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Strong Normalization as Safe Interaction

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Abstract

The elimination rule of union types (denoted by $(\vee E)$) can break strong normalization (i.e. is unsafe) in presence of non-determinism. This is quite disturbing, since union types are often required in presence of non-deterministic computations. Surprisingly, there are also confluent systems for which this rule is unsafe. It appears that studying the safety of $(\vee E)$ amounts to the characterization in a term of safe interactions between some of its subterms.

In this paper, we study the safety of the elimination rule of union types for an extension of the lambda calculus with simple rewrite rules. We prove that the union and intersection types discipline is complete w.r.t. strong normalization. This allows to show that $(\vee E)$ is safe iff an interpretation of types based on biorthogonals is sound for it.

We then discuss two sufficient conditions for the safety of $(\vee E)$. The first one is issued from the closure by union of reducibility candidates while the second is the closure by union of a biorthogonality operator. We show that closure by union of reducibility candidates implies completeness of intersection type assignment.

Finally, we discuss an alternative biorthogonality relation, based on the observation of the least reducibility candidate.

1. Introduction

In this paper, we study the soundness of the elimination rule of union types with respect to strong normalization.

In presence of non-determinism, this rule (denoted by $(\vee E)$) can break strong normalization. What is more surprising is that it is not a matter of confluence: the rule can be unsafe with some confluent systems, while being harmless with some non-confluent ones.

The author showed in [15] that Girard's reducibility candidates are stable by union iff non-normal elimination terms have a reduct, called principal, such that every value of the

term is a value of this reduct. This condition, which is satisfied by the pure lambda-calculus (where values are terms headed by an abstraction), allows to interpret union types by the union of their interpretation. It therefore gives a sufficient condition for the safety of $(\vee E)$. However, when this condition is not met it is difficult to extract interesting information from the l.u.b. of two candidates.

On the other hand, semantics of types based on biorthogonals (see [13, 5]) give very informative interpretations of $(\vee E)$, as shown in [18, 17]. But [18] considers a call-by-value calculus, where biorthogonals are build on the observation of reduction without error, possibly involving infinite computations. Moreover, it seems not easy to have the full power of $(\vee E)$ in a calculus allowing call-by-name evaluation. In [17] the elimination rule of union is used in a call-by-name setting and appears in a restricted form. Therefore, regarding strong normalization, it was up to now unclear how to handle the biorthogonal interpretation of the full rule $(\vee E)$.

In this paper, we solve the problem for the lambda-calculus enriched with simple rewrite rules. Our main result is that the biorthogonal interpretation is sound for $(\vee E)$ iff $(\vee E)$ is safe, that is, iff it can be added to the type system while preserving strong normalization.

The key point is that biorthogonals allow to bring the observation at an arbitrary candidate back to the observation of strong normalization. By proving that intersection and union types are sound and complete w.r.t. strong normalization, we are able to extract all the information we need from the observation of strong normalization.

It appears that this hides an *interaction* problem. Indeed, $(\vee E)$ is safe iff for all elimination term, if each one-step reduct of this term can be safely plugged in a capture-avoiding context, then this term can itself be safely plugged in that context. But, since we consider the full rule $(\vee E)$, contexts are possibly duplicating. It follows that the above must be read as: if each one-step reduct of an elimination term can be safely duplicated in a capture-avoiding context, then this term can be safely duplicated in that context. Hence, its *different* reducts have to interact safely *with each other* in that context. Characterizing the safety of $(\vee E)$

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amounts to characterize systems having this safe interaction property. Intuitively, such systems have a kind of *uniform computations*.

Then we consider sufficient conditions for the safety of $(\vee E)$. Besides the stability by union of reducibility candidates, we consider the interpretation of types arising as the closure by union of a biorthogonality operator. In this case, types are interpreted by non-empty sets upward-closed w.r.t. the observational preorder issued from the orthogonality relation. We show that these sets are reducibility candidates iff each elimination term is greater w.r.t. that preorder than one of its immediate reducts.

A natural question is whether $(\vee E)$ is safe with rewrite systems for which intersection types are sufficient for the completeness of type assignment w.r.t. strong normalization, i.e. when unions are not needed. We show that this is not the case. However, it is interesting to note that when reducibility candidates are stable by union, intersection types are sufficient to type strongly normalizing terms.

We conclude by a discussion on an alternative orthogonality relation build on the observation of the least reducibility candidate. It amounts to observe strongly normalizing reduction to error rather than strong normalization only. This induces a biorthogonal type interpretation having a better adequacy with the type system and would allow for a more natural subtyping relation. However, it is not clear whether these biorthogonals are equivalent, for the soundness of $(\vee E)$, to those issued from strong normalization.

Related Works Intersection and union types are extensively studied in [6, 7, 8] as the logical intermediate to build fully abstract filter models of non-deterministic lambda-calculi. These works consider *must* normalization of (weak) head reduction. Here, *must* normalization of a reduction relation means convergence of any reduction with this relation. This makes sense in non-deterministic calculi even if the considered relation is not the full reduction. In [6, 7] it is remarked that $(\vee E)$ makes the soundness of the type system to fail w.r.t. the considered property. Because they are in a *must* setting, we think that problems caused in their cases by $(\vee E)$ are in essence similar to ours.

Recent applications of union types are the XML processing languages XDuce [11] and CDuce [9].

Concerning strong normalization, union types are used extensively in the constrained type system of [3]. This work motivated our study of stability by union of Girard's reducibility candidates.

Our integration of rewriting with intersection types is inspired by that of [4]. Comparing to this work, we use simpler rewrite rules, since we do not allow pattern matching. Moreover, we use uncurried symbols with a fixed arity. This is the key for the completeness of type assignment.

Our presentation of biorthogonals is inspired from [5],

see also [13, 18, 14]. For properties on lambda-calculus and (union and intersection) types, we refer to [12, 10, 2, 6, 1].

Outline We present the calculus in Sec. 2, with a discussion on $(\vee E)$ and examples of its unsafety. Section. 3 is devoted to the soundness and completeness of the type systems (without $(\vee E)$). We present our analysis of $(\vee E)$ in Sec. 4, and discuss sufficient conditions for its safety in Sec. 5. Finally, in Sec. 6, we briefly discuss the orthogonality relation build on the observation of the least reducibility candidate.

2. Preliminaries

2.1. Types and Terms

Terms are those of the lambda-calculus enriched with symbols of fixed arity:

$$t, u \in \Lambda ::= x \in \mathcal{X} \mid tu \mid \lambda x.t \mid f(t_1, \dots, t_n)$$

where $f \in \mathcal{S}$ is a symbol of arity n . As usual, terms are considered modulo α -conversion. Let $\mathcal{X}(t)$ be the set of variables occurring free in t . By \vec{t} we mean a sequence of terms of length $|\vec{t}|$ and use the same notation for types, contexts, etc.

We consider a set \mathcal{R} of rewrite rules of the form

$$f(\vec{x}) \mapsto_{\mathcal{R}} r$$

where \vec{x} is made of distinct variables and $\mathcal{X}(r) \subseteq \vec{x}$. Write $r \in \mathcal{R}(f)$ if $f(\vec{x}) \mapsto_{\mathcal{R}} r$ and let $\mathcal{S} = \mathcal{F} \cup \mathcal{C}$ where $f \in \mathcal{F}$ iff $\mathcal{R}(f) \neq \emptyset$. Define \rightarrow to be the smallest rewrite relation on Λ containing $\mapsto_{\mathcal{R}}$ and $(\lambda x.t)u \mapsto_{\beta} t[u/x]$, where $t[u/x]$ is the capture-avoiding substitution of u for x in t . We generalize substitutions to functions $\sigma : \mathcal{X} \rightarrow \Lambda$ with $t\sigma =_{\text{def}} t[\sigma(x)/x \mid x \in \text{dom}(\sigma)]$.

We assume that \rightarrow is finitely branching, hence that $\mathcal{R}(f)$ is finite for each $f \in \mathcal{F}$. Define $(t)_{\rightarrow} =_{\text{def}} \{u \mid t \rightarrow u\}$ and let \rightarrow^* be the reflexive transitive closure of \rightarrow . We write $(t_1, \dots, t_n) \rightarrow (t'_1, \dots, t'_n)$ iff there is i such that $t_i \rightarrow t'_i$ and $t_j = t'_j$ for all $j \neq i$.

A term t is *strongly normalizing* iff every reduction sequence issued from t is finite. Let \mathcal{SN} be the set of strongly-normalizing terms. Note that $t \in \mathcal{SN}$ iff either t is not reducible or all its reducts are in \mathcal{SN} . Hence \mathcal{SN} is the smallest set such that for all t ,

$$(\forall u (t \rightarrow u \Rightarrow u \in \mathcal{SN})) \Rightarrow t \in \mathcal{SN}.$$

Types are the following, where o is the base type:

$$T, U \in \mathcal{T} ::= o \mid T \Rightarrow U \mid T \wedge U \mid T \vee U.$$

$$\begin{array}{c}
\frac{}{T \leq T} \quad \frac{U_2 \leq U_1 \quad T_1 \leq T_2}{U_1 \Rightarrow T_1 \leq U_2 \Rightarrow T_2} \quad \frac{T \leq U \quad U \leq V}{T \leq V} \\
\\
\frac{}{(T \Rightarrow U_1) \wedge (T \Rightarrow U_2) \leq T \Rightarrow (U_1 \wedge U_2)} \\
\\
\frac{T_1, T_2 \leq U}{T_1 \vee T_2 \leq U} \quad \frac{}{T_1, T_2 \leq T_1 \vee T_2} \\
\\
\frac{}{T_1 \wedge T_2 \leq T_1, T_2} \quad \frac{T \leq U_1, U_2}{T \leq U_1 \wedge U_2}
\end{array}$$

Figure 1. Subtyping

Subtyping rules are in Fig. 1. They axiomatize the fact that $(\mathcal{T}, \leq, \wedge, \vee)$ is a preorder with all finite non-empty g.l.b.'s and l.u.b.'s respectively given by \wedge and \vee . Note that contrary to [6, 7], $(\mathcal{T}, \leq, \wedge, \vee)$ is not distributive.

Contexts are functions $\Gamma : \mathcal{X} \rightarrow \mathcal{T}$. We use the notations $(x : T) \in \Gamma$ when $\Gamma(x) = T$ and $x \in \Gamma$ when $x \in \text{dom}(\Gamma)$. Given Γ_0 and Γ_1 , we let $\Gamma_0 \wedge \Gamma_1$ be the context such that

$$\Gamma_0 \wedge \Gamma_1(x) =_{\text{def}} \begin{cases} \Gamma_0(x) \wedge \Gamma_1(x) & \text{if } x \in \Gamma_0 \cap \Gamma_1, \\ \Gamma_i(x) & \text{if } x \in \Gamma_i \setminus \Gamma_{1-i}. \end{cases}$$

Typing rules are given in Fig. 2. The rule (SY) is inspired from [4]. It is easy to see that if $f \in \mathcal{C}$ and moreover $\Gamma \vdash_{\wedge\vee} \vec{t} : \vec{T}$ and $\Gamma \vdash_{\wedge\vee} \vec{v} : \vec{V}$, then for all U we have $\Gamma \vdash_{\wedge\vee} f(\vec{t})\vec{v} : U$.

We denote by $\Gamma \vdash_{\wedge} t : T$ typing judgments in the system without \vee and design by \mathcal{T}_{\wedge} the corresponding set of types. Note that for $ty \in \{\wedge, \wedge\vee\}$, if $\Gamma \vdash_{ty} t : T$, then for all Γ' we have $\Gamma \wedge \Gamma' \vdash_{ty} t : T$.

