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# Meromorphic and Multipoint Padé Approximants to Complex Cauchy Transforms with Polar Singularities

L. Baratchart and M. Yattselev

## Abstract

We study the asymptotic pole distribution and the convergence in capacity of AAK-type meromorphic approximants as well as multipoint Padé approximants to functions of the form

$$F(z) = \int \frac{d\lambda(t)}{z-t} + R(z),$$

where  $R$  is a rational function and  $\mu$  a complex measure with compact regular support included in  $\mathbb{R}$ , whose argument has bounded variation on the support.

*Key words:* meromorphic approximation, AAK-theory, rational approximation, Padé approximation, multipoint approximation, orthogonal polynomials, Hardy spaces, critical points.

*AMS Classification (MSC2000):* primary 41A20, 41A21, 41A30, 42C05; secondary 30D50, 30D55, 30E10, 31A15.

## 1 Introduction

This paper is concerned with the asymptotic behavior of certain meromorphic and rational approximants to functions of the form

$$F(z) = \int \frac{d\lambda(t)}{z-t} + R(z), \tag{1.1}$$

where  $R$  is a rational function, holomorphic at infinity, and  $\lambda$  is a complex measure compactly and regularly supported on the real line.

The meromorphic approximants that we consider are optimal, for fixed number of poles in the unit disk, with respect to a  $L^p$ -norm on the unit circle. When studying them, we assume that  $\text{supp}\lambda$  and all poles of  $R$  lie in the open unit disk, so that  $F$  is indeed  $p$ -summable on the unit circle. The asymptotics are then understood when the number of poles goes large. In the case where  $p = \infty$ , this type of approximant was introduced by V. M. Adamyan, D. Z. Arov, and M. G. Krein in their famous paper [1]. Here we deal with their natural generalization to  $L^p$ , although we restrict ourselves to the range  $2 \leq p \leq \infty$  for technical reasons to be explained later. The meromorphic approximation problem also has a conformally invariant formulation on Jordan domains with rectifiable boundary, to which the results of the present paper transpose with obvious modifications if  $\text{supp}\lambda$  is contained in a closed hyperbolic geodesic arc rather than a segment. The interested reader will have no difficulty to carry out this generalization using the construction of [10, Sec. 5].

As for multipoint Padé approximants, we deal with rational interpolants of type  $(n, n)$  where, for each  $n$ , a set of  $2n + 1$  interpolation points has been prescribed one of which is infinity. Moreover, we assume that the interpolation points are conjugate-symmetric and chosen in such a manner that, when  $n$  goes large, they have a limit distribution whose support is disjoint from the poles of  $R$  and the convex hull of  $\text{supp}\lambda$ .

One reason for treating jointly best meromorphic approximants and Padé approximants is that essentially the same technique will be used. This technique rests on recent results on (non-Hermitian) orthogonality on a segment obtained in [8]. A second reason is that when  $p = 2$  best meromorphic approximation to  $F$  reduces to best *rational* approximation of fixed degree in  $L^2$  of the circle. Such rational approximants turn out to be

multipoint Padé approximants as well (although this time the interpolation points are not known *a priori*) hence the two theories make contact for  $p = 2$ .

A few words of comparison between these two types of approximants are perhaps in order. On the one hand classical and multipoint Padé approximants were historically first to be studied, becoming of great importance both in function theory and in number theory, and they offer *linear* computational schemes with remarkable properties that are exploited, *e.g.* for convergence acceleration in numerical analysis [18]. On the other hand, the phenomenon of spurious poles and the poor numerical stability when dealing with inaccurate data have impeded the use of such approximants for modeling and identification in engineering sciences. In addition, given some criterion, the optimal or suboptimal choice of the interpolation points when their number is fixed (a crucial issue in practice) is a difficult nonconvex optimization problem which is still not well-understood as most studies of the approximation rate are asymptotic in nature. In contrast, best meromorphic approximants on the circle have a guaranteed region of holomorphy (the complement of the closed unit disk in our case) and, by definition, they make in some sense best possible use of the freedom to choose the poles. Moreover, their optimality has important interpretations in control and identification of linear dynamical systems [6, 17, 37, 39, 7] and gives rise in specific cases to near-best *rational* approximation schemes [20, 35]. Their main drawback is that they are not so easy to compute, since when  $p = \infty$  the best approximation projection is generically not continuous with respect to the  $L^\infty$  norm, but only with respect to stronger norms [26, 38], while for  $p < \infty$  one has to rely on a numerical search that may get trapped in local minima. In this respect, let us point out that the results of the present paper are valid, not only for best meromorphic approximants, but more generally for critical points of the  $L^p$ -criterion thus also for *local* best approximants.

To put our results into perspective, let us begin with an account of the existing literature. When  $\lambda$  is a positive measure and  $R \equiv 0$  (in this case  $F$  is referred to as a Markov function), the study of diagonal Padé approximants to  $F$  at infinity goes back to A. A. Markov who showed (see [32]) that they converge uniformly to  $F$  on compact subsets of  $\overline{\mathbb{C}} \setminus I$ , where  $I$  is the convex hull of  $\text{supp } \lambda$ . Later this work was extended to multipoint Padé approximants with conjugate-symmetric interpolation schemes by A. A. Gonchar and G. López Lagomasino in [23]. A cornerstone of the theory is the close relationship between diagonal Padé (resp. multipoint Padé) approximants to Markov functions and orthogonal polynomials, since the denominator of the  $n$ -th approximant is the  $n$ -th orthogonal polynomial in  $L^2(d\lambda)$  (resp.  $L^2(d\lambda/p)$  where  $p$  is a polynomial vanishing at finite interpolation points). For further references and sharp error rates, we refer the reader to the monographs [50, 49].

Another generalization of Markov's result was obtained by A. A. Gonchar on adding polar singularities, i.e. on making  $R \neq 0$ . He proved in [22] that Padé approximants still converge to  $F$  locally uniformly in  $\overline{\mathbb{C}} \setminus (S' \cup I)$ , where  $S'$  is the set of poles of  $R$ , provided that  $\lambda$  is a positive measure with singular part supported on a set of logarithmic capacity zero. Subsequently, it was shown by E. A. Rakhmanov in [44] that weaker assumptions on  $\lambda$  can spoil the convergence, but at the same time that if the coefficients of  $R$  are real, then the locally uniform convergence holds for any positive  $\lambda$ . Although it is not a concern to us here, let us mention that one may also relax the assumption that  $\text{supp } \lambda$  is compact. In particular, Padé and multipoint Padé approximants to Cauchy transforms of positive measures supported in  $[0, \infty]$  (such functions are said to be of Stieltjes type) were investigated by G. López Lagomasino in [28, 29].

The case of a complex measure was taken up by G. Baxter in [14] and by J. Nuttall and S. R. Singh in [34], who established strong asymptotics of non-Hermitian orthogonal polynomials on a segment for measures that are absolutely continuous with respect to the (logarithmic) equilibrium distribution of that segment, and whose density satisfy appropriate conditions expressing, in a way or another, that it is smoothly invertible. These results entail that the Padé approximants to  $F$  converge uniformly to the latter on compact subsets of  $\overline{\mathbb{C}} \setminus I$  when  $R \equiv 0$  and  $d\lambda/d\mu_I$  meets these conditions (here  $\mu_I$  indicates the equilibrium distribution on  $I$ ). For instance Baxter's condition is that  $\log d\lambda/d\mu_I$ , when extended periodically, has an absolutely summable Fourier series. When  $d\lambda/d\mu_I$  is holomorphic and nonvanishing on a neighborhood of  $I$ , still stronger asymptotics, allowing to handle multipoint Padé approximants as well, were recently obtained by A. I. Aptekarev and W. Van Assche in [5], using the Riemann-Hilbert approach pioneered by P. Deift and X. Zhou (see *e.g.* [16]).

Meanwhile H. Stahl opened up new perspectives in his pathbreaking papers [47, 48], where he studied diagonal Padé approximants to (branches of) multiple-valued functions that can be continued analytically without restriction except over a set of capacity zero (typical examples are functions with poles and branchpoints).

By essentially representing the “main” singular part of the function as a Cauchy integral over a system of cuts of minimal capacity, and through a deep analysis of zeros of non-Hermitian orthogonal polynomials on such systems of cuts, he established the asymptotic distribution of poles and subsequently the convergence in capacity of the Padé approximants on the complement of the cuts. In [24] this construction was generalized to certain carefully chosen multipoint Padé approximants by A. A. Gonchar and E. A. Rakhmanov, who used it to prove the sharpness of O. G. Parfenov’s theorem (formerly Gonchar’s conjecture) on the rate of approximation by rational functions over compact subsets of the domain of holomorphy, see [35]<sup>1</sup>. Of course the true power of this method lies with the fact that it allows one to deal with measures supported on more general systems of arcs than a segment, which is beyond the scope of the present paper. However, since a segment is the simplest example of an arc of minimal logarithmic capacity connecting two points, the results we just mentioned apply in particular to functions of the form (1.1) where  $\lambda$  is a complex measure supported on a segment which is absolutely continuous there with continuous density that does not vanish outside a set of capacity zero. By different, operator-theoretic methods, combined with a well-known theorem of E. A. Rakhmanov on ratio asymptotics (see [43]), A. Magnus further showed that the diagonal Padé approximants to  $F$  converge *uniformly* on compact subsets of  $\overline{\mathbb{C}} \setminus I$  when  $R \equiv 0$  and  $d\lambda/dt$  is non-zero almost everywhere with continuous argument [31].

In contrast with previous work, the present approach allows for the complex measure  $\lambda$  to vanish on a large subset of  $I$ . Specifically, we require that the total variation measure  $|\lambda|$  has compact regular support and that it is not too thin, say, larger than a power of the radius on relative balls *of the support* (see the definition of the class **BVT** in Section 3). Although fairly general, these conditions could be further weakened, for instance down to the  $\Lambda$ -criterion introduced by H. Stahl and V. Totik in [49]<sup>2</sup>. However, our most stringent assumption bears on the argument of  $\lambda$ , as we require the Radon-Nikodym derivative  $d\lambda/d|\lambda|$  to be of bounded variation on  $\text{supp}\lambda$ . This assumption, introduced in [27, 8], unlocks many difficulties and will lead us to the weak convergence of the poles and to the convergence in capacity on  $\overline{\mathbb{C}} \setminus (S' \cup \text{supp}\lambda)$  of multipoint Padé approximants to functions of the form (1.1). Moreover we shall prove that each pole of  $R$  attracts at least as many poles of the approximants as its multiplicity, and not much more. In fact, our hypotheses give rise to an explicit upper bound on the number of poles of the approximants that may lie outside a given neighborhood of the singular set of  $F$ . Hence, on each compact subset  $K$  of  $\overline{\mathbb{C}} \setminus (S' \cup \text{supp}\lambda)$ , every sequence of approximants contains a subsequence that converges *uniformly* to  $F$  locally uniformly on  $K \setminus E$ , where  $E$  consists of boundedly many (unknown) points. When  $\text{supp}\lambda$  is a finite union of intervals, results of this type were obtained under stronger assumptions in [33] for classical Padé approximants.

As compared to Padé approximants, the study of best meromorphic approximants is quite recent. After the development of the Adamyan-Arov-Krein theory for  $p = \infty$  in [1], the latter was extended to the range  $1 \leq p < \infty$  by F. Seyfert and the first author in [12], and independently by V. A. Prokhorov in [42]. However, it is only for  $p \geq 2$  that the authors of [12] were able to express the error in terms of (generalized) singular vectors of a Hankel operator and subsequently to obtain integral formulas for that error when the approximated function is represented as a Cauchy integral. These formulas make connection with non-Hermitian orthogonality, and form the basis of the present approach.

When  $F$  is a Markov function, best meromorphic approximants have been instrumental in [4, 15] to construct *rational* approximants, and were studied *per se* by E. B. Saff, V. Prokhorov and the first author in [11]. Using results from [4] to make connection with orthogonality, these authors prove (and give error rates for) the uniform convergence of such approximants, locally uniformly on  $\overline{\mathbb{C}} \setminus I$ , whenever  $1 \leq p \leq \infty$  provided that  $\lambda$  satisfies the Szegő condition:  $\log d\lambda/dt \in L^1(I)$ . When  $p = 2$ , the uniform convergence for any Markov function follows from the work [13] by H. Stahl, F. Wielonsky, and the first author.

Apart from [12, Sec. 10] where asymptotics of the poles are established, for  $p \geq 2$ , when  $F$  has exactly two branchpoints and no pole, the present paper seems first to deal with the convergence of best meromorphic approximants to (1.1) in case  $\lambda$  is a complex measure. The results that we prove are similar to those already indicated for Padé approximants, except that logarithmic potentials are replaced by Green potentials throughout.

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<sup>1</sup>It is interesting to note that Parfenov’s proof makes decisive use of the Adamyan-Arov-Krein theory; later, the result was carried over to multiply connected domains by V. A. Prokhorov in [41].

<sup>2</sup>This depends on the corresponding generalization of the results in [8] to be found in [27], as yet unpublished.

## 2 Preliminaries and Notations

Let  $\lambda$  be a complex Borel measure whose support  $S := \text{supp}(\lambda) \subset \mathbb{R}$  is compact and consists of infinitely many points. Denote by  $|\lambda|$  the total variation measure. Clearly  $\lambda$  is absolutely continuous with respect to  $|\lambda|$ , and we shall assume that its Radon-Nikodym derivative (which is of unit modulus  $|\lambda|$ -a.e.) is of bounded variation. In other words,  $\lambda$  is of the form

$$d\lambda(t) = e^{i\varphi(t)} d|\lambda|(t), \quad (2.2)$$

for some real-valued argument function  $\varphi$  such that<sup>3</sup>

$$V(\varphi, S) := \sup \left\{ \sum_{j=1}^N |\varphi(x_j) - \varphi(x_{j-1})| \right\} < \infty, \quad (2.3)$$

where the supremum is taken over all finite partitions  $x_0 < x_1 < \dots < x_N$  of  $S$  as  $N$  ranges over  $\mathbb{N}$ .

For convenience, we extend the definition of  $\varphi$  to the whole of  $\mathbb{R}$  as follows. Let  $I := [a, b]$  be the convex hull of  $S$ . It is easy to see that if we interpolate  $\varphi$  linearly in each component of  $I \setminus S$  and if we set  $\varphi(x) := \lim_{t \rightarrow a, t \in S} \varphi(t)$  for  $x < a$  and  $\varphi(x) := \lim_{t \rightarrow b, t \in S} \varphi(t)$  for  $x > b$  (the limits exist by (2.3)), the variation of  $\varphi$  will remain the same. In other words, we may arrange things so that the extension of  $\varphi$ , still denoted by  $\varphi$ , satisfies

$$V(\varphi, S) = V(\varphi, \mathbb{R}) =: V(\varphi).$$

Denote by  $\mathcal{P}_n$  the space of algebraic polynomials of degree at most  $n$  and by  $\mathcal{M}_n$  the subset consisting of monic polynomials of degree  $n$  whose zeros lie in the open unit disk,  $\mathbb{D}$ .

Define

$$F(z) := \int \frac{d\lambda(\xi)}{z - \xi} + R_s(z), \quad (2.4)$$

with  $R_s \in \mathcal{R}_{s,s}$ , where

$$\mathcal{R}_{m,n} := \{p_m/q_n : p_m \in \mathcal{P}_m, q_n \in \mathcal{M}_n\}$$

is the set of rational functions of type  $(m, n)$  with all their poles in  $\mathbb{D}$ . Hereafter we shall denote by  $Q_s$  the denominator of  $R_s$ , assumed to be in irreducible form, which is a monic polynomial with zeros in  $\mathbb{D}$  of the form

$$Q_s(z) = \prod_{\eta_i \in S'} (z - \eta_i)^{m(\eta_i)}, \quad (2.5)$$

where  $S'$  is the set of poles of  $R_s$  and  $m(\eta)$  stands for the multiplicity of  $\eta \in S'$ . Thus,  $F$  is a meromorphic function in  $\mathbb{C} \setminus S$  with poles at each point of  $S'$ . Note in passing that  $F$  does not reduce to a rational function since  $S$  consists of infinitely many points, cf. [10, Sec. 5.1] for a detailed argument.

To alleviate notations, it is convenient to formally rewrite the right-hand side of (2.4) as a single Cauchy integral plus a constant. For this, we introduce for  $\eta = x_\eta + iy_\eta \in \mathbb{C}$  the distribution  $\Phi_\eta = \chi(x - x_\eta) \otimes \delta(y - y_\eta)$  where  $\delta$  is the Dirac delta at 0 and  $\chi$  the characteristic function of the non-negative semi-axis. For each  $k \in \mathbb{Z}_+$  (the set of nonnegative integers), the partial derivative  $\partial_x^{k+1} \Phi_\eta$  is an analytic functional (although  $\Phi_\eta$  itself is not), acting on any function  $h$  holomorphic in a neighborhood of  $\eta$  according to the rule

$$\langle \partial_x^{k+1} \Phi_\eta, h \rangle = h^{(k)}(\eta),$$

where  $h^{(k)}$  indicates the  $k$ -th derivative. Therefore, if we define  $\Delta_\eta^{(k)}$  to be  $\partial_x^k \Phi_\eta / k!$ , we can formally write

$$\int \frac{d\Delta_\eta^{(k)}(t)}{z - t} = \frac{1}{(z - \eta)^{k+1}},$$

<sup>3</sup>Note that  $e^{i\varphi}$  has bounded variation if and only if  $\varphi$  can be chosen of bounded variation.

and on rewriting  $R_s(z)$  as

$$R_s(z) = R(\infty) + \sum_{\eta_i \in S'} \sum_{k=0}^{m(\eta_i)-1} \frac{r_{i,k}}{(z - \eta_i)^{k+1}}, \quad r_{i,k} \in \mathbb{C},$$

we get

$$R_s(z) = R(\infty) + \int \frac{d\lambda'(\xi)}{z - \xi},$$

where  $\lambda'$  is given by

$$\lambda' := \sum_{\eta_i \in S'} \sum_{k=0}^{m(\eta_i)-1} r_{i,k} \Delta_{\eta_i}^{(k)}, \quad \text{supp}(\lambda') = S'. \quad (2.6)$$

This way  $F$  can be put in the form

$$F(z) = F(\infty) + \int \frac{d\tilde{\lambda}(\xi)}{z - \xi}$$

with

$$\tilde{\lambda} := \lambda + \lambda', \quad \tilde{S} := \text{supp} \tilde{\lambda} = S \cup S', \quad (2.7)$$

which makes for a convenient notation.

