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*An analytical framework for numerical
homogenization - Part II: the case of oversampling*

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*Rapport
de recherche*



An analytical framework for numerical homogenization - Part II: the case of oversampling

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Abstract: In a recent paper (Multiscale Model. Simul., 5 (2006), No. 3, pp. 996-1043), the author has introduced an analytical framework to study the convergence properties of some numerical homogenization methods for elliptic problems. In the applications however, these methods are coupled with oversampling techniques. In the present work, the author addresses this issue within the latter framework and proves the convergence of the methods with oversampling, for convex and quasiconvex energies, in the context of general heterogeneities. This analysis provides us with an interesting variational interpretation of the Petrov-Galerkin formulation of the nonconforming multiscale finite element method for periodic problems.

Key-words: homogenization, finite elements, oversampling

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Un cadre analytique pour l'homogénéisation numérique - Partie II : le cas du sur-échantillonnage

Résumé : Dans un article récent (Multiscale Model. Simul., 5 (2006), No. 3, pp. 996-1043), l'auteur a introduit un cadre analytique qui permet d'étudier les propriétés de convergence de certaines méthodes d'homogénéisation numérique des problèmes aux limites elliptiques. Dans les applications, ces méthodes sont cependant couplées à des techniques de sur-échantillonnage. Dans ce rapport, l'auteur analyse ces méthodes avec sur-échantillonnage, pour des énergies convexes et quasiconvexes, pour des hétérogénéités générales. Au cours de cette analyse, nous donnons une interprétation variationnelle intéressante de la formulation Petrov-Galerkin de la méthode des éléments finis multiéchelles.

Mots-clés : homogénéisation, éléments finis, sur-échantillonnage

In a recent paper (Multiscale Model. Simul., 5 (2006), No. 3, pp. 996-1043), the author has introduced an analytical framework to study the convergence properties of some numerical homogenization methods for elliptic problems. In the applications however, these methods are coupled with oversampling techniques. In the present work, the author addresses this issue within the latter framework and proves the convergence of the methods with oversampling, for convex and quasiconvex energies, in the context of general heterogeneities. This analysis provides us with an interesting variational interpretation of the Petrov-Galerkin formulation of the nonconforming multiscale finite element method for periodic problems.

1 Introduction

The goal of this paper is to continue the analysis of multiscale methods for the numerical homogenization of elliptic equations initiated in [15] and [14], where the convergence and the numerical analysis of some numerical homogenization methods are addressed under quite general hypotheses on the heterogeneities (general spatial dependence of the operator) and on the nature of the operator (convex or quasiconvex associated energy density). In practice however, these methods are usually combined with more sophisticated techniques such as oversampling - which is referred to in [1], [16], [17], [7], [8], [10], [12], and theoretically analyzed in [18], [9] and [6] in the linear and periodic or stochastic cases. In the latter, error estimates are also provided. In the present work, we focus on proving the convergence of some numerical homogenization methods with oversampling under the same general hypotheses as in [14], concerning both the approximation of the homogenized solution and the corrector whenever the notion is well-established. The paper is organized as follows. In the first Section, we very briefly recall the context, the main results of [14] and two numerical methods. Then we discuss the issue of oversampling in the periodic case, before addressing it in a more general setting in Section 4. We also give an energetic interpretation of the nonconforming Petrov-Galerkin formulation of the multiscale finite element method that allows us to prove its convergence in a general setting. Some arguments and proofs are only sketched whenever they are mainly based on the corresponding ones in [14]. We refer the reader to [14, Section 1] and the references therein for further details on notations and useful results that may not be extensively recalled in Section 2.

2 Numerical homogenization methods

Let us recall the analytical framework introduced in [14], and two numerical methods to which the analysis applies: the Heterogeneous Multiscale Method (HMM) and the Multiscale Finite Element Method (MsFEM).

2.1 Minimization problem

In what follows, we consider minimization problems, or the associated Euler-Lagrange equations whenever the two approaches are equivalent, e.g. for monotone operators. The problem

under investigation is

$$\inf \left\{ \int_{\Omega} W_{\epsilon}(x, \nabla u) dx, u \in W^{1,p}(\Omega, \mathbb{R}^d) + BC \right\}, \quad (1)$$

where W_{ϵ} is a family of energy densities, Ω is an open bounded domain of \mathbb{R}^n and BC denotes classical boundary conditions (let say Dirichlet or mixed Dirichlet-Neumann boundary conditions, see [14, pp 999]) in weak form.

Denoting by $I_{\epsilon} : W^{1,p}(\Omega) \rightarrow \mathbb{R}, u \mapsto I_{\epsilon}(u) = \int_{\Omega} W_{\epsilon}(x, \nabla u) dx$, we want to study the behaviour of I_{ϵ} when ϵ vanishes and to numerically approximate the minimizers of I_{ϵ} on sets of prescribed boundary conditions.

Under the following sets of hypotheses on W_{ϵ} , both issues can be answered positively:

The convex case

- H1: W_{ϵ} is a Carathéodory function;
- H2: for almost every $x \in \mathbb{R}^n$, $W_{\epsilon}(x, \cdot)$ is convex on $\mathbb{R}^{d \times n}$;
- H3: there exist $0 < c \leq C$ and $p \geq 1$ such that

$$c|\xi|^p \leq W_{\epsilon}(x, \xi) \leq C(1 + |\xi|^p)$$

for almost all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$.

The quasiconvex case: $n > 1$ and $d > 1$

- H1;
- H4: for almost every $x \in \mathbb{R}^n$, $W_{\epsilon}(x, \cdot)$ is quasiconvex on $\mathbb{R}^{d \times n}$;
- H3.

Assumption H4 generalizes Assumption H2. An energy density satisfying H1-H4-H3 will be referred to as a standard energy density. The direct method of the calculus of variations shows that the minimization problem (1) has at least one solution $u_{\epsilon} \in W^{1,p}(\Omega, \mathbb{R}^d)$.

In addition, there exist a standard energy density W_{hom} and the associated energy functional $I_{hom} : W^{1,p}(\Omega) \rightarrow \mathbb{R}, u \mapsto I_{hom}(u) = \int_{\Omega} W_{hom}(x, \nabla u) dx$, such that, up to extraction, $I_{hom} = \Gamma(L^p) - \lim_{\epsilon \rightarrow 0} I_{\epsilon}$. For every sequence of minimizers u_{ϵ} , there exists a minimizer u_{hom} of I_{hom} on the same set of prescribed boundary conditions such that $u_{\epsilon} \rightharpoonup u_{hom}$ in $W^{1,p}(\Omega)$. In what follows we will consider Γ -converging energies, without loss of generality up to extraction.

The aim of the following section is to recall the definition of an averaged energy density $I_{\eta, \epsilon}$ that approximates I_{hom} and u_{hom} in the sense of Γ -convergence.

2.2 Averaged energy densities

For convex standard energy densities W_ϵ , we set the following

Definition 1 For any $\eta > 0$, denoting by $B(x, \eta)$ the ball of radius η centered at point $x \in \mathbb{R}^n$, we define the energy density

$$W_{\eta, \epsilon}(x, \xi) = \inf \{ \langle W_\epsilon(\cdot, \nabla v(\cdot)) \rangle_{B(x, \eta)} \mid v \in W^{1,p}(B(x, \eta)), \langle \nabla v \rangle_{B(x, \eta)} = \xi \} \quad (2)$$

from $\mathbb{R}^n \times \mathbb{R}^{d \times n}$ to \mathbb{R} and the associated energy functional

$$I_{\eta, \epsilon}(u) = \int_{\Omega} W_{\eta, \epsilon}(x, \nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

whereas for quasiconvex standard energy densities, we set

Definition 2 For $\eta > 0$, let us denote by $C(x, \eta)$ the hypercube of \mathbb{R}^n centered in $x \in \mathbb{R}^n$ and of length η . We then define the averaged energy density by

$$\mathcal{W}_{\eta, \epsilon}(x, \xi) = \inf \left\{ \langle W_\epsilon(\cdot, \xi + \nabla v(\cdot)) \rangle_{C(x, \eta)} \mid v \in W_{\#}^{1,p}(C(x, \eta), \mathbb{R}^d) \right\} \quad (3)$$

from $\mathbb{R}^n \times \mathbb{R}^{d \times n}$ to \mathbb{R} and the energy functional associated with its quasiconvex envelope $Q\mathcal{W}_{\eta, \epsilon}$

$$I_{\eta, \epsilon}(u) = \int_{\Omega} Q\mathcal{W}_{\eta, \epsilon}(x, \nabla u) \quad \text{for all } u \in W^{1,p}(\Omega, \mathbb{R}^d).$$