2.2. The Rule (\vee E)

In this section, we discuss the rule (\vee E). This making, we may anticipate on some results presented later in the paper. The elimination rule of union is the following:

$$(\vee E) \frac{\Gamma \vdash t : T_1 \vee T_2 \quad \begin{array}{c} \Gamma, x : T_1 \vdash c : C \\ \Gamma, x : T_2 \vdash c : C \end{array}}{\Gamma \vdash c[t/x] : C}$$

We denote by $\vdash_{\wedge\vee}$ the type system $\vdash_{\wedge\vee}$ in which we added the rule (\vee E).

The rule can be read as follows: if $t : T_1 \vee T_2$ and for all $i \in \{1, 2\}$ ($v : T_i \Rightarrow c[v/x] : C$) then $c[t/x] : C$. Intuitively, this can be problematic if \vee is not a union, i.e. if there is t such that $t : T_1 \vee T_2$ but neither $t : T_1$ nor $t : T_2$. Such a situation can occur with non-determinism. Indeed, consider the rewrite system

$$t_1 + t_2 \rightarrow t_1 \quad t_1 + t_2 \rightarrow t_2.$$

Assume that $t =_{\text{def}} t_1 + t_2$, where t_1 can be given the type T_1 but not T_2 , and vice-versa for t_2 . Then, t is not in the union of T_1 and T_2 , since it is neither in T_1 nor in T_2 .

Example 2.1. We now give an example of unsoundness of (\vee E). Let $t_1 =_{\text{def}} \lambda z.zy\delta$ and $t_2 =_{\text{def}} \lambda z.\delta$ where $\delta =_{\text{def}} \lambda x.xx$. It is clear that $t_1 t_1$ and $t_2 t_2$ are strongly normalizable. However, $t_1 t_2 \rightarrow^* \delta\delta \notin \mathcal{SN}^1$.

By completeness of type assignment in \vdash_{\wedge} (see [12, 10]), for $i = 1, 2$ there are T_i, U_i, V_i such that $y : V_i \vdash_{\wedge} t_i : T_i$ and $y : V_i, x : T_i \vdash_{\wedge} xx : U_i$. Hence we have

$$\begin{array}{c}
y : V_1 \wedge V_2 \quad \vdash_{\wedge\vee} t_1 + t_2 : T_1 \vee T_2 \\
y : V_1 \wedge V_2, x : T_1 \vdash_{\wedge\vee} \quad xx : U_1 \vee U_2 \\
y : V_1 \wedge V_2, x : T_2 \vdash_{\wedge\vee} \quad xx : U_1 \vee U_2 \\
(\vee E) \frac{}{y : V_1 \wedge V_2 \vdash_{\wedge\vee} (t_1 + t_2)(t_1 + t_2) : U_1 \vee U_2}
\end{array}$$

but $(t_1 + t_2)(t_1 + t_2) \rightarrow^* t_1 t_2 \rightarrow^* \delta\delta \notin \mathcal{SN}$.

Example 2.2. What is more surprising is that this can also occur with confluent systems, such as

$$f \rightarrow \lambda xy.g(xa\delta) \quad f \rightarrow \lambda xy.g(yy) \quad g(x) \rightarrow a.$$

Let $u_1 =_{\text{def}} \lambda xy.g(xa\delta)$ and $u_2 =_{\text{def}} \lambda xy.g(yy)$. Since we have $u_1 u_1 \in \mathcal{SN}$ and $u_2 u_2 \in \mathcal{SN}$, by Completeness (Thm. 3.11) and Interpolation (Prop. 3.8), there are T_1, T_2 and U such that

$$(\vee E) \frac{\begin{array}{c} \vdash_{\wedge\vee} f : T_1 \vee T_2 \quad x : T_1 \vdash_{\wedge\vee} \quad xx : U \\ \vdash_{\wedge\vee} f : T_1 \vee T_2 \quad x : T_2 \vdash_{\wedge\vee} \quad xx : U \end{array}}{\vdash_{\wedge\vee} ff : U}$$

but $ff \rightarrow^* u_1 u_2 \rightarrow^* \lambda y.g(g(\delta\delta)) \notin \mathcal{SN}$.

These two examples also show the necessity of union types for the completeness of type assignment w.r.t. strong normalization.

3. Soundness and Completeness

In this section, we prove soundness and completeness of typing with union and intersection types with respect to strong normalization.

This is the occasion to introduce basic notions on reducibility, that will be presented more deeply in Sec. 4.3.

We also prove a few consequences of completeness. These results, which are important for our analysis of (\vee E), seem not easy to obtain without intersection and union types. Moreover, even in the case of the pure lambda-calculus, they seem to have not been published before.

¹We thank Philippe De Groote for this example.

$$\begin{array}{c}
(\text{AX}) \frac{}{\Gamma, x : T \vdash_{\wedge\vee} x : T} \\
(\Rightarrow \text{I}) \frac{\Gamma, x : U \vdash_{\wedge\vee} t : T}{\Gamma \vdash_{\wedge\vee} \lambda x. t : U \Rightarrow T} \\
(\wedge \text{I}) \frac{\Gamma \vdash_{\wedge\vee} t : T_1 \quad \Gamma \vdash_{\wedge\vee} t : T_2}{\Gamma \vdash_{\wedge\vee} t : T_1 \wedge T_2} \\
(\text{SY}) \frac{\Gamma \vdash_{\wedge\vee} \vec{t} : \vec{T} \quad \forall r \in \mathcal{R}(f), \Gamma, \vec{x} : \vec{T} \vdash_{\wedge\vee} r : U}{\Gamma \vdash_{\wedge\vee} f(\vec{t}) : U} \\
(\Rightarrow \text{E}) \frac{\Gamma \vdash_{\wedge\vee} t : U \Rightarrow T \quad \Gamma \vdash_{\wedge\vee} u : T}{\Gamma \vdash_{\wedge\vee} tu : T} \\
(\text{SUB}) \frac{\Gamma \vdash_{\wedge\vee} t : T \quad T \leq U}{\Gamma \vdash_{\wedge\vee} t : U}
\end{array}$$

Figure 2. Typing

3.1. Reducibility

We introduce well-known basic tools for reducibility. This presentation is consistent with [15], where more details can be found.

As advocated in [18, 17], it turns out to be very convenient to interpret types as closed sets of terms. Recall that a *closure operator* on a partial order (D, \leq) is a function $\bar{\cdot} : D \rightarrow D$ which is idempotent: $\bar{\bar{x}} = \bar{x}$; extensive: $x \leq \bar{x}$; and monotone: $x \leq y \Rightarrow \bar{x} \leq \bar{y}$. It is well-known that the greatest lower bound of a family of closed elements is closed.

Definition 3.1 (Neutral terms). *Terms not headed by an abstraction are called neutral. Let \mathcal{N} be the set of neutral terms.*

Let \mathcal{HN} , the set of hereditary neutral terms, be the smallest set such that for all $t \in \mathcal{N}$, if $(\forall u (t \rightarrow u \Rightarrow u \in \mathcal{HN}))$ then $t \in \mathcal{HN}$.

Note that $\mathcal{HN} \subseteq \mathcal{SN}$.

Definition 3.2 (Reducibility Candidates). *The set \mathcal{CR} of reducibility candidates is the set of all $C \subseteq \mathcal{SN}$ such that*

- (CR0) *if $t \in C$ and $t \rightarrow u$ then $u \in C$,*
- (CR1) *if $t \in \mathcal{N}$ and $(\forall u (t \rightarrow u \Rightarrow u \in C))$ then $t \in C$.*

The property (CR1) is also called the *neutral term property*. It is easy to define a function $\bar{\cdot} : \mathcal{P}(\mathcal{SN}) \rightarrow \mathcal{P}(\mathcal{SN})$ such that \bar{A} is the smallest reducibility candidate containing A . This is a closure operator on $(\mathcal{P}(\mathcal{SN}), \subseteq)$.

Proposition 3.3 (Candidates Lattice). *$(\mathcal{CR}, \subseteq)$ is a complete lattice with least element \mathcal{HN} , greatest element \mathcal{SN} and whose g.l.b.'s are given by \bigcap .*

We turn to the interpretation of arrow types.

Proposition 3.4 (Arrow Type Constructor). *The arrow type constructor $\Rightarrow : \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$, defined as*

$$A \Rightarrow B \quad =_{\text{def}} \quad \{t \mid \forall u (u \in A \Rightarrow tu \in B)\}$$

maps $A, B \in \mathcal{CR}$ to a reducibility candidate.

We interpret $T \in \mathcal{T}$ by $\llbracket T \rrbracket \in \mathcal{CR}$ as follows:

$$\begin{array}{l}
\llbracket o \rrbracket \quad =_{\text{def}} \quad \mathcal{SN} \\
\llbracket T \Rightarrow U \rrbracket \quad =_{\text{def}} \quad \llbracket T \rrbracket \Rightarrow \llbracket U \rrbracket \\
\llbracket T \wedge U \rrbracket \quad =_{\text{def}} \quad \llbracket T \rrbracket \cap \llbracket U \rrbracket \\
\llbracket T \vee U \rrbracket \quad =_{\text{def}} \quad \overline{\llbracket T \rrbracket \cup \llbracket U \rrbracket}.
\end{array}$$

Another possibility is $\llbracket o \rrbracket = \mathcal{HN}$, motivated by the fact that terms typable with o are exactly the hereditary neutral terms (see Sec. 6 and Theorems 3.12 and 6.1).

3.2. Soundness

We show that $\Gamma \vdash_{\wedge\vee} t : T$ implies $t \in \mathcal{SN}$.

Proposition 3.5 (Soundness of Subtyping). *If $T \leq U$ then $\llbracket T \rrbracket \subseteq \llbracket U \rrbracket$.*

Given a substitution $\sigma : \mathcal{X} \rightarrow \Lambda$ and a context Γ , we write $\sigma \models_{\llbracket \cdot \rrbracket} \Gamma$ when $\sigma(x) \in \llbracket T \rrbracket$ for all $(x : T) \in \Gamma$. Recall that the rule (\vee E) is not present in $\vdash_{\wedge\vee}$.

Theorem 3.6 (Soundness of Typing). *If $\Gamma \vdash_{\wedge\vee} t : T$ and $\sigma \models_{\llbracket \cdot \rrbracket} \Gamma$ then $t\sigma \in \llbracket T \rrbracket$.*

Proof. By induction on $\Gamma \vdash_{\wedge\vee} t : T$, using Prop. 3.5 for (SUB). We detail the case of (SY).

Let $\sigma \models_{\llbracket \cdot \rrbracket} \Gamma$ and $\vec{t}' =_{\text{def}} \vec{t}\sigma$. By induction hypothesis, $\vec{t}' \in \llbracket \vec{T} \rrbracket$. We have to show that $t' =_{\text{def}} f(\vec{t}') \in \llbracket T \rrbracket$. Since this term is neutral, it suffices to show that $(t')_{\rightarrow} \subseteq \llbracket T \rrbracket$. We reason by induction on $\vec{t}' \in \mathcal{SN}$. Let $v \in (t')_{\rightarrow}$. If $v = f(\vec{u})$ with $\vec{t}' \rightarrow \vec{u}$, then by (CR0), $\vec{u} \in \llbracket \vec{T} \rrbracket$ and we conclude by induction hypothesis on $\vec{u} \in \mathcal{SN}$. Otherwise, there is a rule $f(\vec{x}) \mapsto_{\mathcal{R}} r$ such that $v = r[\vec{t}'/\vec{x}]$ and since $\vec{t}' \in \llbracket \vec{T} \rrbracket$, by induction hypothesis on $\Gamma, \vec{x} : \vec{T} \vdash_{\wedge\vee} r : T$ we have $r[\vec{t}'/\vec{x}] \in \llbracket T \rrbracket$. \square

Corollary 3.7. *If $\Gamma \vdash_{\wedge\vee} t : T$ then $t \in \mathcal{SN}$.*

3.3. Completeness

The main result of this section is the completeness of intersection and union types with respect to strong normalization: if $t \in \mathcal{SN}$, then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$. The result is proved in [12, 10] for the pure lambda calculus with intersection types.

