} (e^{-\lambda T_k} (1 - e^{-\lambda T_k}))$$

Since for $x \geq 0$, $e^{-x}(1 - e^{-x}) \leq x$, we easily deduce that for all $n \geq 1$

$$S(n; \lambda) \leq \sum_{k=0}^{n-1} \mathbb{E} (Z_{n-k}) \mathbb{E} (\lambda T_k) \leq \frac{n\lambda m^{n+1}}{m-1}$$

and then

$$\sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \frac{m-1}{m^n} S(n; \lambda) \leq \frac{m}{2} \lambda \log_m(\varepsilon/\lambda) (\log_m(\varepsilon/\lambda) + 1).$$

The second term in the right hand side of Equation (14) can be written as

$$(15) \quad \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{m-1}{m^n} S(\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1; \lambda) + \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{m-1}{m^n} (S(n; \lambda) - S(\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1; \lambda)).$$

By using the fact that for all $x > 0$, $x e^{-\lambda x} \leq 1/\lambda$ and for all $k \geq 0$, $T_k \geq 1$, we deduce that

$$S(\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1; \lambda) \leq \frac{1}{\mathbb{E}(e^{-\lambda T_k})} \sum_{k=0}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \frac{\lambda \mathbb{E}(T_k)}{-\log \mathbb{E}(e^{-\lambda T_k})}.$$

For $k \leq \lfloor \log_m(\varepsilon/\lambda) \rfloor$, we have by Jensen's inequality $\mathbb{E}(e^{-\lambda T_k}) \geq e^{-\lambda \mathbb{E}(T_k)} \geq e^{-m\varepsilon/(m-1)}$. In addition,

$$\mathbb{E}(T_n^2) \leq \left(\sum_{i=0}^n \sqrt{\mathbb{E}(Z_i^2)} \right)^2 \leq \frac{m^{2n}}{(m-1)^2} \left(\frac{\sigma^2}{m-1} + 1 \right),$$

where σ^2 is the variance of the random variable G , and then for $k \leq \lfloor \log_m(\varepsilon/\lambda) \rfloor$

$$(16) \quad \frac{\lambda \mathbb{E}(T_k^2)}{\mathbb{E}(T_k)} \leq \frac{\lambda m^k m^k}{(m-1)(m m^k - 1)} \left(\frac{\sigma^2}{m-1} + 1 \right) \leq \frac{\varepsilon}{(m-1)^2} \left(\frac{\sigma^2}{m-1} + 1 \right).$$

Since for $x \geq 0$, $e^{-x} \leq 1 - x + \frac{x^2}{2}$, we have

$$\mathbb{E}(e^{-\lambda T_k}) \leq 1 - \lambda \mathbb{E}(T_k) + \frac{\mathbb{E}((\lambda T_k)^2)}{2} \leq 1 - \lambda \mathbb{E}(T_k) \left(1 - \frac{\varepsilon}{(m-1)^2} \left(\frac{\sigma^2}{m-1} + 1 \right) \right).$$

and then for $k \leq \lfloor \log_m(\varepsilon/\lambda) \rfloor$, which implies $\lambda \mathbb{E}(T_k) \leq \frac{m\varepsilon}{(m-1)}$,

$$\frac{\lambda \mathbb{E}(T_k)}{-\log \mathbb{E}(e^{-\lambda T_k})} \leq \frac{\frac{m\varepsilon}{(m-1)}}{-\log \left(1 - \frac{m\varepsilon}{(m-1)} \left(1 - \frac{\varepsilon}{(m-1)^2} \left(\frac{\sigma^2}{m-1} + 1 \right) \right) \right)} \stackrel{\text{def}}{=} \kappa(\varepsilon)$$

as long as

$$\varepsilon < \min \left(\frac{(m-1)}{m}, \frac{(m-1)^2}{\frac{\sigma^2}{m-1} + 1} \right) \stackrel{\text{def}}{=} \varepsilon_1.$$

