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Guarding curvilinear art galleries with vertex or point guards

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Abstract: One of the earliest and most well known problems in computational geometry is the so-called *art gallery problem*. The goal is to compute the minimum possible number guards placed on the vertices of a simple polygon in such a way that they cover the interior of the polygon. We consider the problem of guarding an art gallery which is modeled as a polygon with curvilinear walls. Our main focus is on polygons the edges of which are convex arcs pointing towards the exterior or interior of the polygon (but not both), named piecewise convex and piecewise concave polygons. We prove that, in the case of piecewise convex polygons, if we only allow vertex guards, $\lfloor \frac{4n}{7} \rfloor - 1$ guards are sometimes necessary, and $\lfloor \frac{2n}{3} \rfloor$ guards are always sufficient. Moreover, an $O(n \log n)$ time and $O(n)$ space algorithm is described that produces a vertex guarding set of size at most $\lfloor \frac{2n}{3} \rfloor$. When we allow point guards the afore-mentioned lower bound drops down to $\lfloor \frac{n}{2} \rfloor$. In the special case of monotone piecewise convex polygons we can show that $\lfloor \frac{n}{2} \rfloor + 1$ vertex or $\lfloor \frac{n}{2} \rfloor$ point guards are always sufficient and sometimes necessary. In the case of piecewise concave polygons, we show that $2n - 4$ point guards are always sufficient and sometimes necessary, whereas it might not be possible to guard such polygons by vertex guards. We conclude with bounds for other types of curvilinear polygons and future work.

Key-words: art gallery, curvilinear polygons, vertex guards, point guards, piecewise convex polygons, piecewise concave polygons

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Garder les galeries d'art curvilignes avec des gardes de sommet ou de point

Résumé : Un des plus anciens et plus célèbres problèmes en géométrie algorithmique est le problème dit de *galerie d'art* qui consiste à calculer le nombre minimal des gardians qui sont nécessaires afin de couvrir l'intérieur d'un polygone simple s'ils sont placés sur ses sommets.

Dans cet article, nous considérons le problème de garde d'une galerie d'art qui est modélisée par un polygone avec des murs courbés. En particulier, nous considérons des polygones dont les bords sont des arcs convexes qui se dirigent vers l'extérieur ou l'intérieur du polygone (mais pas les deux au même temps), appelés polygones par morceaux convexes ou concaves respectivement. Nous montrons que, dans le cas de polygones par morceaux convexes, si nous permettons seulement des gardians de sommets, alors $\lfloor \frac{4n}{7} \rfloor - 1$ gardians sont parfois nécessaires, et $\lfloor \frac{2n}{3} \rfloor$ gardians sont toujours suffisants. D'ailleurs, nous décrivons un algorithme, $O(n \log n)$ en temps et $O(n)$ en espace, qui produit l'ensemble des sommets de garde de taille au plus $\lfloor \frac{2n}{3} \rfloor$. Dans le cas particulier de polygones monotones et par morceaux convexes nous pouvons montrer que $\lfloor \frac{n}{2} \rfloor + 1$ gardians de sommets ou $\lfloor \frac{n}{2} \rfloor$ gardians de points sont toujours suffisants et parfois nécessaires.

Dans le cas des polygones par morceaux concaves, nous prouvons que $2n - 4$ gardians de points sont toujours suffisantes et parfois nécessaires, tandis qu'il ne peut être possible de garder de tels polygones par des gardians de sommets. Nous concluons avec des limites pour d'autres types de polygones courbés et perspectives pour le futur.

Mots-clés : galerie d'art; polygones courbés, gardians de sommet, gardians de point, polygones par morceau convexes, polygones par morceau concaves

1 Introduction

Consider a simple polygon P with n vertices. What is the worst-case bound on the number of guards that we need to place on the vertices of P so that any point in P is visible from at least one guard — two points in P are visible from each other if the line segment connecting them lies entirely inside P. This problem, the minimization version of which is known as the *art gallery problem*, has been one of the earliest problems in Computational Geometry. Applications areas include robotics [21, 36], motion planning [24, 28], computer vision and pattern recognition [32, 37, 2, 33], graphics [26, 7], CAD/CAM [4, 15] and wireless networks [16]. In the late 1980's to mid 1990's interest moved from linear polygonal objects to curvilinear objects [35, 9, 11, 10] — see also the paper by Dobkin and Souvaine [13] that extends linear polygon algorithms to curvilinear polygons, as well as the recent book edited by Boissonnat and Teillaud [3] for a collection of results on computational geometry for curves and surfaces. In this context this paper addresses the classical art gallery problem for various classes of polygonal regions the edges of which are arcs of curves. To the best of our knowledge this is the first time that the art gallery problem is considered in this context.

The first results on the art gallery problem or its variations date back to the 1970's. Chvátal [8] was the first to prove that any simple polygon with n vertices can always be monitored with $\lfloor \frac{n}{3} \rfloor$ vertices; this bound is tight in the worst case. The proof by Chvátal was quite tedious and Fisk [18] gave a much simpler proof by means of triangulating the polygon and coloring its vertices using three colors in such a way so that no triangle in the triangulation of the polygon contains two vertices of the same color. The algorithm proposed by Fisk runs in $O(T(n) + n)$ time, where $T(n)$ is the time to triangulate a simple polygon. Following Chazelle's linear-time algorithm for triangulating a simple polygon [5, 6], the algorithm proposed by Fisk runs in $O(n)$ time. Lee and Lin [22] showed that computing the minimum number of vertex guards for a given simple polygon is NP-hard, which was extended to point guards by Aggarwal [1]. Soon afterwards other types of polygons were considered. Kahn, Klawe and Kleitman [19] showed that simple orthogonal polygons of size n, i.e., simple polygons with axes-aligned edges, can always be monitored with $\lfloor \frac{n}{4} \rfloor$ vertex guards, which is also the worst-case lower bound. Several $O(n)$ algorithms have been proposed for this variation of the problem, notably by Sack [30], who gave the first such algorithm, and later on by Lubiw [25]. Edelsbrunner, O'Rourke and Welzl [14] gave a linear time algorithm for monitoring orthogonal polygons with $\lfloor \frac{n}{4} \rfloor$ point guards.

Beside simple polygons and simple orthogonal polygons, polygons with holes, and orthogonal polygons with holes have been investigated. As far as the type of guards is concerned, *edge guards* and *mobile guards* have been considered. An edge (resp. mobile) guard is an edge (resp. edge or diagonal) of the polygon, and a point is visible from it if it is visible from at least one point on the edge (resp. edge or diagonal). Other types of guarding problems have also been studied in the literature, notably, the *fortress problem* (monitor the exterior of the polygon against enemy raids) and the *prison yard problem* (monitor both the interior and the exterior of the polygon which represents a prison: prisoners must be monitored in the interior of the prison and should not be allowed to escape out of the prison). For a detailed discussion of these variations and the corresponding results the interested reader should refer to the book by O'Rourke [29], the survey paper by Shermer [31] and the book chapter by Urrutia [34].

In this paper we are primarily interested in the case of vertex guards, although results about point guards are also described. In our case, polygons are not required to have linear edges. On the contrary we consider polygons that have smooth curvilinear edges. Clearly, the minimization versions of these problems are NP-hard, since they are direct generalizations of the corresponding art gallery problems for polygons with linear edges. The main focus of this paper is the class of polygons that are either locally convex or locally concave (except possibly at the vertices), the edges of which are convex arcs; we call such polygons *piecewise-convex* and *piecewise-concave polygons*, respectively.

For the first class of polygons we show that it is always possible to monitor them with $\lfloor \frac{2n}{3} \rfloor$ vertex guards, where n is the number of polygon vertices. On the other hand we describe families of piecewise-convex polygons that require a minimum of $\lfloor \frac{4n}{7} \rfloor - 1$ vertex guards and $\lfloor \frac{n}{2} \rfloor$ point guards. Aside from the worst-case upper (resp. lower) bound on the number of guards that are sufficient (resp. necessary) in order to monitor a piecewise-convex polygon, we describe an $O(n \log n)$ time and $O(n)$ space algorithm which computes a vertex guard set of size at most $\lfloor \frac{2n}{3} \rfloor$. Our algorithm should be viewed as a generalization of Fisk's algorithm [18]; in fact, when applied to polygons with linear edges, it produces a guard set of size at most $\lfloor \frac{n}{3} \rfloor$. For the purposes of our complexity analysis and results, we assume, throughout the paper, that the curvilinear edges of our polygons are arcs of algebraic curves of constant degree; as a result all predicates required by the algorithms described in this paper take $O(1)$ time in the Real RAM computation model. The central idea for both obtaining the worst-case upper bound, as well as for designing our algorithm, is to approximate the piecewise-convex polygon by a linear polygon (a polygon with line segments as edges). Additional Steiner vertices are added on the boundary of the curvilinear polygon in order to achieve this. The resulting linear polygon is a simple polygon and captures the essentials of the geometry of the piecewise-convex polygon; for obvious reasons we term this linear polygon the *polygonal approximation*. Once the polygonal approximation has been constructed, we compute a guard set for it by applying a slight modification of Fisk's algorithm [18]. The guard set just computed for the polygonal approximation turns out to be a guard set for the original curvilinear polygon. The final step of both the proof and our algorithm consists in mapping the guard set of the polygonal approximation to a vertex guard set consisting of vertices of the original polygon only.