We begin by two important properties, that are characteristic of intersection types. They are the key properties for completeness.

Proposition 3.8 (Interpolation). *If $\Gamma \vdash_{\wedge} t[u/x] : T$ and $\Gamma \vdash_{\wedge} u : U$ with $x \notin \Gamma$, then there is a type V such that $\Gamma, x : V \vdash_{\wedge} t : T$ and $\Gamma \vdash_{\wedge} u : V$.*

Proof. By induction on t . \square

Lemma 3.9 (Weak Head Expansion).

(i) *Assume that $\Gamma \vdash_{\wedge} u : U$ and $\Gamma \vdash_{\wedge} t[u/x]\vec{v} : T$. Then $\Gamma \vdash_{\wedge} (\lambda x.t)u\vec{v} : T$.*

(ii) *For all $f \in \mathcal{F}$, if $\Gamma \vdash_{\wedge} \vec{t} : \vec{T}$ and $\Gamma \vdash_{\wedge} r[\vec{t}/\vec{x}]\vec{v} : T$ and for all $r \in \mathcal{R}(f)$, then $\Gamma \vdash_{\wedge} f(\vec{t})\vec{v} : T$.*

Proof. The two points are similar: the property is proved by induction on $|\vec{v}|$, and the base case is obtained using Prop. 3.8. \square

For the proof of completeness itself, we use an induction on a preorder that combine reduction and subterm.

Definition 3.10. *We let \prec be the smallest preorder such that $t \prec u$ if either $u \rightarrow t$ or t is a strict subterm of u .*

This preorder is well-founded on \mathcal{SN} .

Theorem 3.11 (Completeness). *If $t \in \mathcal{SN}$, then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$.*

Proof. The proof is by induction on \prec and uses Lem. 3.9. We only detail the case of $t = f(\vec{t})\vec{v}$ with $f \in \mathcal{F}$.

First, note that $\vec{t} \prec t$. For all $r \in \mathcal{R}(f)$ we have $r[\vec{t}/\vec{x}]\vec{v} \prec t$ and by induction hypothesis there are Γ_r, \vec{T}^r and V_r such that $\Gamma_r \vdash_{\wedge} \vec{t} : \vec{T}^r$ and $\Gamma_r \vdash_{\wedge} r[\vec{t}/\vec{x}]\vec{v} : V_r$. Now, taking $\Gamma =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} \Gamma_r$, $\vec{T} =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} \vec{T}^r$ and $V =_{\text{def}} \bigvee_{r \in \mathcal{R}(f)} V_r$, we have $\Gamma \vdash_{\wedge} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{\wedge} r[\vec{t}/\vec{x}]\vec{v} : V$. We conclude that $\Gamma \vdash_{\wedge} f(\vec{t})\vec{v} : V$ thanks to Lem. 3.9.(ii). \square

Note that without further assumptions on \mathcal{R} , union types are required for Thm. 3.11.

The next result says that it would have been complete to interpret \circ by \mathcal{HN} , the least element of \mathcal{CR} .

Theorem 3.12 (\mathcal{HN} -Completeness). *If $t \in \mathcal{HN}$ then for all $T \in \mathcal{T}$ there is Γ such that $\Gamma \vdash_{\wedge} t : T$.*

Proof. Similar to Thm. 3.11. We reason by induction on \prec , using Thm. 3.11 and Lem. 3.9. \square

3.4. Two Interesting Consequences

We now prove two consequences of soundness and completeness of \vdash_{\wedge} . These results seem not easy to obtain without \vdash_{\wedge} , and seem not to have been published before, even for the pure lambda-calculus. They play an important role in our analysis of $(\vee E)$.

The first one says that β -reduction leads to uniform computations.

Theorem 3.13. *If $(\lambda x.t)u \in \mathcal{SN}$ and $v[t[u/x]/y] \in \mathcal{SN}$ then $v[(\lambda x.t)u/y] \in \mathcal{SN}$.*

Proof. Since $(\lambda x.t)u \in \mathcal{SN}$, we have also $u \in \mathcal{SN}$ and $t[u/x] \in \mathcal{SN}$. It follows from Thm. 3.11 that there are Γ', T and U such that $\Gamma' \vdash_{\wedge} u : U$ and $\Gamma' \vdash_{\wedge} t[u/x] : T$.

On the other hand, still thanks to Thm. 3.11, there are Γ'', V such that $\Gamma'' \vdash_{\wedge} v[t[u/x]/y] : V$. Let $\Gamma =_{\text{def}} \Gamma' \wedge \Gamma''$. Since $\Gamma \vdash_{\wedge} t[u/x] : T$, we can use Lem. 3.9.(i) to obtain $\Gamma \vdash_{\wedge} (\lambda y.v)(t[u/x]) : V$. It follows that there is T' such that $\Gamma \vdash_{\wedge} \lambda y.v : T' \Rightarrow V$ and $\Gamma \vdash_{\wedge} t[u/x] : T'$.

Furthermore, since $\Gamma \vdash_{\wedge} u : U$, using Lem. 3.9.(i) we have $\Gamma \vdash_{\wedge} (\lambda x.t)u : T'$. Then, $\Gamma \vdash_{\wedge} (\lambda y.v)((\lambda x.t)u) : V$, and it follows that $v[(\lambda x.t)u/y] \in \mathcal{SN}$ by Cor. 3.7. \square

The analogous of this property for $\mapsto_{\mathcal{R}}$ will be shown to be equivalent to the safety of $(\vee E)$ in Sec. 4.

Note that the capture-avoiding substitution is essential here. Indeed, the property fails if we replace v by a context $C[\]$ able to capture variables.

For example (see [16]), with $C[\] =_{\text{def}} (\lambda y.[\])\delta$, and $(\lambda x.t)u =_{\text{def}} (\lambda x.z)(yy)$, we have $C[t[u/x]] = (\lambda y.z)\delta$ which is in \mathcal{SN} , but $C[(\lambda x.t)u] = (\lambda y.(\lambda x.z)(yy))\delta \rightarrow (\lambda x.z)(\delta\delta) \notin \mathcal{SN}$.

Now, we show that hereditary neutral terms are really *neutral*, in the sense that they can be safely substituted in any strongly normalizing term.

Theorem 3.14. *If $t \in \mathcal{HN}$ and $v \in \mathcal{SN}$ then $v[t/x] \in \mathcal{SN}$.*

Proof. First, assume that $x \notin \mathcal{X}(t)$. Since $v \in \mathcal{SN}$, by Thm. 3.11, there are Γ'', T and V such that $\Gamma'', x : T \vdash_{\wedge} v : V$. Moreover, since $t \in \mathcal{HN}$, by Thm. 3.12, there is Γ' such that $\Gamma' \vdash_{\wedge} t : T$. Hence, taking $\Gamma =_{\text{def}} \Gamma' \wedge \Gamma''$ we have $\Gamma, x : T \vdash_{\wedge} v : V$ and $\Gamma \vdash_{\wedge} t : T$. It follows that $\Gamma \vdash_{\wedge} (\lambda x.v)t : V$, hence $v[t/x] \in \mathcal{SN}$ by Cor. 3.7.

Now, assume that $x \in \mathcal{X}(t)$. Let $y \notin \mathcal{X}(t, v)$ and $t' =_{\text{def}} t[y/x]$. Then we have $t' \in \mathcal{HN}$ hence $v[t'/x] \in \mathcal{SN}$ and $v[t/x] = (v[t'/x])[x/y] \in \mathcal{SN}$. \square

4. Safe Interaction

We now address the problem of the safety of the elimination rule of union:

$$(\vee E) \frac{\Gamma, x : T_1 \vdash c : C \quad \Gamma, x : T_2 \vdash c : C}{\Gamma \vdash c[t/x] : C}$$

Recall that $\vdash_{\wedge\vee}$ is the type system $\vdash_{\wedge\vee}$ in which we added the rule $(\vee E)$. Since we have proved in Sec. 3.2 that typability in $\vdash_{\wedge\vee}$ implies strong normalization, proving the safety of $(\vee E)$ reduces to proving strong normalization of terms typable in $\vdash_{\wedge\vee}$.

In this section, an interpretation $(\llbracket \cdot \rrbracket) : \mathcal{T} \rightarrow \mathcal{CR}$ is given so that the following points are equivalents:

$(\vee E)$ is safe: If $\Gamma \vdash_{\wedge\vee} t : T$ then $t \in \mathcal{SN}$.

(IP) For all $f \in \mathcal{F}$, if $f(\vec{t}) \in \mathcal{SN}$ and $v[r[\vec{t}/\vec{x}]/y] \in \mathcal{SN}$ for all $r \in \mathcal{R}(f)$, then $v[f(\vec{t})/y] \in \mathcal{SN}$.

$(\llbracket \cdot \rrbracket)$ is sound: If $\Gamma \vdash_{\wedge\vee} t : T$ and $\sigma \models_{(\llbracket \cdot \rrbracket)} \Gamma$ then $t\sigma \in \llbracket T \rrbracket$.

4.1. The Interaction Principle

Let us discuss (IP). Since $v' =_{\text{def}} v[f(\vec{t})/x]$ may duplicate $f(\vec{t})$, we can see it as $D[f(\vec{t}), \dots, f(\vec{t})]$, where $D[\dots]$ is a n -holed capture-avoiding context.

The point is that in D , different occurrences of $f(\vec{t})$ may reduce to different $r[\vec{t}/\vec{x}]$. The property (IP) states that if for all $i \in \{1, \dots, n\}$ we have $D[r_i[\vec{t}/\vec{x}], \dots, r_i[\vec{t}/\vec{x}]] \in \mathcal{SN}$, then we have $D[r_1[\vec{t}/\vec{x}], \dots, r_n[\vec{t}/\vec{x}]] \in \mathcal{SN}$.

In words, if each $r_i[\vec{t}/\vec{x}]$ interact safely with D and with itself in D , then the terms $r_1[\vec{t}/\vec{x}], \dots, r_n[\vec{t}/\vec{x}]$ interact safely with *each other* in D .

Proposition 4.1. For all $n \geq 1$, the rule

$$\frac{\Gamma \vdash_{\wedge\vee} t : \bigwedge_{1 \leq i \leq n} (U_i \Rightarrow T)}{\Gamma \vdash_{\wedge\vee} \lambda x. tx : (\bigvee_{1 \leq i \leq n} U_i) \Rightarrow T} \quad (x \notin \mathcal{X}(t))$$

is derivable in $\vdash_{\wedge\vee}$.

Theorem 4.2. If $(\vee E)$ is safe, then (IP) holds.

Proof. Let $f(\vec{t}) \in \mathcal{SN}$ and v such that for all $r \in \mathcal{R}(f)$, $v[r[\vec{t}/\vec{x}]/y] \in \mathcal{SN}$. We reason as in Thm. 3.13, using Thm. 3.11 and Lem. 3.9: there are Γ, V and $(U_r)_{r \in \mathcal{R}(f)}$ such that $\Gamma \vdash_{\wedge\vee} f(\vec{t}) : \bigvee_{r \in \mathcal{R}(f)} U_r$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{\wedge\vee} \lambda y. v : U_r \Rightarrow V$.

By Prop. 4.1, we have $\Gamma \vdash_{\wedge\vee} (\lambda y. v)f(\vec{t}) : V$, hence $v[f(\vec{t})/y] \in \mathcal{SN}$ since $(\vee E)$ is safe by assumption. \square

4.2. Orthogonality

We will show that the maximal method for the soundness of $(\vee E)$ is given by biorthogonals.