We then have the following convergence theorem for strictly convex energy densities

Theorem 1 Let W_ϵ satisfy H1, H2 (strictly), and H3 uniformly for $p > 1$, then the energy densities $W_{\eta, \epsilon}$ also satisfy H1, H2 (strictly), and H3, and the energy $I_{\eta, \epsilon} \Gamma(L^p)$ - and $\Gamma(W^{1,p})$ -converges to I_{hom} as ϵ and η go to 0. Therefore, the unique sequence $u_{\eta, \epsilon}$ of minimizers of $\inf\{I_{\eta, \epsilon}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}) + BC\}$ converges to the unique minimizer u_{hom} of $\inf\{I_{hom}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}) + BC\}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$.

and the corresponding one for quasiconvex energy densities

Theorem 2 Let W_ϵ satisfy H1, H4, and H3 uniformly for $p > 1$, then the energy densities $Q\mathcal{W}_{\eta, \epsilon}$ are standard energy densities and $I_{\eta, \epsilon} \Gamma(L^p)$ - and $\Gamma(W^{1,p})$ -converges to I_{hom} as ϵ and η go to 0. Therefore, for any sequence $u_{\eta, \epsilon}$ of minimizers of $\inf\{I_{\eta, \epsilon}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}^d) + BC\}$, there exists a minimizer u_{hom} of $\inf\{I_{hom}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}^d) + BC\}$ such that

$$\lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} u_{\eta, \epsilon} = u_{hom} \quad \text{weakly in } W^{1,p}(\Omega, \mathbb{R}^d), \quad (4)$$

up to extraction.

Remark 1 It may be stressed that the trial space $W_{\#}^{1,p}(C(x, \eta), \mathbb{R}^d)$ can be replaced by $W_0^{1,p}(C(x, \eta), \mathbb{R}^d)$ without changing the convergence results of Theorem 1 and 2.

2.3 Numerical corrector and fine scale features

The approximation of the homogenized solution u_{hom} in $W^{1,p}(\Omega)$ is not enough to approximate u_ϵ in $W^{1,p}(\Omega)$ since u_ϵ only converges weakly to u_{hom} in $W^{1,p}(\Omega)$. To this aim, numerical correctors have been widely introduced and used to approximate ∇u_ϵ in $L^p(\Omega)$, which describes the fine scale features of the solution. Their convergence properties have been analyzed for general heterogeneities and monotone operators in [14].

Definition 3 Let $\{Q_{H,i}\}_{i \in \llbracket 1, I_H \rrbracket}$ be a partition of Ω in disjoint subdomains of diameter of order H . We define a family (M_H) of approximations of identity on $L^p(\Omega)$ associated with $Q_{H,i}$: for every $w \in L^p(\Omega)$ and $H > 0$,

$$M_H(w) = \sum_{i=1}^{I_H} \langle w \rangle_{Q_{H,i}} 1_{Q_{H,i}}.$$

Keeping the notation of Theorem 1, we define the numerical correctors $v_{\eta,\epsilon}^{H,i}$ for a strictly convex energy density as the unique minimizers (up to a constant) of

$$\inf \left\{ \int_{Q_{H,i}} W_\epsilon(x, \nabla v) \mid v \in W^{1,p}(Q_{H,i}), \langle \nabla v \rangle_{Q_{H,i}} = \langle \nabla u_{\eta,\epsilon} \rangle_{Q_{H,i}} \right\}. \quad (5)$$

We have the convergence result ([14, Theorem 2]):

Theorem 3 In addition to H1, H2, and H3, let us assume that $p \geq 2$, that $W_\epsilon(x, \cdot)$ is continuously differentiable for almost all $x \in \Omega$ and $a_\epsilon(\cdot, 0) = \frac{\partial W_\epsilon}{\partial \xi}(\cdot, 0)$ is bounded, and that the following monotonicity and continuity properties hold:

$$\begin{aligned} \exists 0 \leq \alpha \leq p-1, C > 0 \quad & \mid \quad \text{for almost all } x \in \mathbb{R}^n, \text{ for all } \xi_1, \xi_2 \in \mathbb{R}^n, \\ |a_\epsilon(x, \xi_1) - a_\epsilon(x, \xi_2)| & \leq C(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha, \end{aligned} \quad (6)$$

$$\begin{aligned} \exists 2 \leq \beta < +\infty, c > 0 \quad & \mid \quad \text{for almost all } x \in \mathbb{R}^n, \text{ for all } \xi_1, \xi_2 \in \mathbb{R}^n, \\ (a_\epsilon(x, \xi_1) - a_\epsilon(x, \xi_2), \xi_1 - \xi_2) & \geq c(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta. \end{aligned} \quad (7)$$

Then, denoting by u_ϵ the unique minimizer of I_ϵ on $W^{1,p}(\Omega) + BC$, we have

$$\lim_{\eta, H \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left\| \nabla u_\epsilon - \sum_{i=1}^{I_H} \nabla v_{\eta,\epsilon}^{H,i} 1_{Q_{H,i}} \right\|_{L^p(\Omega)} = 0. \quad (8)$$

Let us briefly discuss the link between the original Tartar's correctors and the numerical correctors of Definition 3. We first recall the definition of Tartar's correctors [21] in the linear case and make some comments about the numerical interest of such a result.

Let $A_{hom} \in L^\infty(\Omega, \mathcal{M}_n(\mathbb{R}))$ be the homogenized matrix of an H -converging sequence A_ϵ , and u_{hom} be the solution of the homogenized problem $\inf \{ \int_\Omega A_{hom} \nabla u \nabla u - fu, u \in$

$H^1(\Omega) + BC\}$. The convergence of u_ϵ to u_{hom} is only weak in $H^1(\Omega)$. The corrector matrices are designed to approximate the gradient of u_ϵ by a function depending on u_{hom} . Given compactly supported subsets $\omega \subset\subset \omega_1 \subset\subset \Omega$, a corrector matrix $P_\epsilon \in H^1(\omega)^n$ is defined by $(P_\epsilon)_j$, the restriction on ω of the unique solution of

$$\inf \left\{ \int_{\omega_1} A_\epsilon \nabla v_\epsilon \nabla v_\epsilon - A_{hom} M_j \nabla v_\epsilon, v_\epsilon(y) = M_j \cdot y + w_\epsilon, w_\epsilon \in H_0^1(\omega_1) \right\}, \quad (9)$$

for M_j describing the canonical basis of \mathbb{R}^n . A corrector for u_ϵ is then given on ω by $C_\epsilon = \sum_j (P_\epsilon)_j (\nabla u_{hom})_j$ and we have $\lim_{\epsilon \rightarrow 0} \|\nabla C_\epsilon - \nabla u_\epsilon\|_{L^2(\omega)} = 0$ [21, Theorem 3 pp. 39]. In particular, correctors are not unique.

From a computational point of view, Tartar's correctors are too abstract since the precise knowledge of A_{hom} is required to calculate the correctors, whereas A_{hom} is in principle unknown. However, if A_{hom} is constant, then (9) turns out to be solvable in practice (the term depending on A_{hom} is constant in the energy). As pointed out by Allaire and Brizzi in [1], the simplest approximation of A_{hom} is the piecewise constant approximation.

The numerical corrector amounts to taking $\omega = \omega_1 = Q_{H,i}$ and approximating A_{hom} by $\langle A_{\eta,\epsilon} \rangle_{Q_{H,i}}$. It should be noticed that ω is not compactly supported in ω_1 . In addition to the convergence in ϵ (and η), there is an error linked to H and the piecewise constant approximation of A_{hom} . Up to an error which depends on H , the numerical corrector may be interpreted as an approximation of some Tartar's corrector in $H^1(Q_{H,i})$. Theorem 3 shows that this also provides an approximation of the gradient of the corrector in $L^2(\Omega)$. Imposing $\omega \subset\subset \omega_1$ in the definition of a numerical corrector is a way to introduce oversampling, as it will be seen in Section 4.