For piecewise-concave polygons vertex guards may not be sufficient in order to monitor the interior of the polygon (see Fig. 15(a)). We thus turn our attention to point guards, and we show that $2n - 4$ point guards are always sufficient and sometimes necessary. Our method for showing the sufficiency result is similar to the technique used to illuminate sets of disjoint convex objects on the plane [17]. Given a piecewise-concave polygon P, we construct a new locally concave polygon Q, contained inside P, and such that the tangencies between edges of Q are maximized. The problem of monitoring P then reduces to the problem of monitoring Q, which essentially consists of a number of faces with pairwise disjoint interiors. The faces of Q require, each, two point guards in order to be monitored, and are in 1–1 correspondence with the triangles of an appropriately defined triangulation graph $\mathcal{T}(R)$ of a polygon R with n vertices. Thus the number of point guards required to monitor P is at most two times the number of faces of $\mathcal{T}(R)$, i.e., $2n - 4$.

The rest of the paper is structured as follows. In Section 2 we introduce some notation and provide various definitions. In Section 3 we present our algorithm for computing a vertex guard set,

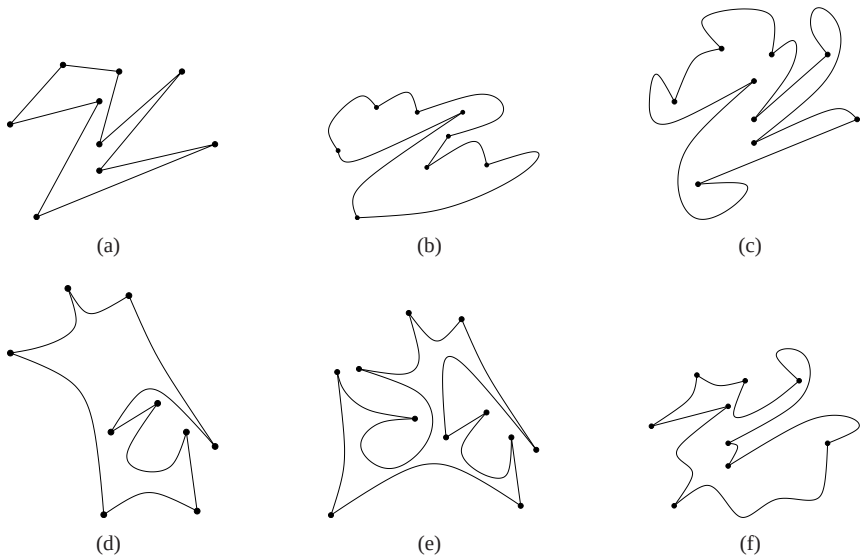

Figure 1: Different types of curvilinear polygons: (a) a linear polygon, (b) a piecewise-convex polygon, (c) a locally convex polygon, (d) a piecewise-concave polygon, (e) a locally concave polygon and (f) a general polygon.

of size $\lfloor \frac{2n}{3} \rfloor$, for a piecewise-convex polygon with n vertices, and present families of piecewise-convex polygons that require a minimum of $\lfloor \frac{4n}{7} \rfloor - 1$ vertex or $\lfloor \frac{n}{2} \rfloor$ point guards in order to be monitored. In Section 4 we present our results for piecewise-concave polygons, namely, that $2n - 4$ point guards are always necessary and sometimes sufficient for this class of polygons. The final section of the paper, Section 5, discusses further results and states open problems.

2 Definitions

Types of curvilinear polygons. Let V be a sequence of points $v_1, \ldots, v_n$, $n \geq 2$, and A a set of curvilinear arcs $a_1, \ldots, a_n$, such that a_i has as endpoints the points v_i and v_{i+1}[1]. We will assume that the arcs a_i and a_j, $i \neq j$, do not intersect, except when $j = i - 1$ or $j = i + 1$, in which case they intersect only at the points v_i and v_{i+1}, respectively. We define a *curvilinear polygon* P to be the closed region delimited by the arcs a_i. The points v_i are called the vertices of P. An arc a_i is a *convex arc* if every line on the plane intersects a_i at either at most two points or along a linear segment.

[1]Indices are considered to be evaluated modulo n.

A polygon P is a *linear polygon* if its edges are line segments (see Fig. 1(a)). A polygon P is *locally convex* (see Fig. 1(c)), (resp. *locally concave* (see Fig. 1(e))), if for every point p on the boundary of P, with the possible exception of P's vertices, there exists disk with its center on p, say D_p, such that $P \cap D_p$ is convex (resp. concave). A polygon P is *piecewise-convex* (see Fig. 1(b)), (resp. *piecewise-concave* (see Fig. 1(d))), if it is locally convex (resp. concave), and the portion of the boundary between every two consecutive vertices is a convex arc. Finally, a polygon is said to be a *general polygon* if we impose no restrictions on the type of its edges (see Fig. 1(f)). We will use the term *curvilinear polygon* to refer to a polygon the edges of which are either line or curve segments.

Guards and guard sets. In our setting, a *guard* or *point guard* is a point in the interior or on the boundary of a curvilinear polygon P. A guard of P that is also a vertex of P is called a *vertex guard*. We say that a curvilinear polygon P is *monitored* by a set G of guards if every point in P is visible from at least one point in G, where two points p and q inside P are visible from each other of the line segment pq lies entirely inside P. The set G that has this property is called a *guard set* for P. A guard set that consists solely of vertices of P is called a *vertex guard set*.

3 Piecewise-convex polygons

In this section we present an algorithm which, given a piecewise-convex polygon P of size n, computes a vertex guard set G of size $\lfloor \frac{2n}{3} \rfloor$. The basic steps of the algorithm are as follows:

1. Compute the polygonal approximation $\tilde{P}$ of P.
2. Compute a constrained triangulation $\mathcal{T}(\tilde{P})$ of $\tilde{P}$.
3. Compute a guard set $G_{\tilde{P}}$ for $\tilde{P}$, by coloring the vertices of $\mathcal{T}(\tilde{P})$ using three colors.
4. Compute a guard set G_P for P from the guard set $G_{\tilde{P}}$.

3.1 Polygonalization of a piecewise-convex polygon

Let a_i be a convex arc with endpoints v_i and v_{i+1}. We call the convex region r_i delimited by a_i and the line segment $v_i v_{i+1}$ a *room*. A room is called degenerate if the arc a_i is a line segment. A line segment pq, where $p, q \in a_i$ is called a *chord*, and the region delimited by the chord pq and a_i is called a *sector*. The chord of a room r_i is defined to be the line segment $v_i v_{i+1}$ connecting the endpoints of the corresponding arc a_i. A degenerate sector is a sector with empty interior. We distinguish between two types of rooms (see Fig. 2):

1. *empty rooms*: these are non-degenerate rooms that do not contain any vertex of P in their interior or in the interior of their chord.
2. *non-empty rooms*: these are non-degenerate rooms that contain at least one vertex of P in their interior or in the interior of their chord.

In order to polygonalize P we are going to add new vertices in the interior of non-linear convex arcs. To distinguish between the two types of vertices, the n vertices of P will be called *original vertices*, whereas the additional vertices will be called *Steiner vertices*.

More specifically, for each empty room r_i we add a vertex $w_{i,1}$ (anywhere) in the interior of the arc a_i (see Fig. 3). For each non-empty room r_i, let X_i be the set of vertices of P that lie in the interior of the chord $v_i v_{i+1}$ of r_i, and R_i be the set of vertices of P that are contained in the interior of r_i or belong to X_i (by assumption $R_i \neq \emptyset$). If $R_i \neq X_i$, let C_i be the set of vertices on the convex

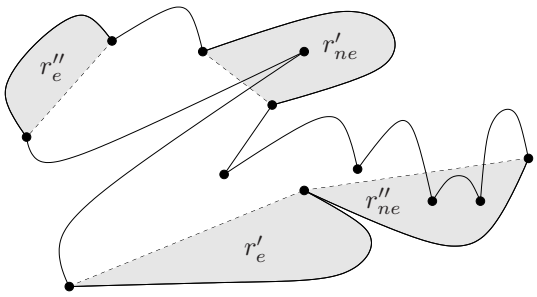

Figure 2: The two types of rooms in a piecewise-convex polygon: r'_e and r''_e are empty rooms, whereas r'_{ne} and r''_{ne} are non-empty rooms.

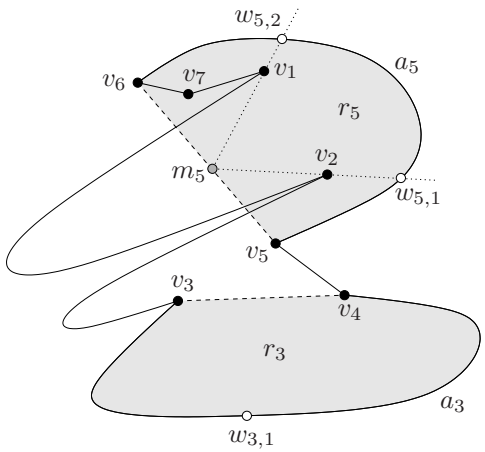

Figure 3: The Steiner vertices (white points) for rooms r_3 (empty) and r_5 (non-empty). $w_{3,1}$ is a point in the interior of a_3. m_5 is the midpoint of v_5 and v_6, whereas $w_{5,1}$ and $w_{5,2}$ are the intersections of the lines $m_5 v_2$ and $m_5 v_1$ with the arc a_5, respectively. In this example $R_5 = \{v_1, v_2, v_7\}$, whereas $C_5^* = \{v_1, v_2\}$.