In this paper we consider the behavior of certain meromorphic (AAK-type) and rational (multipoint Padé) approximants to a function  $F$  of the form (2.4). *When dealing with meromorphic approximants*, in addition to our standing hypothesis that  $S = \text{supp} \lambda \subset \mathbb{R}$ , we always assume that  $\tilde{S} \subset \mathbb{D}$ , namely that  $S \subset (-1, 1)$  and that  $S' \subset \mathbb{D}$ . In contrast, this extra-requirement is irrelevant and will not be made when dealing with multipoint Padé approximants.

Of particular importance will be the asymptotic behavior of the poles of the above-mentioned approximants. The latter will be quantified in terms of the weak\* convergence, when the number of poles increases indefinitely, of the *counting measures* of these poles. By definition, the counting measure of the poles of a meromorphic or rational function is the discrete probability measure with equal mass at each finite pole, counting multiplicities. The weak\* convergence is understood in the usual sense where measures, endowed with the norm of total variation, are regarded as the dual space to continuous functions with compact support.

We denote by  $H^p$ ,  $p \in [1, \infty]$ , the *Hardy space* of the unit disk  $\mathbb{D}$  consisting of holomorphic functions such that

$$\begin{aligned} \|f\|_p^p &:= \sup_{0 < r < 1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(r\xi)|^p |d\xi| < \infty & \text{if } p \in [1, \infty), \\ \|f\|_\infty &:= \sup_{0 < r < 1} \sup_{\xi \in \mathbb{T}} |f(r\xi)| < \infty & \text{if } p = \infty, \end{aligned} \quad (2.8)$$

$\mathbb{T}$  being the unit circle. It is known ([19, Thm. I.5.3]) that a function in  $H^p$  is uniquely determined by its trace (nontangential limit) on  $\mathbb{T}$ , and that the  $L^p(\mathbb{T})$ -norm of this trace is equal to the supremum in (2.8), where  $L^p(\mathbb{T})$  is the space of  $p$ -summable functions on  $\mathbb{T}$ . This way  $H^p$  can be regarded as a closed subspace of  $L^p(\mathbb{T})$ . Let us recall the well-known fact [19] that a nonzero  $f \in H^p$  can be uniquely factored as  $f = jw$  where

$$w(z) = \exp \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log |f(\xi)| |d\xi| \right\}$$

belongs to  $H^p$  and is called the *outer factor* of  $f$ , while  $j \in H^\infty$  has modulus 1 a.e. on  $\mathbb{T}$  and is called the *inner factor* of  $f$ . The latter may be further decomposed as  $j = bS$ , where

$$b(z) = cz^k \prod_{z_l \neq 0} \frac{-\bar{z}_l}{|z_l|} \frac{z - z_l}{1 - \bar{z}_l z}$$

is the normalized *Blaschke product*, with multiplicity  $k \geq 0$  at the origin, associated to a sequence of points  $z_l \in \mathbb{D} \setminus \{0\}$  and a constant  $c \in \mathbb{T}$ , while

$$S(z) = \exp \left\{ -\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi) \right\}$$

is the *singular inner factor* associated with  $\mu$ , a positive measure on  $\mathbb{T}$  which is singular with respect to Lebesgue measure. The  $z_l$  are of course the zeros of  $f$  in  $\mathbb{D}$ , counted with their multiplicities. If there are infinitely many zeros, the convergence of the product  $b(z)$  in  $\mathbb{D}$  is ensured by the condition  $\sum_l (1 - |z_l|) < \infty$  which holds automatically when  $f \in H^p$ . That  $w(z)$  is well-defined rests on the fact that  $\log |f| \in L^1(\mathbb{T})$  if  $f \in H^p \setminus \{0\}$ . For simplicity, we often say that a function is outer (resp. inner) if it is equal to its outer (resp. inner) factor.

Now, the meromorphic approximants that we deal with are defined as follows. For  $p \in [1, \infty]$  and  $n \in \mathbb{N}$ , the *class of meromorphic functions of degree  $n$*  in  $L^p(\mathbb{T})$  is

$$H_n^p := H^p + \mathcal{R}_{n-1,n}$$

which is a closed subset of  $L^p(\mathbb{T})$  (it is in fact weakly closed if  $1 < p < \infty$  and weak\* closed if  $p = \infty$ , see [12, Lemma 5.1.]).

We need also introduce the *set of Blaschke products of degree at most  $n$* , denoted by  $B_n$ , consisting of rational functions of the form

$$b(z) = e^{ic} \prod_{j=1}^m \frac{z - z_j}{1 - \bar{z}_j z}, \quad m \leq n, \quad z_j \in \mathbb{D}, \quad c \in \mathbb{R}.$$

The members of  $B_n$  are those rational functions of degree at most  $n$  holomorphic in  $\mathbb{D}$  and having modulus 1 everywhere on  $\mathbb{T}$ . Note the alternative description:

$$H_n^p = B_n^{-1} H^p. \quad (2.9)$$

Our *best- $L^p(\mathbb{T})$  meromorphic approximation problem* can now be stated as follows.

**MA( $p$ )** : Given  $p \in [1, \infty]$ ,  $f \in L^p(\mathbb{T})$ , and  $n \in \mathbb{N}$ , find  $g_n \in H_n^p$  such that

$$\|f - g_n\|_p = \inf_{g \in H_n^p} \|f - g\|_p. \quad (2.10)$$

This problem has a solution [12, Sec. 5]. It is known to be unique when  $p = \infty$ , provided that  $f$  belongs to the *Douglas algebra*  $H^\infty + C(\mathbb{T})$ , where  $C(\mathbb{T})$  denotes the space of continuous functions on  $\mathbb{T}$  [1]. In particular the solution to **MA**( $\infty$ ) is unique when  $f$  is of type (2.4) since the latter is analytic in some neighborhood of the unit circle (*i.e.* in the complement of  $\tilde{S}$ ). When  $p < \infty$ , a solution needs not be unique even if  $f$  is very smooth [12, Sec. 5]. Therefore, when making a statement about a sequence  $\{g_n\}$  of solutions to **MA**( $p$ ), it is understood that a particular solution has been selected for each  $n$  and that the statement holds true regardless the selection. Note that if  $f$  gets replaced by  $f + c$  for some constant  $c \in \mathbb{C}$ , then  $g_n$  gets replaced by  $g_n + c$ . Therefore, when dealing with **MA**( $p$ ) for  $f = F$  as in (2.4), we may safely assume  $F(\infty) = 0$  for this preliminary reduction will not affect the poles of the approximants nor their convergence properties. Actually, as far as **MA**( $p$ ) is concerned, we shall restrict ourselves to the range  $2 \leq p \leq \infty$  for these are the only values of  $p$  for which a concrete characterization of the solutions is known so far [12, Sec. 8].

As will be explained shortly, for functions of type (2.4), **MA**(2) reduces to best rational approximation in  $L^2(\mathbb{T})$  from  $\mathcal{R}_{n,n}$ . In particular best approximants are holomorphic in  $\overline{\mathbb{C}} \setminus \mathbb{T}$ , and it turns out that they interpolate  $F$  at the reflections of their poles across  $\mathbb{T}$  and also at infinity. Thus one can regard **MA**(2) as an interpolation problem where the interpolation points are *implicitly defined* by the solution.

In contrast, multipoint Padé approximants are rational functions that interpolate a given function at a *prescribed* system of points. More precisely, pick  $n \in \mathbb{N}$  and let  $A_n = \{\zeta_{1,n}, \dots, \zeta_{2n,n}\}$  be a set of  $2n$  interpolation points, where the  $\zeta_{j,n} \in \overline{\mathbb{C}} \setminus \tilde{S}$  need not be distinct nor finite. With  $A_n$  we form the monic polynomial

$$w_{2n}(z) = \prod_{\zeta_{j,n} \in A_n \cap \mathbb{C}} (z - \zeta_{j,n}) \quad (2.11)$$

(note that  $w_{2n}$  retains only the interpolation points at finite distance thus it needs not have exact degree  $2n$ ).

Given  $F$  of type (2.4) and  $A_n$  as above, the *diagonal multipoint Padé approximant to  $F$  associated with  $A_n$*  is the unique rational function  $\Pi_n = p_n/q_n$  where the polynomials  $p_n$  and  $q_n$  satisfy:

- (i)  $\deg p_n \leq n$ ,  $\deg q_n \leq n$ , and  $q_n \not\equiv 0$ ;
- (ii)  $(q_n(z)F(z) - p_n(z))/w_{2n}(z)$  is analytic in  $\mathbb{C} \setminus \tilde{S}$ ;
- (iii)  $(q_n(z)F(z) - p_n(z))/w_{2n}(z) = O(1/z^{n+1})$ , as  $z \rightarrow \infty$ .

A multipoint Padé approximant always exists since the conditions for  $p_n$  and  $q_n$  amount to solving a system of  $2n + 1$  homogeneous linear equations with  $2n + 2$  unknown coefficients, no solution of which can be such that  $q_n \equiv 0$  (we may thus assume that  $q_n$  is monic); note that (iii) entails at least one interpolation condition at infinity.

If we let now  $\mathbf{A} := \{A_n\}_{n \in \mathbb{N}}$  be an *interpolation scheme*, i.e. a sequence indexed by  $n \in \mathbb{N}$  of sets  $A_n$  as above, we get a corresponding sequence  $\{p_n/q_n\}$  of diagonal Padé approximants whose asymptotic behavior can be studied when  $n$  goes large. In this paper we will restrict ourselves to conjugate symmetric interpolation schemes, namely those for which  $\overline{A_n} = A_n$  for all  $n$ .

Given  $F$  as in (2.4) (and some interpolation scheme if we deal with Padé approximants), we shall be interested in three types of questions:

- (a) *What is the asymptotic distribution of the poles of best- $L^p(\mathbb{T})$  meromorphic or multipoint Padé approximants to  $F$  as  $n$  tends to  $\infty$ ?*
- (b) *Do some of these poles converge to the polar singularities of  $F$ ?*
- (c) *What can be said about the convergence of such approximants to  $F$ ?*

As is typical in complex approximation, some answers can be given in terms of logarithmic potential theory, a brief account of which is sketched below for the convenience of the reader. We refer him to the monographs [45, 46] for an extensive treatment.

The *logarithmic potential* and the *logarithmic energy* of a finite positive measure  $\mu$ , compactly supported in  $\mathbb{C}$ , are defined by

$$U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t), \quad z \in \mathbb{C}, \quad (2.12)$$

and

$$I[\mu] := \int U^\mu(z) d\mu(z) = \int \int \log \frac{1}{|z - t|} d\mu(t) d\mu(z), \quad (2.13)$$

respectively. Clearly  $U^\mu$  is a superharmonic function with values in  $(-\infty, +\infty]$  which is not identically  $+\infty$ . In particular  $U^\mu$  is lower semicontinuous, hence it attains its lower bound on  $\text{supp } \mu$  so that  $I[\mu] \in (-\infty, +\infty]$ .

Let now  $E \subset \mathbb{C}$  be compact and  $\Lambda(E)$  denote the set of all probability measures supported on  $E$ . If the logarithmic energy of every measure in  $\Lambda(E)$  is infinite, we say that  $E$  is *polar*. Otherwise, there exists a unique  $\mu_E \in \Lambda(E)$  that minimizes the logarithmic energy over all measures in  $\Lambda(E)$ . This measure is called the *equilibrium distribution* on  $E$ . It is characterized by the property that  $U^{\mu_E}$  is constant on  $E$  except perhaps on a polar subset of  $E$ . The *logarithmic capacity*, or simply the capacity, of  $E$  is defined as

$$\text{cap}(E) = \exp\{-I[\mu_E]\}.$$

By definition, the capacity of an arbitrary subset of  $\mathbb{C}$  is the *supremum* of the capacities of its compact subsets. We agree that the capacity of a polar set is zero.

Polar sets are very small and look very bad; for instance a polar set has Lebesgue measure zero and a closed polar set is totally disconnected. A property is said to hold *quasi everywhere* (abbreviated “q.e.”) if it holds everywhere except on a polar set.

Logarithmic potentials enjoy weak continuity and semi-continuity properties with respect to their defining measures that are worth recording. More precisely, if  $\mu_n$  is a sequence of positive measures supported on a fixed compact set that converges weakly to  $\mu$ , the *principle of descent* ([46, Thm. I.6.8]) asserts that

$$U^\mu(z) \leq \liminf_{n \rightarrow \infty} U^{\mu_n}(z), \quad z \in \mathbb{C},$$



while the *lower envelope theorem* ([46, Thm. I.6.9]) says that

$$U^\mu(z) = \lim_{n \rightarrow \infty} U^{\mu_n}(z), \quad \text{for q.e. } z \in \mathbb{C}.$$

Another important concept is the *regularity* of a compact set. The outer boundary  $\partial_e E$  of a nonpolar compact set  $E$  is said to be *regular* if the equilibrium potential is continuous on  $E$ , and then it is in fact continuous on  $\mathbb{C}$ . For  $\Omega$  the unbounded connected component of the complement of  $E$ , let  $g_\Omega(\cdot, \infty)$  be the Green function of  $\Omega$  with pole at  $\infty$ , i.e. the unique function such that

- (i)  $g_\Omega(z, \infty)$  is a positive harmonic function in  $\Omega$ , which is bounded when  $z$  stays away from  $\infty$ ;
- (ii)  $g_\Omega(z, \infty) - \log |z|$  is bounded near  $\infty$ ;
- (iii)  $\lim_{z \rightarrow \xi, z \in \Omega} g_\Omega(z, \infty) = 0$  for quasi every  $\xi \in \partial\Omega$ .

It is classical fact (cf. [45, Sec. 4.4]) that

$$U^{\mu_E}(z) = \log \left( \frac{1}{\text{cap}(E)} \right) - g_\Omega(z, \infty), \quad z \in \Omega,$$

and if we set  $g_\Omega(z, \infty) \equiv 0$  in  $\mathbb{C} \setminus \overline{\Omega}$  one can see that  $\partial_e E$  is regular if and only if  $g_\Omega(z, \infty)$  is continuous on  $\mathbb{C}$ . Points of continuity of  $g_\Omega(\cdot, \infty)$  on  $\partial_e E$  are called *regular*, other points are called *irregular*; the latter form a polar set. If a boundary point  $\xi \in \partial E$  is not in  $\partial_e E$  but in the closure of some bounded component  $\Omega'$  of  $\overline{\mathbb{C}} \setminus E$ , it is called *regular* if and only if, after a conformal map  $\Phi$  of  $\overline{\mathbb{C}}$  sending some point of  $\Omega'$  to infinity, the point  $\Phi(\xi)$  is regular on  $\partial_e \Phi(E)$ . The definition does not depend on which conformal map is used, and if all points of  $\partial E$  are regular we say that  $E$  is regular.