2.4 HMM

The application of the Heterogeneous Multiscale Method to elliptic problems introduced in [7] can be interpreted as the minimization of a discretization of $I_{\eta,\epsilon}$ plus a lower order term $f \in L^{p'}(\Omega)$ on a finite element basis, which reads

$$\inf \left\{ \sum_{i=1}^{I_H} \sum_{j=1}^{N_{GP}} q_j (W_{\eta,\epsilon}(x_{ij}, \nabla u_H(x_{ij})) - f(x_{ij})u_H(x_{ij})), u_H \in V_H + BC \right\}, \quad (10)$$

where I_H is the number of mesh elements, N_{GP} is the number of Gauss points per element, x_{ij} are the Gauss points, q_j are the weights, and V_H is an FE space. Then the computation of the FE minimizer of (10) requires only evaluations of derivatives of $W_{\eta,\epsilon}(x_{ij}, \xi)$ for particular ξ at Gauss points x_{ij} . We refer the reader to [7, 14] for details on the method and its analysis.

2.5 MsFEM

The Multiscale Finite Element Method may also be interpreted as the minimization of a (different) discretization of $I_{\eta,\epsilon}$ on a $P1$ -finite element space V_H associated to a triangulation

$\{Q_{H,i}\}_i$ of Ω , namely:

$$\inf \left\{ \sum_{i=1}^{I_H} |Q_{H,i}| W_{H,\epsilon}^{MsFEM}(x_i, \nabla u_H(x_i)) - \sum_{i=1}^{I_H} \sum_{j=1}^{N_{GP}} q_j f(x_{ij}) u_H(x_{ij}), u_H \in V_H + BC \right\}, \quad (11)$$

where the second term of the energy has been integrated by a quadrature rule associated to the triangulation, and

$$W_{H,\epsilon}^{MsFEM}(x_i, \xi) = \inf \left\{ \frac{1}{|Q_{H,i}|} \int_{Q_{H,i}} W_\epsilon(y, \nabla v) \mid v(y) = \xi \cdot y \text{ on } \partial Q_{H,i} \right\}, \quad (12)$$

which is a particular energy density of type $W_{\eta,\epsilon}$.

In the analysis of the MsFEM in [14, Section 3.2], we have extensively used the following rewriting of the problem:

$$\inf \left\{ \int_{\Omega} W_{H,\epsilon}^{MsFEM}(x, \nabla u_H) - f u_H \mid u_H \in V_H + BC \right\}, \quad (13)$$

extending the definition of $W_{H,\epsilon}^{MsFEM}$ on Ω by

$$W_{H,\epsilon}^{MsFEM}(x, \xi) = \sum_i W_{H,\epsilon}^{MsFEM}(x_i, \xi) 1_{Q_{H,i}}(x),$$

since ∇u_H is constant on each $Q_{H,i}$. With this formulation, estimates on the energy are easy to obtain due to the inclusion $V_H \subset W^{1,p}(\Omega)$.

3 Oversampling in the periodic case

3.1 Setting of the problem

Since the homogenized equation is explicitly known when $W_\epsilon(\cdot, \cdot) = W(\frac{\cdot}{\epsilon}, \cdot)$ and W is 1-periodic in space, the application of the MsFEM or the HMM strategies to this case allows us to perform a numerical analysis of the methods.

Doing so ([18] for MsFEM, [6] for HMM e.g.), the error between the numerical approximation and the solution of the homogenized problem is proved to exhibit some term called the cell resonance and boundary layer error. This error is linked to two phenomena:

- when $C(x, \eta)$ is not a multiple of the periodic cell on the one hand (cell resonance), and
- when linear Dirichlet boundary conditions are used on the other hand (the cell problem in periodic homogenization is completed by periodic boundary conditions).

These phenomena are the sources of a boundary layer, whose effect is of order $\frac{\epsilon}{\eta}$ in the linear case (see [14] for the monotone case) for the $H^1(\Omega)$ norm of the homogenized solution (and for the homogenized energy) and of order $\sqrt{\frac{\epsilon}{\eta}}$ for the $L^2(\Omega)$ norm of the gradient of the corrector.

The aim of oversampling is twofold: improve the convergence rate, and if not, at least, improve the prefactor, by reducing both sources of error.

In the linear periodic case, oversampling restores a convergence of order $\frac{\epsilon}{\eta}$ for the $L^2(\Omega)$ norm of the gradient of the corrector and reduces the prefactor of the error of order $\frac{\epsilon}{\eta}$ for the homogenized energy ([6] and [16] e.g.). The strategy consists in introducing bigger volume elements $C(x, \eta + \zeta)$, where $\zeta = \zeta(\epsilon, \eta) > 0$, and using the information only on $C(x, \eta)$ to avoid the boundary layer of presumed order ζ . The application of oversampling is different for the MsFEM and the HMM. In particular, several choices for the MsFEM have been analyzed in great details in [18] in the linear periodic case. The mathematical formulation of oversampling is introduced in the following section.

3.2 Mathematical formulation

Let $\zeta = \zeta(\epsilon, \eta) \in \mathbb{R}_+$. We define an oversampled energy density

$$W_{\eta, \epsilon, \zeta}^{over}(x, \xi) = \langle W_\epsilon(y, \xi + \nabla v_{\eta, \epsilon, \zeta}^{over}(y)) \rangle_{C(x, \eta)}, \quad (14)$$

where $v_{\eta, \epsilon, \zeta}^{over}$ is the restriction on $C(x, \eta)$ of the solution $\tilde{v}_{\eta, \epsilon, \zeta}^{over}$ of the following minimization problem posed on $C(x, \eta + \zeta)$

$$\inf \left\{ \langle W_\epsilon(\cdot, \xi + \nabla v(\cdot)) \rangle_{C(x, \eta + \zeta)} \mid v \in W_0^{1,p}(C(x, \eta + \zeta)) \right\}. \quad (15)$$

We can naturally extend this definition to balls $B(x, \eta)$ and other boundary conditions ($\langle \nabla v \rangle = \xi$, periodic boundary conditions). Since the arguments and results are the same, we will focus on Dirichlet boundary conditions in what follows.

We will not make a full error analysis of the cancellations that occur due to oversampling, and we refer the reader to [18] for a deep analysis in the linear case. We will however make two remarks that may explain why numerical errors are reduced by the use of oversampling methods. The aim of the present work is to prove the convergence of numerical homogenization methods with oversampling for general energy densities and general heterogeneities. In this setting, we are not able to exhibit error estimates, but we will prove convergence results and relate oversampling methods to Tartar's correctors in homogenization.

3.3 A remark on boundary conditions

The mismatch between boundary conditions may be understood as follows. Let us consider a domain $C(x, \eta)$ with $\eta/\epsilon \in \mathbb{N}$. The domain $C(x, \eta)$ is then exactly obtained by the concatenation of a given number of periodic cells. If periodic boundary conditions were used

Table 1: Error on the approximated homogenized coefficient (performed with [13, FreeFEM]).

| Number of periodic cells per dimension | without oversampling | | | with oversampling | | |
|--|----------------------|---------------------|--------------------|-------------------|---------------------|--------------------|
| | error | rate of convergence | prefactor (rate=1) | error | rate of convergence | prefactor (rate=1) |
| 1 | 0.15707 | - | 0.15707 | 0.15707 | - | 0.15707 |
| 2 | 0.08449 | 0.8945552 | 0.16898 | 0.02100 | 2.9029464 | 0.04200 |
| 4 | 0.04335 | 0.9628402 | 0.17340 | 0.01177 | 0.8353076 | 0.04708 |
| 8 | 0.02193 | 0.9830069 | 0.17544 | 0.00597 | 0.9793258 | 0.04776 |
| 12 | 0.01456 | 1.0101425 | 0.17472 | 0.00397 | 1.0062045 | 0.04764 |
| 16 | 0.01103 | 0.9651599 | 0.17648 | 0.00299 | 0.9854375 | 0.04784 |
| 20 | 0.00876 | 1.0326219 | 0.17520 | 0.00239 | 1.0037486 | 0.04780 |

on $C(x, \eta)$, $W_{\eta, \epsilon}$ would exactly be W_{hom} . If Dirichlet boundary conditions are used, far from the boundary, the solution is almost periodic. Close to the boundary the solution is very different from the periodic solution as illustrated in [1, Fig. 3]. In order to reduce the error, it is then natural to use the solution on periodic cells contained in $C(x, \eta)$ that are far from the boundary $\partial C(x, \eta)$. We refer the reader to the example of the half space dealt with in [2]. This remark is of great interest for the reconstruction of the fine scales features which highly suffers from this boundary layer.