It follows that for $\varepsilon < \varepsilon_1$,

$$S(\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1; \lambda) \leq (1 + \log_m(\varepsilon/\lambda)) e^{m\varepsilon/(m-1)} \kappa(\varepsilon).$$

and as a consequence

$$\sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{m-1}{m^n} S(\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1; \lambda) \leq \frac{\lambda}{\varepsilon} \log_m(\varepsilon/\lambda) e^{m\varepsilon/(m-1)} \kappa(\varepsilon),$$

which is $o(\lambda(\log \lambda)^2)$ when $\lambda \rightarrow 0$. In addition, the second term in the right hand side of Equation (15) can be rewritten as

$$\sum_{k=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{m-1}{m^k} \sum_{n=1}^{\infty} \frac{1}{m^n} \mathbb{E} \left(e^{-\lambda T_{n-1}} Z_n \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \right) \mathbb{E} \left(e^{-\lambda T_k} (1 - e^{-\lambda T_k}) \right).$$

We first note that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{m^n} \mathbb{E} \left(e^{-\lambda T_{n-1}} Z_n \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \right) &= \\ &= \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \frac{1}{m^n} \mathbb{E} \left(e^{-\lambda T_{n-1}} Z_n \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \right) \\ &\quad + \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{1}{m^n} \mathbb{E} \left(e^{-\lambda T_{n-1}} Z_n \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \right). \end{aligned}$$

The first term in the right hand side of the above equation is less than or equal to the quantity $\log_m(\varepsilon/\lambda)$ since $e^{-\lambda T_{n-1}} \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \leq 1$ and $\mathbb{E}(Z_n) = m^n$. The second term can be upper bounded as

$$\begin{aligned} \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{1}{m^n} \mathbb{E} \left(e^{-\lambda T_{n-1}} Z_n \mathbb{E}(e^{-\lambda T_k})^{Z_n-1} \right) &\leq \\ &\leq \sum_{n=\lfloor \log_m(\varepsilon/\lambda) \rfloor + 1}^{\infty} \frac{1}{m^n} \mathbb{E} \left(Z_n e^{-\lambda Z_n} \right) \leq \frac{1}{(m-1)\varepsilon}, \end{aligned}$$

where we have used the fact that $T_k \geq 1$ for all $k \geq 0$ and $x^{-\lambda x} \leq 1/(e\lambda)$ for all $x > 0$. It follows that the second term in the right hand side of Equation (15) is upper bounded by the quantity

$$\frac{\lambda}{\varepsilon} \left(\log_m(\varepsilon/\lambda) + \frac{1}{(m-1)\varepsilon} \right),$$

which is $o(\lambda(\log_m \lambda)^2)$ when $\lambda \rightarrow 0$.

By using the above inequalities, we come up with the conclusion that for every $\varepsilon > 0$,

$$(17) \quad \limsup_{\lambda \rightarrow 0} \frac{f_b(\lambda)}{\lambda(\log_m \lambda)^2} \leq \frac{m}{2}.$$

For establishing a lower bound for $f_b(\lambda)$, we introduce the size-biased Galton-Watson branching process. The sequence of random variables (Z_n/m^n) being a positive martingale, it induces a probability distribution $\tilde{\mathbb{P}}$ such that, for any $n \geq 1$ and any random variable Y measurable with respect to the random variables Z_1, \dots, Z_n ,

$$\int Y d\tilde{\mathbb{P}} = \mathbb{E} \left(Y \frac{Z_n}{m^n} \right).$$

It is known, see Lyons and Peres [7], that under the probability $\tilde{\mathbb{P}}$, the sequence (Z_n) has the same distribution as a branching process with immigration (\tilde{Z}_n) where the number of children has the same distribution as G and the number of new immigrants is distributed as \tilde{G} such that $\mathbb{P}(\tilde{G} = n) = n\mathbb{P}(G = n)/m$. If $\tilde{Z}_0 = 1$, it is easy to check that

$$\tilde{\mathbb{E}} \left(\tilde{Z}_n \right) = m^n + \frac{m^n - 1}{m(m-1)} \mathbb{E}(G^2).$$