Throughout we use the concept of *balayage* of a measure ([46, Sec. II.4]). In particular, we make extensive use of the following fact. Let  $D$  be a domain (connected open set) with compact boundary  $\partial D$  whose complement has positive capacity, and  $\mu$  be a finite Borel measure with compact support in  $D$ . Then there exists a unique Borel measure  $\hat{\mu}$  supported on  $\partial D$ , with total mass is equal to that of  $\mu$ :  $\|\mu\| = \|\hat{\mu}\|$ , whose potential  $U^{\hat{\mu}}$  is bounded on  $\partial D$  and satisfies for some constant  $c(\mu; D)$

$$U^{\hat{\mu}}(z) = U^\mu(z) + c(\mu; D) \quad \text{for q.e. } z \in \mathbb{C} \setminus D. \quad (2.14)$$

Necessarily then, we have that  $c(\mu; D) = 0$  if  $D$  is bounded and  $c(\mu; D) = \int g_D(t, \infty) d\mu(t)$  otherwise. Equality in (2.14) holds for all  $z \in \mathbb{C} \setminus \overline{D}$  and also at all regular points of  $\partial D$ . The measure  $\hat{\mu}$  is called the *balayage* of  $\mu$  onto  $\partial D$ . It has the property that

$$U^{\hat{\mu}}(z) \leq U^\mu(z) + c(\mu; D) \quad \text{for every } z \in \mathbb{C}, \quad (2.15)$$

and also that

$$\int h d\mu = \int h d\hat{\mu} \quad (2.16)$$

for any function  $h$  which is harmonic in  $D$  and continuous in  $\overline{D}$  (including at infinity if  $D$  is unbounded). From its defining properties  $\hat{\mu}$  has finite energy, therefore it cannot charge polar sets. Consequently, on solving the generalized Dirichlet problem [45, Thm. 4.1.5] for an arbitrary positive continuous function on  $\partial D$ , it follows from (2.16) that  $\hat{\mu}$  is positive if  $\mu$  is positive; in particular the balayage of a probability measure is a probability measure. This entails a weak-continuity principle that we record for later use: *if  $\mu_n$  is sequence of probability measures compactly supported in  $D$  converging weak\* to some measure  $\mu$ , and if there is a compact set  $K \subset D$  such that  $\mu_n(D \setminus K) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\hat{\mu}_n$  converges weak\* to  $\hat{\mu}$ .* Indeed, it is easily checked that  $\mu$  is a probability measure supported on  $K$  and that any weak\* limit point  $\nu$  of  $\hat{\mu}_n$  is a probability measure supported on  $\partial D$ . Next, if we denote with superscripts 1 and 2 the restrictions to  $K$  and  $D \setminus K$  respectively, we have that  $\mu_n^1$  converges weak\* to  $\mu$  and we see by the positive linearity of balayage that  $\nu$  is a weak\* limit point of  $\hat{\mu}_n^1$  since  $\|\hat{\mu}_n^1\| = \|\mu_n^1\| \rightarrow 0$ . Then, applying (2.15) to  $\mu_n^1$  and using the principle of descent, we get that  $U^\nu$  is

bounded on  $\partial D$  since  $U^{\mu_n^1}$  is uniformly bounded there; moreover, by the lower envelope theorem for  $\mu_n^1$ , (2.14) is satisfied with  $\nu$  in place of  $\hat{\mu}$ .

The minimal energy problem can also be formulated for signed measures [46, Thm. VIII.1.4]. In particular for  $E_1, E_2$  two disjoint compact sets of positive capacity, there exists a unique measure  $\mu^* = \mu_1^* - \mu_2^*$ , with  $\mu_1^* \in \Lambda(E_1)$  and  $\mu_2^* \in \Lambda(E_2)$ , that minimizes the energy integral

$$I[\mu_1 - \mu_2] = \int \log \frac{1}{|z - t|} d(\mu_1 - \mu_2)(t) d(\mu_1 - \mu_2)(z), \quad \mu_j \in \Lambda(E_j), \quad j = 1, 2. \quad (2.17)$$

It can be proved ([46, Lemma 1.8]) that  $I[\mu^*]$  is positive and finite. The value  $C(E_1, E_2) = 1/I[\mu^*]$  is called the *condenser capacity* of the pair  $(E_1, E_2)$ . Further, it holds that  $\mu_1 = \hat{\mu}_2$  and  $\mu_2 = \hat{\mu}_1$  where  $\hat{\mu}_1$  (resp.  $\hat{\mu}_2$ ) indicates the balayage of  $\mu_1$  (resp.  $\mu_2$ ) onto  $\partial(\bar{\mathbb{C}} \setminus E_2)$  (resp.  $\partial(\bar{\mathbb{C}} \setminus E_1)$ ); this property in fact characterizes  $\mu^*$ , see [46, Thm. VIII.2.6].

In analogy to the logarithmic case, one can define the *Green potential* and the *Green energy* of a positive measure  $\mu$  supported in a domain  $D$  with compact non-polar boundary. The only difference is now that, in (2.12)-(2.13), the logarithmic kernel  $\log(1/|z - t|)$  gets replaced by  $g_D(z, t)$ , the *Green function* for  $D$  with pole at  $t \in D$ . The latter is the unique function defined on  $D$  such that (cf. [45])

- (i)  $g_D(z, t)$  is a positive harmonic function in  $D \setminus \{t\}$ , which is bounded outside each neighborhood of  $t$ ;
- (ii)  $g_D(z, t) - \log \frac{1}{|z - t|}$  is bounded near  $t$ ;
- (iii)  $\lim_{z \rightarrow \xi} g_D(z, t) = 0$  for quasi every  $\xi \in \partial D$ .

Those points at which limit (iii) exists are precisely the regular points of  $\partial D$  [45, Thm. 4.4.9]. For example, it is easily checked that in the case of the unit disk we have

$$g_{\mathbb{D}}(z, t) = \log \left| \frac{1 - z\bar{t}}{z - t} \right|.$$

The Green potential relative to the domain  $D$  of a finite positive measure  $\mu$  compactly supported in  $D$  is given by

$$U_G^\mu(z; D) = \int g_D(z, t) d\mu(t).$$

It can be re-expressed in terms of the logarithmic potentials of  $\mu$  and of its balayage  $\hat{\mu}$  onto  $\partial D$  by the formula [46, Thm. II.4.7 and Thm. II.5.1]

$$U^{\hat{\mu} - \mu}(z) = c(\mu; D) - U_G^\mu(z; D), \quad z \in D \quad (2.18)$$

where  $c(\mu; D)$  was defined after equation (2.14). Moreover, (2.18) continues to hold at every regular point of  $\partial D$ ; in particular, it holds q.e. on  $\partial D$ .

Exactly as in the logarithmic case, if  $E$  is a compact nonpolar subset of  $D$ , there exists a unique measure  $\mu_G^E \in \Lambda(E)$  which minimizes the Green energy among all measures in  $\Lambda(E)$ . This measure is called the *Green equilibrium distribution* on  $E$  relative to  $D$ . By (2.18) we have that

$$U_G^{\mu_G^E}(z; D) = U^{\mu_G^E}(z) - U^{\hat{\mu}_G^E}(z) + c(\mu_G^E; D), \quad z \in D, \quad \text{and q.e. } z \in \partial D,$$

where  $\hat{\mu}_G^E$  is the balayage of  $\mu_G^E$  onto  $\partial D$ . In addition the Green equilibrium distribution satisfies

$$U_G^{\mu_G^E}(z; D) = \frac{1}{C(E, \partial D)}, \quad \text{for q.e. } z \in E. \quad (2.19)$$

Moreover, equality in (2.19) holds at all regular points of  $E$ .

Green potentials have two important properties not shared by logarithmic potentials, namely they are positive and they transform naturally under conformal mappings since this is true of Green functions. In particular, one can define the Green equilibrium distribution with respect to  $D$  of a compact set  $E$  in  $\overline{\mathbb{C}}$ , because one can use conformal mapping to bring the situation back to the case where  $E$  is compact in  $\mathbb{C}$ . The same is true of the condenser capacity.

From the physical viewpoint, the equilibrium measure  $\mu_E$  can be interpreted as the equilibrium distribution of a unit electric charge placed on the conductor  $E$ , and finding such a measure is called the electrostatics problem. A natural way of generalizing this problem is to add an external field acting on the charge. Specifically, let  $E$  be a closed subset of  $\mathbb{C}$ , and  $Q$  a lower semi-continuous function on  $E$  which is less than infinity on a nonpolar subset; if  $E$  is unbounded, one requires in addition that  $Q(z) - \log |z|$  tends to infinity when  $|z|$  tends to infinity in  $E$ . Then there exists a unique measure  $\mu_w^E \in \Lambda(E)$ , called the *weighted equilibrium distribution*, that minimizes the weighted energy integral

$$I_w[\mu] := \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z) + 2 \int Q(t) d\mu(t)$$

over all  $\mu \in \Lambda(E)$ . Moreover, there is a constant  $F_w$  ([46, Thm. I.1.3]) such that

$$\begin{aligned} U^{\mu_w^E}(z) + Q(z) &\geq F_w && \text{for q.e. } z \in E, \\ U^{\mu_w^E}(z) + Q(z) &\leq F_w && \text{for every } z \in \text{supp}(\mu_w^E), \end{aligned}$$

$F_w$  is called the *modified Robin constant* for  $Q$  and it is equal to

$$F_w = I_w[\mu_w^E] - \int Q(t) d\mu_w^E(t).$$

Coarser discussions of potentials, energy, Green functions, and balayage are held, for example, in [46] and [30]. For the reader's convenience, we formulate below two propositions that will be of particular use to us. The first one has to do with the particular geometry of the disk, and we prove it in detail as we could not find an appropriate reference for it in the literature.

**Proposition A** *Let  $E \subset \mathbb{D}$  be a compact set of positive capacity not containing 0 with connected complement, and  $E^*$  stand for its reflection across the unit circle, i.e.  $E^* := \{z \in \mathbb{C} : 1/\bar{z} \in E\}$ . Further, let  $\mu \in \Lambda(E)$  and  $\sigma \in \Lambda(E^*)$  solve the signed energy problem for the condenser  $(E, E^*)$ . Then, we have that*

- (a)  $\sigma$  is reflected from  $\mu$  across the unit circle, i.e.  $\sigma(B) = \mu(B^*)$  for any Borel set  $B$ , and likewise  $\mu$  is reflected from  $\sigma$ ;
- (b)  $\mu$  is the Green equilibrium distribution on  $E$  relative to  $\overline{\mathbb{C}} \setminus E^*$  and  $\sigma$  is the Green equilibrium distribution on  $E^*$  relative to  $\overline{\mathbb{C}} \setminus E$ ;
- (c)  $\tilde{\mu} = \tilde{\sigma}$ , where  $\tilde{\lambda}$  denotes the balayage of the measure  $\lambda$  onto the unit circle. Moreover, the balayage of  $\tilde{\mu}$  onto  $E$  is  $\mu$  and the balayage of  $\tilde{\mu}$  onto  $E^*$  is  $\sigma$ ;
- (d)  $\tilde{\mu}$  is the Green equilibrium distribution on  $\mathbb{T}$  relative to both  $\overline{\mathbb{C}} \setminus E$  and  $\overline{\mathbb{C}} \setminus E^*$ ;
- (e)  $\mu$  is the Green equilibrium distribution on  $E$  relative to  $\mathbb{D}$  and  $\sigma$  is the Green equilibrium distribution on  $E^*$  relative to  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . Further,

$$U_G^{\tilde{\mu}}(z; \mathbb{C} \setminus E) = \frac{1}{C(E, \mathbb{T})}, \quad z \in \mathbb{C} \setminus \mathbb{D}, \quad \text{and} \quad U_G^{\tilde{\mu}}(z; \mathbb{C} \setminus E^*) = \frac{1}{C(E^*, \mathbb{T})}, \quad z \in \overline{\mathbb{D}},$$

where in fact

$$C(E, \mathbb{T}) = C(E^*, \mathbb{T}).$$

*Proof:* (a) First of all we observe that  $E^*$  is compact since  $E$  does not contain 0. Then, if we denote by  $\mu'$  and  $\sigma'$  respectively the reflections of  $\mu$  and  $\sigma$  across  $\mathbb{T}$ , and likewise by  $z'$  the reflection of the point  $z \in \mathbb{C}$ , we get on changing variables in (2.17)

$$\begin{aligned} I[\mu - \sigma] &= \int \log \frac{|z't'|}{|t' - z'|} d(\mu' - \sigma')(t') d(\mu' - \sigma')(z') \\ &= I[\sigma' - \mu'] + \int \log |z'| d(\sigma' - \mu')(t') d(\sigma' - \mu')(z') + \int \log |t'| d(\sigma' - \mu')(t') d(\sigma' - \mu')(z') \\ &= I[\sigma' - \mu'], \end{aligned} \quad (2.20)$$

where the last equality uses that  $\log |z|$  is bounded on  $E \cup E^*$  together with Fubini's theorem and the fact that  $\sigma' - \mu'$  has total mass zero. Since  $\sigma' \in \Lambda(E)$  and  $\mu' \in \Lambda(E^*)$  we get that  $\mu = \sigma'$  and  $\sigma = \mu'$  by uniqueness of the solution to the signed energy problem on  $(E, E^*)$ .

(b) This follows immediately from [46, Cor. VIII.2.7] as applied to both  $\mu - \sigma$  and  $\sigma - \mu$ .

(c) Let  $z_0 \in \mathbb{D}$  and  $\delta_{z_0}$  be the Dirac measure with a unit mass at  $z_0$ . Then, by the properties of balayage (see 2.14), we get for each  $z \in \mathbb{T}$  since the latter is regular

$$U^{\tilde{\delta}_{z_0}}(z) = -\log |z - z_0| = -\log |z - 1/\bar{z}_0| - \log |z_0| = U^{\tilde{\delta}_{1/\bar{z}_0}}(z) - g_{\mathbb{C} \setminus \mathbb{D}}(1/\bar{z}_0, \infty) - \log |z_0|.$$

Therefore the potentials of  $\tilde{\delta}_{z_0}$  and  $\tilde{\delta}_{1/\bar{z}_0}$  differ by a constant on their common support  $\mathbb{T}$ , and since these two measures have equal mass and finite energy we get from the unicity theorem [46, Thm. II.4.6] that  $\tilde{\delta}_{z_0} = \tilde{\delta}_{1/\bar{z}_0}$ . Using the linearity of balayage [46, Eq. II.4.13] and the definition of  $\sigma$ , we now obtain

$$\tilde{\mu} = \int \tilde{\delta}_t d\mu(t) = \int \tilde{\delta}_{1/\bar{t}} d\mu(t) = \int \tilde{\delta}_t d\sigma(t) = \tilde{\sigma}.$$

Since balayage onto  $\partial E$  from outside  $\mathbb{D}$  can be done in two steps (we first balayage onto  $\mathbb{T}$  and then onto  $\partial E$ ), we get on denoting this balayage with a “ $\hat{\phantom{x}}$ ” that

$$\hat{\tilde{\mu}} = \hat{\tilde{\sigma}} = \hat{\sigma} = \mu,$$

where the last equality follows from the properties of equilibrium signed measures mentioned after (2.17). The equality  $\hat{\tilde{\sigma}} = \sigma$  is obtained by the same argument.

(d) By (c) the pairs  $(\tilde{\mu}, \mu)$  and  $(\tilde{\mu}, \sigma)$  satisfy the mutual balayage property and therefore solve the signed energy problem for the condensers  $(\mathbb{T}, E)$  and  $(\mathbb{T}, E^*)$ , respectively. The assertion now follows from [46, Cor. VIII.2.7].

(e) We have that

$$U_G^\mu(z; \mathbb{D}) = U^\mu(z) - U^{\tilde{\mu}}(z) = \int g_{\mathbb{C} \setminus E}(t, \infty) d\tilde{\mu}(t), \quad \text{q.e. } z \in E,$$

where the first equality is a general property of balayage onto the boundary of a bounded domain (in our case  $D$  since we balayage out  $\mu$  onto  $\mathbb{T}$ ) [46, Thm. II.5.1] and the second equality follows from (c) and (2.14) applied with  $D = \mathbb{C} \setminus E$  and  $\tilde{\mu}$  in place of  $\mu$ . Then, since  $\mu$  has finite energy and  $U_G^\mu(z; \mathbb{D})$  is constant q.e. on  $E$ , we deduce that  $\mu$  is the Green equilibrium measure on  $E$  relative to  $\mathbb{D}$  [46, Thm. II.5.12] and moreover that

$$\frac{1}{C(E, \mathbb{T})} = \int g_{\mathbb{C} \setminus E}(t, \infty) d\tilde{\mu}(t). \quad (2.21)$$

Now, using again (c) and the general relation between Green potentials and balayage [46, Thm. II.5.1], this time for the measure  $\tilde{\mu}$  over the unbounded domain  $\mathbb{C} \setminus E$ , we get for any  $z \in \mathbb{C} \setminus \mathbb{D}$

$$U_G^{\tilde{\mu}}(z; \mathbb{C} \setminus E) = U^{\tilde{\mu}}(z) - U^\mu(z) + \int g_{\mathbb{C} \setminus E}(t, \infty) d\tilde{\mu}(t) = \frac{1}{C(E, \mathbb{T})}$$

where the last equality comes from (2.21) and the fact that  $U^{\tilde{\mu}}$  and  $U^{\mu}$  coincide on  $\mathbb{C} \setminus \mathbb{D}$  by the regularity of  $\mathbb{T}$ . The assertions on  $\sigma$  and on  $U_G^{\tilde{\mu}}(z; \mathbb{C} \setminus E^*)$  are proved in the same manner. Finally, it follows from the proof of (d) that

$$\frac{1}{C(E, \mathbb{T})} = I[\mu - \tilde{\mu}], \quad \frac{1}{C(E^*, \mathbb{T})} = I[\sigma - \tilde{\mu}],$$

and since  $\tilde{\mu}$  is its own reflection across  $\mathbb{T}$  we get from (a) and a computation similar to (2.20) that these two quantities indeed coincide. ■

**Proposition B** *Let  $E$  be a compact set of positive capacity and  $\Omega$  be the unbounded component of  $\overline{\mathbb{C}} \setminus E$ . Let further  $\sigma$  be a probability measure such that  $\text{supp}(\sigma) \subset \Omega$ . Then, if  $\hat{\sigma}$  denotes the balayage of  $\sigma$  onto  $\partial\Omega$ , it holds that*

$$\mu_w^E = \hat{\sigma} \quad \text{and} \quad F_w = \int g_{\mathbb{C} \setminus E}(z, \infty) d\sigma(t),$$

where  $\mu_w^E$  is the weighted equilibrium distribution on  $E$  for the external field  $-U^\sigma$  and  $F_w$  is the modified Robin constant. Moreover, if  $E$  is regular, then  $\text{supp}(\mu_w^E) = \partial\Omega = \partial_e E$ , the so-called exterior boundary of  $E$ .