The technique of oversampling is unlikely to change the order of convergence of the homogenized energy, as it can be easily seen in the periodic one-dimensional case (oversampling simply does not change anything). However, not taking into account the boundary layer may improve the prefactor of the error. For the two-dimensional numerical examples considered in [18] and [9], namely a periodic linear operator defined by

$$a_\epsilon = \left(\frac{2 + 1.8 \sin(2\pi x/\epsilon)}{2 + 1.8 \cos(2\pi y/\epsilon)} + \frac{2 + \sin(2\pi y/\epsilon)}{2 + 1.8 \cos(2\pi x/\epsilon)} \right) I, \quad (16)$$

the associated energy density is $W_\epsilon(x, \xi) = \frac{1}{2} a_\epsilon(x) |\xi|^2$. To be more precise Tables 1 and 2 report on some simple numerical tests that show the significant effect of oversampling in the illustrative case (16). In Table 1 the error between the homogenized operator (one of the two nonzero coefficients) and the approximated homogenized operator is reported on, using Dirichlet boundary conditions on an increasing number of periodic cells. The first approximated operator is obtained without oversampling, whereas the second one is obtained by considering only the periodic cell which is at the center of the domain of computation. The convergence rate is clearly proportional to the inverse of the number of periodic cells per dimension in both cases, however the prefactor is four times smaller in the second case.

The interest of oversampling for the reconstruction of the fine scale features of the solution u_ϵ , namely the numerical corrector, is the improvement of the rate of convergence in the H^1 norm from $\sqrt{\frac{\epsilon}{\eta}}$ to $\frac{\epsilon}{\eta}$, as illustrated in Table 2. This issue is dealt with theoretically in [18] and [6]. It relies on the two-scale expansion of the solution and will not be detailed here

Table 2: L^2 -norm of the error on the gradient of the corrector (performed with [13, FreeFEM]).

| Number of periodic cells per dimension | without oversampling | | | with oversampling | | |
|--|----------------------|---------------------|----------------------|-------------------|---------------------|--------------------|
| | error | rate of convergence | prefactor (rate=0.5) | error | rate of convergence | prefactor (rate=1) |
| 1 | 0.2100420 | - | 0.2100420 | 0.2100420 | - | 0.2100420 |
| 2 | 0.1564490 | 0.4249854 | 0.2212523 | 0.01158010 | 0.8934616 | 0.0231602 |
| 4 | 0.1130700 | 0.4684762 | 0.2261400 | 0.00360518 | 1.6835045 | 0.0144207 |
| 8 | 0.0808329 | 0.4842017 | 0.2286300 | 0.00181738 | 0.9882112 | 0.0145390 |
| 12 | 0.0662452 | 0.4908463 | 0.2294801 | 0.00121064 | 1.0019278 | 0.0145277 |
| 16 | 0.0574405 | 0.4957851 | 0.2297620 | 0.00091015 | 0.9917024 | 0.0145624 |
| 20 | 0.0514685 | 0.4919688 | 0.2301741 | 0.00072566 | 1.0151664 | 0.0145132 |

since it cannot be generalized to other heterogeneities. In [14, Section 3], the interest of oversampling for nonlinear operators is addressed in terms of error contributions: the error made on the homogenized energy could be greater than the error due to the boundary layers of the corrector. Therefore the effect of oversampling on the correctors may not change the order of the error for the fine scales. In this case however, the prefactor of the error can still be reduced by oversampling.

Remark 2 *The same type of boundary layers occurs if $C(x, \eta)$ is not exactly a multiple of the periodic cell, both with periodic and Dirichlet boundary conditions. Numerical tests show the same behaviour as in Tables 1 and 2.*

3.4 A remark on the volume element $C(x, \eta)$

Assuming that the mismatch due to boundary conditions is reduced, one still has to deal with another source of error: the mismatch between $C(x, \eta)$ and the periodic cell. The domain $C(x, \eta)$ may not be exactly a multiple of the periodic cell. Therefore the mean of the energy on a periodic cell with a given periodic function does not coincide with the mean on $C(x, \eta)$ of the energy with the same periodic function. This error is more subtle than the previous one and of the same order, namely $\frac{\epsilon}{\eta}$ in the linear periodic case. This source of error does not exist for the Petrov-Galerkin formulation of the MsFEM [18], as will be discussed in the following section. This solution is intimately linked to the particular formulation (or discretization of $I_{\eta, \epsilon}$ in other terms) of the MsFEM. It does not apply to the HMM for example. In the latter case, something else has to be done and may be a numerical issue of interest, at least in the periodic case.

3.5 Interpretation of the MsFEM in Petrov-Galerkin formulation

The reason why one can get rid of the error discussed in the previous section for the MsFEM is a characteristic spatial feature implied by $\eta = H$ in the formulation: for all $u_H \in V_H$,

there exists a function $u_H^{MsFEM} \in L^p(\Omega)$ such that $u_H^{MsFEM}|_{Q_{H,i}} \in W^{1,p}(Q_{H,i})$ for all i , and

$$\int_{\Omega} W_{H,\epsilon}^{MsFEM}(x, \nabla u_H) = \int_{\Omega} W_{\epsilon}(x, \nabla u_H^{MsFEM}), \quad (17)$$

where ∇u_H^{MsFEM} abusively denotes $\sum_i \nabla u_H^{MsFEM}|_{Q_{H,i}}$.

In other terms, if W_{ϵ} is the periodic composition of two materials A and B , the ratio of A and B in W_{ϵ} is exactly preserved in $W_{H,\epsilon}^{MsFEM}$ whereas it is only preserved up to an error of order $\frac{\epsilon}{\eta}$ for a generic averaged energy density $W_{\eta,\epsilon}$. In the light of the discussion of Section 3.4, each periodic cell of the material Ω is exactly accounted for once in the MsFEM: if only half of a given periodic cell belongs to some $Q_{H,i}$ then the other half belongs to some other $Q_{H,j}$. For a generic $W_{\eta,\epsilon}$, this may not be true.

The correspondence (17) between u_H and u_H^{MsFEM} defines a (nonlinear) mapping from V_H to $\oplus_i W^{1,p}(Q_{H,i})$, as introduced in [10]. Without oversampling, the mapping takes values in $W^{1,p}(\Omega)$, whereas with oversampling the restrictions of u_H^{MsFEM} are in $W^{1,p}(Q_{H,i})$ but $u_H^{MsFEM} \notin W^{1,p}(\Omega)$. In the latter case, the MsFEM is called nonconforming. By extension the variational interpretation of the oversampled MsFEM will also be called nonconforming although $V_H \subset W^{1,p}(\Omega)$, the nonlinear mapping providing us with a relationship of duality between the points of view $(W_{H,\epsilon}^{MsFEM}, V_H)$ and $(W_{\epsilon}, \{u_H^{MsFEM}\})$.

Let us give an interpretation of the cell resonance error defined in [18] for the nonconforming MsFEM. Within the present analytical framework, the classical MsFEM only suffers from a boundary condition mismatch, whereas the general averaged energies also suffer from a geometric mismatch ($C(x, \eta)$ versus the periodic cell). In a way, the MsFEM is "geometric error free" by construction in the present language. When using oversampling via a nonconforming Galerkin method ([9]) to reduce the error due to the boundary conditions, one then introduces another error by adding indirectly a geometric error. The use of a Petrov-Galerkin formulation ([18]) with oversampling preserves the geometric error free character of the classical MsFEM.