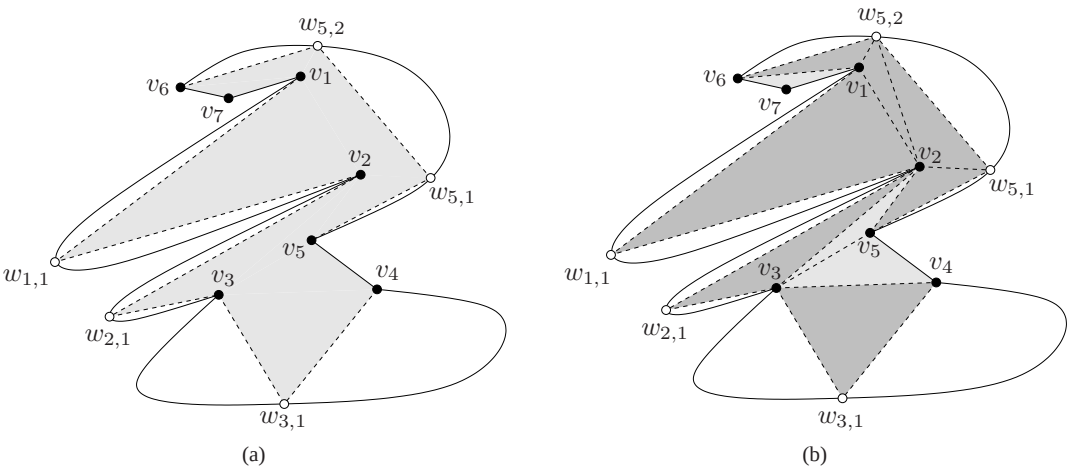

Figure 4: (a) The polygonal approximation $\tilde{P}$, shown in gray, of the piecewise-convex polygon P with vertices v_i, $i = 1, \ldots, 7$. (b) The constrained triangulation $\mathcal{T}(\tilde{P})$ of $\tilde{P}$. The dark gray triangles are the constrained triangles. The polygonal region $v_5 w_{5,1} w_{5,2} v_6 v_1 v_2 v_5$ is a crescent. The triangles $w_{5,1} v_2 v_5$ and $v_1 w_{5,2} v_6$ are boundary crescent triangles. The triangle $v_2 w_{5,2} v_1$ is an upper crescent triangle, whereas the triangle $v_2 w_{5,1} w_{5,2}$ is a lower crescent triangle.

hull of the vertex set $(R_i \setminus X_i) \cup \{v_i, v_{i+1}\}$; if $R_i = X_i$, let $C_i = X_i \cup \{v_i, v_{i+1}\}$. Finally, let $C_i^* = C_i \setminus \{v_i, v_{i+1}\}$. Clearly, v_i and v_{i+1} belong to the set C_i and, furthermore, $C_i^* \neq \emptyset$.

Let m_i be the midpoint of $v_i v_{i+1}$ and $\ell_i^{\perp}(p)$ the line perpendicular to $v_i v_{i+1}$ passing through a point p. If $C_i^* \neq X_i$, then, for each $v_k \in C_i^*$, let w_{i,j_k}, $1 \leq j_k \leq |C_i^*|$, be the (unique) intersection of the line $m_i v_k$ with the arc a_i; if $C_i^* = X_i$, then, for each $v_k \in C_i^*$, let w_{i,j_k}, $1 \leq j_k \leq |C_i^*|$, be the (unique) intersection of the line $\ell_i^{\perp}(v_k)$ with the arc a_i.

Now consider the sequence $\tilde{V}$ of the original vertices of P augmented by the Steiner vertices added to empty and non-empty rooms; the order of the vertices in $\tilde{V}$ is the order in which we encounter them as we traverse the boundary of P counterclockwise. The linear polygon defined by the sequence $\tilde{V}$ of vertices is denoted by $\tilde{P}$ (see Fig. 4(a)). It is easy to show that:

Lemma 1 *The linear polygon $\tilde{P}$ is a simple polygon.*

Proof. It suffices show that the line segments replacing the curvilinear segments of P do not intersect other edges of P or $\tilde{P}$.

Let r_i be an empty room, and let $w_{i,1}$ be the point added in the interior of a_i. The interior of the line segments $v_i w_{i,1}$ and $w_{i,1} v_{i+1}$ lie in the interior of r_i. Since P is a piecewise-convex polygon, and r_i is an empty room, no edge of P could potentially intersect $v_i w_{i,1}$ or $w_{i,1} v_{i+1}$. Hence replacing a_i by the polyline $v_i w_{i,1} v_{i+1}$ gives us a new piecewise-convex polygon.

Let r_i be a non-empty room. Let $w_{i,1}, \ldots, w_{i,K_i}$ be the points added on a_i, where K_i is the cardinality of C_i^*. By construction, every point $w_{i,k}$ is visible from $w_{i,k+1}$, $k = 1, \ldots K_i - 1$, and every point $w_{i,k}$ is visible from $w_{i,k-1}$, $k = 2, \ldots K_i$. Moreover, $w_{i,1}$ is visible from v_i and w_{i,K_i} is visible from v_{i+1}. Therefore, the interior of the segments in the polyline $v_i w_{i,1} \ldots w_{i,K_i} v_{i+1}$ lie in the interior of r_i and do not intersect any arc in P. Hence, substituting a_i by the polyline $v_i w_{i,1} \ldots w_{i,K_i} v_{i+1}$ gives us a new piecewise-convex polygon.

As a result, the linear polygon $\tilde{P}$ is a simple polygon. □

We call the linear polygon $\tilde{P}$, defined by $\tilde{V}$, the *straight-line polygonal approximation* of P, or simply the *polygonal approximation* of P. An obvious result for $\tilde{P}$ is the following:

Corollary 2 *If P is a piecewise-convex polygon the polygonal approximation $\tilde{P}$ of P is a linear polygon that is contained inside P.*

We end this subsection by proving a tight upper bound on the size of the polygonal approximation of a piecewise-convex polygon. We start by stating and proving an intermediate result, namely that the sets C_i^* are pairwise disjoint.

Lemma 3 *Let i, j, with $1 \leq i < j \leq n$. Then $C_i^* \cap C_j^* = \emptyset$.*

Proof. Consider an arc a_i of P, delimited by the vertices v_i and v_{i+1} and let π_i denote the shortest path between them. Note that the internal vertices of π_i are vertices of C_i^* , and so π_i is also a convex arc. The convex arcs a_i and π_i bound a (curvilinear) polygon, that we denote by Q_i, for which π_i is locally concave. That is, every point in C_i^* is a reflex vertex of Q_i, and so every point in C_i^* is a reflex (i.e., locally concave) vertex of P as well. At every vertex $w \in C_i^*$, the bisector of the internal angle of P enters the polygon Q_i and leaves Q_i (and P) at some point along a_i.

Consider the bisector of the internal angle at every reflex vertex w of P. If the bisector intersects some arc a_j, then w can belong to the set C_j^*. Since every bisector intersects at most one arc a_j, every vertex w belongs to at most one set C_j^*. □

An immediate consequence of Lemma 3 is the following corollary that gives us a tight bound on the size of the polygonal approximation $\tilde{P}$ of P.

Corollary 4 *If n is the size of a piecewise-convex polygon P, the size of its polygonal approximation $\tilde{P}$ is at most $3n$. This bound is tight (up to an additive constant).*

Proof. Let a_i be a convex arc of P, and let r_i be the corresponding room. If r_i is an empty room, then $\tilde{P}$ contains one Steiner vertex due to a_i. Hence $\tilde{P}$ contains at most n Steiner vertices attributed to empty rooms in P. If a_i is a non-empty room, then $\tilde{P}$ contains $|C_i^*|$ Steiner vertices due to a_i. By Lemma 3 the sets C_i^*, $i = 1, \ldots, n$ are pairwise disjoint, which implies that $\sum_{i=1}^{n} |C_i^*| \leq |P| = n$. Therefore $\tilde{P}$ contains the n vertices of P, contains at most n vertices due to empty rooms in P, and at most n vertices due to non-empty rooms in P. We thus conclude that the size of $\tilde{P}$ is at most $3n$.

The upper bound of the paragraph above is tight up to an additive constant. Consider the piecewise-convex polygon P of Fig. 5. It consists of $n - 1$ empty rooms and one non-empty room r_1, such that $|C_1^*| = n - 2$. It is easy to see that $|\tilde{P}| = 3n - 3$. □

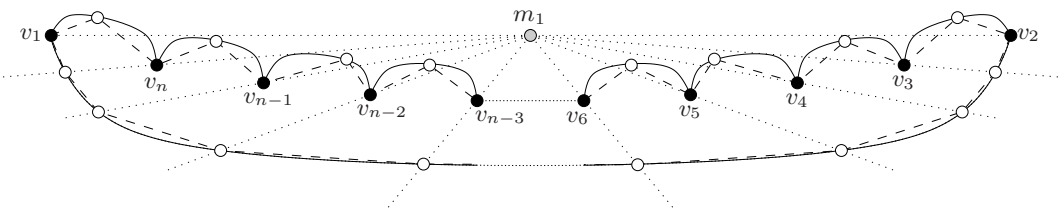

Figure 5: A piecewise-convex polygon P of size n (solid curve), the polygonal approximation $\tilde{P}$ of which consists of $3n-3$ vertices (dashed polyline).