Given two sets \mathcal{A} and Π , and $\perp \subseteq \mathcal{A} \times \Pi$, let

$$\begin{aligned} \forall A \subseteq \mathcal{A}, \quad A^\perp &=_{\text{def}} \{\pi \in \Pi \mid \forall a (a \in A \Rightarrow a \perp \pi)\}; \\ \forall P \subseteq \Pi, \quad P^\perp &=_{\text{def}} \{a \in \mathcal{A} \mid \forall \pi (\pi \in P \Rightarrow a \perp \pi)\}. \end{aligned}$$

Let us discuss a few properties of \perp . First, it is easy to see that $(\cdot)^\perp$ is anti-monotonic: $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$. It follows that $X = X^{\perp\perp}$ iff there is Y such that $X = Y^\perp$. Moreover, $(\cdot)^{\perp\perp}$ is a closure operator on $\mathcal{P}(\mathcal{A})$ (resp. $\mathcal{P}(\Pi)$).

For the interpretation of $(\vee E)$, the important point is the De Morgan laws. We have

$$\begin{aligned} X^\perp \cap Y^\perp &= (X \cup Y)^\perp, \\ X^\perp \cup Y^\perp &\subseteq (X \cap Y)^\perp. \end{aligned}$$

Note that in general, $(X \cap Y)^\perp \not\subseteq X^\perp \cup Y^\perp$. Indeed, if x is orthogonal to every element of $X \cap Y$, then there is no reason that x is orthogonal to every element of X or to every element of Y .

4.3. Biorthogonal Reducibility

We now introduce a family of biorthogonals that arises from the observation of \mathcal{SN} , the top element of \mathcal{CR} .

For the interpretation of $(\vee E)$, we use extended evaluation contexts $E[\] \in \mathcal{E}$ that allow call-by-value evaluation [18]. It is interesting to see them both as terms and contexts. Therefore, we let $[] \in \mathcal{X}$ be a distinguished variable and define \mathcal{E} as follows:

$$E[\] \in \mathcal{E} ::= [\] \mid E[\]t \mid tE[\].$$

We let $E[t] =_{\text{def}} (E[\])[t/[]]$.

Definition 4.3. Let $t \top E[\]$ iff $E[t] \in \mathcal{SN}$.

Note that since $\mathcal{E} \not\subseteq \mathcal{SN}$, we have $\emptyset^{\top\top} = \emptyset$. It is easy to see that $\mathcal{SN}^\top = \{[\]\}$, hence $\mathcal{SN}^{\top\top} = \mathcal{SN}$. Therefore, by monotonicity of $(\cdot)^{\top\top}$, $A \subseteq \mathcal{SN}$ implies $A^{\top\top} \subseteq \mathcal{SN}$.

Since we use call-value-evaluation contexts in \mathcal{E} , it needs some work to prove that \top -biorthogonals are reducibility candidates. The main point is to prove the neutral term property, for which we use completeness of type assignment and the axiom (IP).

Proposition 4.4 (Neutral Term Property). Let $E[\] \in \mathcal{SN}$ and $t \in \mathcal{N}$. If (IP) holds and $\forall u (t \rightarrow u \Rightarrow E[u] \in \mathcal{SN})$ then $E[t] \in \mathcal{SN}$.

Proof. First, if $t \in \mathcal{HN}$, since $E[\] \in \mathcal{SN}$, by Thm. 3.14 we have $E[t] \in \mathcal{SN}$.

Otherwise, since $t \in \mathcal{N}$, we have $(t)_{\rightarrow} \neq \emptyset$ and there are two cases. In the first one $t = (\lambda x.t_1)t_2\vec{v}$ and we conclude by Thm. 3.13. In the second one $t = f(\vec{t})\vec{v}$ with $f \in \mathcal{F}$ and the result follows from (IP). \square

Then, we obtain that biorthogonals of non-empty subsets of \mathcal{SN} are reducibility candidates.

Lemma 4.5. *If $A \subseteq \mathcal{SN}$ is not empty, then (IP) implies $A^{\top\top} \in \mathcal{CR}$.*

Proof. Since $A \subseteq \mathcal{SN}$, we have $A^{\top\top} \subseteq \mathcal{SN}$. Stability by reduction is trivial. Since $A \neq \emptyset$ we have $A^{\top} \subseteq \mathcal{SN}$, hence the neutral term property is insured by Prop. 4.4. applied using (IP). \square

Hence, the set $\{A^{\top\top} \mid \emptyset \neq A \subseteq \mathcal{SN}\}$ is a subset of \mathcal{CR} . Moreover, thanks to the idempotence of $(\cdot)^{\top\top}$, it is exactly the set of $A^{\top\top}$ with $A \in \mathcal{CR}$. Therefore, we can consistently denote it by $\mathcal{CR}^{\top\top}$.

On the other hand, it is interesting to note that the reducibility candidates involved in the interpretation of $T \in \mathcal{T}_{\wedge}$ are biorthogonals. This observation seems to originate from [13], and to be the starting point of the utilization of biorthogonals in reducibility. If $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{E}$ let $A \cdot B =_{\text{def}} \{E[\]a \mid a \in A \ \& \ E[\] \in B\}$.

Proposition 4.6 (Types as Biorthogonals). *For all $T \in \mathcal{T}_{\wedge}$, $\llbracket T \rrbracket = \llbracket T \rrbracket^{\top\top}$.*

Proof. Indeed, we have

$$\begin{aligned} \llbracket \circ \rrbracket &= \mathcal{SN} = \{\{\ \}\}^{\top} \\ \llbracket U \Rightarrow V \rrbracket &= (\llbracket U \rrbracket \cdot \llbracket V \rrbracket^{\top})^{\top} \\ \llbracket U \wedge V \rrbracket &= (\llbracket U \rrbracket^{\top} \cup \llbracket V \rrbracket^{\top})^{\top}. \end{aligned}$$

\square

4.4. Completeness of Biorthogonals

Biorthogonals are not stable by union because the De Morgan law $A^{\perp} \cup B^{\perp} = (A \cap B)^{\perp}$ is in general not satisfied. However, since $A^{\perp} \cap B^{\perp} = (A \cup B)^{\perp}$ we have

$$(A \cup B)^{\perp\perp} = (A^{\perp} \cap B^{\perp})^{\perp}.$$

Therefore, the closure of union is quite informative: if $a \in (A \cup B)^{\perp\perp}$ then $a \perp \pi$ for all $\pi \in A^{\perp} \cap B^{\perp}$.

We take advantage of this fact for the interpretation of $(\vee E)$, and from now on, the interpretation of types with biorthogonals will differ from that of Sec. 3.1.

Given $T \in \mathcal{T}$, we define $\llbracket T \rrbracket$ as follows:

$$\begin{aligned} \llbracket \circ \rrbracket &=_{\text{def}} \{\{\ \}\}^{\top} (= \mathcal{SN}) \\ \llbracket U \Rightarrow V \rrbracket &=_{\text{def}} (\llbracket U \rrbracket \cdot \llbracket V \rrbracket^{\top})^{\top} \\ \llbracket U \wedge V \rrbracket &=_{\text{def}} (\llbracket U \rrbracket^{\top} \cup \llbracket V \rrbracket^{\top})^{\top} \\ \llbracket U \vee V \rrbracket &=_{\text{def}} (\llbracket U \rrbracket^{\top} \cap \llbracket V \rrbracket^{\top})^{\top}. \end{aligned}$$

Lemma 4.7. *If (IP) then for all $T \in \mathcal{T}$, $\llbracket T \rrbracket \in \mathcal{CR}$.*

Proof. By induction on T , using Lem. 4.5 for $T = T_1 \vee T_2$.

Note that we cannot avoid the induction on T and directly use Lem. 4.5, since it requires $\emptyset \neq \llbracket T \rrbracket \subseteq \mathcal{SN}$. \square

It is directly in the soundness proof that we use the possibility of call-by-value evaluation with \mathcal{E} .

Theorem 4.8. *Let $\Gamma \vdash_{\wedge\vee} t : T$. If (IP) and $\sigma \models_{(\cdot)} \Gamma$ then $t\sigma \in \llbracket T \rrbracket$.*

Proof. By induction on $\Gamma \vdash_{\wedge\vee} t : T$. Thanks to Lem. 4.7, using (IP), we have $\llbracket U \rrbracket \in \mathcal{CR}$ for all $U \in \mathcal{T}$. Then, the proof is identical to that of Thm. 3.6, except for the case of the rule $(\vee E)$. We only detail this case.

$$\frac{\Gamma \vdash_{\wedge\vee} t : T_1 \vee T_2 \quad \begin{array}{l} \Gamma, x : T_1 \vdash_{\wedge\vee} c : C \\ \Gamma, x : T_2 \vdash_{\wedge\vee} c : C \end{array}}{\Gamma \vdash_{\wedge\vee} c[t/x] : C}$$

Let $\sigma \models_{(\cdot)} \Gamma$, $t' =_{\text{def}} t\sigma$ and $c' =_{\text{def}} c\sigma$. Recall that we can assume that $x \notin \mathcal{X}(\sigma)$. Hence, we have to show that $c'[t'/x] \in \llbracket C \rrbracket$.

Let $E[\] \in \llbracket C \rrbracket^{\top}$. By induction hypothesis, for all $v \in \llbracket T_1 \rrbracket \cup \llbracket T_2 \rrbracket$ we have $c'[v/x] \top E[\]$. Moreover, since $v \in \mathcal{SN}$, we have $(\lambda x.c')v \top E[\]$ by Thm. 3.13. It follows that $E[(\lambda x.c')[\]] \in \llbracket T_1 \rrbracket^{\top} \cap \llbracket T_2 \rrbracket^{\top}$.

On the other hand, by induction hypothesis we have $t' \in (\llbracket T_1 \rrbracket^{\top} \cap \llbracket T_2 \rrbracket^{\top})^{\top}$. Therefore $t' \top E[(\lambda x.c')[\]]$, hence $(\lambda x.c')t' \top E[\]$. We deduce that $c'[t'/x] \top E[\]$. \square

Theorem 4.9. *The following are equivalents:*

- (i) *If $\Gamma \vdash_{\wedge\vee} t : T$ then $t \in \mathcal{SN}$.*
- (ii) *For all $f \in \mathcal{F}$, if $f(\vec{t}) \in \mathcal{SN}$ and $v[r[\vec{t}/\vec{x}]/y] \in \mathcal{SN}$ for all $r \in \mathcal{R}(f)$, then $v[f(\vec{t})/y] \in \mathcal{SN}$.*
- (iii) *The interpretation $\llbracket \cdot \rrbracket$ is sound for $(\vee E)$.*

4.5. Comparison with Reducibility Candidates

We have shown that the biorthogonal interpretation is sound and complete w.r.t. the safety of $(\vee E)$. We now compare it to the impredicative interpretation of $(\vee E)$ defined in \mathcal{CR} . Given $A, B \in \mathcal{CR}$, let $A \vee B$ be

$$\{t \mid \forall C \in \mathcal{CR}, \forall c \in (A \Rightarrow C) \cap (B \Rightarrow C), ct \in C\}$$

In general, it is unclear whether $A, B \in \mathcal{CR}$ implies $A \vee B \in \mathcal{CR}$. Indeed, given $t \in \mathcal{N}$, $C \in \mathcal{CR}$ and knowing that for all $u \in (t)_{\rightarrow}$, $cu \in C$ it is not clear why $ct \in C$. On the other hand, a subtle modification to $A \vee B$ makes it much easier to handle. Let $A \vee_{\top} B$ be

$$\{t \mid \forall C \in \mathcal{CR}^{\top\top}, \forall c \in (A \Rightarrow C) \cap (B \Rightarrow C), ct \in C\}.$$

The point is that in observing $ct \in C$ with $C \in \mathcal{CR}^{\top\top}$, in fact we observe \mathcal{SN} since $ct \in C$ holds iff for all $E[\] \in C^\top$, $E[ct] \in \mathcal{SN}$. Thanks to soundness of completeness of $\vdash_{\wedge\vee}$, we are able to extract the information we need from the observation of \mathcal{SN} .