*Proof:* This is a consequence of [46, Thm IV.1.10(g)]. ■

This paper is organized as follows. Section 3 deals with meromorphic approximants that are solutions (more generally: critical points) of  $\mathbf{MA}(p)$  when  $f$  is of the form (2.4). We discuss the asymptotics of poles as being the limit zero distribution of polynomials satisfying certain non-Hermitian orthogonality relations with respect to varying measures. We apply the results to the convergence in capacity of these approximants, and to the convergence of some of their poles to the polar singularities of  $f$ . Section 4 deals with the corresponding matters for multipoint Padé approximants. Some computational results are adduced in Section 5.

### 3 Meromorphic Approximation

We start this section with a brief insight to the solution of  $\mathbf{MA}(p)$ , needed to understand the results of this section and providing a motivation for them.

Originally this problem was solved for the case  $p = \infty$  by V. M. Adamyan, D. Z. Arov, and M. G. Krein in [1] and the solution came through operator theory. The most accessible reference to this result is perhaps [51]. Later F. Seyfert and the first author [12], and independently V. A. Prokhorov [42], generalized it to the case  $1 \leq p \leq \infty$ , but it is only in [12] and when  $p \in [2, \infty]$  that concrete equations were obtained for the approximants. These form the basis of our approach, and presently limit its scope to  $p \geq 2$ .

As it is the case of interest here, we shall restrict our discussion to the situation where the approximated function is of the form (2.4), and accordingly we write  $F$  instead of  $f$ . We should note that when  $p = 2$  the problem reduces to rational approximation. Indeed, since  $H^2 \subset L^2(\mathbb{T})$  as a closed subspace,  $L^2(\mathbb{T})$  can be decomposed into the orthogonal sum of  $H^2$  and its orthogonal complement  $\bar{H}_0^2$ , which consists of analytic functions in  $\mathbb{C} \setminus \overline{\mathbb{D}}$  vanishing at  $\infty$  with norm  $\sup_{r>1} \|f(r \cdot)\|_2$  (compare (2.8)). Now, since  $F$  is the Cauchy transform of a measure supported in  $\mathbb{D}$ , it belongs to  $\bar{H}_0^2$ . Thus for any  $g = (h + p_{n-1}/q_n) \in H_n^2$  with  $h \in H^2$  and  $p_{n-1}/q_n$  a rational function in  $\bar{H}_0^2$ , we get by orthogonality

$$\|F - g\|_2^2 = \|h\|_2^2 + \|F - p_{n-1}/q_n\|_2^2.$$

Clearly then, for  $g$  to be a best approximant  $h$  must be zero, and we arrive at the following approximation problem:

*Given  $f \in \bar{H}_0^2$  and  $n \in \mathbb{N}$ , minimize  $\|f - p_{n-1}/q_n\|_2$  over  $p_{n-1} \in \mathcal{P}_{n-1}$  and  $q_n \in \mathcal{M}_n$ .*

Moreover, thanks to Parseval's formula,  $p_{n-1}/q_n$  will be a best approximant to  $f$  not only from  $\mathcal{R}_{n-1,n}$  but

also from  $\mathcal{R}_{n,n}$  since  $f$  vanishes at infinity. Despite this, we shall not distinguish  $p = 2$  from the other cases but rather keep a unified operator approach.

Let us denote by  $\mathbf{P}_+$  and  $\mathbf{P}_-$  the analytic and anti-analytic projections acting on Fourier series as

$$\mathbf{P}_+ \left( \sum_{k=-\infty}^{+\infty} a_k e^{ik\theta} \right) = \sum_{k=0}^{+\infty} a_k e^{ik\theta} \quad \mathbf{P}_- \left( \sum_{k=-\infty}^{+\infty} a_k e^{ik\theta} \right) = \sum_{k=-\infty}^{-1} a_k e^{ik\theta}.$$

By a well-known theorem of M. Riesz [19],  $\mathbf{P}_+ : L^p(\mathbb{T}) \rightarrow H^p$  and  $\mathbf{P}_- : L^p(\mathbb{T}) \rightarrow \bar{H}_0^p$  are bounded when  $1 < p < \infty$ , and when  $p = 2$  they are just the orthogonal projections associated to the orthogonal decomposition:  $L^2(\mathbb{T}) = H^2 \oplus \bar{H}_0^2$ . When  $f \in L^1(\mathbb{T})$ , we simply regard  $\mathbf{P}_+(f)$  and  $\mathbf{P}_-(f)$  as Fourier series of distributions.

For each  $p \in [2, \infty]$ , the *Hankel operator with symbol  $F$*  is given by

$$\begin{aligned} A_F : H^{p'} &\rightarrow \bar{H}_0^2 \\ u &\mapsto \mathbf{P}_-(Fu), \end{aligned}$$

where  $p'$  is conjugate to  $p$  modulo 2, i.e.  $1/p + 1/p' = 1/2$ .

For  $n = 0, 1, 2, \dots$ , the  $n$ -th *singular number* of the operator  $A_F$  is defined to be

$$\sigma_n(A_F) := \inf \left\{ \|\Gamma - A_F\|, \quad \Gamma : H^{p'} \rightarrow \bar{H}_0^2 \text{ a linear operator of rank } \leq n \right\},$$

where  $\|\cdot\|$  stands for the operator norm; when  $p = 2$  we assume in addition that  $\Gamma$  is weak\* continuous.

Note that if  $p = \infty$  then  $p' = 2$ , hence  $A_F$  operates between Hilbert spaces, and since it is compact<sup>4</sup> the  $\sigma_n(A_F)$  are just the singular values of  $A_F$ , that is, the square-roots of the eigenvalues of  $A_F^* A_F$  arranged in nonincreasing order; throughout  $A_F^*$  indicates the adjoint of  $A_F$ . When  $2 \leq p < \infty$ , the usual eigenvector equation gets replaced by a *nonlinear* equation of Hammerstein type. More precisely, to each  $\sigma_n(A_F)$  there exists (at least one)  $v_n \in H^{p'}$  of unit norm, whose inner factor is a Blaschke product of degree at most  $n$ , such that

$$\begin{aligned} A_F^* A_F(v_n) &= \sigma_n^2(A_F) \mathbf{P}_+ \left( |v_n|^{p'-2} v_n \right) & \text{if } p > 2, \\ A_F^* A_F(v_n) &= \mathbf{P}_+ (|A_F(v_n)|^2 v_n) \quad \text{and} \quad \|A_F(v_n)\|_2 = \sigma_n(A_F) & \text{if } p = 2. \end{aligned} \quad (3.1)$$

Such a  $v_n$  will be called a  $n$ -th singular vector for  $A_F$ . From the definition it follows that a  $n$ -th singular vector can be factored as

$$v_n = b_n w_n \quad (3.2)$$

where  $b_n \in B_n$  and  $w_n$  is an outer function of unit norm in  $H^{p'}$ . Actually, upon pairing with  $v_n$ , equation (3.1) implies that  $v_n \in B_n$  if  $p = 2$ , so that  $w_n \equiv 1$  in this case.

The solutions to  $\mathbf{MA}(p)$  turn out to be exactly the functions of the form ([12, Thm. 8.2])

$$g_n = F - \frac{A_F(v_n)}{v_n} = \frac{\mathbf{P}_+(Fv_n)}{v_n}, \quad (3.3)$$

where  $v_n$  is some  $n$ -th singular vector for  $A_F$ . Moreover, we have that  $\|F - g_n\|_p = \sigma_n(A_F)$ . Such a  $g_n$  is called a best meromorphic  $L^p$ -approximant of order  $n$  to  $F$ .

When  $p \in [2, \infty)$  a best meromorphic approximant  $g_n$  is, in general, not unique, but it has exactly  $n$  poles; this implies that  $b_n$  in (3.2) has exact degree  $n$  for any  $n$ -th singular vector  $v_n$ . To the contrary, if  $p = \infty$ , then  $g_n$  is unique but may have less than  $n$  poles and  $b_n$  could have degree less than  $n$ . In the latter case the span of  $n$ -th singular vectors has dimension greater than 1, so that  $v_n$  is far from being unique up to a multiplicative constant, but it is remarkable that all the  $v_n$  give rise to the same  $g_n$  through formula (3.3); in fact there always exist  $n$ -th singular vectors for which  $b_n$  has exact degree  $n$ , but some of the poles may cancel with zeros of  $\mathbf{P}_+(Fv_n)$  in (3.3).

<sup>4</sup>This can be deduced from the fact that  $F \in C(\mathbb{T})$ , see [39, Thm. I.5.5].

The notion of a best approximant can be further weakened to the notion of a *critical point*. By definition, a function  $g_n$  is a critical point of order  $n$  in  $\mathbf{MA}(p)$  if and only if it assumes the form

$$g_n = F - \frac{A_F(v_n)}{v_n},$$

where  $v_n$  is a  $H^{p'}$  function of unit norm (a Blaschke product if  $p = 2$ ) whose inner factor lies in  $B_n \setminus B_{n-1}$ , which is such that

$$\begin{aligned} A_F^* A_F(v_n) &= \gamma_n \mathbf{P}_+ \left( |v_n|^{p'-2} v_n \right), & \gamma_n \in \mathbb{R}, & \text{ if } p > 2, \\ A_F^* A_F(v_n) &= \mathbf{P}_+ (|A_F(v_n)|^2 v_n) & & \text{ if } p = 2. \end{aligned} \quad (3.4)$$

The difference with (3.1) is that here  $\gamma_n$  or  $\|A_F(v_n)\|_2$  needs not be equal to  $\sigma_n(A_F)$ . With a slight abuse of language, we will continue to say that  $v_n$  is a singular vector associated to  $g_n$  although  $\gamma_n$  may no longer be a singular value. Note that, as for best meromorphic approximants,  $v_n$  is just a Blaschke product when  $p = 2$ . Thus it has an inner-outer factorization of type (3.2) where  $b_n$  has exact degree  $n$  and  $w_n \equiv 1$  if  $p = 2$ .

A critical point  $g_n$  of order  $n$  may have less than  $n$  poles, even though we insisted in the definition that  $v_n$  has exactly  $n$  zeros. This happens when some zeros of  $v_n$  (which are, of course, zeros of  $b_n$ ) cancel with some zeros of  $\mathbf{P}_+(Fv_n)$ . In this case  $b_n$  can be further factorized as

$$b_n = b_{n_1} \cdot b_{n_2}, \quad (3.5)$$

where  $b_{n_1}$  is the greatest common inner divisor of  $\mathbf{P}_+(Fv_n)$  and  $v_n$ . Then ([12, Prop. 9.1])

$$\begin{aligned} A_F(v_n)(\xi) &= \gamma_n^{1/2} \bar{\xi} \overline{(b_{n_2} j_n w_n^{p'/2})} (\xi) = \gamma_n^{1/2} \left( b_{n_2} j_n w_n^{p'/2} \right)^\sigma (\xi), & p > 2 \\ A_F(v_n)(\xi) &= \bar{\xi} \overline{(b_{n_2} u_n)} (\xi) = (b_{n_2} u_n)^\sigma (\xi), & p = 2, \end{aligned} \quad (3.6)$$

for a.e.  $\xi \in \mathbb{T}$ , where  $j_n$  is some inner function,  $u_n \in H^2$ , and  $h^\sigma(z) := z^{-1} \overline{h(1/\bar{z})}$ . Note that if  $h \in H^2$  then  $h^\sigma \in \bar{H}_0^2$  and vice versa. We remark that for the case  $p = 2$  equation (3.6) is an interpolation condition saying that  $g_n$  interpolates  $F$  with order 2 at the reflection of its poles. The same can be said when  $p \in (2, \infty]$ , provided  $g_n$  is analytic at the reflection of its poles, but this is no longer automatic because it may no longer be rational.

Although their definition is a little technical, critical points are just those  $g_n \in H_n^p$  for which the derivative of  $\|F - g_n\|_p$  with respect to  $b_n \in B_n$  and  $h \in H^p$  in factorization (2.9) does vanish. Beyond best approximants, the most important critical points are *local best approximant* [12, Prop. 9.3.]. By definition, a local best approximant is some  $g_n \in H_n^p$  for which there exists  $\delta > 0$  such that

$$g \in H_n^p \quad \text{and} \quad \|g - g_n\|_p \leq \delta \quad \text{imply} \quad \|F - g_n\|_p \leq \|F - g\|_p.$$

The reason why we introduce critical points is that all a numerical search can yield in general is a local best approximant, and we feel it is important that our results should apply to computable objects.

When  $p \in [2, \infty)$ , best and local best approximants have exactly  $n$  poles, counting multiplicities, hence they are *a fortiori* critical points of order  $n$  ([12, Prop. 9.2 and 9.3]); such critical points are called *irreducible*. For  $p = \infty$  the critical points are just the best meromorphic approximants, so the notion is nothing new, but it may happen that a best approximant out of  $H_n^\infty$  has less than  $n$  poles. However, each time the number of poles of  $g_n$  increases with  $n$ , it jumps to the maximum value  $n$ , in particular there exists a subsequence of natural numbers, say  $\mathbb{N}_0 = \mathbb{N}_0(F)$ , such that for each  $n \in \mathbb{N}_0$  the best approximant  $g_n$  has exactly  $n$  poles in  $\mathbb{D}$ , *i.e.* it is irreducible ([12, p. 114]). Since the behavior of the poles of best approximants from  $H_n^\infty$  is entirely characterized by this subsequence, hereafter we say “a sequence of irreducible critical points of order  $n$ ” to mean if  $p = \infty$  that we pass to a subsequence if needed. Let  $\{g_n\}$  be such a sequence. As usual, we denote by  $v_n$  an associated singular vector to  $g_n$ . According to (2.9), each  $g_n$  can be decomposed as

$$g_n = b_n^{-1} \cdot h_n,$$

where  $h_n \in H^p$  and  $b_n = b_{n_2}$ , i.e.  $b_{n_1} \equiv 1$  in (3.5). Moreover, we can write  $b_n$  as  $q_n/\tilde{q}_n$ , where  $q_n \in \mathcal{M}_n$  and  $\tilde{q}_n(t) := t^n \overline{q_n(1/\bar{t})}$  is the reciprocal polynomial of  $q_n$ . Arguing like in ([12, Sec. 10]), where  $R_s$  is not present, equation (3.6) implies easily the following orthogonality relations

$$\int t^k q_n(t) \frac{w_n(t)}{\tilde{q}_n^2(t)} d\tilde{\lambda}(t) = 0, \quad k = 0, \dots, n-1, \quad (3.7)$$

where  $w_n$  is the outer factor of  $v_n$  and  $\tilde{\lambda}$  is given by (2.7). Upon rewriting (3.7) as

$$\int P_{n-1}(t) q_n(t) \frac{w_n(t)}{\tilde{q}_n^2(t)} d\lambda(t) + \sum_{\eta_i \in S'} \sum_{k=0}^{m(\eta_i)-1} \frac{r_{i,k}}{k!} \left( P_{n-1}(t) q_n(t) \frac{w_n(t)}{\tilde{q}_n^2(t)} \right)^{(k)} \Big|_{t=\eta_i} = 0,$$

for all  $P_{n-1} \in \mathcal{P}_{n-1}$  and taking  $P_{n-1}$  to be a multiple of  $Q_s$ , these relations yield for  $n > s$

$$\int t^k Q_s(t) q_n(t) \frac{w_n(t)}{\tilde{q}_n^2(t)} d\lambda(t) = 0, \quad k = 0, \dots, n-s-1 \quad (3.8)$$

where  $Q_s$  was defined in (2.5). When  $p \in (2, \infty]$ , the  $w_n$  are *a priori* holomorphic in the unit disk only, but due to the analyticity of  $F$  it is possible to say more. Since for  $|z| > 1$

$$\begin{aligned} A_F(v_n)(z) &= \mathbf{P}_-(Fv_n)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(Fv_n)(\xi)}{z - \xi} d\xi \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \int_{\tilde{S}} \frac{v_n(\xi)}{(z - \xi)(\xi - t)} d\tilde{\lambda}(t) d\xi = \int_{\tilde{S}} \frac{v_n(t)}{z - t} d\tilde{\lambda}(t), \end{aligned} \quad (3.9)$$

we obtain from (3.6) that  $(b_n j_n w_n^{p'/2})^\sigma$  can be analytically extended to  $\overline{\mathbb{C}} \setminus \tilde{S}$  by the formula

$$\gamma_n^{1/2} (b_n j_n w_n^{p'/2})^\sigma(z) = A_F(v_n)(z) = \int \frac{v_n(t)}{z - t} d\tilde{\lambda}(t), \quad z \in \overline{\mathbb{C}} \setminus \tilde{S}. \quad (3.10)$$

Recall that  $\gamma_n \neq 0$  since  $F$  is not rational. This, in particular, implies that  $b_n j_n w_n^{p'/2}$  can be extended to a holomorphic function in

$$\Omega_{\tilde{S}} := \overline{\mathbb{C}} \setminus \tilde{S}^*, \quad (3.11)$$

where  $\tilde{S}^*$  denotes, as before, the reflection of  $\tilde{S}$  across  $\mathbb{T}$ . This extension, in turn, implies that  $j_n$  must be a finite Blaschke product, because the zeros of an analytic function are isolated in the domain of analyticity. Multiplying (3.10) by  $b_n j_n$  and applying  $\mathbf{P}_-$ , we get from the definition of “ $\sigma$ ”, since  $w_n^{p'/2} \in H^2$  and  $|b_n j_n| = 1$  on  $\mathbb{T}$  that the left-hand side is

$$\gamma_n^{1/2} \mathbf{P}_- \left( b_n j_n (b_n j_n w_n^{p'/2})^\sigma \right) = \gamma_n^{1/2} \mathbf{P}_- \left( (w_n^{p'/2})^\sigma \right) = \gamma_n^{1/2} (w_n^{p'/2})^\sigma$$

and computing the right-hand side as in (3.9) we get for  $|z| > 1$

$$\begin{aligned} \gamma_n^{1/2} (w_n^{p'/2})^\sigma(z) &= \int \frac{b_n(t) j_n(t) v_n(t)}{z - t} d\tilde{\lambda}(t) = \int \frac{b_n(t) j_n(t) v_n(t)}{z - t} d\lambda(t) \\ &+ \sum_{\eta_i \in S'} \sum_{k=0}^{m(\eta_i)-1} \frac{r_{i,k}}{k!} \left( \frac{b_n(t) j_n(t) v_n(t)}{z - t} \right)^{(k)} \Big|_{t=\eta_i}. \end{aligned}$$

This shows that  $(w_n^{p'/2})^\sigma$  has a meromorphic extension to  $\overline{\mathbb{C}} \setminus S$  and may only have a pole of order at most  $m(\eta)$  at each  $\eta \in S'$ , though possible cancellations may occur due to the zeros of  $b_n j_n$ . Equivalently,  $w_n^{p'/2}$  can be meromorphically extended to  $\Omega_{\tilde{S}}$  and may only have poles at the points  $1/\bar{\eta}$ , of respective multiplicities at most  $m(\eta)$  for each  $\eta \in S'$ .