3.2 Triangulating the polygonal approximation

Let P be a piecewise-convex polygon, $\tilde{P}$ its polygonal approximation, and $S_{\tilde{P}}$ the set of Steiner vertices in $\tilde{P}$. We are going to construct a *constrained triangulation* of $\tilde{P}$, i.e., we are going to triangulate $\tilde{P}$, while enforcing some triangles to be part of this triangulation. More precisely, we are going to enforce the way the triangles of $\mathcal{T}(\tilde{P})$ are created in the neighborhoods of the vertices in $S_{\tilde{P}}$. By enforcing the triangles in these neighborhoods, we effectively triangulate parts of $\tilde{P}$. The remaining untriangulated parts of $\tilde{P}$ consist of one or more interior disjoint polygons, which can then be triangulated in linear time and space. The pre-specified triangles in $\mathcal{T}(\tilde{P})$ will be referred to as *constrained triangles*. The rationale behind the way this particular constrained triangulation is constructed is to ensure that:

1. every triangle of $\mathcal{T}(\tilde{P})$, that contains a vertex in $S_{\tilde{P}}$, also contains at least one vertex of P, i.e., no triangles contain only Steiner vertices,
2. every vertex in $S_{\tilde{P}}$ belongs to at least one triangle in $\mathcal{T}(\tilde{P})$ the other two vertices of which are both vertices of P, and
3. the triangles of $\mathcal{T}(\tilde{P})$ that contain vertices of $\tilde{P}$ can be monitored by vertices of P.

These properties are going to be exploited in Step 4 of the algorithm presented later in this subsection.

Let us proceed to define the constrained triangles in $\mathcal{T}(\tilde{P})$. If r_i is an empty room, and $w_{i,1}$ is the point added on a_i, add the edges v_iv_{i+1}, $v_iw_{i,1}$ and $w_{i,1}v_{i+1}$, thus forming the constrained triangle $v_iw_{i,1}v_{i+1}$ (see Fig. 4(b)). If r_i is a non-empty room, $c_1,\ldots,c_{K_i}$ the vertices in C_i^*, $K_i = |C_i^*|$, and $w_{i,1},\ldots,w_{i,K_i}$ the vertices added on a_i ($w_{i,j}$ has been added on a_i due to c_j), add the following edges, if they do not already exist:

1. c_k, c_{k+1}, for $k = 1,\ldots,K_i-1$, and v_ic_1, $c_{K_i}v_{i+1}$;
2. $c_kw_{i,k}$, for $k = 1,\ldots,K_i$;
3. $c_kw_{i,k+1}$, for $k = 1,\ldots,K_i-1$;

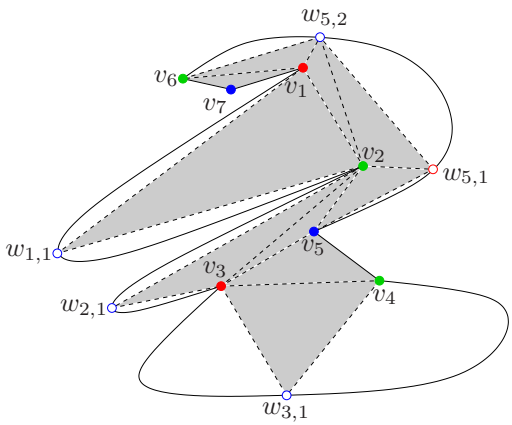

Figure 6: The three guard sets for $\tilde{P}$, are also guard sets for P, as Lemma 5 suggests.

4. $w_{i,k}, w_{i,k+1}$, for $k = 1, \ldots, K_i - 1$, and $v_i w_{i,1}$, $w_{i,K_i} v_{i+1}$.

These edges form $2K_i$ constrained triangles: $c_k c_{k+1} w_{i,k+1}$, for $k = 1, \ldots, K_i - 1$; $c_k w_{i,k} w_{i,k+1}$, for $k = 1, \ldots, K_i - 1$; $v_i c_1 w_{i,1}$ and $v_{i+1} c_{K_i} w_{i,K_i}$. We call the polygonal region delimited by these triangles a *crescent*. The triangles $v_i c_1 w_{i,1}$ and $v_{i+1} c_{K_i} w_{i,K_i}$ are called *boundary crescent triangles*, the triangles $c_k c_{k+1} w_{i,k+1}$, $k = 1, \ldots, K_i - 1$ are called *upper crescent triangles* and the triangles $c_k w_{i,k} w_{i,k+1}$, $k = 1, \ldots, K_i - 1$ are called *lower crescent triangles*.

Note that the points $w_{i,j}$, $j < K_i$ (resp. w_{i,K_i}) are vertices of exactly one triangle (resp. exactly two triangles) in $\mathcal{T}(\tilde{P})$, such that the other two vertices of the triangle (resp. of each of the two triangles) belong to P.

3.3 Computing a guard set for the original polygon

Assume that we have colored the vertices $\tilde{V}$ of $\tilde{P}$ with three colors, so that no triangle in $\mathcal{T}(\tilde{P})$ contains two vertices of the same color. This can be easily done by the standard three-coloring algorithm for linear polygons presented in [27, 18]. Let red, green and blue be the three colors, and let K_A, Π_A and M_A be the set of vertices of red, green and blue color, respectively, in a subset A of $\tilde{P}$. Clearly, all three sets $K_{\tilde{P}}$, $\Pi_{\tilde{P}}$ and $M_{\tilde{P}}$ are guard sets for $\tilde{P}$. In fact, they are also guard sets for P, as the following lemma suggests (see also Fig. 6).

Lemma 5 *Each one of the sets $K_{\tilde{P}}$, $\Pi_{\tilde{P}}$ and $M_{\tilde{P}}$ is a guard set for P.*

Proof. Let $G_{\tilde{P}}$ be one of $K_{\tilde{P}}$, $\Pi_{\tilde{P}}$ and $M_{\tilde{P}}$. By construction, $G_{\tilde{P}}$ monitors all triangles in $\mathcal{T}(\tilde{P})$. To show that $G_{\tilde{P}}$ is a guard set for P, it suffices to show that $G_{\tilde{P}}$ also monitors the non-degenerate sectors defined by the edges of $\tilde{P}$ and the corresponding convex subarcs of P.

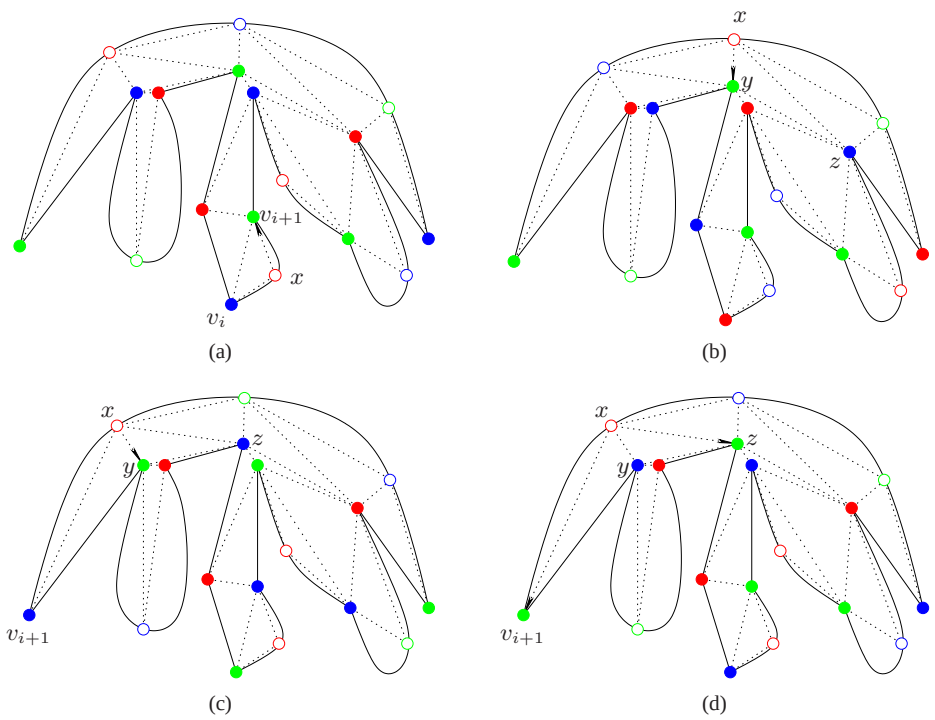

Figure 7: The three cases in the definition of the mapping f. Case (a): x is a Steiner vertex in an empty room. Case (b): x is an Steiner vertex in a non-empty room and is not the last Steiner vertex added on the curvilinear arc. Cases (c) and (d): x is the last Steiner vertex added on the curvilinear arc of a non-empty room (in (c) $|f(x)| = 1$, whereas in (d) $|f(x)| = 2$).

Indeed, let s be a non-degenerate sector associated with the convex arc a_i, and let $T \in \mathcal{T}(\tilde{P})$ be the triangle incident to the chord of s. If r_i is an empty room, each of the three vertices of T monitors r_i (and therefore also s). If r_i is a non-empty room, the vertex of T that is not an endpoint of the chord of s is a vertex in C_i^* and monitors s by construction. Clearly, one of the three vertices of T belongs to $G_{\tilde{P}}$. $\square$

Let as now assume, without loss of generality, that $|K_P| \leq |\Pi_P| \leq |M_P|$. Define the mapping f from $K_{S_{\tilde{P}}}$ to the power set 2^{Π_P} of Π_P by mapping a vertex x in $K_{S_{\tilde{P}}}$ to all the neighboring vertices of x in $\mathcal{T}(\tilde{P})$ that belong to Π_P (see Fig. 7 for the three possible cases for x). Notice that $1 \leq |f(x)| \leq 2$.

Finally, define the set $G_P = K_P \cup f(K_{S_{\tilde{P}}})$, where $f(K_{S_{\tilde{P}}}) = \bigcup_{x \in K_{S_{\tilde{P}}}} f(x)$. We claim that G_P is a guard set for P.

Lemma 6 *The set $G_P = K_P \cup f(K_{S_{\tilde{P}}})$ is a guard set for P.*

Proof. The regions in P^α are sectors defined by a curvilinear arc, which is a subarc of an edge of P and the corresponding chord connecting the endpoints of this subarc. To show that G_P is a guard set for P, it suffices show that every triangle in $\mathcal{T}(\tilde{P})$ and every sector in P^α is monitored by at least one vertex in G_P.