For the conforming MsFEM (described in Section 2.5), due to the very definition of the averaged energy density, we have for all $u_H, v_H \in V_H$,

$$\begin{aligned} \int_{\Omega} \partial_{\xi} W_{H,\epsilon}^{MsFEM}(x, \nabla u_H) \nabla v_H &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{H,\epsilon}^{MsFEM}(x, \langle \nabla u_H \rangle_i) \langle \nabla v_H \rangle_i \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \langle \nabla v_H \rangle_i \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H^{MsFEM} \\ &= \int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H^{MsFEM}, \end{aligned}$$

where we have formally used the Euler-Lagrange equation associated with (12). In particular we also have:

$$\int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H^{MsFEM} = \int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H,$$

which shows that the Galerkin and Petrov-Galerkin (for which the test space is V_H) formulations are equivalent for the MsFEM without oversampling considered in [18].

When using oversampling techniques, the multiscale finite element space (or more precisely manifold) $\{v_H^{MsFEM}\}$ is not a subspace of $W^{1,p}(\Omega)$, the method is therefore nonconforming. In this case, the Petrov-Galerkin and Galerkin methods are different since the Euler-Lagrange equation of (15) is not defined on $Q_{H,i}$ anymore but on a larger domain which prevents us from writing the previous decomposition as a sum of Euler-Lagrange equations on the $Q_{H,i}$. Thus

$$\int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_{H,over}^{MsFEM}) \nabla v_{H,over}^{MsFEM} \neq \int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_{H,over}^{MsFEM}) \nabla v_H,$$

As will be made clear hereafter, the Petrov-Galerkin formulation is more natural from a variational point of view than the Galerkin formulation. Before going to this, let us quickly go back to the motivation of the oversampling techniques. Oversampling aims at getting rid of the effects of the boundary layer on $Q_{H,i}$ by computing the minimum (15) on a domain $\tilde{Q}_{H,i}^{\zeta}$ of diameter of order $H + \zeta$ and by only considering the restriction of the associated solution $v_{\eta,\epsilon,over}^{H,i} \in W^{1,p}(\tilde{Q}_{H,i}^{\zeta})$ on $Q_{H,i}$. In the following calculation, we show that the nonconforming Galerkin formulation still indirectly uses the multiscale finite element near the boundary whereas this is not the case for the Petrov-Galerkin formulation. For all $v_H \in V_H$ and the associated test function v_H^{MsFEM} by (17), we have

$$\begin{aligned} & \int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H^{MsFEM} \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H^{MsFEM} \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) (\langle \nabla v_H \rangle_i + \nabla v_{\eta,\epsilon,over}^{H,i}) \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \langle \nabla v_H \rangle_i \\ &\quad - \sum_i \int_{\tilde{Q}_{H,i}^{\zeta} \setminus Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_{\eta,\epsilon,over}^{H,i}, \end{aligned}$$

using the Euler-Lagrange equation of (15). The last equality clearly shows that the Galerkin formulation uses the restriction of the multiscale finite element on $\tilde{Q}_{H,i}^{\zeta} \setminus Q_{H,i}$, which contains a part of the boundary layer that was supposed to be cancelled by the oversampling method and introduces a geometric error. Erasing this term at the very last line and going backward,

we recover the Petrov-Galerkin formulation and the equivalent variational interpretation

$$\begin{aligned} \int_{\Omega} \partial_{\xi} W_{H,\epsilon}^{MsFEM}(x, \nabla u_H) \nabla v_H &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{H,\epsilon}^{MsFEM}(x, \langle \nabla u_H \rangle_i) \langle \nabla v_H \rangle_i \\ &= \sum_i \int_{Q_{H,i}} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \langle \nabla v_H \rangle_i \\ &= \int_{\Omega} \partial_{\xi} W_{\epsilon}(x, \nabla u_H^{MsFEM}) \nabla v_H, \end{aligned}$$

which shows that the Petrov-Galerkin formulation is more natural than the Galerkin formulation with respect to the analytical framework of Section 2. In Section 4, we will prove the convergence of the Petrov-Galerkin formulation of the oversampled MsFEM for rather general elliptic operators and general heterogeneities using the energetic formulation of Section 2.

4 Oversampling for general heterogeneities

In this section, we first define the oversampling method for general heterogeneities by making precise the dependence of the oversampling upon the characteristic lengthscales ϵ and η . Provided a right scaling, we then prove the convergence of numerical homogenization methods with oversampling, within the framework of Section 2.

There are numerical evidence that show the practical interest of oversampling for non-periodic problems. There is also another motivation that is related to Tartar's corrector. In Section 2.3, the numerical corrector has been recast in the frame of Tartar's corrector provided an approximation depending on H and provided $\omega = \omega_1$. In Tartar's original work however, the correctors are proved to exist using $\omega \subset\subset \omega_1$, which is oversampling in the present language. Numerical correctors with oversampling are therefore approximations of Tartar's correctors that may seem more natural than the numerical correctors of Section 2.3. The use of oversampling allows us to recover all the diversity of the original Tartar's correctors.

4.1 Scaling of the oversampling

The aim of oversampling is to reduce the mismatch between the free oscillations of an unconstrained solution at fixed ϵ and the boundary conditions on domains $C(x, \eta)$. A major assumption concerns the convergence of the energies I_{ϵ} to a homogenized energy I_{hom} which is supposed not to exhibit oscillations at small scales. The oversampling for general heterogeneities should match the scales of the oscillations. Therefore it has to vanish with η , but it may also already vanish with ϵ . We set

Definition 4 Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. For all $x \in \Omega$, $\eta > 0$ and $\epsilon < \eta$, ζ defines an ϵ -admissible oversampling domain $C(x, \eta + \zeta(\epsilon))$ if

$$\lim_{\tau \rightarrow 0^+} \zeta(\tau) = 0^+,$$

and an η -admissible oversampling domain $C(x, \eta + \zeta(\eta))$ if

$$\lim_{\tau \rightarrow 0^+} \frac{\zeta(\tau)}{\tau} = 0^+.$$

By admissible, we mean that the scaling of the oversampling does not affect the convergence properties of the algorithm, but only its rate. It should be also emphasized that the parameter ϵ is arbitrary and may not have a simple interpretation in terms of the heterogeneities. Therefore the definition of an ϵ -admissible oversampling gives no hint on the practical definition of an efficient oversampling. Let us consider e.g. periodic homogenization and instead of parametrizing by the size ϵ of the period let us parametrize the energy by its square ϵ^2 . This does not change the limit. The oversampling $\zeta(\tau) = \tau$ (of order ϵ^2) is ϵ -admissible, however it does not allow to get rid of the boundary layer effect whose order is greater than $\epsilon \gg \epsilon^2$. On the contrary, η -admissible oversamplings should yield improvements on the boundary layer effects as ϵ vanishes, however they are not optimal and overestimate the size of the boundary layer since $\lim_{\epsilon \rightarrow 0} \zeta(\eta)/\epsilon = +\infty$. It may also be noticed that ϵ -admissible oversamplings are a particular case of η -admissible oversamplings.

More heuristically, if ϵ is related to the "size" of the heterogeneities, ζ should satisfy a property of the type $\lim_{\tau \rightarrow 0} \frac{\zeta(\tau)}{\tau} = +\infty$ in order to see the effect of oversampling. The prototype is given by the linear periodic case in Section 3.3 for which the boundary layer is of order ϵ . This property has not been imposed in the definition of an ϵ -admissible oversampling since it is not stable under a change of parametrization.

Remark 3 The oversamplings introduced in Definition 4 are the two extreme cases. One can also introduce particular oversamplings depending both on ϵ and η and providing us with suitable regimes for given applications. They can also be seen as particular cases of the η -admissible oversampling.

4.2 Convergence results

In this section, we prove the convergence of numerical homogenization with η -admissible oversampling, which also implies the convergence with ϵ -admissible oversampling. We first address the convergence of a continuous oversampled energy density whose FE-discretization leads to the HMM. We then show the convergence of two versions of the nonconforming Petrov-Galerkin MsFEM.