Now we are ready to analyze the behavior of the poles of the irreducible critical points. To do so, we shall restrict the class of measures under consideration as follows. We say that a complex measure  $\mu$  of type (2.2), with argument of bounded variation, supported on  $(-1, 1)$ , belongs to the class **BVT** if



- (1)  $\text{supp}(\mu)$  is a regular set;
- (2) there exist positive constants  $c$  and  $L$  such that, for any  $x \in \text{supp}(\mu)$  and  $\delta \in (0, 1)$ , the total variation of  $\mu$  satisfies  $|\mu|([x - \delta, x + \delta]) \geq c\delta^L$ .

Let us first state the 4 theorems in this section before proceeding with the proofs. We start with a result on the zero distribution of polynomials satisfying certain nonlinear orthogonality relations. It is the working tool of our approach to the asymptotic behavior of poles of irreducible critical points, in particular of poles of best or local best approximants. We continue to denote by  $I$  the convex hull of  $S = \text{supp}(\lambda)$ .

**Theorem 3.1** *Let  $\{q_n\}_{n \in \mathbb{N}}$  be a sequence of polynomials of exact degree  $n$  with all zeros in  $\mathbb{D}$  satisfying the orthogonality relations*

$$\int t^k q_n(t) \frac{\omega_n(t)}{\bar{q}_n^2(t)} d\tilde{\lambda}(t) = 0, \quad k = 0, \dots, n-1, \quad (3.12)$$

where  $\tilde{\lambda} = \lambda + \lambda'$  is given by (2.2)-(2.6) with  $S \subset (-1, 1)$  and  $S' \subset \mathbb{D}$ , while  $\mathcal{W} = \{\omega_n\}_{n=1}^\infty$  is a family of complex measurable functions on the union of  $S'$  and  $I$ , whose moduli are uniformly bounded above and below, and whose arguments are smooth with uniformly bounded derivatives on  $I$ . Suppose further that  $\lambda \in \mathbf{BVT}$ . Then the counting measures of the zeros of  $q_n(z) = \prod_{j=1}^n (z - \xi_{j,n})$ , namely  $\nu_n := (1/n) \sum_{j=1}^n \delta_{\xi_{j,n}}$ , converge in the weak\* sense to  $\mu_G^S$ , the Green equilibrium distribution on  $S$  relative to  $\mathbb{D}$ .

The above stated theorem is a direct generalization of Theorem 5.1 and Corollary 6.2 in [8]. The main difference is that here we add a distribution of the form (2.6) to the measure (3.12), i.e.  $F$  may have polar singularities inside of the unit disk. As compared to the above reference, we simplified the proof somewhat by using the two-constant theorem instead of weighted potential theory. To the author's knowledge these are the first published results about the zero distribution of polynomials satisfying nonlinear orthogonality equations like (3.12) that are typical of rational or meromorphic approximation with free poles. In the Ph.D. thesis of R. Küstner [27], an analog of Theorem 5.1 in [8] is given when the measure  $\lambda$ , instead of belonging to  $\mathbf{BVT}$ , satisfies the so-called  $\Lambda$ -criterion introduced in [49, Sec. 4.2]:

$$\text{cap} \left( \left\{ t \in S : \limsup_{r \rightarrow 0} \frac{\text{Log}(1/\mu[t-r, t+r])}{\text{Log}(1/r)} < +\infty \right\} \right) = \text{cap}(S).$$

Paralleling the arguments in [27], all the results in this section could be obtained under this weaker assumption, but the exposition would be heavier and we leave it to the interested reader to carry out the details.

**Theorem 3.2** *Let  $p \in [2, \infty]$ ,  $p'$  the conjugate exponent modulo 2, and  $\{g_n\}_{n \in \mathbb{N}}$  a sequence of irreducible critical points of order  $n$  of  $\mathbf{MA}(p)$  for  $F$ , where  $F$  is given by (2.4) with  $\lambda \in \mathbf{BVT}$ . Then the counting measures of the poles of  $g_n$  converge to  $\mu_G^S$  in the weak\* sense.*

The previous theorem gives one answer to question (a) raised in Section 2. The next one addresses question (c) by stating that approximants behave rather nicely toward the approximated function, namely they converge in capacity to  $F$  on  $\mathbb{D} \setminus S$  and locally uniformly on  $\mathbb{C} \setminus \mathbb{D}$ . Moreover,  $n$ -th root estimates for the error are provided. Recall that a sequence of functions  $\{g_n\}$  converges in capacity to a function  $g$  on a compact set  $K \subset \mathbb{C}$  if, for any  $\epsilon > 0$ ,

$$\text{cap}(\{z \in K : |g_n(z) - g(z)| \geq \epsilon\}) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We say that  $\{g_n\}$  converges in capacity to  $g$  on a domain  $D$  if  $\{g_n\}$  converges in capacity on compact subsets of  $D$ . Hereafter we denote by  $\|\cdot\|_E$  the  $\sup$  norm on a set  $E$ .

**Theorem 3.3** *Let  $\{g_n\}_{n \in \mathbb{N}}$ ,  $F$ , and  $\lambda$  be as in Theorem 3.2. Then  $\{g_n\}$  converges to  $F$  in capacity on  $\mathbb{D} \setminus S$ ; more precisely, for any compact  $K \subset \mathbb{D} \setminus S$  there exist positive constants  $\beta$  and  $\gamma$  such that, for  $n$  large enough, one has*

$$\text{cap} \{z \in K : |r_n(z)| \geq e^{-\beta n}\} \leq e^{-\gamma n},$$

where  $r_n := (F - g_n)$ .

In addition, in the case of rational approximation (i.e. when  $p = 2$ ), we have that for any compact  $K \subset \mathbb{C} \setminus (\mathbb{D} \cup \mathcal{K})$

$$\limsup_{n \rightarrow \infty} \|r_n\|_K^{1/2n} \leq \exp \left\{ -\frac{1}{C(S, \mathbb{T})} \right\}.$$

where  $C(S, \mathbb{T})$  is the condenser capacity and  $\mathcal{K}$  is the set of limit points of poles of  $\{g_n\}$ . In particular,  $\{g_n\}$  converges locally uniformly in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ .

To approach question (b), we need to introduce some more notation. For any  $\xi \neq 0 \in \mathbb{C}$ , we let  $\text{Arg}(\xi) \in (-\pi, \pi]$  be the principal branch of the argument and for  $\xi = 0$  we set  $\text{Arg}(0) = \pi$ . With this definition,  $\text{Arg}(\cdot)$  becomes a left continuous function on  $\mathbb{R}$ . Now, for any interval  $[a, b] \subset \mathbb{R}$  we can define the angle in which this interval is seen at  $\xi \in \mathbb{C}$  by

$$\text{Angle}(\xi, [a, b]) := |\text{Arg}(a - \xi) - \text{Arg}(b - \xi)|.$$

Note that  $0 \leq \text{Angle}(\xi, [a, b]) \leq \pi$  and  $\text{Angle}(\xi, [a, b]) = \pi$  if and only if  $\xi \in [a, b]$ . We define additively this angle for a system of disjoint closed intervals: if  $\{[a_j, b_j]\}_{j=1}^m$  is such a system, then the angle in which it is seen at  $\xi$  is defined by

$$\theta(\xi) := \sum_{j=1}^m \text{Angle}(\xi, [a_j, b_j]). \quad (3.13)$$

The notation  $\theta(\xi)$  does not reflect the dependency on the system of intervals, but the latter will always be made clear. The forthcoming theorem implies that each pole of  $F$  attracts at least as many poles of meromorphic approximants as its multiplicity and not much more. This is one answer to question (b).

**Theorem 3.4** *Let  $\{g_n\}_{n \in \mathbb{N}}$ ,  $F$ , and  $\lambda$  be as in Theorem 3.2. Then for each  $\eta \in S' \setminus S$  there exist  $D_\eta$ , an open neighborhood of  $\eta$ , a constant  $C_\eta = C_\eta(F)$ , and a number  $N = N(\eta) \in \mathbb{N}$  such that for any  $n \geq N$*

$$m(\eta) \leq \#\{S_n \cap D_\eta\} \leq m(\eta) + C_\eta, \quad (3.14)$$

where  $m(\eta)$  is the multiplicity of  $\eta$ .

In fact, the proof of the above theorem is constructive in the sense that we obtain an upper bound for  $C_\eta$ . Namely, let  $I_{S,m} := \{[a_j, b_j]\}_{j=1}^m$  be any finite system of intervals covering  $S$  and let  $w_n$  be the outer function in the inner-outer factorization of the singular vector  $v_n$  associated to  $g_n$  for each  $n \in \mathbb{N}$ . Then

$$C_\eta \leq \frac{V(\varphi) + V_W + (m-1)\pi + 2 \sum_{\eta_i \in S'} m(\eta_i) \theta(\eta_i)}{\pi - \theta(\eta)},$$

where

$$V_W := \sup_{n \in \mathbb{N}} V(\text{Arg}(w_n), S) \quad (3.15)$$

and  $\theta(\cdot)$  is the angle function defined in (3.13) applied to  $I_{S,m}$ . We shall prove in due course that indeed  $V_W < +\infty$ .

The proof of Theorem 3.1 relies on several auxiliary lemmas.

**Lemma 3.5** *With the previous notation the following statements hold true*

- (a) *Let  $\nu$  be a positive measure which has infinitely many points in its support and assume the latter is covered by finitely many disjoint intervals:  $\text{supp}(\nu) \subseteq \cup_{j=1}^m [a_j, b_j]$ . Let further  $\psi$  be a function of bounded variation on  $\text{supp}(\nu)$ . If for some integer  $l$  we have*

$$\int P_{l-1}(t) e^{i\psi(t)} d\nu(t) = 0, \quad \forall P_{l-1} \in \mathcal{P}_{l-1},$$

then

$$\sum_{j=1}^m V(\psi, [a_j, b_j]) \geq (l - m + 1)\pi.$$

(b) Let  $[a, b] \subset (-1, 1)$  and  $\xi \in \mathbb{D}$ . Define

$$g(\xi, t) := \operatorname{Arg}(t - \xi) - 2\operatorname{Arg}(t - 1/\bar{\xi}), \quad (3.16)$$

where the term  $2\operatorname{Arg}(t - 1/\bar{\xi})$  is omitted if  $\xi = 0$ . Then

$$V(g(\xi, \cdot), [a, b]) \leq \operatorname{Angle}(\xi, [a, b]). \quad (3.17)$$

(c) Let  $\psi$  be a real function of bounded variation on an interval  $[a, b]$ ,  $\{a_n(x)\}$  a sequence of continuously differentiable real functions with uniformly bounded derivatives on  $[a, b]$ , and  $Q$  a polynomial. Then there exists a polynomial  $T \neq 0$  and a constant  $\beta \in (0, \pi/32)$  such that

$$\left| \operatorname{Arg} \left( e^{i(\psi(x) + a_n(x))} Q(x) T(x) \right) \right| \leq \pi/2 - 2\beta \quad (3.18)$$

for all  $x \in [a, b]$  such that  $T(x)Q(x) \neq 0$  and all  $n$  from some infinite sequence  $\mathbb{N}_1 \subset \mathbb{N}$ .

(d) Assume  $I \subset (0, 1)$  and  $\{q_n\}$  is a sequence of polynomials of degree  $m_n$  whose roots  $\{\xi_{1,n}, \dots, \xi_{m_n,n}\}$  lie in  $\mathbb{D}$  and satisfy

$$\sum_{j=1}^{m_n} (\pi - \operatorname{Angle}(\xi_{j,n}, I)) \leq C$$

where the constant  $C$  is independent of  $n$ . Then, to every  $\epsilon > 0$  there exists an integer  $l$  such that, for all  $n$  large enough, there is a polynomial  $T_{l,n}$  of degree at most  $l$  satisfying:

$$\left| \frac{\tilde{q}_n(x)}{|\tilde{q}_n(x)|} - T_{l,n}(x) \right| < \epsilon, \quad x \in I.$$

In particular, the argument of  $T_{l,n}(x)/\tilde{q}_n(x)$  lies in the interval  $(-2\epsilon, 2\epsilon)$  when  $n$  is large enough.

*Proof:* (a) This assertion follows from the proof of Lemma 3.2 in [8] if we put  $d_n = n$  there.

(b) When  $\xi \notin I$ , the proof of this statement is contained in that of Lemma 5.2 in [8]. In the remaining cases, one can see by inspection that (3.17) reduces to  $0 \leq \pi$  when  $\xi = b$  and to  $\pi \leq \pi$  when  $\xi \in [a, b]$ .

(c) Observe that  $\varphi(x) = \psi(x) + \operatorname{Arg}(Q(x))$  is a real function of bounded variation on  $I$ . Therefore by Lemma 3.4 in [8], there exist a polynomial  $T^* \neq 0$  and a constant  $\beta^* \in (0, \pi/16)$  such that

$$\left| \operatorname{Arg} \left( e^{i\psi(x)} Q(x) T^*(x) \right) \right| = \left| \operatorname{Arg} \left( e^{i\varphi(x)} T^*(x) \right) \right| \leq \pi/2 - 2\beta^* \quad \text{for } x \in I, Q(x)T^*(x) \neq 0.$$

For later use we also record that, by the very construction of  $T^*$  the cited lemma, its zeros belong to  $I$  and are discontinuity points of  $\varphi$ .

Let  $K$  be such that  $|a'_n(x)| \leq K$  for all  $n \in \mathbb{N}$  and  $x \in I$ , where the superscript “prime” indicates the derivative. By Jackson’s theorem (cf. e.g. [40]) there is a constant  $C > 0$  and there are polynomials  $\{T_{n,l}\}$  of degree at most  $l$  such that

$$\left| e^{-ia_n(x)} - T_{n,l}(x) \right| \leq \frac{CK}{l}.$$

Fix  $l$  so large that  $CK/l \leq \beta^*/3$ . Being bounded of bounded degree, the sequence  $\{T_{n,l}\}$  has a subsequence converging uniformly on  $I$  to a polynomial  $T_l$  of degree at most  $l$ . Therefore, for some subsequence  $\mathbb{N}_1$  we obtain

$$\left| 1 - e^{ia_n(x)} T_l(x) \right| \leq \frac{\beta^*}{2}, \quad n \in \mathbb{N}_1,$$

which implies that

$$\left| \operatorname{Arg} \left( e^{ia_n(x)} T_l(x) \right) \right| \leq \frac{\beta^* \pi}{4} < \beta^*, \quad n \in \mathbb{N}_1.$$

Now inequality (3.18) follows by taking  $T = T^* T_l$ ,  $\beta = \beta^*/2$ , and using that  $|\operatorname{Arg}(\xi_1 + \xi_2)| \leq |\operatorname{Arg}(\xi_1)| + |\operatorname{Arg}(\xi_2)|$  for any  $\xi_1, \xi_2 \in \mathbb{C}$ .

(d) This is exactly what is proved in Lemma 5.4 of [8]. ■

**Lemma 3.6** Let  $q_n(z) = \prod_{j=1}^n (z - \xi_{j,n})$  satisfy (3.12) for  $|\xi_{j,n}| < 1$  for  $j = 1, \dots, n$ , where  $\tilde{\lambda} = \lambda + \lambda'$  is given by (2.2)-(2.6) with  $S \subset (-1, 1)$  and  $S' \subset \mathbb{D}$ , while  $\omega_n$  is a complex-valued measurable function on  $S \cup S'$ . Consider a covering of  $S$  by finitely many disjoint closed intervals:  $S \subseteq I_{S,m} := \bigcup_{j=1}^m [a_j, b_j]$ . Then<sup>5</sup>

$$\sum_{j=1}^n (\pi - \theta(\xi_{j,n})) \leq V(\varphi) + V(\arg(\omega_n), S) + \sum_{\eta_j \in S'} m(\eta_j) \theta(\eta_j) + (m + s - 1)\pi, \quad (3.19)$$

where  $\arg(\omega_n)$  is any argument function for  $\omega_n$  on  $S$  and  $m(\eta_j)$  is the multiplicity of  $\eta_j$ .