If $\tilde{T}_n = \tilde{Z}_0 + \tilde{Z}_1 + \dots + \tilde{Z}_n$, we have by Jensen inequality

$$\begin{aligned} \mathbb{E} \left(e^{-\lambda T_{n-k-1}} \frac{Z_{n-k}}{m^{n-k}} \mathbb{E} \left(e^{-\lambda T_k} \right)^{Z_{n-k-1}} \right) &= \tilde{\mathbb{E}} \left(e^{-\lambda \tilde{T}_{n-k-1}} \mathbb{E} \left(e^{-\lambda T_k} \right)^{\tilde{Z}_{n-k-1}} \right) \\ &\geq \tilde{\mathbb{E}} \left(e^{-\lambda \tilde{T}_{n-k}} \mathbb{E} \left(e^{-\lambda T_k} \right)^{\tilde{T}_{n-k}} \right) \\ &\geq e^{-\lambda(1+\mathbb{E}(T_k))\tilde{\mathbb{E}}(\tilde{T}_{n-k})} \\ &\geq e^{-\frac{\lambda m}{(m-1)} \left(m^{n-k} + \frac{m^{n+1}}{m-1} \right) \left(1 + \frac{g_2}{m} \right)} \end{aligned}$$

since

$$\tilde{\mathbb{E}}(\tilde{T}_n) \leq \frac{m^{n+1}}{m-1} \left(1 + \frac{g_2}{m} \right),$$

where $g_2 = \mathbb{E}(G^2)$. In addition, by using the fact that $e^{-x}(1 - e^{-x}) \geq x - 2x^2$ holds for $x > 0$, we have

$$\begin{aligned} &\sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \frac{m-1}{m^n} S(n; \lambda) \\ &\geq \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} (m-1) \sum_{k=0}^{n-1} \frac{1}{m^k} e^{-\frac{\lambda m}{(m-1)} (m^{n-k} + m^{n+1}/(m-1)) (1+g_2/m)} \mathbb{E}(\lambda T_k - 2(\lambda T_k)^2) \\ &\geq e^{-\frac{\varepsilon m}{(m-1)} (1+m/(m-1)) (1+g_2/m)} \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} (m-1) \sum_{k=0}^{n-1} \frac{\lambda \mathbb{E}(T_k)}{m^k} \left(1 - \frac{\lambda \mathbb{E}(T_k^2)}{\mathbb{E}(T_k)} \right). \end{aligned}$$

By using Inequality (16), we have

$$\begin{aligned} &\sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} (m-1) \sum_{k=0}^{n-1} \frac{\lambda \mathbb{E}(T_k)}{m^k} \left(1 - \frac{\lambda \mathbb{E}(T_k^2)}{\mathbb{E}(T_k)} \right) \\ &\geq \lambda (1 - k(\varepsilon)) \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \sum_{k=0}^{n-1} \frac{m^{k+1} - 1}{m^k} \end{aligned}$$

where

$$k(\varepsilon) \stackrel{\text{def}}{=} \frac{\varepsilon}{(m-1)^2} \left(\frac{\sigma^2}{m-1} + 1 \right),$$

which is in $(0, 1]$ long as $\varepsilon < \varepsilon_1$. Since

$$\begin{aligned} \sum_{n=1}^{\lfloor \log_m(\varepsilon/\lambda) \rfloor} \sum_{k=0}^{n-1} \frac{m^{k+1} - 1}{m^k} &= \frac{m}{2} [\log_m(\varepsilon/\lambda)] ([\log_m(\varepsilon/\lambda)] + 1) + \frac{m \lfloor \log_m(\varepsilon/\lambda) \rfloor}{m-1} \\ &\quad - \frac{m}{(m-1)^2} \left(\frac{1}{m^{\lfloor \log_m(\varepsilon/\lambda) \rfloor}} - 1 \right) \end{aligned}$$

and since we already know that the second term in the right hand side of Equation (14) is $o(\lambda(\log_m(\lambda))^2)$ when $\lambda \rightarrow 0$, we then deduce that for all $\varepsilon \in (0, \varepsilon_1)$

$$\liminf_{\lambda \rightarrow 0} \frac{f_b(\lambda)}{\lambda(\log_m \lambda)^2} \geq \frac{m}{2} e^{-\frac{\varepsilon m}{(m-1)}(1+m/(m-1))(1+g_2/m)} (1 - k(\varepsilon))$$

and hence,

$$(18) \quad \liminf_{\lambda \rightarrow 0} \frac{f_b(\lambda)}{\lambda(\log_m \lambda)^2} \geq \frac{m}{2}.$$

Combining Equations (13), (17) and (18), Equation (12) follows. \square

From Proposition 1, one gets that the size of the sampled tree scales with the size of the original tree. The same phenomenon is true for the variance from Proposition 3. When $\lambda \rightarrow 0$, the rescaled variance is much larger than the squared of the first moment. This indicates that the dispersion of the exploration process is very high when only a few number of nodes are selected for exploring the whole tree.