Lemma 4.10. *For all $A, B \subseteq \mathcal{SN}$,*

$$(A^\top \cap B^\top)^\top = A \vee_{\top} B.$$

Proof. If $c \in (A \Rightarrow C) \cap (B \Rightarrow C)$ and $E[\] \in C^\top$, then $E[c[\]] \in A^\top \cap B^\top$. This implies $(A^\top \cap B^\top)^\top \subseteq A \vee_{\top} B$.

Conversely, if $E[\] \in A^\top \cap B^\top$ then $\lambda x.E[x] \in (A \Rightarrow \mathcal{SN}) \cap (B \Rightarrow \mathcal{SN})$. \square

In conclusion, the interest, and strength, of biorthogonals, is that they bring observation at an arbitrary $C \in \mathcal{CR}^{\top\top}$ back to the observation of \mathcal{SN} , that we can manage, thanks to completeness of type assignment.

5. Sufficient Conditions for Safe Interaction

In this section, we address the question of finding sufficient conditions for the safety of $(\vee E)$.

We begin by studying two conditions, arising when closing by union respectively reducibility candidates and biorthogonals (involving applicative contexts only). These conditions follow a common scheme that we present first.

On the other hand, it is natural to ask whether typability in a subsystem of $\vdash_{\wedge\vee}$ can imply safe interaction (i.e. the safety of $(\vee E)$). In particular, rewrite systems for which intersection types are sufficient for the completeness of type assignment may have a form of uniformity in their computational behavior. We show that this is not sufficient for the safety of $(\vee E)$. It is interesting to note that, however, stability by union of reducibility candidates implies completeness of \vdash_{\wedge} w.r.t. strong normalization.

5.1. Stability by Union

One possibility is to use a family of reducibility candidates that is stable by union. We address this question in general terms.

Theorem 5.1. *Let $\mathcal{U} \subseteq \mathcal{CR}$ be a collection of sets such that $\mathcal{SN} \in \mathcal{U}$ and $A, B \in \mathcal{U}$ implies $A \Rightarrow B, A \cap B, A \cup B \in \mathcal{U}$. Given $T \in \mathcal{T}$, define $\llbracket T \rrbracket_{\mathcal{U}}$ as*

$$\begin{aligned} \llbracket o \rrbracket_{\mathcal{U}} &=_{\text{def}} \mathcal{SN} \\ \llbracket T \Rightarrow U \rrbracket_{\mathcal{U}} &=_{\text{def}} \llbracket T \rrbracket_{\mathcal{U}} \Rightarrow \llbracket U \rrbracket_{\mathcal{U}} \\ \llbracket T \wedge U \rrbracket_{\mathcal{U}} &=_{\text{def}} \llbracket T \rrbracket_{\mathcal{U}} \cap \llbracket U \rrbracket_{\mathcal{U}} \\ \llbracket T \vee U \rrbracket_{\mathcal{U}} &=_{\text{def}} \llbracket T \rrbracket_{\mathcal{U}} \cup \llbracket U \rrbracket_{\mathcal{U}}. \end{aligned}$$

If $\Gamma \vdash_{\wedge\vee} t : T$ and $\sigma \models_{\llbracket \cdot \rrbracket_{\mathcal{U}}} \Gamma$ then $t\sigma \in \llbracket T \rrbracket_{\mathcal{U}}$

The next point is to build such a $\mathcal{U} \subseteq \mathcal{CR}$. We can gain some insight by looking at collections of sets arising as the closure by union of some closure operator. This motivates the following proposition, whose proof is not difficult and can be found in [15]. If $\bar{\cdot} : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$ is a closure operator, write \bar{x} for $\overline{\{x\}}$ and $\bar{\mathcal{P}}^*(D)$ for $\{\bar{X} \mid \emptyset \neq X \subseteq D\}$.

Proposition 5.2. *Given a closure operator $\bar{\cdot} : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$, let $\bar{\Omega}$ be the set of non-empty $X \subseteq D$ such that $\bar{X} = \bigcup \{\bar{x} \mid x \in X\}$. Then $\bar{\Omega}$ is the smallest set such that $\bar{\mathcal{P}}^*(D) \subseteq \bar{\Omega}$ and $\emptyset \neq \mathcal{C} \subseteq \bar{\Omega} \Rightarrow \bigcup \mathcal{C}, \bigcap \mathcal{C} \in \bar{\Omega}$.*

5.2. The Principal Reduct Property

We first consider the closure by union of \mathcal{CR} .

Definition 5.3. *Let $t \sqsubseteq_{\mathcal{SN}} u$ iff $t, u \in \mathcal{SN}$ and for all $v \notin \mathcal{N}$, if $t \rightarrow^* v$ then $u \rightarrow^* v$.*

Note that if $t \sqsubseteq_{\mathcal{SN}} u$ and $t\vec{t}, u\vec{t} \in \mathcal{SN}$, then $t\vec{t} \sqsubseteq_{\mathcal{SN}} u\vec{t}$. In [15], it is shown that $\bar{t} = \{u \mid u \sqsubseteq_{\mathcal{SN}} t\}$ for all $t \in \mathcal{SN}$ (where $\bar{\cdot}$ is the closure operator of \mathcal{CR} defined in Sec. 3.1). Then, it follows from Prop. 5.2, that the closure by union of \mathcal{CR} is the set of non-empty $C \subseteq \mathcal{SN}$ which are downward closed w.r.t. $\sqsubseteq_{\mathcal{SN}}$.

Moreover, we have shown that the obtained collection of sets is a collection of reducibility candidates iff for every $t \in \mathcal{N} \cap \mathcal{SN}$ which is not a normal form there is $u \in (t)_{\rightarrow}$ such that $u = \max_{\sqsubseteq_{\mathcal{SN}}} (t)_{\rightarrow}$. We say that such an u is a *principal reduct* of t . When $t \in \mathcal{N} \cap \mathcal{SN}$ has a principal reduct or is a normal form, we say that it has the *principal reduct property* (p.r.p.). We say that \mathcal{R} has the p.r.p. when every $f(\vec{t}) \in \mathcal{SN}$ with $f \in \mathcal{F}$ has the p.r.p..

In our case, the principal reduct property of all $t \in \mathcal{N} \cap \mathcal{SN}$ is equivalent to the p.r.p. of \mathcal{R} .

Proposition 5.4. *Every $t \in \mathcal{N} \cap \mathcal{SN}$ has the p.r.p. iff \mathcal{R} has the p.r.p..*

Proof. Easy, using Weak Standardization:

If $t \mapsto_{\beta} u$ and $t\vec{t} \rightarrow v$ with $v \neq u\vec{t}$, then $v = t'\vec{t}'$ with $(t, \vec{t}) \rightarrow (t', \vec{t}')$ and there is u' such that $t' \mapsto_{\beta} u'$ and $u\vec{t} \rightarrow^* u'\vec{t}'$. \square

To summarize, we obtain that the p.r.p. of \mathcal{R} implies that for all $T, U \in \mathcal{T}$ we have $\llbracket T \rrbracket \cup \llbracket U \rrbracket = \llbracket T \rrbracket \cup \llbracket U \rrbracket$ (recall from Sec. 3.1 that $\llbracket T \vee U \rrbracket =_{\text{def}} \llbracket T \rrbracket \cup \llbracket U \rrbracket$). Then, the safety of $(\vee E)$ follows from Thm. 5.1.

Theorem 5.5. *Assume that \mathcal{R} has the principal reduct property. If $\Gamma \vdash_{\wedge\vee} t : T$ and $\sigma \models_{\llbracket \cdot \rrbracket} \Gamma$ then $t\sigma \in \llbracket T \rrbracket$.*

5.3. Closure by Union of Biorthogonals

We now turn to the closure by union of a family of biorthogonals. Let $\perp\!\!\!\perp \subseteq \mathcal{A} \times \Pi$ and $a \leq b$ iff $a^{\perp\!\!\!\perp} \subseteq b^{\perp\!\!\!\perp}$.

For all $a \in A$, we have $a^{\perp\perp} = \{b \mid a \leq b\}$. Hence, by Prop. 5.2, the closure by union of \perp -biorthogonals is the collection of non-empty subsets of \mathcal{A} (resp. Π) that are upward closed w.r.t. \leq .

Definition 5.6. Let $t \triangleleft \vec{t}$ iff $t\vec{t} \in \mathcal{SN}$ and $t \lesssim u$ iff $t \triangleleft u \triangleleft$.
Let \mathcal{O} be the set of all non-empty $C \subseteq \mathcal{SN}$ such that if $t \in C$ and $t \lesssim u$, then $u \in C$.

Hence, \mathcal{O} is the closure by union of $\mathcal{CR}^{\triangleleft\triangleleft}$. Note that $t \sqsubseteq_{\mathcal{SN}} u$ implies $u \lesssim t$. It is also easy to see that $t \lesssim u$ implies $t\vec{t} \lesssim u\vec{t}$ for all \vec{t} . The next proposition easily follows.

Proposition 5.7 (Type Constructions in \mathcal{O}). Let $A, B \subseteq \mathcal{O}$. Then, $A \Rightarrow B, A \cap B, A \cup B \in \mathcal{O}$.

The interesting point is the necessary and sufficient condition for $\mathcal{O} \subseteq \mathcal{CR}$: for all $f(\vec{t}) \in \mathcal{SN}$ with $f \in \mathcal{F}$ there is $d \in \mathcal{R}(f)$ such that $d[\vec{t}/\vec{x}] = \min_{\lesssim} \{r[\vec{t}/\vec{x}] \mid r \in \mathcal{R}(f)\}$. This property is called the weak principal reduct property (w.p.r.p.).

Lemma 5.8. $\mathcal{O} \subseteq \mathcal{CR}$ iff \mathcal{R} has the w.p.r.p.

Proof. Using Weak Standardization (see Prop. 5.4). \square

Theorem 5.9. Assume that \mathcal{R} has the weak principal reduct property. If $\Gamma \vdash_{\wedge} t : T$ and $\sigma \models_{[\cdot]_{\mathcal{O}}} \Gamma$, then $t\sigma \in \llbracket T \rrbracket_{\mathcal{O}}$.

5.4. Typability in \vdash_{\wedge}

A natural question is whether $(\vee E)$ is safe with rewrite systems for which intersection types are sufficient for the completeness w.r.t. strong normalization.

Indeed, one could expect to have that if for all $t \in \mathcal{SN}$ there are Γ, T such that $\Gamma \vdash_{\wedge} t : T$, then $(\vee E)$ is safe. This is not the case, as shown in the following example.

Example 5.10. Consider the system of Ex. 2.2. Let $T_S =_{\text{def}} o \wedge (o \Rightarrow o)$, hence $\Gamma \vdash_{\wedge} t : T_S$ implies $\Gamma \vdash_{\wedge} tt : o$. Then we have:

$$(SY) \frac{\begin{array}{l} \vdash_{\wedge} \lambda xy.g(xa\delta) : (o \Rightarrow T_S \Rightarrow o) \Rightarrow T_S \Rightarrow o \\ \vdash_{\wedge} \lambda xy.g(yy) : (o \Rightarrow T_S \Rightarrow o) \Rightarrow T_S \Rightarrow o \end{array}}{\vdash_{\wedge} f : (o \Rightarrow T_S \Rightarrow o) \Rightarrow T_S \Rightarrow o}$$

Moreover, it is easy to see that with this system, if $t \in \mathcal{SN}$ then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$. Since by Ex. 2.2 this system breaks the safety of $(\vee E)$, it follows that completeness of typability in \vdash_{\wedge} does not imply safe interaction.