*Proof:* If  $\omega_n$  has no argument function of bounded variation on  $S$ , there is nothing to prove. Otherwise, we pick one and extend it to the whole of  $\mathbb{R}$  without increasing the variation, as explained in Section 2. In particular, we get

$$V(\arg(\omega_n), S) = \sum_{j=1}^m V(\arg(\omega_n), [a_j, b_j]).$$

As in the case of (3.8), equation (3.12) yields

$$\int P_{n-s-1}(t) Q_s(t) q_n(t) \frac{\omega_n(t)}{\tilde{q}_n^2(t)} d\lambda(t) = 0, \quad (3.20)$$

where  $P_{n-s-1}$  is any polynomial in  $\mathcal{P}_{n-s-1}$ .

Denote by  $\psi_n(t)$  an argument function for  $e^{i\varphi(t)} Q_s(t) q_n(t) \omega_n(t) / \tilde{q}_n^2(t)$ , say

$$\psi_n(t) = \varphi(t) + \arg(\omega_n(t)) + \sum_{\eta_i \in S'} \text{Arg}(t - \eta_i) + \sum_{i=1}^n (\text{Arg}(t - \xi_{i,n}) - 2\text{Arg}(t - 1/\bar{\xi}_{i,n})),$$

where it is understood that  $\text{Arg}(t - 1/\bar{\xi}_{i,n})$  is omitted when  $\xi_{i,n} = 0$ . It is easy to see that  $\psi_n$  is of bounded variation. Then Lemma 3.5(a) with

$$\psi = \psi_n, \quad d\nu(t) = \left| \frac{Q_s(t) q_n(t) \omega_n(t)}{\tilde{q}_n^2(t)} \right| d|\lambda|(t), \quad \text{and} \quad l = n - s$$

implies that

$$\sum_{j=1}^m V(\psi_n, [a_j, b_j]) \geq (n - s - m + 1)\pi.$$

So, we are left to show that

$$\sum_{j=1}^m V(\psi_n, [a_j, b_j]) \leq V(\varphi) + \sum_{j=1}^m V(\arg(\omega_n), [a_j, b_j]) + \sum_{\eta_i \in S'} m(\eta_i) \theta(\eta_i) + \sum_{i=1}^n \theta(\xi_{i,n}).$$

By the definition of  $\psi_n$ , we have

$$\begin{aligned} \sum_{j=1}^m V(\psi_n, [a_j, b_j]) &\leq \sum_{j=1}^m V(\varphi, [a_j, b_j]) + \sum_{j=1}^m V(\arg(\omega_n), [a_j, b_j]) \\ &\quad + \sum_{j=1}^m \sum_{\eta_i \in S'} m(\eta_i) V(\text{Arg}(\cdot - \eta_i), [a_j, b_j]) \\ &\quad + \sum_{j=1}^m \sum_{i=1}^n V(g(\xi_{i,n}, \cdot), [a_j, b_j]), \end{aligned}$$

where  $g(\xi, t)$  was defined in (3.16). The assertion of the lemma now follows from Lemma 3.5(b) and the fact that by the monotonicity of  $V(\text{Arg}(\cdot - \xi), [a, b]) = \text{Angle}(\xi, [a, b])$ . ■

<sup>5</sup>Note that the hypothesis  $\lambda \in \mathbf{BVT}$  is not required for this lemma to hold.

**Corollary 3.7** *Let  $\varphi$  and  $\arg(\omega_n)$  have bounded variation on  $S$ , with  $V(\arg(\omega_n), S) < C$  where  $C$  is independent on  $n$ . Then, to each neighborhood  $U$  of  $S$ , there exists a constant  $k_U \in \mathbb{N}$  such that each  $q_n$  has at most  $k_U$  zeros outside of  $U$  for  $n$  large enough.*

*Proof:* Since  $U$  is open, its intersection with  $(-1, 1)$  is a countable union of intervals. By compactness, a finite number of them will cover  $S$ , say  $\cup_{j=1}^m (a_j, b_j)$ . Apply Lemma 3.6 to the closure of these intervals and observe that any zero of  $q_n$  which lies outside of  $U$  will contribute to the left-hand side of (3.19) by more than some positive fixed constant which depends only on  $U$ . Since the right-hand side of (3.19) does not depend on  $n$  and is finite we can have only finitely many such zeros. ■

*Proof of Theorem 3.1:* Hereafter we are going to use (3.20) rather than (3.12) and we set  $q_n(z) = \prod_{j=1}^n (z - \xi_{j,n})$ .

We start by observing that we may suppose  $S \subset (0, 1)$ . Indeed, if this is not the case, take a negative number  $w$  such that  $-1 < w < a$ , where  $[a, b] = I$  denotes the convex hull of  $S$ . Then  $M_w(S)$ , the image of  $S$  under the Möbius transformation  $M_w(z) := (z - w)/(1 - zw)$ , is a subset of  $(0, 1)$ . Moreover, observe that the Green equilibrium measure is invariant under Möbius transformations, i.e., for any Borel set  $E \subset M_w(S)$  we have that

$$\mu_G^{M_w(S)}(E) = \mu_G^S(M_{-w}(E))$$

( $M_{-w}$  is the inverse function of  $M_w$ ). This implies that the weak\* convergence of  $\nu_n$  to  $\mu_G^S$  is equivalent to that of  $\nu_n^w$  to  $\mu_G^{M_w(S)}$ , where  $\nu_n^w$  is the counting measure of the images of the zeros of  $q_n$  under  $M_w$ . Now, if we let

$$\begin{aligned} \ell_n(\tau) &= q_n(M_{-w}(\tau))(1 + w\tau)^n, \\ L_s(\tau) &= Q_s(M_{-w}(\tau))(1 + w\tau)^s, \\ p_{n-s-1}(\tau) &= P_{n-s-1}(M_{-w}(\tau))(1 + w\tau)^{n-s-1}, \\ \omega_n^*(\tau) &= \omega_n(M_{-w}(\tau)), \end{aligned}$$

then  $\ell_n$  is a polynomial of degree  $n$  with zeros at  $M_w(\xi_{j,n})$ ,  $j = 1, \dots, n$ . In addition, since  $M_x(1/\bar{\xi}) = 1/\overline{M_x(\xi)}$ ,  $x \in (-1, 1)$ , we have that

$$\tilde{\ell}_n(\tau) = \tilde{q}_n(M_{-w}(\tau))(1 + w\tau)^n.$$

Analogously,  $L_s$  is a polynomial of degree  $s$  with zeros at  $M_w(\eta_j)$ ,  $j = 1, \dots, s$ , and  $p_{n-s-1}$  is an arbitrary polynomial of degree at most  $n - s - 1$ . Let us show that  $\ell_n$  satisfies orthogonality relations of type (3.20) for a new measure, supported this time in  $(0, 1)$ , that still belongs to **BVT**.

With the above notation equation (3.20) implies

$$\begin{aligned} 0 &= \int_S P_{n-s-1}(t) Q_s(t) q_n(t) \frac{\omega_n(t)}{\tilde{q}_n^2(t)} e^{i\varphi(t)} d|\lambda|(t) \\ &= \int_{M_w(S)} (P_{n-s-1} Q_s q_n)(M_{-w}(\tau)) \frac{\omega_n(M_{-w}(\tau))}{\tilde{q}_n^2(M_{-w}(\tau))} e^{i\varphi(M_{-w}(\tau))} d|\lambda|(M_{-w}(\tau)) \\ &= \int_{M_w(S)} p_{n-s-1}(\tau) L_s(\tau) \ell_n(\tau) \frac{\omega_n^*(\tau)}{\tilde{\ell}_n^2(\tau)} e^{i\varphi^*(\tau)} d|\lambda^*|(\tau), \end{aligned}$$

where  $d|\lambda^*|(\tau) = (1 + w\tau) d|\lambda|(M_{-w}(\tau))$  is a positive measure supported on  $M_w(S)$ ,  $\varphi^*(\tau) = \varphi(M_{-w}(\tau))$  is a function of bounded variation, and  $\{\omega_n^*\}$  is a sequence of measurable functions whose moduli are uniformly bounded above and below, and whose arguments are smooth with uniformly bounded derivatives. Further, since Green functions are conformally invariant,  $M_w(S)$  is regular so clearly  $\lambda^* \in \mathbf{BVT}$ . This allows us to assume that  $S \subset (0, 1)$ .

First we will suppose that *all zeros of the polynomials  $q_n$  lie outside some fixed neighborhood of zero*.

For each  $n$  denote by  $\sigma_n$  the counting measure of zeros of  $\tilde{q}_n$ . By the assumption that we just made there exists a compact set  $K$  such that  $0 \notin K$  and  $\text{supp}(\nu_n) \subset K$ . Then  $\text{supp}(\sigma_n) \subset \bar{K}^{-1}$  for all  $n \in \mathbb{N}$ , and  $\bar{K}^{-1}$  is also compact. By Helly's selection theorem (cf. [46, Thm. 0.1.3]) there exists a subsequence of natural numbers,  $\mathbb{N}_1$ , such that  $\nu_n \xrightarrow{*} \nu$  for  $n \in \mathbb{N}_1$ , where  $\xrightarrow{*}$  stands for weak\* convergence. Denote by  $\sigma$  the reflection of  $\nu$  across

the unit circle, i.e.  $d\sigma(t) = d\nu \circ (1/\bar{t})$ . It is easy to check that  $\sigma_n \xrightarrow{*} \sigma$ . Observing that the assumptions on  $\omega_n$  imply that the variation of its argument on  $I$ , thus *a fortiori* on  $S$ , is bounded independently of  $n$ , it follows from Corollary 3.7 that  $\nu$  and  $\sigma$  are probability measures such that  $\text{supp}(\nu) \subset S \subset (0, 1)$  and  $\text{supp}(\sigma) \subset \bar{S}^{-1} = S^{-1}$ .

*Claim:* it is enough to show that the logarithmic potential of  $\nu - \sigma$  is a constant q.e. on  $S$ . Indeed, let  $U^{\nu-\sigma} = D_1$  q.e. on  $S$ , where

$$U^{\nu-\sigma} = \int \log \frac{1}{|z-t|} d(\nu-\sigma)(t).$$

Then, since  $U^\sigma$  is harmonic outside of  $S^{-1}$ , we have that  $U^\nu$  is bounded quasi everywhere on  $S$ , hence everywhere by lower semi-continuity of potentials. Thus,  $\nu$  has finite energy and by reflection so does  $\sigma$ . Moreover, for quasi every  $z \in S^{-1}$ , we have

$$\begin{aligned} U^{\nu-\sigma}(z) &= \int \log \frac{1}{|z-t|} d(\nu-\sigma)(t) = \int \log \frac{1}{|z-1/\bar{x}|} d[(\nu-\sigma) \circ (1/x)] \\ &= \int \log \left| \frac{x/z}{x-1/\bar{z}} \right| d(\sigma-\nu)(x) = \int \log |x/z| d(\sigma-\nu)(x) - U^{\nu-\sigma}(1/z) \\ &= \int \log |x| d(\sigma-\nu)(x) - D_1 =: D_2, \end{aligned} \tag{3.21}$$

where we used that  $\nu - \sigma$  has total mass zero. Now, denote by  $\hat{\sigma}$  the balayage of  $\sigma$  onto  $S$ . Then

$$U^{\hat{\sigma}}(t) = U^\sigma(t) + c(\sigma; \mathbb{C} \setminus S)$$

for quasi every  $t$  on  $S$ . Thus, as  $(\nu - \hat{\sigma})(\mathbb{C}) = 0$  and since  $\nu$  and  $\hat{\sigma}$  have finite energy, we get

$$\begin{aligned} 0 &= \int D_1 d(\nu - \hat{\sigma})(z) = \int U^{\nu-\sigma}(z) d(\nu - \hat{\sigma})(z) = \int U^{\nu-\hat{\sigma}}(z) d(\nu - \hat{\sigma})(z) \\ &= \int \log \frac{1}{|z-t|} d(\nu - \hat{\sigma})(t) d(\nu - \hat{\sigma})(z). \end{aligned}$$

But the energy of a signed measure is equal to zero if and only if the measure is zero ([46, Lemma I.1.8]), provided that this measure is the difference of two positive measures with finite energy; thus,  $\nu = \hat{\sigma}$ . Using (3.21), we can obtain in a similar fashion that  $\sigma = \hat{\nu}$ , where  $\hat{\nu}$  is the balayage of  $\nu$  onto  $S^{-1}$ . Hence, we proved that  $\sigma - \nu$  is the equilibrium signed measure for the condenser  $(S, S^{-1})$  ([46, Thm. VIII.2.6]). Then Proposition A points (b) and (e) ensures that  $\nu = \mu_G^S$  is the Green equilibrium distribution relative to both  $\mathbb{D}$  and  $\mathbb{C} \setminus S^{-1}$ . Since  $\{\nu_n\}_{n \in \mathbb{N}_1}$  was an arbitrary subsequence, the whole sequence  $\{\nu_n\}_{n \in \mathbb{N}}$  converges to  $\mu_G^S$  in the weak\* sense. *This proves the claim.*

Being left to prove that  $U^{\nu-\sigma}$  is a constant q.e. on  $S$ , suppose to the contrary that this is not true. Then there exist nonpolar Borel subsets of  $S$ , say  $E_-$  and  $E_+$ , and two constants  $d$  and  $\tau > 0$  such that

$$U^{\nu-\sigma}(x) \geq d + \tau, \quad x \in E_+, \quad U^{\nu-\sigma}(x) \leq d - 2\tau, \quad x \in E_-.$$

In this case we claim that there exists  $y_0 \in \text{supp}(\nu)$  such that

$$U^{\nu-\sigma}(y_0) > d. \tag{3.22}$$

Indeed, otherwise we would have that

$$U^\nu(x) \leq U^\sigma(x) + d, \quad x \in \text{supp}(\nu). \tag{3.23}$$

Then the principle of domination ([46, Thm. II.3.2]) would yield that (3.23) is true for all  $z \in \mathbb{C}$ , but this would contradict the existence of  $E_+$ .

Since all  $\sigma_n$  are supported outside of the closed unit disk, the sequence of potentials  $\{U^{\sigma_n}\}_{n \in \mathbb{N}_1}$  converges to  $U^\sigma$  uniformly on compact subsets of  $\mathbb{D}$ . This implies that for any given sequence of points  $\{y_n\} \subset \mathbb{D}$  such that  $y_n \rightarrow y_0$  as  $n \rightarrow \infty$ ,  $n \in \mathbb{N}_1$ , we have

$$\lim_{n \rightarrow \infty, n \in \mathbb{N}_1} U^{\sigma_n}(y_n) = U^\sigma(y_0). \tag{3.24}$$

On the other hand all  $\nu_n$ ,  $n \in \mathbb{N}_1$ , have their support in  $\mathbb{D}$ . So, by applying the principle of descent for the above sequence  $\{y_n\}$ , we obtain

$$\liminf_{n \rightarrow \infty, n \in \mathbb{N}_1} U^{\nu_n}(y_n) \geq U^\nu(y_0). \quad (3.25)$$

Combining (3.22), (3.24), and (3.25) we get

$$\liminf_{n \rightarrow \infty, n \in \mathbb{N}_1} U^{\nu_n - \sigma_n}(y_n) \geq U^{\nu - \sigma}(y_0) > d. \quad (3.26)$$

Since  $\{y_n\}$  was an arbitrary sequence in  $\mathbb{D}$  converging to  $y_0$ , we deduce from (3.26) that there exists  $\rho > 0$  such that, for any  $y \in [y_0 - 2\rho, y_0 + 2\rho]$  and  $n \in \mathbb{N}_1$  large enough, the following inequality holds

$$U^{\nu_n - \sigma_n}(y) \geq d. \quad (3.27)$$

Since

$$U^{\nu_n - \sigma_n}(y) = \frac{1}{n} \log \left| \frac{1}{\alpha_n} \frac{\tilde{q}_n(y)}{q_n(y)} \right|,$$

where  $\alpha_n := \prod_{j=1}^n |\xi_{j,n}|$ , inequality (3.27) can be rewritten as

$$\left| \alpha_n \frac{q_n(y)}{\tilde{q}_n(y)} \right| \leq e^{-nd} \quad (3.28)$$

for any  $y \in [y_0 - 2\rho, y_0 + 2\rho]$  and  $n \in \mathbb{N}_1$  large enough.

Here we notice for later use that the above reasoning does not really require the polynomials to have exact degree  $n$ . Specifically, let  $\{p_n\}$  be a sequence of monic polynomials of degree  $d_n = n + o(n)$  where  $o$  denotes the Landau symbol “little oh”. Moreover, suppose that the counting measures of their zeros normalized by  $1/n$  rather than  $1/d_n$  (so it may no longer be a probability measure) are supported on a fixed compact set of the complex plane. Call  $\mu_n$  these measures and assume that they converge to  $\nu$  in the weak\* topology. In this case (3.25) and (3.26) still hold with  $\nu_n$  replaced by  $\mu_n$ , at the cost perhaps of dropping finitely many terms of  $\mathbb{N}_1$  and making  $\rho$  smaller. Thus, we obtain that

$$\left| \alpha_n \frac{p_n(y)}{\tilde{q}_n(y)} \right| \leq e^{-nd} \quad (3.29)$$

for any  $y \in [y_0 - 2\rho, y_0 + 2\rho]$  and  $n \in \mathbb{N}_1$  large enough.