If T is a triangle in $\mathcal{T}(\tilde{P})$ that is defined over vertices of P, one of its vertices is in $K_P \subseteq G_P$. If T is a triangle inside an empty room, or a boundary crescent triangle, either the unique Steiner vertex of T is in $K_{\tilde{P}}$, in which case one of the other two vertices of T belongs to $f(K_{S_{\tilde{P}}})$, or the Steiner vertex of T is not in K_P, in which case one of the other two vertices of T belongs to K_P. Moreover, the sector/sectors adjacent to an edge of T in r_i is/are visible by both vertices of T in P and thus monitored by one of them. Finally, upper and lower crescent triangles come in pairs. Let T be an upper crescent triangle in a non-empty room r_i. Let x, y be the vertices of T in P, and let z be its vertex in $S_{\tilde{P}}$; it is assumed here that z is the intersection of $m_i y$ with a_i. Let T' be the lower crescent triangle adjacent to T along the edge xz, w be the third vertex of T', and s be the sector in P^α adjacent to zw. Since x and y belong to C_i^*, either x or y monitors T, T' and s. We end the proof by claiming that either x or y belongs to G_P: if x or y belongs to K_P the claim is obvious; if neither x nor y belongs to K_P, then $z \in K_P$ in which case one of x and y belongs to $f(K_{S_{\tilde{P}}})$. □

Since $f(K_{S_{\tilde{P}}}) \subseteq \Pi_P$ we get that $G_P \subseteq K_P \cup \Pi_P$. Since K_P and Π_P are the two sets of smallest cardinality among K_P, Π_P and M_P, we conclude that $|G_P| \leq |K_P| + |\Pi_P| \leq \lfloor \frac{2n}{3} \rfloor$, and thus arrive at the following theorem.

Theorem 7 *Let P be a piecewise-convex polygon with $n \geq 2$ vertices. P can be monitored with at most $\lfloor \frac{2n}{3} \rfloor$ vertex guards.*

We close this subsection by making two remarks:

Remark 1 *The bound on the size of the vertex guard set in Theorem 7 is tight: our algorithm will produce a vertex guard set of size exactly $\lfloor \frac{2n}{3} \rfloor$ when applied to the piecewise-convex polygon of Fig. 5 or the crescent-like piecewise-convex polygon of Fig. 11.*

Remark 2 *When the input to our algorithm is a linear polygon all rooms are degenerate; consequently, no Steiner vertices are created, and the guard set computed corresponds to the set of colored vertices of smallest cardinality, hence producing a vertex guard set of size at most $\lfloor \frac{n}{3} \rfloor$. In that respect, our algorithm can be viewed as a generalization of Fisk's algorithm [18] to the class of piecewise-convex polygons.*

Remark 3 *Given a linear polygon P with $r \geq 2$ reflex vertices, we can view P as a piecewise-convex polygon the edges of which are c convex polylines, where $c \geq r$. In this context Theorem 7 can be "translated" as follows:*

> *Consider a linear polygon P with $r \geq 2$ reflex vertices. If P can be decomposed into $c \geq r$ convex polylines pointing towards the exterior of the polygon, then P can be monitored with at most $\lfloor \frac{2c}{3} \rfloor$ vertex guards.*

3.4 Time and space complexity

In this subsection we will show how to compute the vertex guard set G_P in $O(n \log n)$ time and $O(n)$ space. It is straightforward to show that Steps 2–4 of our algorithm (see beginning of Section 3) can be implemented in linear time and space. To complete our time and space complexity analysis, we need to show how to compute the polygonal approximation $\tilde{P}$ of P in $O(n \log n)$ time and linear space. In order to compute the polygonal approximation $\tilde{P}$ of P, it suffices to compute for each room r_i the set of vertices C_i^*. If $C_i^* = \emptyset$, then r_i is empty, otherwise we have the set of vertices we wanted. From C_i^* we can compute the points $w_{i,k}$ and the linear polygon $\tilde{P}$ in $O(n)$ time and space.

The underlying idea is to split P into y-monotone piecewise-convex subpolygons. For each room r_i within each such y-monotone subpolygon we will then compute the corresponding set C_i^*. This will be done by first computing a subset S_i of the set R_i of the points inside the room r_i, such that $S_i \supseteq C_i^*$, and then applying an optimal time and space convex hull algorithm to the set $S_i \cup \{v_i, v_{i+1}\}$ in order to compute C_i, and subsequently from that C_i^*. In the discussion that follows, we assume that for each convex arc a_i of P we associate a set S_i, which is initialized to be the empty set. The sets S_i will be progressively filled with vertices of P, so that in the end they fulfill the containment property mentioned above.

Splitting P into y-monotone piecewise-convex subpolygons is done in two steps:

1. First we split each convex arc a_i into y-monotone pieces. Let P' be the piecewise-convex polygon we get by introducing the y-extremal points for each a_i. Since each a_i can yield up to three y-monotone convex pieces, we conclude that $|P'| \leq 3n$. Obviously splitting the convex arcs a_i into y-monotone pieces takes $O(n)$ time and space. A vertex added to split a convex arc into y-monotone pieces will be called an *added extremal vertex*.

2. Second, we apply to P' the standard algorithm for computing y-monotone subpolygons of a linear polygon (cf. [23] or [12]). The algorithm in [23] (or [12]) is valid not only for line segments, but also for piecewise-convex polygons consisting of y-monotone arcs (such as P'). Since $|P'| \leq 3n$, we conclude that computing the y-monotone subpolygons of P' takes $O(n \log n)$ time and requires $O(n)$ space.

Note that a non-split arc of P belongs to exactly one y-monotone subpolygon. y-monotone pieces of a split arc of P may belong to at most three y-monotone subpolygons (see Fig. 8).

Suppose now that we have a y-monotone polygon Q. The edges of Q are either convex arcs of P, or pieces of convex arcs of P, or line segments between mutually visible vertices of P, added in order to form the y-monotone subpolygons of P; we call these line segments *bridges* (see Fig. 8). For each non-bridge edge e_i of Q, we want to compute the set C_i^*. This is done by sweeping Q in the negative y-direction (i.e., by moving the sweep line from $+\infty$ to $-\infty$). The events of the sweep correspond to the y coordinates of the vertices of Q, which are all known before-hand and can be put in a decreasing sorted list. There are four different types of events:

1. the first event: corresponds to the top-most vertex of Q,

2. the last event: corresponds to the bottom-most vertex of Q,

3. a left event: corresponds to a vertex of the left y-monotone chain of Q, and

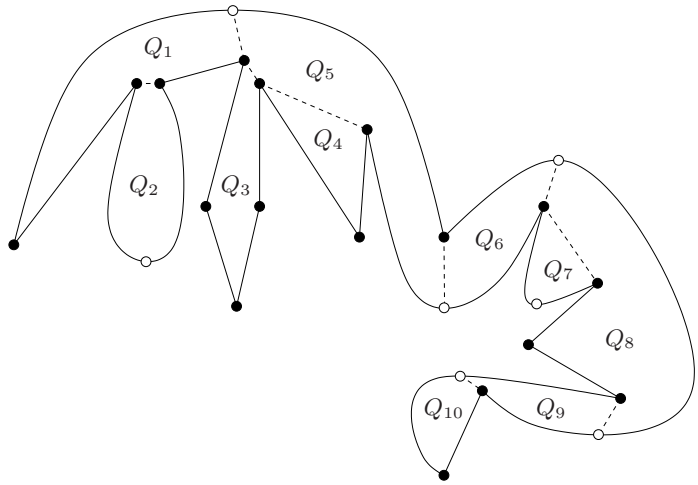

Figure 8: Decomposition of a piecewise-convex polygon into ten y-monotone subpolygons. The white points are added extremal vertices that have been added in order to split non-y-monotone arcs to y-monotone pieces. The bridges are shown as dashed segments.

4. a right event: corresponds to a vertex of the right y-monotone chain of Q.

Our sweep algorithm proceeds as follows. Let ℓ be the sweep line parallel to the x-axis at some y. For each y in between the y-maximal and y-minimal values of Q, ℓ intersects Q at two points which belong to either a left edge e_l or a left vertex v_l (i.e., an edge or vertex on the left y-monotone chain of Q), and either a right edge e_r or a right vertex v_r (i.e., a edge or vertex on the right y-monotone chain of Q). We associate the current left edge e_l at position y to a point set S_L and the current right edge at position y to a point set S_R. If the edge e_l (resp. e_r) is a non-bridge edge, the set S_L (resp. S_R) will contain vertices of Q that are inside the room of the convex arc of P corresponding e_l (resp. e_r).

When the y-maximal vertex v_{max} is encountered, i.e., during the first event, we initialize S_L and S_R to be the empty set. When a left event is encountered due a vertex v_l, let $e_{l,up}$ be the left edge above v_l and $e_{l,down}$ be the left edge below v_l and let e_r be the current right edge. If $e_{l,up}$ is an non-bridge edge, and a_i is the corresponding convex arc of P, we augment the set S_i by the vertices in S_L. Then, irrespectively of whether or not $e_{l,up}$ is a bridge edge, we re-initialize S_L to be the empty set. Finally, if e_r is a non-bridge edge, and a_k is the corresponding convex arc in P, we check if v_l is inside the room r_k or lies in the interior of the chord of r_k; if this is the case we add v_l to S_R. When a right event is encountered our sweep algorithm behaves symmetrically. When the last event is encountered due to the y-minimal vertex v_{min}, let e_l (resp. e_r) be the left (resp. right) edge of Q above v_{min}. If e_l (resp. e_r) is a non-bridge edge, let a_i (resp. a_j) be the corresponding convex arc in P. In this case we simply augment S_i (resp. S_j) by the vertices in S_L (resp. S_R).