However, it is interesting to note that the p.r.p. implies completeness of \vdash_{\wedge} . That is, if \mathcal{R} has the p.r.p. and $t \in \mathcal{SN}$ then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$.

Theorem 5.11 (Completeness). Assume that \mathcal{R} has the p.r.p.. If $t \in \mathcal{SN}$, then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$.

6. \mathcal{HN} -Biorthogonality

In this section, we briefly discuss an orthogonality relation based on the observation of \mathcal{HN} (the bottom element of \mathcal{CR}) rather than \mathcal{SN} (its top element). This semantics induces a better adequacy with the type system. However, it is not clear whether it is complete w.r.t. the safety of $(\vee E)$.

We interpret $T \in \mathcal{T}$ by $\llbracket T \rrbracket_{\mathcal{HN}} \in \mathcal{CR}$ as in Sec. 3.1, except that $\llbracket o \rrbracket_{\mathcal{HN}} =_{\text{def}} \mathcal{HN}$. The properties of Sec. 3.2 holds also for $\llbracket T \rrbracket_{\mathcal{HN}}$. This way we get the soundness of o -typability w.r.t. \mathcal{HN} .

Theorem 6.1. If $\Gamma \vdash_{\wedge} t : o$ then $t \in \mathcal{HN}$.

Since Thm. 3.12 says that any hereditary neutral term is typable by any $T \in \mathcal{T}$, it follows that a term is typable by o iff it can be given any type (in different contexts), suggesting that o should be the least element of \mathcal{T} . This agrees with $\llbracket \cdot \rrbracket_{\mathcal{HN}}$, since $\llbracket o \rrbracket_{\mathcal{HN}} \subseteq \llbracket T \rrbracket_{\mathcal{HN}}$ for all $T \in \mathcal{T}$, but contradicts $\llbracket o \rrbracket = \mathcal{SN}$.

A development similar to that of Sec. 4 goes through with $\llbracket \cdot \rrbracket_{\mathcal{HN}}$. First, we obtain the analogous of Thm. 3.13 and Thm. 3.14 for \mathcal{HN} . Again, these results seem not to have been published before.

Theorem 6.2. If $(\lambda x.t)u \in \mathcal{SN}$ and $v[t[u/x]/y] \in \mathcal{HN}$ then $v[(\lambda x.t)u/y] \in \mathcal{HN}$.

Proof. As in Thm. 3.13, from $(\lambda x.t)u \in \mathcal{SN}$, thanks to Thm. 3.11 we get Γ', T and U such that $\Gamma' \vdash_{\wedge} u : U$ and $\Gamma' \vdash_{\wedge} t[u/x] : T$. On the other hand, thanks to Thm. 3.12 there is Γ'' such that $\Gamma'' \vdash_{\wedge} v[t[u/x]/y] : o$. Now, reasoning as in Thm. 3.13, we obtain that $\Gamma' \wedge \Gamma'' \vdash_{\wedge} (\lambda y.v)((\lambda x.t)u) : o$, hence $v[(\lambda x.t)u/y] \in \mathcal{HN}$ by Thm. 6.1. \square

Theorem 6.3. If $t \in \mathcal{HN}$ and $v \in \mathcal{HN}$ then $v[t/x] \in \mathcal{HN}$.

Proof. As for Thm. 3.14, using Thm. 3.12 instead of Thm. 3.11 and Thm. 6.1 instead of Cor. 3.7. \square

In the same way that \top -biorthogonals where in correspondence with $\llbracket \cdot \rrbracket$, we can define \perp -biorthogonals in correspondence with $\llbracket \cdot \rrbracket_{\mathcal{HN}}$.

Definition 6.4. Let $t \perp E[\]$ iff $E[t] \in \mathcal{HN}$.

In fact, reduction to an hereditary neutral term corresponds to reduction to error in [18, 17]. Since in these papers biorthogonals are based on the observation of non-reduction to error, they are in some sense dual to our \mathcal{HN} -biorthogonals.

Note that $t^{\perp} \subseteq u^{\perp}$ implies $t^{\top} \subseteq u^{\top}$, but the converse is false: $\lambda x.x^{\top} \subseteq \lambda yx.yx^{\top}$, but $(\lambda x.x)z \in \mathcal{HN}$ while $(\lambda yx.yx)z \notin \mathcal{HN}$. As in Prop. 4.6, if $U, V \in \mathcal{T}_{\wedge}$ we have

$$\begin{aligned} \llbracket o \rrbracket_{\mathcal{HN}} &= \mathcal{HN} = \{[\]\}^{\perp} \\ \llbracket U \Rightarrow V \rrbracket_{\mathcal{HN}} &= (\llbracket U \rrbracket_{\mathcal{HN}} \cdot \llbracket V \rrbracket_{\mathcal{HN}})^{\perp} \\ \llbracket U \wedge V \rrbracket_{\mathcal{HN}} &= (\llbracket U \rrbracket_{\mathcal{HN}} \cup \llbracket V \rrbracket_{\mathcal{HN}})^{\perp}. \end{aligned}$$

In order to get an interesting interpretation of $(\vee E)$, we define $(\cdot)_{\mathcal{HN}}$ analogously as (\cdot) . Note that the only change in the definition is the orthogonality relation: we deduce $(o)_{\mathcal{HN}} = \mathcal{HN}$ from $(o)_{\mathcal{HN}} =_{\text{def}} \{[\]\}^\perp$. Again, the important case is that of $(\vee E)$:

$$((T_1 \vee T_2)_{\mathcal{HN}}) =_{\text{def}} ((T_1)_{\mathcal{HN}}^\perp \cap ((T_2)_{\mathcal{HN}}^\perp)^\perp.$$

With the same method as in Sec. 4.4, we obtain the dual of Thm. 4.9:

Theorem 6.5. *The following are equivalent:*

- (i) *If $\Gamma \vdash_{\wedge\vee} t : o$ then $t \in \mathcal{HN}$.*
- (ii) *(IP $_{\mathcal{HN}}$): For all $f \in \mathcal{F}$, if $f(\vec{t}) \in \mathcal{SN}$ and for all $r \in \mathcal{R}(f)$, $v[r[\vec{t}/\vec{x}]/y] \in \mathcal{HN}$ then $v[f(\vec{t})/y] \in \mathcal{HN}$.*
- (iii) *$(\cdot)_{\mathcal{HN}}$ is sound for $(\vee E)$.*

We conclude by showing that we indeed obtained a sufficient condition for the safety of $(\vee E)$.

Lemma 6.6. $(\text{IP}_{\mathcal{HN}}) \Rightarrow (\text{IP})$.

Proof. Let $f(\vec{t}) \in \mathcal{SN}$, such that $v[r[\vec{t}/\vec{x}]/y] \in \mathcal{SN}$ for each $r \in \mathcal{R}(f)$. Then, since $\mathcal{R}(f)$ is finite, there is \vec{u} such that for all $r \in \mathcal{R}(f)$, $v[r[\vec{t}/\vec{x}]/y]\vec{u} \in \mathcal{HN}$. By $(\text{IP}_{\mathcal{HN}})$ we obtain that $v[f(\vec{t})/y]\vec{u} \in \mathcal{HN}$ hence $v[f(\vec{t})/y] \in \mathcal{SN}$. \square

The converse is unclear because we do not have subject reduction in $\vdash_{\wedge\vee}$. It would require, at least, to add the subtyping rule $U \Rightarrow (T_1 \vee T_2) \leq (U \Rightarrow T_1) \vee (U \Rightarrow T_2)$, which is clearly unsound in our setting. Subject reduction in presence of $(\vee E)$ is extensively studied in [1].

7. Conclusion

We provided necessary and sufficient conditions for the safety of the elimination rule of union types. These conditions express that the rule necessitates a form of interaction property. Our main result is the proof that in our framework, biorthogonals are complete w.r.t. the safety of $(\vee E)$.

We considered a very simple form of rewriting, with the objective of concentrating ourselves on the very problem of $(\vee E)$. However, as future work, it is important to study the case of rewrite rules with pattern matching.

Our results can be summarized in the following diagram:

$$\begin{array}{ccc} (\vee E) \text{ is safe} & \iff & (\text{IP}) \longleftarrow \text{w.p.r.p. (Sec.5.3)} \\ & & \uparrow \uparrow \\ & & (\text{IP}_{\mathcal{HN}}) \text{ (Sec. 6)} \quad \text{p.r.p. (Sec.5.2)} \end{array}$$

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A. Derivability in $\vdash_{\wedge\vee}$ and \vdash_{\wedge}

In this appendix, we present results on derivability in the type systems $\vdash_{\wedge\vee}$ and \vdash_{\wedge} .

We begin by some fundamental properties of the type system. They are well-known in the case of the pure lambda-calculus with intersection types [12, 10, 2]. Concerning union types, they have been proved in [6, 7] when $(\mathcal{T}, \leq, \wedge, \vee)$ is a distributive lattice. Proving them in our framework does not bring any difficulty.

We then detail the proof of completeness of type assignment in $\vdash_{\wedge\vee}$ w.r.t. strong normalization.

Excepted Thm. A.5, all the properties presented in this section are common to \vdash_{\wedge} and $\vdash_{\wedge\vee}$. Let $ty \in \{\wedge, \wedge\vee\}$.

Proposition A.1 (Inversion).

- (i) $\Gamma \vdash_{ty} x : T$ iff $(x : U) \in \Gamma$ with $U \leq T$.
- (ii) $\Gamma \vdash_{ty} tu : T$ iff there is U such that $\Gamma \vdash_{ty} t : U \Rightarrow T$ and $\Gamma \vdash_{ty} u : U$.
- (iii) $\Gamma \vdash_{ty} \lambda x.t : T$ iff there exists $n \geq 1$ and $U_1, \dots, U_n, V_1, \dots, V_n$ such that $\bigwedge_i (U_i \Rightarrow V_i) \leq T$ and for all $i \in \{1, \dots, n\}$, $\Gamma, x : U_i \vdash_{ty} t : V_i$.
- (iv) $\Gamma \vdash_{ty} f(\vec{t}) : V$ iff there are \vec{T} such that $\Gamma \vdash_{ty} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma, \vec{x} : \vec{T} \vdash_{ty} r : V$.

Proof. All cases are proved by trivial inductions on the typing derivations. \square

The next property is fundamental for type systems. It corresponds to cut-elimination.

Lemma A.2 (Substitution). *If $\Gamma, x : U \vdash_{ty} t : T$ and $\Gamma \vdash_{ty} u : U$ then $\Gamma \vdash_{ty} t[u/x] : T$.*

Proof. By a trivial induction on $\Gamma, x : U \vdash_{ty} t : T$. \square

In particular, if $x \notin \mathcal{X}(t)$ then $\Gamma, x : U \vdash_{ty} t : T$ implies $\Gamma \vdash_{ty} t : T$. This property is called Contraction.

We now turn to the key properties for completeness of type assignment. They are characteristic properties of intersection types (recall that in our case, union types are needed to type function symbols, not for the lambda-calculus itself).

Proposition A.3 (Interpolation). *Let $x \notin \Gamma$. If $\Gamma \vdash_{ty} t[u/x] : T$ and $\Gamma \vdash_{ty} u : U$, then there is V such that $\Gamma, x : V \vdash_{ty} t : T$ and $\Gamma \vdash_{ty} u : V$.*

Proof. Given $v \in \Lambda$, we let $v' =_{\text{def}} v[u/x]$. We reason by induction on t . We only detail the case of $t = f(\vec{t})$.