In another connection, since  $U^{\nu - \sigma}(x) \leq d - 2\tau$  on  $E_-$ , applying the lower envelope theorem gives us

$$\liminf_{n \rightarrow \infty, n \in \mathbb{N}_1} U^{\nu_n - \sigma_n}(x) = U^{\nu - \sigma}(x) \leq d - 2\tau, \quad \text{for q.e. } x \in E_-. \quad (3.30)$$

Let  $Z$  be a finite system of points from  $(-1, 1)$ , to be specified later. Then by [2, 3] there exists<sup>6</sup>  $S_0 \subset S$  such that  $S_0$  is regular,  $\text{cap}(E_- \cap S_0) > 0$  and  $\text{dist}(Z, S_0) > 0$ , where  $\text{dist}(Z, S_0) := \min_{z \in Z} \text{dist}(z, S_0)$ . Put for simplicity  $b_n(x) = q_n(x)/\tilde{q}_n(x)$ , which is a finite Blaschke product. Then

$$U^{\nu_n - \sigma_n}(x) = -\frac{1}{n} \log |\alpha_n b_n(x)|$$

and by (3.30), as  $|\alpha_n| < 1$ , there exist  $\mathbb{N}_2 \subset \mathbb{N}_1$  and  $x \in E_- \cap S_0$  such that

$$|b_n(x)| \geq |\alpha_n b_n(x)| \geq e^{-n(d-\tau)}$$

for any  $n \in \mathbb{N}_2$ . Let  $x_n$  be a point where  $|b_n|$  reaches its maximum on  $S_0$ . Then

$$M_n := \max_{x \in S_0} |b_n(x)| = |b_n(x_n)| \geq |\alpha_n| M_n = |\alpha_n b_n(x_n)| \geq e^{-n(d-\tau)}. \quad (3.31)$$

<sup>6</sup>In [8]  $S_0$  is not introduced, which makes there for a slightly incorrect argument in Theorem 3.1. An alternative remedy in that reference would be to apply the lower envelope theorem to  $Tq_n$  rather than  $q_n$ , as they have the same asymptotic zero distribution.

Note that  $M_n < 1$  and therefore  $d - \tau$  is necessarily positive. For simplicity, we shall denote  $\mathbb{D} \setminus S_0$  by  $D$ . Since the modulus of a Blaschke product is bounded by 1 in the unit disk and  $\log |b_n|$  is a subharmonic function, the two-constant theorem ([45, Thm. 4.3.7]) yields the following pointwise estimate

$$\log |b_n(z)| \leq \omega_D(z, S_0) \log |M_n| \quad (3.32)$$

for any  $z \in D$ , where  $\omega_D(z, S_0)$  is the harmonic measure on  $D$  (cf. [45, Sec. 4.3]). Combining the last two inequalities we get

$$|b_n(z)| \leq (M_n)^{\omega_D(z, S_0)} = M_n \left( \frac{1}{M_n} \right)^{1 - \omega_D(z, S_0)} \leq M_n e^{n(d - \tau)(1 - \omega_D(z, S_0))} \quad (3.33)$$

for  $z \in D$ , and obviously also when  $z \in S_0$ , where  $\omega_D(\cdot, S_0) = 1$  for  $S_0$  is regular. Moreover, by the regularity of  $S_0$  again, it is known ([45, Thm. 4.3.4]) that for any  $x \in S_0$

$$\lim_{z \rightarrow x} \omega_D(z, S_0) = 1$$

uniformly with respect to  $x \in S_0$ . Thus, for any  $\delta > 0$  there exists  $r(\delta) < \text{dist}(S_0, \mathbb{T})$  such that for  $z$  satisfying  $\text{dist}(z, S_0) \leq r(\delta)$  we have

$$1 - \omega_D(z, S_0) \leq \frac{\delta}{d - \tau}.$$

This, together with (3.33), implies that for fixed  $\delta$ , to be adjusted later, we have

$$|b_n(z)| \leq M_n e^{n\delta}, \quad |x_n - z| \leq r(\delta).$$

Note that  $b_n$  is analytic in  $\mathbb{D}$ , which, in particular, yields for  $|z - x_n| < r(\delta)$

$$b'_n(z) = \frac{1}{2\pi i} \int_{|\xi - x_n| = r(\delta)} \frac{b_n(\xi)}{(\xi - z)^2} d\xi.$$

Thus, for any  $z$  such that  $|z - x_n| \leq r(\delta)/2$  we get

$$|b'_n(z)| \leq \frac{1}{2\pi} \cdot \frac{4M_n e^{n\delta}}{r^2(\delta)} \cdot 2\pi r(\delta) = \frac{4M_n e^{n\delta}}{r(\delta)}.$$

Now, for any  $x$  such that

$$|x - x_n| \leq \frac{r(\delta)}{8e^{n\delta}} \quad (3.34)$$

the mean value theorem yields

$$|b_n(x) - b_n(x_n)| \leq \frac{4M_n e^{n\delta}}{r(\delta)} |x - x_n| = \frac{M_n}{2}.$$

Thus, for  $x$  satisfying (3.34) we have

$$|b_n(x)| \geq |b_n(x_n)| - |b_n(x) - b_n(x_n)| \geq M_n - \frac{M_n}{2} = \frac{M_n}{2}$$

and by (3.31)

$$|\alpha_n b_n(x)| \geq \frac{|\alpha_n| M_n}{2} \geq \frac{1}{2} e^{-n(d - \tau)}. \quad (3.35)$$

The estimates (3.29), (3.35), together with the relation (3.20) are the main ingredients in proving the claim that  $U^{\nu - \sigma}$  is constant q.e. on  $S$ . To combine them we shall use a specific choice of  $P_{n-s-1}$  in (3.20).

First, we pick a polynomial  $T$  such that Lemma 3.5-(c) holds with  $\psi = \varphi$ ,  $a_n = \text{Arg}(\omega_n)$ ,  $Q = Q_s$ , and  $[a, b] = I$  for  $n \in \mathbb{N}_3 \subset \mathbb{N}_2$ . We denote by  $k$  the degree of  $T$ . Second, for each  $n \in \mathbb{N}_3$ , we choose  $T_{l,n}$



as in Lemma 3.5-(d) with  $\epsilon = \delta/9$ . Note that the use of Lemma 3.5-(d) is legitimate by Lemma 3.6 and our assumptions on  $S$ ,  $\varphi$  and  $\omega_n$ . Since all  $T_{l,k}$  are bounded and have degree at most  $l$ , which does not depend on  $n$ , there exists  $\mathbb{N}_4 \subset \mathbb{N}_3$  such that sequence  $\{T_{l,n}\}_{n \in \mathbb{N}_4}$  converges uniformly to some polynomial  $T_l$  on  $I$ . In particular, we have that  $\deg(T_l) \leq l$  and the argument of  $T_l(x)/\tilde{q}_n(x)$  lies in  $(-\delta/4, \delta/4)$  for  $n \in \mathbb{N}_4$  large enough. Denote by  $2\alpha$  the smallest even integer strictly greater than  $2l + k + s$ . As soon as  $n$  is large enough, since  $y_0 \in \text{supp}(\nu)$ , there exist  $\beta_{1,n}, \dots, \beta_{2\alpha,n}$ , zeros of  $q_n$ , lying in

$$O_\gamma([y_0 - \rho, y_0 + \rho]) := \{z \in \mathbb{C} : \text{dist}(z, [y_0 - \rho, y_0 + \rho]) \leq \gamma/2\},$$

where  $\gamma$ ,  $0 < \gamma < \rho/2$ , will be specified later. Define

$$P_n^*(z) = \frac{\overline{q_n(\bar{z})} T(z) T_l^2(z)}{\prod_{j=1}^{2\alpha} (z - \bar{\beta}_{j,n})}.$$

The polynomial  $P_n^*$  has degree  $n - s - 1$  or  $n - s - 2$ , depending on the parity of  $k + s$ .

Denote by  $I_n \subset (0, 1)$  the interval defined by (3.34). By comparing (3.28) with (3.35) and (3.31), it is clear that  $I_n$  and  $[y_0 - 2\rho, y_0 + 2\rho]$  are disjoint when  $n \in \mathbb{N}_4$  is large enough.

Now, we choose  $\gamma = \gamma(\rho)$  so small that

$$\left| \sum_{j=1}^{2\alpha} \text{Arg} \left( \frac{1}{x - \bar{\beta}_{j,n}} \right) \right| = \left| \sum_{j=1}^{2\alpha} \text{Arg} (x - \bar{\beta}_{j,n}) \right| \leq \delta/2, \quad x \in \mathbb{R} \setminus [y_0 - 2\rho, y_0 + 2\rho].$$

Letting  $\delta$  be such that  $\delta < \beta$ , the choices of  $T$ ,  $T_l$ , and  $P_n^*$  together imply

$$\begin{aligned} & \left| \text{Arg} \left( P_n^*(x) Q_s(x) q_n(x) \frac{\omega_n(x)}{\tilde{q}_n^2(x)} e^{i\varphi(x)} \right) \right| \\ &= \left| \text{Arg} \left( |q_n(x)|^2 \cdot \prod_{j=1}^{2\alpha} \frac{1}{(x - \bar{\beta}_{j,n})} \cdot \frac{T_l^2(x)}{\tilde{q}_n^2(x)} \cdot T(x) Q_s(x) \omega_n(x) e^{i\varphi(x)} \right) \right| \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} + \frac{\pi}{2} - 2\beta \leq \pi/2 - \delta, \end{aligned}$$

for  $x \in I \setminus [y_0 - 2\rho, y_0 + 2\rho]$  except at points where  $T$  or  $Q_s$  are equal to zero. This means that for such  $x$

$$\begin{aligned} \text{Re} \left( |\alpha_n|^2 P_n^*(x) Q_s(x) q_n(x) \frac{\omega_n(x)}{\tilde{q}_n^2(x)} e^{i\varphi(x)} \right) &\geq \sin \delta \left| \alpha_n^2 P_n^*(x) Q_s(x) q_n(x) \frac{\omega_n(x)}{\tilde{q}_n^2(x)} e^{i\varphi(x)} \right| \\ &\geq \sin \delta \left| \frac{\alpha_n^2 b_n^2(x) Q_s(x) T(x) T_l^2(x) \omega_n(x)}{\prod_{j=1}^{2\alpha} (x - \bar{\beta}_{j,n})} \right| \geq 0. \end{aligned}$$

Moreover, if  $x \in S \setminus [y_0 - 2\rho, y_0 + 2\rho]$  satisfies (3.34), then by (3.31) and (3.35) the above quantity is bounded from below by

$$|T(x) Q_s(x)| \frac{\sin \delta \min_{x \in [a,b]} |T_l(x)|^2 \inf_{n \in \mathbb{N}} \min_{x \in [a,b]} |\omega_n(x)|}{(\text{diam}(S) + 2\rho)^{2\alpha}} e^{-2nd+2n\tau} = c_1 |T(x) Q_s(x)| e^{-2nd+2n\tau},$$

where  $\text{diam}(S) := \max_{x,y \in S} |x - y|$  and

$$c_1 := \frac{\sin \delta \min_{x \in [a,b]} |T_l(x)|^2 \inf_{n \in \mathbb{N}} \min_{x \in [a,b]} |\omega_n(x)|}{(\text{diam}(S) + 2\rho)^{2\alpha}} > 0$$

by construction of  $T_l$  and the uniform boundedness of  $\{|\omega_n|\}$  from below. Thus

$$\text{Re} \left( \int_{S \setminus [y_0 - 2\rho, y_0 + 2\rho]} |\alpha_n|^2 P_n^*(t) Q_s(t) q_n(t) \frac{\omega_n(t)}{\tilde{q}_n^2(t)} e^{i\varphi(t)} d|\lambda|(t) \right)$$

$$\begin{aligned}
 &\geq \sin \delta \int_{S \setminus [y_0 - 2\rho, y_0 + 2\rho]} \left| \alpha_n^2 P_n^*(t) Q_s(t) q_n(t) \frac{\omega_n(t)}{\tilde{q}_n^2(t)} e^{i\varphi(t)} \right| d|\lambda|(t) \\
 &\geq c_1 e^{-2nd+2n\tau} \int_{S \cap I_n} |T(t) Q_s(t)| d|\lambda|(t) \geq c_2 e^{-2nd+n(2\tau-L\delta)}.
 \end{aligned} \tag{3.36}$$

The last inequality is true by the following argument. Recall that  $x_n$ , the middle point of  $I_n$ , belongs to  $S_0$ , where  $\text{dist}(S_0, Z) > 0$  and  $Z$  is a finite system of points that we now choose to be the zeros of  $TQ_s$  in  $(-1, 1)$ . Then  $TQ_s$ , which is independent of  $n$ , is uniformly bounded below on  $I_n$  and (3.36) follows from this and hypothesis **BVT** point (2). On the other hand (3.28) and (3.29), applied with  $p_n = P_n^*$ , yield that

$$\left| \int_{[y_0 - 2\rho, y_0 + 2\rho]} |\alpha_n|^2 P_n^*(t) Q_s(t) q_n(t) \frac{\omega_n(t)}{\tilde{q}_n^2(t)} e^{i\varphi(t)} d|\lambda|(t) \right| \leq c_3 e^{-2nd}, \tag{3.37}$$

where we used uniform boundedness of  $\{|\omega_n|\}$  from above. Choosing  $\delta$  so small that  $2\tau - L\delta > 0$ , which is possible, the bound in (3.36) becomes bigger than the bound in (3.37) for  $n$  large enough. But this is impossible, since by (3.20) the sum of these two integrals must be zero.

We just completed the case when all the zeros of polynomials  $q_n$  stay away from the point zero. Now we shall consider the general situation. Let  $\epsilon > 0$  be such that  $U := \mathbb{D} \setminus \{z : |z| \leq \epsilon\}$  is a neighborhood of  $S$ . Corollary 3.7 says that there exists a constant  $k_U \in \mathbb{N}$  such that each  $q_n$  has at most  $k_U$  zeros outside of  $U$ , or equivalently, zeros which have modulus less than  $\epsilon$ . In this case, from any sequence of natural numbers, we can extract a subsequence, say  $\mathbb{N}_0$ , such that for some number  $m \leq k_U$ ,  $q_n$  has exactly  $m$  zeros outside of  $U$  for each  $n \in \mathbb{N}_0$ . Denote these zeros  $\xi_{1,n}, \dots, \xi_{m,n}$ , and consider the polynomials

$$q_n^*(z) := \frac{\tilde{q}_n(z)}{\prod_{j=1}^m (1 - z\bar{\xi}_{j,n})}, \quad n \in \mathbb{N}_0.$$

Then the sequence  $\{q_n\}_{n \in \mathbb{N}_0}$  will satisfy the following weighted orthogonality relation:

$$\int P_{n-s-1}(t) Q_s(t) q_n(t) \frac{\omega_n^*(t)}{(q_n^*(t))^2} d\lambda(t) = 0, \quad P_{n-s-1} \in \mathcal{P}_{n-s-1},$$

where

$$\omega_n^*(t) := \omega_n(t) \prod_{j=1}^m (1 - t\bar{\xi}_{j,n})^{-2}, \quad t \in I.$$

In what follows we are going to stress the modifications needed to adapt the previous proof to the present case. Let, as before,  $\mathbb{N}_1 \subset \mathbb{N}_0$  be a subsequence of natural numbers such that  $\nu_n \xrightarrow{*} \nu$ ,  $n \in \mathbb{N}_1$ . Because we only discarded a fixed number of zeros from  $\tilde{q}_n(z)$  to obtain  $q_n^*$ , the counting measures  $\sigma_n^*$  of the zeros of  $q_n^*$  (normalized by  $1/n$ ), again converge weak\* to  $\sigma$ . Since  $U^{\sigma_n^*}$  enjoys all the relevant properties of  $U^{\sigma_n}$ , inequalities (3.28) and (3.29) remain valid with  $\tilde{q}_n$  replaced by  $q_n^*$ .

Further, define  $b_n(x)$  as  $q_n(x)/q_n^*(x)$ . In this case,  $b_n$  is no longer a Blaschke product, but rather a Blaschke product multiplied by the polynomial  $\prod_{j=1}^m (z - \xi_{j,n})$ . Then we get instead of (3.32) that

$$\log |b_n(z)| \leq \omega_D(z, S_0) \log |M_n| + (1 - \omega_D(z, S_0)) m \log 2,$$

and (3.33) can be replaced by

$$|b_n(z)| \leq M_n e^{n(d-\tau+m \log 2/n)(1-\omega_D(z, S_0))} \leq M_n e^{n(d-\tau+1)(1-\omega_D(z, S_0))},$$

for  $n$  large enough. This yields an insignificant modification of  $r(\delta)$  (we should make  $1 - \omega_D(z, S_0)$  less than  $\delta/(d - \tau + 1)$  rather than just  $\delta/(d - \tau)$ ). Lemma 3.5(d) can be applied to polynomials  $q_n^*$  without change.