We claim that our sweep-line algorithm computes a set S_i such that $S_i \supseteq C_i^*$. To prove this we need the following intermediate result:

Lemma 8 *Given a non-empty room r_i of P, with a_i the corresponding convex arc, the vertices of the set C_i^* belong to the y-monotone subpolygons of P' computed via the algorithm in [23] (or [12]), which either contain the entire arc a_i or y-monotone pieces of a_i.*

Proof. Let u be a vertex of P in C_i^* that is not a vertex of any of the y-monotone subpolygons of P' (computed by the algorithm in [23] or [12]) that contain either the entire arc a_i or y-monotone pieces of a_i. Let v_{max} (resp. v_{min}) be the vertex of P of maximum (resp. minimum) y-coordinate in C_i; ties are broken lexicographically. Let ℓ_u be the line parallel to the x-axis passing through u. Consider the following cases:

1. $u \in C_i^* \setminus \{v_{min}, v_{max}\}$. Without loss of generality we can assume that u is a vertex in the right y-monotone chain of C_i (see Figs. 9(a) and 9(b)). Let u' be the intersection of ℓ_u with a_i. Let Q (resp. Q') be the y-monotone subpolygon of P' that contains u (resp. u'); by our assumption $Q \neq Q'$. Finally, let u_+ (resp. u_-) be the vertex of C_i above (resp. below) u in the right y-monotone chain of C_i.

   The line segment uu' cannot intersect any edges of P, since this would contradict the fact that $u \in C_i^*$. Similarly, uu' cannot contain any vertices of P': if v is a vertex of P in the interior of uu', u would be inside the triangle vu_+u_-, which contradicts the fact that $u \in C_i^*$, whereas if v is a vertex of $P' \setminus P$ in the interior of uu', P would not be locally convex at v, a contradiction with the fact that P is a piecewise-convex polygon. As a result, and since $Q \neq Q'$, there exists a bridge edge e intersecting uu'. Let w_+, w_- be the two endpoints of e in P', where w_+ lies above the line ℓ_u and w_- lies below the line ℓ_u. In fact neither w_+ nor w_- can be a vertex in $P' \setminus P$, since the algorithm in [23] (or [12]) will connect a vertex in $P' \setminus P$ inside a room r_k with either the y-maximal or the y-minimal vertex of C_k only. Let ℓ_+ (resp. ℓ_-) be the line passing through the vertices u and u_+ (resp. u and u_-). Finally, let s be the sector delimited by the lines ℓ_+, ℓ_- and a_i. Now, if w_+ or w_- lies inside s, then u will be inside the triangle $w_+u_+u_-$ or inside the triangle $w_-u_+u_-$, respectively (see Fig. 9(a)). In either case we get a contradiction with the fact that $u \in C_i^*$. If neither w_+ nor w_- lie inside s, then both w_+ and w_- have to be vertices inside r_i, and moreover u will lie inside the convex quadrilateral $w_+u_+u_-w_-$; again this contradicts the fact that $u \in C_i^*$ (see Fig. 9(b)).

2. $u \equiv v_{max}$. By the maximality of the y-coordinate of u in C_i, we have that the y-coordinate of u is larger than or equal to the y-coordinates of both v_i and v_{i+1}. Therefore, the line ℓ_u intersects the arc a_i exactly twice, and, moreover, a_i has a y-maximal vertex of $P' \setminus P$ in its interior, which we denote by v'_{max} (see Fig. 9(c)). Let u' be the intersection of ℓ_u with a_i that lies to the right of u, and let Q (resp. Q') be the y-monotone subpolygon of P' that contains u (resp. u'). By assumption $Q \neq Q'$, which implies that there exists a bridge edge e intersecting the line segment uu'. Notice, that, as in the case $u \in C_i^* \setminus \{v_{min}, v_{max}\}$, the line segment uu' cannot intersect any edges of P, or cannot contain any vertex v of $P' \setminus P$; the former would contradict the fact that $u \in C_i^*$, whereas as the latter would contradict the fact that P is

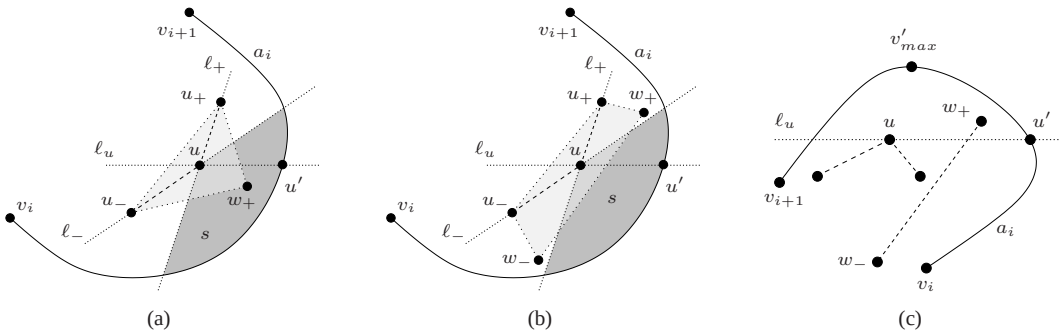

Figure 9: Proof of Lemma 8. (a) The case $u \in C_i^* \setminus \{v_{min}, v_{max}\}$, with $w_+ \in s$. (b) The case $u \in C_i^* \setminus \{v_{min}, v_{max}\}$, with $w_+, w_- \notin s$. (c) The case $u \equiv v_{max}$.

piecewise-convex. Furthermore, uu' cannot contain vertices of P since this would contradict the maximality of the y-coordinate of u in C_i.

Let w_+ and w_- be the endpoints of e above and below ℓ_u, respectively. Notice that e cannot have v'_{max} as endpoint, since the only bridge edge that has v'_{max} as endpoint is the bridge edge $v'_{max}u$. But then w_+ must be a vertex of P lying inside r_i; this contradicts the maximality of the y-coordinate of u among the vertices in C_i.

3. $u \equiv v_{min}$. This case is entirely symmetric to the case $u \equiv v_{max}$. □

An immediate corollary of the above lemma is the following:

Corollary 9 *For each convex arc a_i of P, the set S_i computed by the sweep algorithm described above is a superset of the set C_i^*.*

Let us now analyze the time and space complexity of Step 1 of the algorithm sketched at the beginning of this subsection. Computing the polygonal approximation $\tilde{P}$ of P requires subdividing P into y-monotone subpolygons. This subdivision takes $O(n \log n)$ time and $O(n)$ space. Then we need to compute the sets S_i for each convex arc a_i of P. The sets S_i can be implemented as red-black trees. During the course of our algorithm we only perform insertions on the S_i's. A vertex v of P is inserted at most $deg(v)$ times in some S_i, where $deg(v)$ is the degree of v in the y-monotone decomposition of P. Since the sum of the degrees of the vertices of P in the y-monotone decomposition of P is $O(n)$, we conclude that the total size of the S_i's is $O(n)$ and that we perform $O(n)$ insertions on the S_i's. Therefore we need $O(n \log n)$ time and $O(n)$ space to compute the S_i's and the C_i^*'s. The analysis above thus yields the following:

Theorem 10 *Let P be a piecewise-convex polygon with $n \geq 2$ vertices. We can compute a guard set for P of size at most $\lfloor \frac{2n}{3} \rfloor$ in $O(n \log n)$ time and $O(n)$ space.*

3.5 Lower bound constructions

In this subsection we are going to present a piecewise-convex polygon that requires a minimum of $\lfloor \frac{4n}{7} \rfloor - 1$ vertex (resp. $\lfloor \frac{n}{2} \rfloor$ point) guards in order to be monitored.

Let us first consider the windmill-like piecewise-convex polygon W with seven vertices of Fig. 10(a), a detail of which is shown in Fig. 10(b). The *double ear* defined by the vertices v_3, v_4 and v_5 and the convex arcs a_3 and a_4 is constructed in such a way so that neither v_3 nor v_5 can monitor both rooms r_3 and r_4 by itself. This is achieved by ensuring that a_3 (resp. a_4) intersects the line v_4v_5 (resp. v_3v_4) twice. Note that both a_3 and a_4 intersect the line mv_4 only at v_4, where m is the midpoint of the line segment v_3v_5. The double ear defined by the vertices v_5, v_6 and v_7 and the convex arcs a_5 and a_6 is constructed in an analogous way. Moreover, the vertices v_1, v_2, v_4 and v_6 are placed in such a way so that they do not (collectively) monitor the interior of the triangle $v_3v_5v_7$ (for example the lengths of the edges v_1v_7 and v_2v_3 are considered to be big enough, so that v_2 does not see too much of the triangle $v_3v_5v_7$). As a result of this construction, the windmill-like piecewise-convex polygon W cannot be monitored by two vertex guards, but can be monitored with three. There are actually only five possible guard triplets: $\{v_3, v_4, v_6\}$, $\{v_3, v_5, v_6\}$, $\{v_3, v_5, v_7\}$, $\{v_4, v_5, v_7\}$ and $\{v_4, v_6, v_7\}$. Any guard set that contains either v_1 or v_2 has cardinality at least four. The vertices v_1 and v_2 will be referred to as *base vertices*.