5. THE DEPTH BIASED MODEL

In this section, it is assumed that, for $n \geq 0$, a node at depth n is chosen with probability $(1 - \exp(-(\alpha/m)^n))$ for some $\alpha \in [0, 1]$. The mean number of selected nodes at depth n in the tree is equal to $m^n(1 - \exp(-(\alpha/m)^n)) \sim \alpha^n$ and the total number of selected nodes in the whole tree $\mathcal{N}(\mathcal{T})$ is such that

$$\frac{1}{1-\alpha} - \frac{1}{2(1-\alpha^2/m)} \leq \mathbb{E}(\mathcal{N}(\mathcal{T})) = \sum_{n=0}^{\infty} m^n (1 - e^{-(\alpha/m)^n}) \leq \frac{1}{1-\alpha},$$

in particular the mean number of selected nodes $\mathcal{N}(\mathcal{T}) \sim 1/(1-\alpha)$ when $\alpha \rightarrow 1$. The behavior of the size $R(\alpha)$ of the sampled tree is used to estimate the speed of the exploration process, when the number of selected nodes becomes large. We first give the expression of the mean value $\mathbb{E}(R(\alpha))$ of the size of the sampled tree.

Lemma 1. *The mean value of the size of the sampled tree in the depth biased model is given by*

$$(19) \quad \mathbb{E}(R(\alpha)) = \sum_{n=0}^{\infty} m^n \left(1 - \mathbb{E} \left(\exp \left(- \left(\frac{\alpha}{m} \right)^n \sum_{i=0}^{\infty} \alpha^i \frac{Z_i}{m^i} \right) \right) \right).$$

Proof. As in the previous section, for $n \geq 0$ and $1 \leq \ell \leq Z_n$, the symbol $\mathcal{T}^{n,\ell}$ denotes the sub-tree of \mathcal{T} whose root is (n, ℓ) . The node (n, ℓ) is in the sampled

tree if $\mathcal{N}(\mathcal{T}^{n,\ell}) \neq 0$. Since nodes at a given depth are selected independently one from each other, we have

$$\mathbb{E}(\mathbb{1}_{\{\mathcal{N}(\mathcal{T}^{\ell,n}) \neq 0\}} \mid \mathcal{T}^{n,\ell}) = 1 - \exp\left(-\left(\frac{\alpha}{m}\right)^n \sum_{i=0}^{\infty} \alpha^i \frac{Z_i}{m^i}\right)$$

where we have used the fact that the sub-tree $\mathcal{T}^{n,\ell}$ has the same offspring distribution as the original tree \mathcal{T} . Hence,

$$\mathbb{E}(\mathbb{1}_{\{\mathcal{N}(\mathcal{T}^{\ell,n}) \neq 0\}}) = 1 - \mathbb{E}\left(\exp\left(-\left(\frac{\alpha}{m}\right)^n \sum_{i=0}^{\infty} \alpha^i \frac{Z_i}{m^i}\right)\right).$$

It follows that the size of the sampled tree given by

$$(20) \quad R(\alpha) = \sum_{n=0}^{+\infty} \sum_{\ell=1}^{Z_n} \mathbb{1}_{\{\mathcal{N}(\mathcal{T}^{\ell,n}) \neq 0\}}$$

and its mean value is, by using the independence in the selection of nodes,

$$\mathbb{E}(R(\alpha)) = \sum_{n=0}^{+\infty} \mathbb{E}(Z_n) \mathbb{P}(\mathcal{N}(\mathcal{T}^{\ell,n}) \neq 0),$$

Equation (19) follows. □

The growth rate of the exploration process is defined by the ratio

$$\frac{\mathbb{E}(S_\alpha)}{\mathbb{E}(\mathcal{N}(\mathcal{T}))} = \frac{1}{\eta(\alpha)} \sum_{n=0}^{+\infty} (1-\alpha)m^n \left(1 - \mathbb{E}\left(\exp\left(-\left(\frac{\alpha}{m}\right)^n \sum_{i=0}^{\infty} \alpha^i \frac{Z_i}{m^i}\right)\right)\right),$$

where $\eta(\alpha) = (1-\alpha)\mathbb{E}(\mathcal{N}(\mathcal{T}) \rightarrow 1)$ when $\alpha \rightarrow 1$.