4.2.1 Convergence at the continuous level

The oversampled continuous energy density is given by the following

Definition 5 Let ζ be an η -admissible oversampling and W_ϵ satisfy H1, H4, and H3. For all $\eta, \epsilon > 0$, the associated oversampled energy density is defined by

$$\mathcal{W}_{\eta,\epsilon}^{over}(x, \xi) = \langle W_\epsilon(y, \xi + \nabla v_{\eta,\epsilon}^{over}) \rangle_{C(x,\eta)}$$

where $v_{\eta,\epsilon}^{over}$ is the restriction on $C(x, \eta)$ of $\tilde{v}_{\eta,\epsilon}^{over}$, solution of (15) with $\zeta = \zeta(\eta)$.

We then have the following two convergence results.

Theorem 4 Let W_ϵ satisfy H1, H2 (strictly), and H3 uniformly for $p > 1$, and ζ be an η -admissible oversampling, then the energy densities $C\mathcal{W}_{\eta,\epsilon}^{over}$ also satisfy H1, H2 and H3, and the energy $I_{\eta,\epsilon}^{over} : v \in W^{1,p}(\Omega) \mapsto \int_\Omega C\mathcal{W}_{\eta,\epsilon}^{over}(x, \nabla v) \Gamma(L^p)$ - and $\Gamma(W^{1,p})$ -converges to I_{hom} as ϵ and η go to 0, where CW denotes the convex envelop of \mathcal{W} . Therefore, any sequence $u_{\eta,\epsilon}^{over}$ of minimizers of $\inf\{I_{\eta,\epsilon}^{over}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}) + BC\}$ converges to the unique minimizer u_{hom} of $\inf\{I_{hom}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}) + BC\}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$.

Theorem 5 Let W_ϵ satisfy H1, H4, and H3 uniformly for $p > 1$, and ζ be an η -admissible oversampling, then the energy densities $Q\mathcal{W}_{\eta,\epsilon}^{over}$ are standard energy densities and $I_{\eta,\epsilon}^{over} : v \in W^{1,p}(\Omega) \mapsto \int_\Omega Q\mathcal{W}_{\eta,\epsilon}^{over}(x, \nabla v) \Gamma(L^p)$ - and $\Gamma(W^{1,p})$ -converges to I_{hom} as ϵ and η go to 0. Therefore, for any sequence $u_{\eta,\epsilon}^{over}$ of minimizers of $\inf\{I_{\eta,\epsilon}^{over}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}^d) + BC\}$, there exists a minimizer u_{hom} of $\inf\{I_{hom}(v) \mid v \in W^{1,p}(\Omega, \mathbb{R}^d) + BC\}$ such that

$$\lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} u_{\eta,\epsilon}^{over} = u_{hom} \quad \text{weakly in } W^{1,p}(\Omega, \mathbb{R}^d), \quad (18)$$

up to extraction.

In practice, one does not need to convexify $\mathcal{W}_{\eta,\epsilon}^{over}$ since the minimum is searched in a finite dimensional subspace of $W^{1,p}(\Omega)$, the strict convexity being recovered at the limit $\epsilon \rightarrow 0$ for ϵ -oversamplings and $\eta \rightarrow 0$ for η -oversamplings, in the spirit of [14, Theorem 4] for the quasiconvex case.

Theorems 4 and 5 imply the convergence of the HMM with oversampling in the general case.

Proof of Theorems 4 and 5.

We divide the proof in two steps. We first introduce an averaged energy density for which the strategy used to prove [14, Theorems 1 and 3] holds. We then show the oversampled energy density to be uniformly close to this averaged energy as η goes to zero.

Let us introduce the averaged energy density

$$\tilde{\mathcal{W}}_{\eta,\epsilon}^{over}(x, \xi) = \langle W_\epsilon(y, \xi + \nabla \tilde{v}_{\eta,\epsilon}^{over}) \rangle_{C(x,\eta+\zeta(\eta))}.$$

This energy density is of type (3) (cf. Remark 1) up to denoting by $\tilde{\eta} = \eta + \zeta(\eta)$. Thus Theorems 1 and 2 apply and we denote by $\tilde{I}_{\eta,\epsilon}^{over}$ the associated energy functional.

Let us prove now that the $\Gamma(L^p)$ and $\Gamma(W^{1,p})$ -convergence of $\tilde{I}_{\eta,\epsilon}^{over}$ and $I_{\eta,\epsilon}^{over}$ are equivalent. Due to Meyers' regularity estimate, H1, H4, H3 and possibly a convolution argument

(see [19] and [4, Theorem C.2]), there exist $\alpha > 0$ and $c > 0$, independent of η and ϵ such that

$$\|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p+\alpha}(C(x,\eta+\zeta(\eta)))} \leq c \|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p}(C(x,\eta+\zeta(\eta)))}. \quad (19)$$

The independence of α upon η is a consequence of Meyers' theorem. Let us prove that c does not depend on η either. Up to introducing the scaling $W_{\#}^{1,p}((0,1)^n) \ni v \mapsto \bar{v}(\cdot) = (\eta + \zeta(\eta))v(\frac{\cdot}{\eta+\zeta(\eta)}) \in W_{\#}^{1,p}((0,\eta + \zeta(\eta))^n)$, we have

$$\int_{(0,1)^n} W(x, \nabla_x v(x)) dx = \frac{1}{(\eta + \zeta(\eta))^n} \int_{(0,\eta+\zeta(\eta))^n} W\left(\frac{y}{\eta + \zeta(\eta)}, \nabla_y \bar{v}(y)\right) dy$$

for any standard energy density. Let c_1 denote the coefficient provided by Meyers' theorem on the domain $(0,1)^n$ and by c_2 the constant of the Poincaré-Wirtinger inequality. Let $v \in W_{\#}^{1,p}(0,1)^n$ be a minimizer of the associated energy on a given set. We have for $\eta + \zeta(\eta) \leq 1$:

$$\begin{aligned} \|\bar{v}\|_{W^{1,p+\alpha}((0,\eta+\zeta(\eta))^n)}^{p+\alpha} &= (\eta + \zeta(\eta))^{n+p+\alpha} \|v\|_{L^{p+\alpha}((0,1)^n)}^{p+\alpha} + (\eta + \zeta(\eta))^n \|\nabla v\|_{L^{p+\alpha}((0,1)^n)}^{p+\alpha} \\ &\leq (\eta + \zeta(\eta))^n \|v\|_{W^{1,p+\alpha}((0,1)^n)}^{p+\alpha} \\ &\leq (\eta + \zeta(\eta))^n c_1^{p+\alpha} \|v\|_{W^{1,p}((0,1)^n)}^{p+\alpha} \\ &\leq (\eta + \zeta(\eta))^n c_1^{p+\alpha} (1 + c_2)^{p+\alpha} \|\nabla v\|_{L^p((0,1)^n)}^{p+\alpha} \\ &\leq c_1^{p+\alpha} (1 + c_2)^{p+\alpha} \|\bar{v}\|_{W^{1,p}((0,\eta+\zeta(\eta))^n)}^{p+\alpha}, \end{aligned}$$

which shows that (19) holds with $c = c_1(1 + c_2)$.

Using the growth condition H3 on W_ϵ , (15) and (19) we obtain

$$\|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p+\alpha}(C(x,\eta+\zeta(\eta)))}^p \leq c(\eta + \zeta(\eta))^n (1 + |\xi|^p). \quad (20)$$

The application of Hölder inequality yields

$$\|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p}(C(x,\eta+\zeta(\eta)) \setminus C(x,\eta))}^p \leq C[\eta^{n-1}\zeta(\eta)]^{\frac{\alpha}{p+\alpha}} \|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p+\alpha}(C(x,\eta+\zeta(\eta)))}^p$$

which implies

$$\|\tilde{v}_{\eta,\epsilon}^{over}\|_{W^{1,p}(C(x,\eta+\zeta(\eta)) \setminus C(x,\eta))}^p \leq C(\eta + \zeta(\eta))^n [\eta^{n-1}\zeta(\eta)]^{\frac{\alpha}{p+\alpha}} (1 + |\xi|^p)$$

using (20), and finally

$$|\tilde{\mathcal{W}}_{\eta,\epsilon}^{over}(x, \xi) - \mathcal{W}_{\eta,\epsilon}^{over}(x, \xi)| \leq C \left([\eta^{n-1}\zeta(\eta)]^{\frac{\alpha}{p+\alpha}} + \frac{\zeta(\eta)}{\eta} \right) (1 + |\xi|^p), \quad (21)$$

using the well known quasi-uniform Lipschitz property of rank-one convex functions (see [14, Formula (2.12)] e.g.) and $\frac{\eta}{\eta+\zeta} = \eta(1 - \frac{\zeta}{\eta} + o(\frac{\zeta}{\eta}))$.