By inversion, (Prop. A.1.(iv)) there are \vec{T} such that $\Gamma \vdash_{ty} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma, \vec{x} : \vec{T} \vdash_{ty} r : T$. By induction hypothesis, there is \vec{V} such that for all $i \in \{1, \dots, |\vec{t}|\}$,

$\Gamma, x : V_i \vdash_{ty} t_i : T_i$ and $\Gamma \vdash_{ty} u : V_i$. Taking $V =_{\text{def}} \bigwedge_i V_i$ we get $\Gamma, x : V \vdash_{ty} \vec{t} : \vec{T}$ and $\Gamma \vdash_{ty} u : V$. Moreover, we have for all $r \in \mathcal{R}(f)$ $\Gamma, x : V \vdash_{ty} r : T$ and it follows that $\Gamma, x : V \vdash_{ty} f(\vec{t}) : T$. \square

Note that if $\Gamma \vdash_{ty} t : T$ and $y \notin \Gamma$, then $\Gamma[y/x] \vdash_{ty} t[y/x] : T$.

Lemma A.4 (Weak Head Expansion).

- (i) *If $\Gamma \vdash_{ty} u : U$ and $\Gamma \vdash_{ty} t[u/x]\vec{v} : T$ then $\Gamma \vdash_{ty} (\lambda x.t)u\vec{v} : T$.*
- (ii) *For all $f \in \mathcal{F}$, if $\Gamma \vdash_{ty} \vec{t} : \vec{T}$, and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{ty} r[\vec{t}/\vec{x}]\vec{v} : T$, then $\Gamma \vdash_{ty} f(\vec{t})\vec{v} : T$.*

Proof. We only detail (ii): (i) is similar and simpler.

First, assume that \vec{x} is disjoint from $\text{dom}(\Gamma)$. We reason by induction on $|\vec{v}|$.

In the case $|\vec{v}| = 0$, we have $\Gamma \vdash_{ty} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{ty} r[\vec{t}/\vec{x}] : T$. By Prop. A.3, for all $r \in \mathcal{R}(f)$ there are Γ_r and \vec{T}^r such that $\Gamma_r \vdash_{ty} \vec{t} : \vec{T}^r$ and $\Gamma_r, \vec{x} : \vec{T}^r \vdash_{ty} r : T$. Hence, taking $\vec{T}' =_{\text{def}} \bigwedge_r \vec{T}^r$ and $\Gamma =_{\text{def}} \bigwedge_r \Gamma_r$ we have $\Gamma \vdash_{ty} \vec{t} : \vec{T}'$ and for all $r \in \mathcal{R}(f)$, $\Gamma, \vec{x} : \vec{T}' \vdash_{ty} r : T$. Thus $\Gamma \vdash_{ty} f(\vec{t}) : T$.

Now, assume the property for \vec{v} and let $\Gamma \vdash_{ty} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{ty} r[\vec{t}/\vec{x}]\vec{v} : T$. By Prop. A.1.(ii), for all $r \in \mathcal{R}(f)$ there is V_r such that $\Gamma \vdash_{ty} r[\vec{t}/\vec{x}]\vec{v} : V_r \Rightarrow T$ and $\Gamma \vdash_{ty} \vec{v} : V_r$. Since $\mathcal{R}(f)$ is finite, taking $V = \bigwedge_r V_r$, we get $\Gamma \vdash_{ty} \vec{v} : V$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{ty} r[\vec{t}/\vec{x}]\vec{v} : V \Rightarrow T$. By induction hypothesis we have $\Gamma \vdash_{ty} f(\vec{t})\vec{v} : V \Rightarrow T$ and we conclude that $\Gamma \vdash_{ty} f(\vec{t})\vec{v} : T$.

It remains to consider the case where \vec{x} and $\text{dom}(\Gamma)$ are not disjoint. Let $\vec{x}' =_{\text{def}} \{x_i \mid i \in \{1, \dots, |\vec{x}|\} \ \& \ x_i \in \Gamma\}$. Hence, Γ is of the form $\Gamma', \vec{x}' : \vec{U}$. Consider \vec{y} disjoint from $\text{dom}(\Gamma)$ such that $|\vec{y}| = |\vec{x}'|$, and let $\vec{t}' =_{\text{def}} \vec{t}[\vec{y}/\vec{x}']$ and $\vec{v}' =_{\text{def}} \vec{v}[\vec{y}/\vec{x}']$. Therefore, we have $\Gamma', \vec{y} : \vec{U} \vdash_{ty} \vec{t}' : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma', \vec{y} : \vec{U} \vdash_{ty} r[\vec{t}'/\vec{x}]\vec{v}' : T$. Hence $\Gamma', \vec{y} : \vec{U} \vdash_{ty} f(\vec{t}')\vec{v}' : T$, since \vec{x} is disjoint from $\text{dom}(\Gamma', \vec{y} : \vec{U})$; and we have $\Gamma', \vec{x}' : \vec{U} \vdash_{ty} f(\vec{t}')\vec{v}' : T$. \square

We finish by the full proof of Thm. 3.11 (Completeness).

Theorem A.5 (Completeness). *If $t \in \mathcal{SN}$, then there are Γ and T such that $\Gamma \vdash_{\wedge\vee} t : T$.*

Proof. The proof is by induction on \prec . Recall that $t = \lambda \vec{x}. h\vec{v}$ where h is either a variable, a β -redex or a symbol $f(\vec{t})$.

If $|\vec{x}| \neq 0$, then by induction hypothesis there are Γ and \vec{T} such that $\Gamma, \vec{x} : \vec{T} \vdash_{\wedge\vee} h\vec{v} : T$, and therefore $\Gamma \vdash_{\wedge\vee} \lambda \vec{x}. h\vec{v} : \vec{T} \Rightarrow T$.

Now we assume that $|\vec{x}| = 0$ and reason by cases on h .

$h = x \in \mathcal{X}$. Since $\vec{v} \prec t$, by induction hypothesis there are Γ_i, \vec{U}^i and T_i such that $\Gamma_i, \vec{x} : \vec{U}^i \vdash_{\wedge\vee} v_i : T_i$. Hence, with $\Gamma' =_{\text{def}} \bigwedge_i (\Gamma_i, \vec{x} : \vec{U}^i)$, we have for all i that $\Gamma' \vdash_{\wedge\vee} v_i : T_i$. Let $T \in \mathcal{T}$ and $\Gamma =_{\text{def}} \Gamma' \wedge (x : \vec{T} \Rightarrow T)$. We have $\Gamma \vdash_{\wedge\vee} x\vec{v} : T$.

$h = (\lambda x.u)v$. We have $v \prec t$ and $u[v/x]\vec{v} \prec t$, hence by induction hypothesis there are Γ, T and V such that $\Gamma \vdash_{\wedge\vee} u[v/x]\vec{v} : T$ and $\Gamma \vdash_{\wedge\vee} v : V$. By Lem. A.4.(i), we have $\Gamma \vdash_{\wedge\vee} (\lambda x.u)v\vec{v} : T$.

$h = f(\vec{t})$. First, note that $\vec{t} \prec t$.

If $f \in \mathcal{C}$, since $\vec{v} \prec t$, there are Γ, \vec{T} and \vec{V} such that $\Gamma \vdash_{\wedge\vee} \vec{t} : \vec{T}$ and $\Gamma \vdash_{\wedge\vee} \vec{v} : \vec{V}$. Hence $\Gamma \vdash_{\wedge\vee} f(\vec{t})\vec{v} : T$ for all $T \in \mathcal{T}$.

The interesting case is when $f \in \mathcal{F}$. For all $r \in \mathcal{R}(f)$ we have $r[\vec{t}/\vec{x}]\vec{v} \prec t$ and by induction hypothesis there are Γ_r, \vec{T}^r and V_r such that $\Gamma_r \vdash_{\wedge\vee} \vec{t} : \vec{T}^r$ and $\Gamma_r \vdash_{\wedge\vee} r[\vec{t}/\vec{x}]\vec{v} : V_r$. Now, taking $\Gamma =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} \Gamma_r, \vec{T} =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} \vec{T}^r$ and $V =_{\text{def}} \bigvee_{r \in \mathcal{R}(f)} V_r$, we have $\Gamma \vdash_{\wedge\vee} \vec{t} : \vec{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{\wedge\vee} r[\vec{t}/\vec{x}]\vec{v} : V$. We conclude that $\Gamma \vdash_{\wedge\vee} f(\vec{t})\vec{v} : V$ thanks to Lem. A.4.(ii). \square

B. Typability in \vdash_{\wedge} (Sec 5.4)

In this appendix, we give the proofs of Sec. 5.4.

B.1. Proofs of Example 5.10

In this section, we prove the claim of Ex. 5.10: for the system of Ex. 2.2, intersection types are complete w.r.t. strong normalization.

Since by Ex. 2.2 this system breaks the safety of $(\vee E)$, it follows that completeness of type assignment in \vdash_{\wedge} does not imply safe interaction. We use the preorder \prec defined in Def.3.10.

Proposition B.1. *Let \mathcal{R} be the system of Ex. 2.2 and $f\vec{v} \in \mathcal{SN}$ be such that every $u \prec f\vec{v}$ is typable in \vdash_{\wedge} . Then $f\vec{v}$ is typable in \vdash_{\wedge} .*

Proof. We reason by cases on $|\vec{v}|$. Let $u_1 =_{\text{def}} \lambda xy.g(xa\delta)$, $u_2 =_{\text{def}} \lambda xy.g(yy)$ and recall that $T_S = o \wedge (o \Rightarrow o)$.

$|\vec{v}| = 0$. This case is dealt with in Ex. 5.10.

$|\vec{v}| = 1$. Let $v =_{\text{def}} \vec{v}$. Since $va\delta \prec fv$, we have Γ and T such that $\Gamma \vdash_{\wedge} va\delta : T$. Thanks to Prop. A.1.(ii) there are U_1, U_2 such that $\Gamma \vdash_{\wedge} v : U_1 \Rightarrow U_2 \Rightarrow T$, $\Gamma \vdash_{\wedge} a : U_1$ and $\Gamma \vdash_{\wedge} \delta : U_2$.

It follows that for all $i \in \{1, 2\}$ we have

$$\Gamma \vdash_{\wedge} u_i : (U_1 \Rightarrow U_2 \Rightarrow T) \Rightarrow T_S \Rightarrow o.$$

Hence $\Gamma \vdash_{\wedge} u_i v : T_S \Rightarrow o$ and $\Gamma \vdash_{\wedge} fv : T_S \rightarrow o$.

$|\vec{v}| \geq 2$. In this case, we show by induction on $|\vec{v}|$ that for all $T \in \mathcal{T}$, there exists Γ such that for all $i \in \{1, 2\}$ we have $\Gamma \vdash_{\wedge} u_i \vec{v} : T$. This implies $\Gamma \vdash_{\wedge} f\vec{v} : T$.

The induction step easily follows from the induction hypothesis and Prop. A.1.(ii). We only detail the base case $\vec{v} = v_1 v_2$.

Since $v_1 a \delta \prec f v_1 v_2$, as in the case $|\vec{v}| = 1$, we have Γ_1, U_1, U_2, U_3 such that $\Gamma_1 \vdash_{\wedge} v_1 : U_1 \Rightarrow U_2 \Rightarrow U_3$, $\Gamma_1 \vdash_{\wedge} a : U_1$ and $\Gamma_1 \vdash_{\wedge} \delta : U_2$.

On the other hand, since $v_2 v_2 \prec f v_1 v_2$, there are Γ_2, V_1, V_2 such that $\Gamma_2 \vdash_{\wedge} v_2 : V_1 \wedge (V_1 \Rightarrow V_2)$.

It follows that for all $T \in \mathcal{T}_{\wedge}$ we have

$$\begin{aligned} \Gamma_1, x : U_1 \Rightarrow U_2 \Rightarrow U_3 \vdash_{\wedge} g(xa\delta) : T \\ \text{and } \Gamma_2, y : V_1 \wedge (V_1 \Rightarrow V_2) \vdash_{\wedge} g(yy) : T. \end{aligned}$$

Hence $\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge} u_i v_1 v_2 : T$ for all $i \in \{1, 2\}$. \square

Theorem B.2 (Completeness). *Let \mathcal{R} be the system of Ex. 2.2. If $t \in \mathcal{SN}$ then t is typable in \vdash_{\wedge} .*

Proof. We reason by induction on \prec as for Thm. A.5, except when t is an applied symbol.