Consider now the crescent-like polygon C with n vertices of Fig. 11. The vertices of C are in strictly convex position. This fact has the following implication: if v_i, v_{i+1}, v_{i+2} and v_{i+3} are four consecutive vertices of C, and u is the point of intersection of the lines v_iv_{i+1} and $v_{i+2}v_{i+3}$, then the triangle $v_{i+1}uv_{i+2}$ is monitored if and only if either v_{i+1} or v_{i+2} is in the guard set of C. As

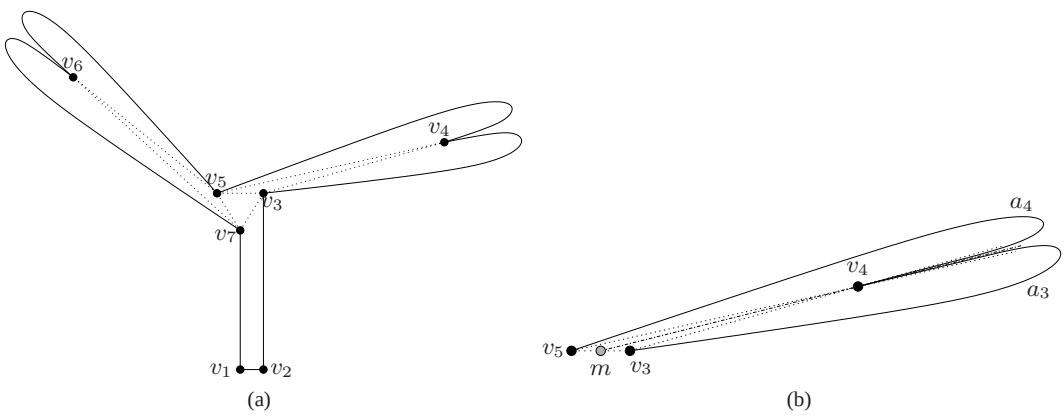

Figure 10: The windmill-like piecewise-convex polygon W that requires at least three vertex guards in order to be monitored. The only triplets of guards that monitor W are $\{v_3, v_4, v_6\}$, $\{v_3, v_5, v_6\}$, $\{v_3, v_5, v_7\}$, $\{v_4, v_5, v_7\}$ and $\{v_4, v_6, v_7\}$.

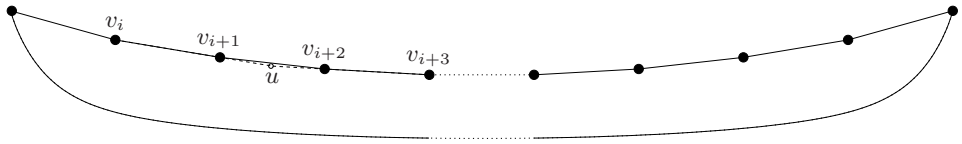

Figure 11: The crescent-like piecewise-convex polygon C, that requires a guard set of at least $\lfloor \frac{n}{2} \rfloor$ vertex guards.

a result, it is easy to see that C cannot be monitored with less than $\lfloor \frac{n}{2} \rfloor$ vertices, since in this case there will be at least one edge both endpoints of which would not be in the guard set for C.

In order to construct the piecewise-convex polygon that gives us the lower bound mentioned at the beginning of this subsection, we are going to merge several copies of W with C. More precisely, consider the piecewise-convex polygon P of Fig. 12 with $n = 7k$ vertices. It consists of copies of the polygon W merged with C at every other linear edge of C, through the base points of the W's.

In order to monitor any of the windmill-like subpolygons, we need at least three vertices per such polygon, none which can be a base point. This gives a total of $3k$ vertices. On the other hand, in order to monitor the crescent-like part of P we need at least $k - 1$ guards among the base points. To see that, notice that there are $k - 1$ linear segments connecting base points; if we were to use less than $k - 1$ guards, we would have at least one such line segment e, both endpoints of which would not participate in the guard set of G. But then, as in the case of C, there would be a triangle, adjacent to e, which would not be monitored. Therefore, in order to monitor P we need a minimum of $4k - 1 = \lfloor \frac{4n}{7} \rfloor - 1$ vertex guards.

The lower bound for point guards can be established much more easily. Consider the comb-like piecewise-convex polygon C shown in Fig. 13. It can be readily seen that we need one point guard per prong of C; since C contains $\lfloor \frac{n}{2} \rfloor$ prongs, we conclude that a minimum of $\lfloor \frac{n}{2} \rfloor$ point guards are required in order to monitor P. We, thus, arrive at the following theorem.

Theorem 11 *There exists family of piecewise-convex polygons with n vertices any vertex (resp. point) guard set of which has cardinality at least $\lfloor \frac{4n}{7} \rfloor - 1$ (resp. $\lfloor \frac{n}{2} \rfloor$).*

4 Piecewise-concave polygons

In this section we deal with the problem of monitoring piecewise-concave polygons using point guards. Monitoring a piecewise-concave polygon with vertex guards may be impossible even for very simple configurations (see Fig. 15(a)). In particular we prove the following:

Theorem 12 *Let P be a piecewise-concave polygon with n vertices. $2n - 4$ point guards are always sufficient and sometimes necessary in order to monitor P.*

Proof. To prove the sufficiency of $2n - 4$ point guards we essentially apply the technique in [17] for illuminating disjoint compact convex sets — please refer to Fig. 14. We denote by A_i the

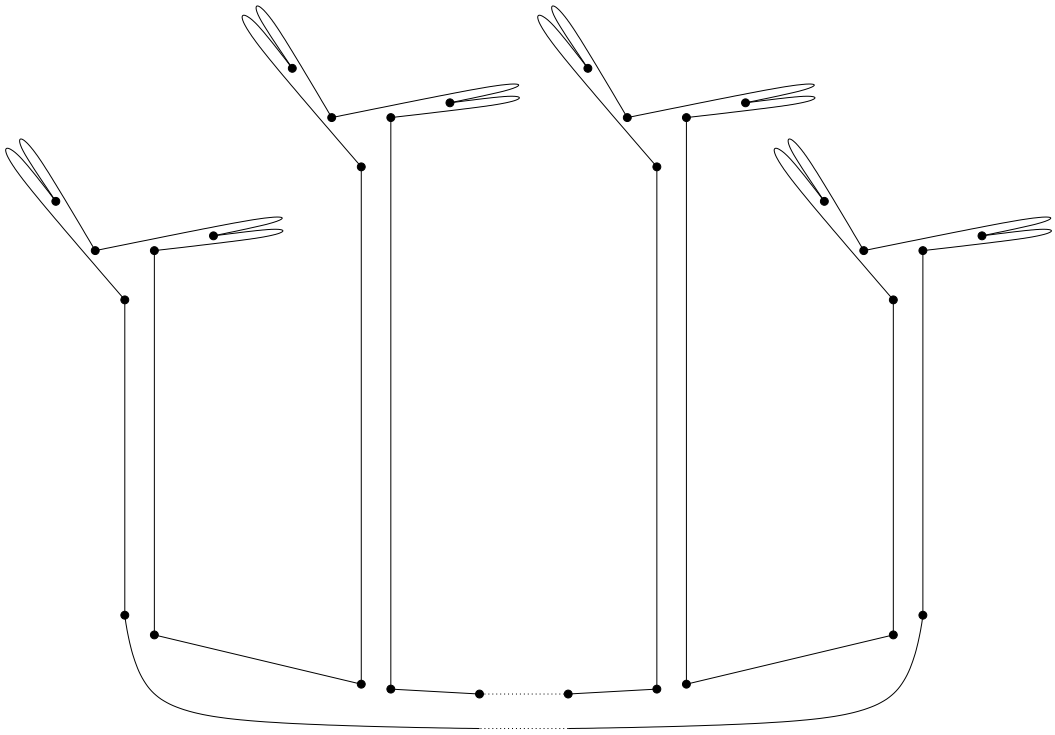

Figure 12: The lower bound construction.

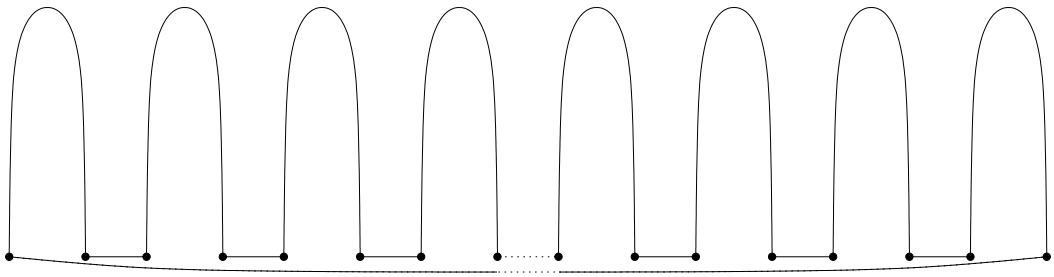

Figure 13: A comb-like piecewise-convex polygon that requires $\lfloor \frac{n}{2} \rfloor$ point guards in order to be monitored: one point guard is required per prong.

convex object delimited by a_i and the chord $v_i v_{i+1}$ of a_i. Let $t_i(v_j)$ be the tangent line to a_i at v_j,

$j = i, i+1$, and let b_{i+1} be the bisecting ray of $t_i(v_{i+1})$, $t_{i+1}(v_{i+1})$ pointing towards the interior of P.

Construct a set of locally convex arcs $\mathcal{K} = \{\kappa_1, \kappa_2, \ldots, \kappa_n\}$ that lie entirely inside P as such that (cf. [17]):

(a) the endpoints of κ_i are v_i, v_{i+1},

(b) κ_i is tangent to b_i (resp. b_{i+1}) at v_i (resp. v_{i+1}),

(c) if S_i is the locally convex object defined by κ_i and its chord $v_i v_{i+1}$, then $A_i \subseteq S_i$, $1 \leq i \leq n$,

(d) the arcs κ_i are pairwise non-crossing, and

(e) the number of tangencies between the elements of $\mathcal{K}$ is maximized.