Theorem 2. *As $\alpha \nearrow 1$, the following equivalence holds*

$$\frac{\mathbb{E}(S_\alpha)}{\mathbb{E}(\mathcal{N}(\mathcal{T}))} \sim \frac{1}{1-\alpha} \kappa(\alpha),$$

where

$$(21) \quad \kappa(\alpha) \stackrel{\text{def.}}{=} \frac{\log m}{m-1} \int_0^1 m^{\{\log_m(\alpha) \log_m(u) - \log_m(1-\alpha)\}} du,$$

and $\{x\}$ is the fractional part of $x \in \mathbb{R}$. In particular, the renormalized growth rate $\kappa(\alpha)$ is an oscillating function such that

$$\liminf_{\alpha \nearrow 1} \kappa(\alpha) = \frac{\log m}{m-1} \quad \text{and} \quad \limsup_{\alpha \nearrow 1} \kappa(\alpha) = \frac{m \log m}{m-1}.$$

Proof. Let us first introduce the function

$$H(\alpha) = \sum_{n=0}^{+\infty} (1-\alpha)m^n \left(1 - \mathbb{E}\left(\exp\left(-\left(\frac{\alpha}{m}\right)^n \frac{W}{1-\alpha}\right)\right)\right),$$

then

$$\begin{aligned}
\left| H(\alpha) - \frac{\mathbb{E}(S_\alpha)}{\mathbb{E}(\mathcal{N}(\mathcal{T}))} \right| &\leq (1-\alpha) \sum_{n=0}^{\infty} \alpha^n \left| \sum_{i=0}^{\infty} \alpha^i \mathbb{E} \left(\frac{Z_i}{m^i} - \frac{W}{1-\alpha} \right) \right| \\
&\leq (1-\alpha) \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{\infty} \alpha^i \mathbb{E} \left(\left| \frac{Z_i}{m^i} - W \right| \right) \\
&\leq (1-\alpha) \text{Var}(1-W) \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{\infty} \left(\frac{\alpha}{\sqrt{m}} \right)^i = \frac{\text{Var}(1-W)}{1 - \frac{\alpha}{\sqrt{m}}},
\end{aligned}$$

where we have used Inequality (10) in the last step. Hence,

$$(22) \quad \left| H(\alpha) - \frac{\mathbb{E}(S_\alpha)}{\mathbb{E}(\mathcal{N}(\mathcal{T}))} \right| \leq \frac{\text{Var}(1-W)}{1 - \frac{1}{\sqrt{m}}}$$

From the above inequality, we observe that the ratio $\mathbb{E}(S_\alpha)/\mathbb{E}(\mathcal{N}(\mathcal{T}))$ and $H(\alpha)$ has up to a constant factor the same behavior as $\alpha \rightarrow 1$. Hence, in the following, we analyze the function $H(\alpha)$.

Since in the supercritical case, $W > 0$ a.s., we can write

$$1 - \mathbb{E} \left(\exp \left(- \left(\frac{\alpha}{m} \right)^n \frac{W}{1-\alpha} \right) \right) = \int_0^{\frac{1}{1-\alpha}} \mathbb{1}_{\{u \leq (\frac{\alpha}{m})^n \frac{1}{1-\alpha}\}} W e^{-uW} du$$

and then by Fubini's theorem

$$\begin{aligned}
H(\alpha) &= (1-\alpha) \int_0^{\frac{1}{1-\alpha}} \sum_{n=0}^{\infty} \mathbb{1}_{\{u \leq (\frac{\alpha}{m})^n \frac{1}{1-\alpha}\}} \mathbb{E}(W e^{-uW}) du \\
&= \frac{1-\alpha}{m-1} \int_0^{\frac{1}{1-\alpha}} \left(m^{\lceil \log_{\frac{m}{\alpha}} 1/(u(1-\alpha)) \rceil} - 1 \right) \mathbb{E}(W e^{-uW}) du.
\end{aligned}$$