The result is trivial if either $t = a\vec{v}$ or $t = g(u)\vec{v}$. Otherwise, we have $t = f\vec{v}$ and since by induction hypothesis every $u \prec t$ is typable in \vdash_{\wedge} , we conclude by Prop. B.1. \square

B.2. The System \vdash_{\wedge} and the P.R.P.

Now, we show that if \mathcal{R} has the p.r.p., then every strongly normalizable term is typable in \vdash_{\wedge} .

In addition to the properties proved in Appendix A we have the following.

Proposition B.3. *If $\bigwedge_{i \in I} (U_i \Rightarrow T_i) \leq U \Rightarrow T$, then there is a non-empty $J \subseteq I$ such that $U \leq \bigwedge_{j \in J} U_j$ and $\bigwedge_{j \in J} T_j \leq T$.*

Proof. See [2]. \square

With intersection and union types, Prop. B.3 is proved in [6, 7] assuming the distributivity of $(\mathcal{T}, \leq, \wedge, \vee)$. In our case, we can not assume the distributivity of $(\mathcal{T}, \leq, \wedge, \vee)$ since it would break the soundness of the biorthogonal type interpretation. Now we can prove:

Proposition B.4 (Subject Reduction). *If $\Gamma \vdash_{\wedge} t : T$ and $t \rightarrow u$ then $\Gamma \vdash_{\wedge} u : T$.*

Proof. By induction on $\Gamma \vdash_{\wedge} t : T$, using Prop. A.1.(iii) and Prop. B.3 when $t \mapsto_{\beta} u$. \square

The important technical property is following lemma. Recall that \prec is defined in Def.3.10.

Lemma B.5. *Let $v \in \mathcal{SN}$ such that every $v' \prec v$ is typable in \vdash_{\wedge} . Then, for all $t, u \prec v$, if $u \sqsubseteq_{\mathcal{SN}} t$ and $\Gamma \vdash_{\wedge} t : T$ then there is Γ' such that $\Gamma' \vdash_{\wedge} u : T$.*

Proof. Let $t \prec v$ such that $\Gamma \vdash_{\wedge} t : T$. We show by induction on \prec that for all u , if $u \sqsubseteq_{\mathcal{SN}} t$ and $u \prec v$ then there exists Γ' such that $\Gamma' \vdash_{\wedge} u : T$.

If $u \notin \mathcal{N}$, then by definition we have $t \rightarrow^* u$, thus $\Gamma \vdash_{\wedge} u : T$ by Subject Reduction (Prop. B.4). Otherwise, u is of the form $h\bar{u}$ where h is either a variable, an abstraction or a function.

$h = x$. In this case, $u \in \mathcal{HN}$. Since $\bar{u} \prec v$, by assumption there are Γ' and \bar{U} such that $\Gamma' \vdash_{\wedge} \bar{u} : \bar{U}$. Hence we have $\Gamma' \wedge (x : \bar{U} \Rightarrow T) \vdash_{\wedge} x\bar{u} : T$.

$h = (\lambda x.u_1)u_2$. In this case, since $u_1[u_2/x]\bar{u} \sqsubseteq_{\mathcal{SN}} t$, and $u_1[u_2/x]\bar{u} \prec u$, by induction hypothesis there is Γ_1 such that $\Gamma_1 \vdash_{\wedge} u_1[u_2/x]\bar{u} : T$. Moreover, since $u_2 \prec v$, by assumption there are Γ_2 and U such that $\Gamma_2 \vdash_{\wedge} u_2 : U$. Therefore, by Lem. A.4.(i) we have $\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge} (\lambda x.u_1)u_2\bar{u} : T$.

$h = f(\bar{t})$. Since for all $r \in \mathcal{R}(f)$ we have $r[\bar{t}/\bar{x}]\bar{u} \prec u$ and $r[\bar{t}/\bar{x}]\bar{u} \sqsubseteq_{\mathcal{SN}} t$, by induction hypothesis there is Γ_1 such that for all $r \in \mathcal{R}(f)$ we have $\Gamma_1 \vdash_{\wedge} r[\bar{t}/\bar{x}]\bar{u} : T$. On the other hand, since $\bar{t} \prec v$, by assumption there are Γ_2, \bar{T} such that $\Gamma_2 \vdash_{\wedge} \bar{t} : \bar{T}$. Hence by Lem. A.4.(ii) we have $\Gamma_1 \wedge \Gamma_2 \vdash_{\wedge} f(\bar{t})\bar{u} : T$.

□

Theorem B.6 (Completeness). *Assume that \mathcal{R} has the p.r.p.. If $t \in \mathcal{SN}$, then there are Γ and T such that $\Gamma \vdash_{\wedge} t : T$.*

Proof. The proof is by induction on \prec . It is the same as the proof of Thm. A.5, except for the case $t = f(\bar{t})\bar{v}$. We only detail this case.

First, note that we have $\bar{t} \prec t$. Since moreover for all $r \in \mathcal{R}(f)$ we have $r[\bar{t}/\bar{x}]\bar{v} \prec t$ by induction hypothesis there are Γ_r, \bar{T}^r and V_r such that $\Gamma_r \vdash_{\wedge} \bar{t} : \bar{T}^r$ and $\Gamma_r \vdash_{\wedge} r[\bar{t}/\bar{x}]\bar{v} : V_r$.

Now, by assumption, there is $d \in \mathcal{R}(f)$ such that for all $r \in \mathcal{R}(f)$, $r[\bar{t}/\bar{x}]\bar{v} \sqsubseteq_{\mathcal{SN}} d[\bar{t}/\bar{x}]\bar{v}$. Since $d[\bar{t}/\bar{x}]\bar{v} \prec t$ and for all $r \in \mathcal{R}(f)$, $r[\bar{t}/\bar{x}]\bar{v} \prec t$, and moreover by induction hypothesis every $v \prec t$ is typable in \vdash_{\wedge} , by Lem. B.5 for all $r \in \mathcal{R}(f)$, there is Γ'_r such that $\Gamma'_r \vdash_{\wedge} r[\bar{t}/\bar{x}]\bar{v} : T_d$.

Let $\Gamma =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} (\Gamma_r \wedge \Gamma'_r)$ and $\bar{T} =_{\text{def}} \bigwedge_{r \in \mathcal{R}(f)} \bar{T}^r$. We have $\Gamma \vdash_{\wedge} \bar{t} : \bar{T}$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{\wedge} r[\bar{t}/\bar{x}]\bar{v} : T_d$. We get $\Gamma \vdash_{\wedge} f(\bar{t})\bar{v} : T_d$ thanks to Lem. A.4.(ii). □

C. \mathcal{HN} -Biorthogonality (Sec. 6)

This appendix is devoted to the proofs of Sec. 6 concerning \mathcal{HN} -biorthogonality. The development is similar to that of Sec. 4.

We show that the following properties are equivalents:

($\forall E$) is \mathcal{HN} -safe: If $\Gamma \vdash_{\wedge \vee} t : o$ then $t \in \mathcal{HN}$.

($\text{IP}_{\mathcal{HN}}$) For all $f \in \mathcal{F}$, if $f(\bar{t}) \in \mathcal{SN}$ and $v[r[\bar{t}/\bar{x}]/y] \in \mathcal{HN}$ for all $r \in \mathcal{R}(f)$, then $v[f(\bar{t})/y] \in \mathcal{HN}$.

(\cdot) $_{\mathcal{HN}}$ is sound: If $\Gamma \vdash_{\wedge \vee} t : T$ and $\sigma \models_{(\cdot)_{\mathcal{HN}}} \Gamma$ then $t\sigma \in (\llbracket T \rrbracket)_{\mathcal{HN}}$.

Theorem C.1. *If ($\forall E$) is \mathcal{HN} -safe, then ($\text{IP}_{\mathcal{HN}}$) holds.*

Proof. The proof is similar to that of Thm. 4.2.

Let $f(\bar{t}) \in \mathcal{SN}$ and v such that for all $r \in \mathcal{R}(f)$, $v[r[\bar{t}/\bar{x}]/y] \in \mathcal{HN}$. Reasoning as in Thm. 6.2, there are $\Gamma, (U_r)_{r \in \mathcal{R}(f)}$ such that $\Gamma \vdash_{\wedge \vee} f(\bar{t}) : \bigvee_{r \in \mathcal{R}(f)} U_r$ and for all $r \in \mathcal{R}(f)$, $\Gamma \vdash_{\wedge \vee} \lambda y.v : U_r \Rightarrow o$.

By Prop. 4.1, we have $\Gamma \vdash_{\wedge \vee} (\lambda y.v)f(\bar{t}) : o$, hence $v[f(\bar{t})/y] \in \mathcal{HN}$ since ($\forall E$) is \mathcal{HN} -safe. □

Proposition C.2 (Neutral Term Property). *Let $E[\] \in \mathcal{SN}$ and $t \in \mathcal{N}$. If ($\text{IP}_{\mathcal{HN}}$) holds and $\forall u(t \rightarrow u \Rightarrow E[u] \in \mathcal{HN})$ then $E[t] \in \mathcal{HN}$.*

Proof. First, if $t \in \mathcal{HN}$, since $E[\] \in \mathcal{SN}$, by Thm. 6.3 we have $E[t] \in \mathcal{SN}$.

Otherwise, since $t \in \mathcal{N}$, we have $(t)_{\rightarrow} \neq \emptyset$ and there are two cases. In the first one $t = (\lambda x.t_1)t_2\bar{v}$ and we conclude by Thm. 6.2. In the second one $t = f(\bar{t})\bar{v}$ with $f \in \mathcal{F}$ and the result follows from ($\text{IP}_{\mathcal{HN}}$). □

Then, as in Sec. 4.3, we obtain that biorthogonals of non-empty subsets of \mathcal{SN} are reducibility candidates.

Lemma C.3. *If $A \subseteq \mathcal{SN}$ is not empty, then ($\text{IP}_{\mathcal{HN}}$) implies $A^{\perp\perp} \in \mathcal{CR}$.*

Proof. As for Lem. 4.5, using Prop. C.2 instead of Prop. 4.4 and ($\text{IP}_{\mathcal{HN}}$) instead of (IP). □

We deduce that types are interpreted as biorthogonals, and the main result easily follows.

Lemma C.4. *If ($\text{IP}_{\mathcal{HN}}$) then for all $T \in \mathcal{T}$, $(\llbracket T \rrbracket)_{\mathcal{HN}} \in \mathcal{CR}$.*

Proof. Reason by induction on T , using Lem. C.3 in the case $T = T_1 \vee T_2$. □

Theorem C.5. *Let $\Gamma \vdash_{\wedge \vee} t : T$. If ($\text{IP}_{\mathcal{HN}}$) and $\sigma \models_{(\cdot)_{\mathcal{HN}}} \Gamma$ then $t\sigma \in (\llbracket T \rrbracket)_{\mathcal{HN}}$.*

Proof. By induction on $\Gamma \vdash_{\wedge \vee} t : T$. Thanks to Lem. C.4, using ($\text{IP}_{\mathcal{HN}}$), we have $(\llbracket U \rrbracket)_{\mathcal{HN}} \in \mathcal{CR}$ for all $U \in \mathcal{T}$. Then, once we have noted that $t' \perp E[(\lambda x.c')[\]]$ implies $c'[t'/x] \perp E[\]$, the proof is exactly the same as for Thm. 4.8, using Thm. 6.2 instead of Thm. 3.13. □