Let Q be the piecewise-concave polygon defined by the sequence of the arcs in $\mathcal{K}$.

Suppose now that κ_i and $\kappa_{\sigma(j)}$ are tangent, $1 \leq j \leq m$, and let $\ell_{i,\sigma(j)}$ be the common tangent to κ_i and $\kappa_{\sigma(j)}$. Let $s_{i,\sigma(j)}$ be the line segment on $\ell_{i,\sigma(j)}$ between the points of intersection of $\ell_{i,\sigma(j)}$ with $\ell_{i,\sigma(j-1)}$ and $\ell_{i,\sigma(j+1)}$. Let Π_i be the polygonal region defined by the chord $v_i v_{i+1}$ and the line segments $s_{i,\sigma(j)}$. Π_i is a linear polygon with at most two reflex vertices (at v_i and/or v_{i+1}). It is easy to see that placing guards on the vertices of the Π_i's monitors both P and Q. Let G_Q be the guard set of P constructed this way. Construct, now, a planar graph Γ with vertex set $\mathcal{K}$. Two vertices κ_i and κ_j of Γ are connected via an edge if κ_i and κ_j are tangent. The graph Γ is a planar graph combinatorially equivalent to the triangulation graph $\mathcal{T}(R)$ of a polygon R with n vertices. The edges of Γ connecting the arcs κ_i, κ_{i+1}, $1 \leq i \leq n$, are the boundary edges of R, whereas all other edges of Γ correspond to diagonals in $\mathcal{T}(R)$. Let Q° denote the interior of Q. Observing that Q° consists of a number of faces that are in 1–1 correspondence with the triangles in $\mathcal{T}(R)$, we conclude that Q° consists of $n-2$ faces, each containing three guards of G_Q. It fact, each face of Q° can actually be monitored by only two of the three guards it contains and thus we can eliminate one of them per face of Q°. The new guard set G of Q constructed above is also a guard set for P and contains $2(n-2)$ point guards.

To prove the necessity, refer to the piecewise-concave polygon P in Fig. 15(b). Each one of the pseudo-triangular regions in the interior of P requires exactly two point guards in order to be monitored. Consider for example the pseudo-triangle τ shown in gray in Fig. 15(b). We need one point along each one of the lines l_1, l_2 and l_3 in order to monitor the regions near the corners of τ, which implies that we need at least two points in order to monitor τ (two out of the three points of intersection of the lines l_1, l_2 and l_3). The number of such pseudo-triangular regions is exactly $n-2$, thus we need a total of $2n-4$ point guards to monitor P. □

5 Discussion and open problems

In this paper we have considered the problem of monitoring a polygonal art gallery, the walls of which are allowed to be arcs of curves. In the case of piecewise-convex polygons with n vertices, it is always possible to monitor the polygon with $\lfloor \frac{2n}{3} \rfloor$ vertex guards, whereas $\lfloor \frac{4n}{7} \rfloor - 1$ vertex

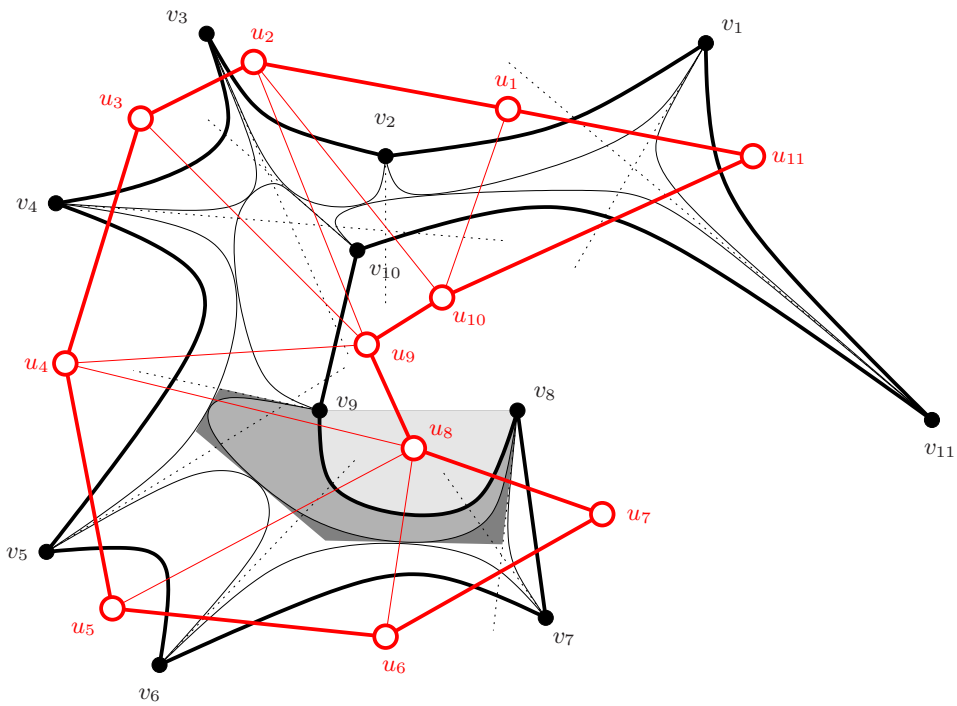

Figure 14: The proof for the upper bound of Theorem 12. The polygon P is shown with thick solid curvilinear arcs. The arcs κ_i are shown as thin solid arcs. The dotted rays are the bisecting rays b_i, whereas the dashed ray is the ray $r_8(v_9)$. The regions A_8, $S_8 \setminus A_8$ and $\Pi_8 \setminus S_8$ are also shown using three levels of gray; note that Π_8 has one reflex vertex at v_9. The graph Γ (i.e., the triangulation graph $\mathcal{T}(R)$) is shown in red: the node u_i corresponds to the arc a_i and the polygon R is depicted via thick segments.

guards are sometimes necessary. Furthermore, we have described an $O(n \log n)$ time and $O(n)$ space algorithm for computing a vertex guard set of size at most $\lfloor \frac{2n}{3} \rfloor$. For piecewise-concave polygons with n vertices we have shown that $2n-4$ point guards are always sufficient and sometimes necessary.

Up to now we have not found a piecewise-convex polygon that requires more than $\lfloor \frac{4n}{7} \rfloor + O(1)$ vertex (resp. $\lfloor \frac{n}{2} \rfloor$ point) guards, nor have we devised a polynomial time algorithm for monitoring a piecewise-convex polygon with less than $\lfloor \frac{2n}{3} \rfloor$ vertex guards. Closing the gap between the upper and lower bounds, for both vertex and point guards, remains an open problem. The worst-case upper and lower bound on the size of guard sets that we have shown for piecewise-concave polygons is of existential nature. Transforming the proof into an algorithm remains elusive. It is possible, however, to compute, in $O(n \log \log n)$ time and linear space, a linear-size point guard set by decomposing

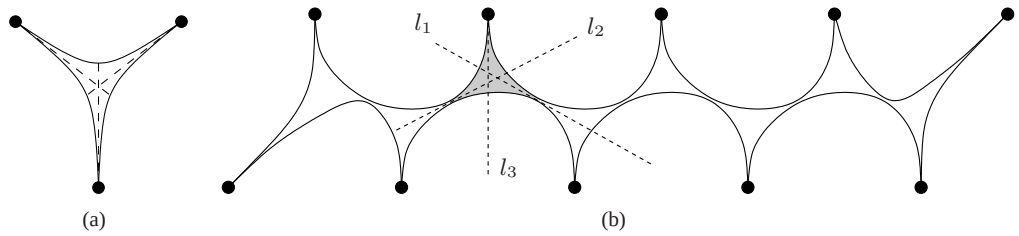

Figure 15: (a) A piecewise-concave polygon P that cannot be monitored solely by vertex guards. Two consecutive edges of P have a common tangent at the common vertex and as a result the three vertices of P see only the points along the dashed segments. (b) A piecewise-concave polygon P that requires $2n - 4$ point guards in order to be monitored.

the piecewise-concave polygon into a linear number of trapezoids (cf. [13]) and then monitor each trapezoid with a constant number of guards. This construction, however, can yield a guard set of size larger than $2n - 4$.

Beyond the two classes of polygons considered in this paper, it is straightforward to prove the following results (the details can be found in the relevant technical report [20]):

1. Given a *monotone* piecewise-convex polygon P with n vertices (i.e., a piecewise-convex polygon P for which there exists a line L such that any line $L^{\perp}$ perpendicular to L intersects the boundary of P at most twice), $\lfloor \frac{n}{2} \rfloor + 1$ vertex (resp. $\lfloor \frac{n}{2} \rfloor$ point) guards are always sufficient and sometimes necessary in order to monitor P.

2. Given a locally convex polygon P with n vertices, n point guards are always sufficient and sometimes necessary in order to monitor P. In particular, the n vertices of P are a guard set for P.

3. Given a *monotone* locally convex polygon (defined in direct analogy to monotone piecewise-convex polygons), $\lfloor \frac{n}{2} \rfloor + 1$ vertex or point guards are always sufficient and sometimes necessary.

4. Finally, there exist *general polygons* that cannot be monitored with a finite number of point guards.

Other types of guarding problems have been studied in the literature, which either differ on the type of guards (e.g., edge or mobile guards), the topology of the polygons considered (e.g., polygons with holes) or the guarding model (e.g., the fortress problem or the prison yard problem, mentioned in Section 1); see the book by O'Rourke [29], the survey paper by Shermer [31] of the book chapter by Urrutia [34] for an extensive list of the variations of the art gallery problem with respect to the types of guards or the guarding model. It would be interesting to extend these results to the families of curvilinear polygons presented in this paper.

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