Therefore the dominated convergence theorem implies the uniform convergence to zero of $\tilde{I}_{\eta,\epsilon}^{over} - I_{\eta,\epsilon}^{over}$ on any bounded subset of $W^{1,p}(\Omega)$ as η goes to zero. This is enough to

ensure the equivalence of the Γ -convergences of the energy functionals, as briefly recalled below (see [3] or [5] for classical definitions related to Γ -convergence).

Let us first notice that the energies are only finite on $W^{1,p}(\Omega)$. For all $v \in W^{1,p}(\Omega)$ and all sequence $v_{\eta,\epsilon} \in W^{1,p}(\Omega)$ such that $\lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} v_{\eta,\epsilon} = v$ in $L^p(\Omega)$, either

$$\lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \tilde{I}_{\eta,\epsilon}^{over}(v_{\eta,\epsilon}) = \lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} I_{\eta,\epsilon}^{over}(v_{\eta,\epsilon}) = +\infty$$

and the Γ -liminf inequality trivially holds, or the sequence $\{v_{\epsilon,\eta}\}$ is bounded in $W^{1,p}(\Omega)$. In the latter case, the sequence belongs to a set on which the convergence of $\tilde{I}_{\eta,\epsilon}^{over} - I_{\eta,\epsilon}^{over}$ to zero is uniform. Thus, $\lim_{\eta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \tilde{I}_{\eta,\epsilon}^{over}(v_{\eta,\epsilon}) - I_{\eta,\epsilon}^{over}(v_{\eta,\epsilon}) = 0$ and the $\Gamma(L^p)$ -liminf (resp. limsup) of $I_{\eta,\epsilon}^{over}$ and $\tilde{I}_{\eta,\epsilon}^{over}$ coincide. Therefore they have the same $\Gamma(L^p)$ -limit. The same reasoning holds for the $\Gamma(W^{1,p})$ -convergence.

As a consequence, the Γ -convergence results obtained for $\tilde{I}_{\eta,\epsilon}^{over}$ hold for $I_{\eta,\epsilon}^{over}$, which concludes the proof of Theorems 4 and 5. \square

4.2.2 Convergence of the nonconforming Petrov-Galerkin MsFEM

Definition 6 Let ζ be an η -admissible oversampling and W_ϵ satisfy H1, H4, and H3. Let $\{Q_{H,i}\}_i$ be a triangulation of Ω . For all $\eta, \epsilon > 0$, the associated MsFEM oversampled energy density is defined by

$$W_{H,\epsilon,over}^{MsFEM}(x, \xi) = \sum_i \langle W_\epsilon(y, \xi + \nabla v_{H,\epsilon}^{over,i}) \rangle_{Q_{H,i}} 1_{Q_{H,i}}(x)$$

where $v_{H,\epsilon}^{over,i}$ is the restriction on $Q_{H,i}$ of $\tilde{v}_{H,\epsilon}^{over,i}$, solution of (15) where $C(x, \eta + \zeta)$ is replaced by $\tilde{Q}_{H,i}^{\zeta(H)}$, the concatenation of $Q_{H,i}$ and of a crown of width $\zeta(H)$.

The associated result is then the following.

Theorem 6 Let W_ϵ satisfy H1, H4 (resp. H2 strictly), and H3 uniformly for $p > 1$, and ζ be an η -admissible oversampling. Let V_H be the space of P1-finite elements on the regular triangulation $\{Q_{H,i}\}_i$. The oversampled Petrov-Galerkin MsFEM reads:

$$\inf \left\{ \sum_{i=1}^{I_H} |Q_{H,i}| W_{H,\epsilon,over}^{MsFEM}(x_i, \nabla u_H(x_i)) - \sum_{i=1}^{I_H} \sum_{j=1}^{N_{GP}} q_j f(x_{ij}) u_H(x_{ij}), u_H \in V_H + BC \right\}, \quad (22)$$

where the second term $f \in L^{p'}(\Omega)$ of the energy has been integrated by a quadrature rule associated with the triangulation. Then any sequence of minimizers $\{u_{\epsilon,H}^{over}\}$ of (22) converges weakly in $W^{1,p}(\Omega)$ up to extraction (resp. converges strongly in $W^{1,p}(\Omega)$) to a minimizer (resp. the unique minimizer) of $\inf\{I_{hom}(v) - \int_\Omega f v, v \in W^{1,p}(\Omega) + BC\}$.

Proof of Theorem 6.

Let us divide the proof in three steps. First we introduce an averaged energy density whose associated energy functional $\Gamma(W^{1,p})$ -converges to the homogenized energy. We then prove the convergence of the associated infima to the infimum of the homogenized energy, following the proof of [14, Section 3.2]. This implies the results of Theorem 6 for this averaged energy. We finally apply the argument of uniform convergence used in the proof of Theorems 4 and 5.

Let us consider the following averaged energy density

$$\tilde{W}_{H,\epsilon,over}^{MsFEM}(x, \xi) = \sum_i \langle W_\epsilon(y, \xi + \nabla \tilde{v}_{H,\epsilon}^{over,i}) \rangle_{\tilde{Q}_{H,i}^{\zeta(H)}} 1_{Q_{H,i}}(x).$$

Let P_H be an equicontinuous family of projectors from $W^{1,p}(\Omega)$ to V_H such that for all $v \in W^{1,p}(\Omega)$, $\lim_{H \rightarrow 0} \|P_H v - v\|_{W^{1,p}(\Omega)} = 0$. We then associate with $\tilde{W}_{H,\epsilon,over}^{MsFEM}$ an energy functional $\tilde{I}_{H,\epsilon,over}^{MsFEM} : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{I}_{H,\epsilon,over}^{MsFEM}(v) = \int_{\Omega} \tilde{W}_{H,\epsilon,over}^{MsFEM}(x, \nabla P_H v). \quad (23)$$

This family of energy functionals is equicontinuous on $W^{1,p}(\Omega)$ (see [14, p 1033]) and converges pointwise on $W^{1,p}(\Omega)$ to I_{hom} as ϵ and H vanish. Thus [14, Lemma 8] (or [5, Theorem 5.9]) implies the $\Gamma(W^{1,p})$ -convergence of $\tilde{I}_{H,\epsilon,over}^{MsFEM}$ to I_{hom} . It remains to prove the convergence of the infima of $\tilde{I}_{H,\epsilon,over}^{MsFEM}$ to the infimum of the homogenized energy to obtain the thesis of Theorem 6 for the family $\tilde{I}_{H,\epsilon,over}^{MsFEM}$. We only treat in detail the new argument (based on Meyers' estimates) with respect to [14, pp 1033–1035]. It is enough to prove that $\tilde{I}_{H,\epsilon,over}^{MsFEM} - I_{H,\epsilon}^{MsFEM}$ converges uniformly to zero on bounded subsets of $W^{1,p}(\Omega)$. To this aim, we can apply Meyers' estimates on each mesh element $Q_{H,i}$. The exponent α in (19) may however depend on H . Due to the regularity of the mesh this is not the case and there exists $\bar{\alpha}$ independent of H such that Meyers' estimate holds on every $Q_{H,i}$ with exponent $\bar{\alpha}$. It suffices to introduce a linear transform $T_{H,i}$ which maps $Q_{H,i}$ to Q , the reference mesh element. Up to a change of variable in the integrals using $T_{H,i}$, the new energy density satisfies a growth condition of order p with constants only depending on c , C and the quotients $\{\frac{\lambda_k}{\lambda_1 \dots \lambda_n}\}_{k \in \llbracket 1, n \rrbracket}$, where (λ_k) denote the eigenvalues of $T_{H,i}$. These quotients are bounded from above and below uniformly in H and i by definition of the regularity of the mesh. Therefore, there exists $\bar{\alpha}$ associated with the reference mesh element and this growth condition, such that Meyers' estimate holds on all $Q_{H,i}$. The strategy used in the proof of Theorem 4 then shows the uniform convergence of $\tilde{I}_{H,\epsilon,over}^{MsFEM} - I_{H,\epsilon}^{over}$ to zero on bounded subsets of $W^{1,p}(\Omega)$, which implies the convergence of the infimum of $\tilde{I}_{H,\epsilon,over}^{MsFEM}$ to the limit of the infima of $I_{H,\epsilon}^{MsFEM}$, which is exactly the infimum of I_{hom} as proved in [14, pp 1033–1035]. The results of Theorem 6 then hold for the energy density $\tilde{W}_{H,\epsilon,over}^{MsFEM}$.