It follows that

$$\begin{aligned}
H(\alpha) &= \frac{(1-\alpha)}{m-1} \int_0^{1/(1-\alpha)} \frac{m^{\{\log_{\frac{m}{\alpha}} 1/(u(1-\alpha))\}}}{m^{\log_{\frac{m}{\alpha}} (1/u(1-\alpha))}} \mathbb{E}(W e^{-uW}) du \\
&\quad - \frac{(1-\alpha)}{(m-1)} \mathbb{E} \left(1 - e^{-W/(1-\alpha)} \right),
\end{aligned}$$

where $\{z\} = \lceil z \rceil - z$. The absolute value of second term in the right hand side of the above equation is bounded by $1/(m-1)$. The first term can be rewritten as

$$\begin{aligned}
&\frac{1-\alpha}{m-1} \int_0^{\frac{1}{1-\alpha}} \frac{m^{\{\log_{\frac{m}{\alpha}} 1/(u(1-\alpha))\}}}{m^{\log_{\frac{m}{\alpha}} u(1-\alpha)}} \mathbb{E}(W e^{-uW}) du \\
&= \frac{(1-\alpha)^{-\log \alpha / (\log m - \log \alpha)}}{m-1} \int_0^{\frac{1}{1-\alpha}} \frac{m^{\{\log_{\frac{m}{\alpha}} 1/(u(1-\alpha))\}}}{u^{\log m / \log(m/\alpha)}} \mathbb{E}(W e^{-uW}) du
\end{aligned}$$

As α goes to 1, the quantity $H(\alpha)$ is equivalent to

$$H(\alpha) \sim \frac{1}{m-1} \int_0^{+\infty} \frac{m^{\{\log_{\frac{m}{\alpha}} 1/(u(1-\alpha))\}}}{u^{\log m / \log(m/\alpha)}} \mathbb{E}(W e^{-uW}) du,$$

and because, only the neighborhood of 0 matters in this diverging integral, one gets that

$$\begin{aligned} H(\alpha) &\sim \frac{1}{m-1} \int_0^1 \frac{m^{\{\log_m \frac{1}{u(1-\alpha)}\}}}{u^{\log m / \log(m/\alpha)}} \mathbb{E}(W e^{-uW}) du \\ &\sim \frac{1}{m-1} \int_0^1 \frac{m^{\{\log_m \frac{1}{u(1-\alpha)}\}}}{u^{\log m / \log(m/\alpha)}} du \\ &= \frac{1}{m-1} \times \frac{1 - \log_m(\alpha)}{-\log_m(\alpha)} \int_0^1 m^{\{\log_m(\alpha) \log_m(u) - \log_m(1-\alpha)\}} du \end{aligned}$$

by using the fact that $\mathbb{E}(W) = 1$ and a change of variable.

To conclude note that, for $\delta \in (0, 1)$, if $\alpha_n = 1 - e^{-\delta-n}$, then

$$\lim_{n \rightarrow +\infty} \int_0^1 m^{\{\log_m(\alpha_n) \log_m(u) - \log_m(1-\alpha_n)\}} du = m^\delta,$$

which illustrates the oscillating behavior of the function $\kappa(\alpha)$ defined by Equation (21). \square

From Proposition 2, one observes that the speed of the exploration process is oscillating when the number of selected nodes is moderately large. Hence, in both the uniform and depth biased models, the exploration of the structure of the tree (namely, the number of nodes) by selecting nodes at random reveals some instability.

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(F. Guillemin) FRANCE TELECOM R&D, F-22300 LANNION
E-mail address: `Fabrice.Guillemin@orange-ftgroup.com`

(Ph. Robert) INRIA-ROCQUENCOURT, RAP PROJECT, DOMAINE DE VOLUCEAU, 78153 LE
CHESNAY, FRANCE
E-mail address: `Philippe.Robert@inria.fr`
URL: `http://www-rocq.inria.fr/~robert`