As for the proof of Theorem 4 we use Meyers' estimate once more to obtain the uniform convergence of $\tilde{I}_{H,\epsilon,over}^{MsFEM} - I_{H,\epsilon,over}^{MsFEM}$ to zero on bounded subsets of $W^{1,p}(\Omega)$, proving Theorem 6.

□

Remark 4 In [18], the oversampled energy density is defined in a slightly different way. In the same spirit, one can replace $v_{H,\epsilon}^{over,i}$ by $\bar{v}_{H,\epsilon}^{over,i}(y) = v_{H,\epsilon}^{over,i}(y) - \langle \nabla v_{H,\epsilon}^{over,i} \rangle_{Q_{H,i}} \cdot y$, e.g.. The present proof easily adapts since

$$\lim_{H \rightarrow 0} \lim_{\epsilon \rightarrow 0} \langle \nabla v_{H,\epsilon}^{over,i} \rangle_{Q_{H,i}} = 0$$

and $W_\epsilon(y, \cdot)$ is uniformly (in space) Lipschitz-continuous.

4.3 Fine scale reconstruction

We now extend the numerical corrector of [14, Definition 5] to the case of oversampling. In the linear periodic case, oversampling improves the approximation a lot since a great part of the error is located in a boundary layer of order ϵ . For general heterogeneities we are not able to show that the approximation is better. Even if it were, in view of [14, Section 3.2], it is not clear whether the order of the global error be reduced. However we prove that the numerical corrector associated with the oversampling method has the same general convergence properties as the numerical corrector without oversampling. The interest of oversampling then relies on the possible reduction of the prefactor term in the error. Its efficiency is illustrated numerically in [11, p. 67] for a linear stochastic case.

Definition 7 Let $\{Q_{H,i}\}_{i \in [1, I_H]}$ and (M_H) be as in Definition 3. Keeping the notation of Theorem 4, we define the numerical correctors $v_{\eta,\epsilon,over}^{H,i}$ for a strictly convex energy density as the restriction on $Q_{H,i}$ of the unique minimizers (up to a constant) of

$$\inf \left\{ \int_{\tilde{Q}_{H,i}^{\zeta(\epsilon)}} W_\epsilon(x, \nabla v) \mid v \in W^{1,p}(\tilde{Q}_{H,i}^{\zeta(\epsilon)}), \langle \nabla v \rangle_{\tilde{Q}_{H,i}^{\zeta(\epsilon)}} = \langle \nabla u_{\eta,\epsilon}^{over} \rangle_{Q_{H,i}} \right\}, \quad (24)$$

where $\tilde{Q}_{H,i}^{\zeta(\epsilon)}$ is the concatenation of $Q_{H,i}$ and of a crown of width $\zeta(\epsilon)$.

We then have the corresponding convergence result of Theorem 3:

Theorem 7 In addition to H1, H2, and H3, let us assume that $p \geq 2$, that $W_\epsilon(x, \cdot)$ is continuously differentiable for almost all $x \in \Omega$ and $a_\epsilon(\cdot, 0) = \frac{\partial W_\epsilon}{\partial \xi}(\cdot, 0)$ is bounded, and satisfies the monotonicity and continuity properties (7) and (6). Then, denoting by u_ϵ the unique minimizer of I_ϵ on $W^{1,p}(\Omega) + BC$, we have

$$\lim_{\eta, H \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left\| \nabla u_\epsilon - \sum_{i=1}^{I_H} \nabla v_{\eta,\epsilon}^{H,i,over} 1_{Q_{H,i}} \right\|_{L^p(\Omega)} = 0. \quad (25)$$

Proof of Theorem 7.

The convergence of the numerical correctors is a direct consequence of Theorem 4 and [14, Theorem 2].

The proof in [14, Section 2.4] is based on two arguments: the strong convergence of $u_{\eta,\epsilon}$ to u_{hom} in $W^{1,p}(\Omega)$ and a passage from local estimates on $Q_{H,i}$ to a global estimate on Ω . The first argument holds for the oversampled method due to Theorem 4. The local estimates are now obtained on $\tilde{Q}_{H,i}$ and also imply a global estimate on Ω since $\lim_{H \rightarrow 0} \frac{\zeta(H)}{H} = 0$. All the details of [14, Section 2.4] adapt straightforwardly to the present case. \square

Remark 5 In Definition 24, one can replace the constraint on the mean by the Dirichlet boundary conditions $u_{\eta,\epsilon}^{over} \cdot y$ on $\partial\tilde{Q}_{H,i}^{\zeta(\epsilon)}$ or by periodic boundary conditions.

Remark 6 An easy corollary of Theorem 7 shows that the family $\{v_{\eta,\epsilon}^{over,i}\}$ associated with Definition 6 and formulation (22) form also a corrector, which completes the convergence result of the nonconforming Petrov-Galerkin formulation of the MsFEM in the monotone case.

5 Conclusion

In numerical homogenization, the choice of the boundary conditions for the problem at the micro scale in order to speed up the convergence of the numerical homogenization process is a difficult issue. It has been discussed a lot in the literature: e.g. in [20] for the community of *applied mathematics* and [22] for the community of *mechanics*. An alternative issue is given by oversampling, whose aim is precisely to minimize the effect of the boundary conditions of the micro scale problem. In the classical periodic and stochastic cases, oversampling has been proved to give better theoretical and numerical results, independently of the boundary conditions used. In the present work, we have extended the convergence results of [14] to the case of oversampling. This has allowed us to prove the convergence of advanced numerical methods such as HMM with oversampling and the nonconforming Petrov-Galerkin formulation of MsFEM in a general setting. To sum up, numerical homogenization methods with oversampling do indeed converge. In addition, oversampling may improve the convergence of the numerical methods in two ways. Concerning the approximation of the homogenized energy, oversampling does not improve the convergence rate in general but may improve the prefactor. For the numerical corrector however, both the convergence rate and the prefactor may be improved.

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 3 |
| 2 | Numerical homogenization methods | 3 |
| 2.1 | Minimization problem | 3 |
| 2.2 | Averaged energy densities | 5 |
| 2.3 | Numerical corrector and fine scale features | 6 |
| 2.4 | HMM | 7 |
| 2.5 | MsFEM | 7 |
| 3 | Oversampling in the periodic case | 8 |
| 3.1 | Setting of the problem | 8 |
| 3.2 | Mathematical formulation | 9 |
| 3.3 | A remark on boundary conditions | 9 |
| 3.4 | A remark on the volume element $C(x, \eta)$ | 11 |
| 3.5 | Interpretation of the MsFEM in Petrov-Galerkin formulation | 11 |
| 4 | Oversampling for general heterogeneities | 14 |
| 4.1 | Scaling of the oversampling | 14 |
| 4.2 | Convergence results | 15 |
| 4.2.1 | Convergence at the continuous level | 15 |
| 4.2.2 | Convergence of the nonconforming Petrov-Galerkin MsFEM | 18 |
| 4.3 | Fine scale reconstruction | 20 |
| 5 | Conclusion | 21 |

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