Therefore we are left to show that  $\{\omega_n^*\}$  is uniformly bounded above and below on  $I$ , and that its arguments are smooth with uniformly bounded derivatives on  $I$  with respect to  $n$ . The uniform boundedness of  $\{\omega_n^*\}$  easily follows from the estimates

$$\left(\frac{1}{2}\right)^{2m} \leq \prod_{j=1}^m \left| \frac{1}{(1 - t\bar{\xi}_{j,n})} \right|^2 \leq \left(\frac{1}{1 - \epsilon}\right)^{2m}.$$

Since none of the  $\xi_{j,n}$ ,  $j = 1, \dots, m$  can come close to  $I^{-1}$ ,  $\text{Arg}(1 - t\bar{\xi}_{j,n})$  is a smooth function on  $I$  whose derivative  $\text{Im}\xi_{j,n}/(1 - t\bar{\xi}_{j,n})$  is uniformly bounded there independently of  $j$  and  $n$ . Then the rest of the assumptions on  $\{\omega_n^*\}$  follows from the representation

$$\arg(\omega_n^*(t)) = \arg(\omega_n(t)) - 2 \sum_{j=1}^m \text{Arg}(1 - t\bar{\xi}_{j,n}).$$

This completes the proof of the theorem. ■

The forthcoming lemma is needed for the proof of Theorem 3.2. Recall that  $\mathcal{H}$ , a family of functions analytic in some fixed domain of the complex plane, is called *normal* if each sequence of functions from  $\mathcal{H}$  contains a locally uniformly convergent subsequence.

**Lemma 3.8** *Let  $p \in (2, \infty]$  and  $\{g_n\}_{n \in \mathbb{N}}$  be a sequence of a sequence of irreducible critical points of order  $n$  of  $\mathbf{MA}(p)$  for  $F$  given by (2.4)-(2.6) with  $\lambda$  satisfying<sup>7</sup> (2.2) and (2.3). Further, let  $v_n$  be an associated singular vector to  $g_n$  with inner-outer factorization given by  $v_n = b_n \cdot w_n$  for some Blaschke product  $b_n$  and  $w_n$  an outer function in  $H^{p'}$ . Then the families  $\mathcal{W} := \{w_n\}$  and  $\mathcal{W}_{p'} := \{w_n^{p'/2}\}$  are normal in  $\mathbb{D}$  and  $\Omega_{\tilde{S}}$  respectively, where  $\Omega_{\tilde{S}}$  was defined in (3.11). Moreover, any limit point of  $\mathcal{W}$  is zero free in  $\mathbb{D}$ .*

*Proof:* The main idea of the proof was given in [12, Thm. 10.1]. The necessary modification for the case of complex measures with argument of bounded variation were given in [10, Prop. 6.3]. Only a simple adjustment is needed in the present case where the approximated function may have polar singularities inside  $\mathbb{D}$ . Nevertheless, we present the whole proof for the ease of the reader.

First we establish the normality of  $\mathcal{W}_{p'}$ . By a classical theorem of Montel it suffices to show that  $\mathcal{W}_{p'}$  is locally uniformly bounded in  $\Omega_{\tilde{S}}$ . We claim it is enough to prove that

$$\mathcal{H} := \left\{ \left( j_n w_n^{p'/2} \right) (z) \overline{(Q_s^2 T)(1/\bar{z})} \right\}$$

is locally uniformly bounded in  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$ , where  $j_n$  is the finite Blaschke product from (3.6) and  $T$  is a fixed polynomial with all zeros in  $S$  such that

$$\left| \text{Arg} \left( e^{i\varphi(x)} T(x) \right) \right| \leq \frac{\pi}{2} - \beta, \quad x \in I, \quad (3.38)$$

for some  $\beta \in (0, \pi/16)$ , where  $\varphi$  is given by (2.2). The existence of such a polynomial is guaranteed by Lemma 3.5-(c) (with  $\psi = \varphi$  and  $a_n = 0$ ) where it should be observed that, in the proof, the polynomial  $T_l$  is not needed since  $a_n = 0$ . Hence, as mentioned in that proof, the zeros of  $T$  belong to  $I$  and are discontinuity points of  $\varphi$ ; but since the latter was extended linearly on each component of  $\mathbb{R} \setminus S$ , the zeros of  $T$  actually belong to  $S$ , as announced. The family  $\mathcal{H}$  is well defined, since  $j_n w_n^{p'/2}$  extends analytically to  $\Omega_{\tilde{S}}$  (see the discussion after equation (3.10)).

Now, the normality of  $\mathcal{H}$  in  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$  is sufficient to establish the lemma, because  $|\overline{(Q_s^2 T)(1/\bar{z})}|$  is bounded away from zero on compact subsets of  $\Omega_{\tilde{S}}$ , since all zeros of  $Q_s^2 T$  lie in  $\tilde{S}$ , hence the sequence

$$\left\{ \left( j_n w_n^{p'/2} \right) (z) \right\}$$

will be locally uniformly bounded in  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$ . Moreover, since  $j_n$  is a finite Blaschke product, we have

$$|j_n(z)| > 1, \quad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}},$$

and therefore  $\mathcal{W}_{p'}$  will be locally uniformly bounded in  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$ . But since each  $w_n^{p'/2}$  is holomorphic in  $\Omega_{\tilde{S}} \supset \overline{\mathbb{D}}$ , the maximum principle yields that this family is, in fact, normal in  $\Omega_{\tilde{S}}$ . *This establishes the claim.*

<sup>7</sup>Note that we do not require the hypothesis  $\lambda \in \mathbf{BVT}$  to hold. It is sufficient to have a measure merely with an argument of bounded variation and infinitely many points in the support.

So, it only remains to prove that  $\mathcal{H}$  is locally uniformly bounded in  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$ . This can be established in the following way.

Consider the family of traces of functions from  $\mathcal{H}$  on  $\mathbb{T}$ :

$$\mathcal{H}_{\mathbb{T}} := \left\{ \left( j_n w_n^{p'/2} \right) (\xi) \overline{(Q_s^2 T)(\xi)} \right\} \subset L^2(\mathbb{T}).$$

Each function of  $\mathcal{H}_{\mathbb{T}}$  is a sum of its analytic and antianalytic parts  $h_{n,1}(\xi)$  and  $h_{n,2}(\xi)$ , where

$$\begin{aligned} h_{n,1}(\xi) &= \mathbf{P}_+ \left( \left( j_n w_n^{p'/2} \right) (\xi) \overline{(Q_s^2 T)(\xi)} \right), \\ h_{n,2}(\xi) &= \mathbf{P}_- \left( \left( j_n w_n^{p'/2} \right) (\xi) \overline{(Q_s^2 T)(\xi)} \right). \end{aligned}$$

We can regard  $h_{n,2}$  as an analytic function in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  that vanishes at infinity. Thus, upon writing

$$h_{n,2}(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\left( j_n w_n^{p'/2} \right) (\xi) \overline{(Q_s^2 T)(\xi)}}{z - \xi} d\xi, \quad |z| > 1, \quad (3.39)$$

we see that  $\{h_{n,2}\}$  is locally uniformly bounded in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , since  $Q_s$  and  $\mathbb{T}$  are fixed,  $|j_n| \equiv 1$  on  $\mathbb{T}$ , and  $\|w_n^{p'/2}\|_2 = 1$ .

Now, multiplying the first line of (3.6) by  $b_{n_2} Q_s^2 T = b_n Q_s^2 T$  and applying  $\mathbf{P}_-$  yields

$$\begin{aligned} A_F(b_n v_n Q_s^2 T)(\xi) &= \gamma_n^{1/2} \mathbf{P}_- \left( \overline{\xi \left( j_n w_n^{p'/2} \right) (\xi) (Q_s^2 T)(\xi)} \right) \\ &= \gamma_n^{1/2} \overline{\xi \mathbf{P}_+ \left( \left( j_n w_n^{p'/2} \right) (\xi) \overline{(Q_s^2 T)(\xi)} \right)} \\ &= \gamma_n^{1/2} \overline{\xi h_{n,1}(\xi)}, \quad \xi \in \mathbb{T}, \end{aligned} \quad (3.40)$$

where we used that

$$\mathbf{P}_-(h(\xi)) = \overline{\xi \mathbf{P}_+ \left( \overline{\xi h(\xi)} \right)}, \quad h \in L^2(\mathbb{T}),$$

and

$$\mathbf{P}_-(\mathbf{P}_-(h)g) = \mathbf{P}_-(hg), \quad h \in L^2(\mathbb{T}), \quad g \in H^2.$$

Following the computations preceding (3.10) we obtain

$$A_F(b_n v_n Q_s^2 T)(\xi) = \int \frac{(b_n v_n Q_s^2 T)(t)}{\xi - t} d\tilde{\lambda}(t) = \int \frac{(b_n v_n Q_s^2 T)(t)}{\xi - t} d\lambda(t), \quad \xi \in \mathbb{T},$$

where the second equality is valid since  $Q_s^2$  has a zero at each  $\eta \in S'$  of multiplicity greater than or equal to  $m(\eta) + 1$ . Thus, using (3.40), we get an analytic extension of  $h_{n,1}$  to  $\overline{\mathbb{C}} \setminus S^{-1}$  by the formula

$$h_{n,1}(z) = \gamma_n^{-1/2} \int \frac{\overline{(b_n v_n Q_s^2 T)(t)}}{1 - tz} d\bar{\lambda}(t). \quad (3.41)$$

Combining (3.39) and (3.41) we see that

$$\left( j_n w_n^{p'/2} \right) (z) \overline{(Q_s^2 T)(1/\bar{z})} = h_{n,1}(z) + h_{n,2}(z), \quad z \in \Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}.$$

So, for  $\mathcal{H}$  to be normal, it remains to show that  $\{h_{n,1}\}$  is uniformly bounded on compact subsets of  $\Omega_{\tilde{S}} \setminus \overline{\mathbb{D}}$ . Due to formula (3.41) this is actually true in  $\overline{\mathbb{C}} \setminus S^{-1}$ . Indeed, let  $K \subset \overline{\mathbb{C}} \setminus S^{-1}$  be compact. Then for any  $z \in K$

equation (3.41) yields the estimate

$$\begin{aligned}
 |h_{n,1}(z)| &\leq \gamma_n^{-1/2} \int \frac{|b_n v_n Q_s^2 T|(t)}{|1-tz|} d|\lambda|(t) \\
 &\leq \frac{\gamma_n^{-1/2}}{\inf_{t \in S, z \in K} |1-tz|} \int |b_n v_n Q_s^2 T|(t) d|\lambda|(t) \\
 &\leq \frac{\gamma_n^{-1/2} (\sin \beta)^{-1}}{\inf_{t \in S, z \in K} |1-tz|} \left| \int |b_n^2 w_n Q_s^2 T|(t) e^{i(\varphi(t) + \text{Arg } T(t))} d|\lambda|(t) \right| \\
 &= \frac{\gamma_n^{-1/2} (\sin \beta)^{-1}}{\inf_{t \in S, z \in K} |1-tz|} \left| \int \overline{(b_n w_n^{1/2} Q_s)}(t) (b_n w_n^{1/2} Q_s T)(t) d\lambda(t) \right|,
 \end{aligned}$$

where we used (3.38). Note that  $w_n^{1/2}$  is well-defined in  $\mathbb{D}$  since  $w_n$  has no zeros there. For simplicity we shall denote

$$\gamma(K) := \frac{(\sin \beta)^{-1}}{\inf_{t \in S, z \in K} |1-tz|}.$$

We remark that this quantity is finite and depends solely on  $K$  and the measure  $\lambda$ .

Define  $G_n(z) := \overline{(b_n Q_s w_n^{1/2}(\bar{z}))}$ ,  $z \in \mathbb{D}$ . Then  $G_n \in H^{2p'}$ , its trace has the same  $L^{2p'}(\mathbb{T})$  norm as  $\overline{(Q_s w_n^{1/2})}$  and  $G_n$  coincides with  $\overline{(b_n Q_s w_n^{1/2})}$  on  $(-1, 1)$ . Further, let  $r < 1$ , dependent on  $n$ , be such that  $\mathbb{D}_r$  contains  $\tilde{S}$  and all zeros of  $b_n j_n$ . Then by the estimates above, the fact that  $Q_s$  vanishes on  $S'$ , the Cauchy formula, equation (3.10), and Fubini-Tonelli's theorem we get for every  $z \in K$

$$\begin{aligned}
 |h_{n,1}(z)| &\leq \gamma(K) \left| \gamma_n^{-1/2} \int \left( G_n Q_s T w_n^{-1/2} \right)(t) v_n(t) d\tilde{\lambda}(t) \right| \\
 &= \gamma(K) \left| \frac{\gamma_n^{-1/2}}{2\pi i} \int_{\tilde{S}} \left( \int_{\mathbb{T}_r} \frac{\left( G_n Q_s T w_n^{-1/2} \right)(\xi) v_n(t)}{\xi - t} d\xi \right) d\tilde{\lambda}(t) \right| \\
 &= \gamma(K) \left| \frac{1}{2\pi i} \int_{\mathbb{T}_r} \left( G_n Q_s T w_n^{-1/2} \right)(\xi) \overline{(b_n j_n w_n^{p'/2})} (1/\bar{\xi}) \frac{d\xi}{\xi} \right| \\
 &= \gamma(K) \left| \frac{1}{2\pi i} \int_{\mathbb{T}_r} (G_n Q_s T)(\xi) \overline{(b_n j_n)} (1/\bar{\xi}) \frac{\overline{w_n^{p'/2} (1/\bar{\xi})}}{w_n^{1/2}(\xi)} \frac{d\xi}{\xi} \right|. \tag{3.42}
 \end{aligned}$$

The integrand

$$(G_n Q_s T)(\xi) \overline{(b_n j_n)} (1/\bar{\xi}) \frac{\overline{w_n^{p'/2} (1/\bar{\xi})}}{w_n^{1/2}(\xi)} \frac{1}{\xi}$$

is meromorphic in  $\mathbb{D} \setminus S$ . Moreover all of its poles, that are necessarily zeros of  $b_n j_n$ , are encapsulated both by  $\mathbb{T}_r$  and  $\mathbb{T}$ . Despite possible zeros of  $w_n$  on the unit circle, the function

$$\frac{\overline{w_n^{p'/2} (1/\bar{\xi})}}{w_n^{1/2}(\xi)}$$

extends continuously there, since  $p' \geq 2$  and  $w_n^{p'/2}$  is analytic on  $\mathbb{T}$ . Therefore, the integration contour  $\mathbb{T}_r$  in the last integral of (3.42) can be replaced by  $\mathbb{T}$  without changing the value of that integral. Thus, for any  $z \in K$ , we get by a straightforward majorization and Hölder's inequality that

$$|h_{n,1}(z)| \leq \gamma(K) \left| \frac{1}{2\pi} \int_{\mathbb{T}} (Q_s T)(\xi) G_n(\xi) \frac{\overline{(b_n j_n w_n^{p'/2})}(\xi)}{w_n^{1/2}(\xi)} d|\xi| \right|$$

$$\begin{aligned} &\leq \gamma(K) \|Q_s\|_\infty \|T\|_\infty \|G_n\|_2 \|w_n^{(p'-1)/2}\|_2 \\ &\leq \gamma(K) \|Q_s\|_\infty \|T\|_\infty \|Q_s w_n^{1/2}\|_2 \|w_n\|_{p'-1}^{(p'-1)/2} \leq \gamma(K) \|Q_s\|_\infty^2 \|T\|_\infty. \end{aligned}$$

This finishes the proof of the local uniform boundedness of  $\mathcal{W}_{p'}$ , since the above estimate does not depend on  $n$ , but only on the set  $K$  itself and the measure  $\tilde{\lambda}$ .

A classical result of Hurwitz states that each limit point of a normal family consisting of zero free function in the domain of the normality is zero free unless it is identically zero. Now,  $\mathcal{W}_{p'}$  has no zero function as a limit point because these functions have unit  $L^2(\mathbb{T})$  norm. Since each  $w_n^{p'/2}$  is outer in  $\mathbb{D}$ , no limit point of  $\mathcal{W}_{p'}$  can have zeros in  $\mathbb{D}$ . The corresponding result for  $\mathcal{W}$  is then obvious.  $\blacksquare$

Note that in the previous theorem the normality of  $\mathcal{W}$  in  $\mathbb{D}$  was clear beforehand by the Cauchy formula, since  $\|w_n\|_{p'} = 1$ , but the nonvanishing of every limit point was not.

*Proof of Theorem 3.2:* Let  $v_n$  be a singular vector associated to  $g_n$  with inner-outer factorization given by  $v_n = b_n \cdot w_n$  for some Blaschke product  $b_n = q_n/\tilde{q}_n$  and some outer function  $w_n \in H^{p'}$ ,  $\|w_n\|_{p'} = 1$ , where  $w_n \equiv 1$  when  $p = 2$ . The poles of  $g_n$  are exactly the zeros of  $q_n$ . Moreover,  $\{q_n\}$  is a sequence of polynomials satisfying the non-Hermitian weighted orthogonality relations (3.7). Thus, the assertion of the theorem will follow from Theorem 3.1 if  $\mathcal{W} = \{w_n\}$  is uniformly bounded above and below on  $I$ , the convex hull of  $S$ , and if it is a family of functions whose arguments are smooth with uniformly bounded derivatives on  $I$ . In the case  $p = 2$  this is trivial since each  $w_n \equiv 1$ . In the case  $p \in (2, \infty]$  Lemma 3.8 says that  $\mathcal{W}$  is a normal family. Thus, it is uniformly bounded above on  $I$ . Moreover, since all limit points of  $\mathcal{W}$  are zero free in  $\mathbb{D}$ , this family is uniformly bounded below on  $I$  (in fact on any compact subset of  $\mathbb{D}$ ). Further, the derivatives again form a normal family and so does the logarithmic derivative  $w'_n/w_n$  in  $\mathbb{D}$ . Since the imaginary part of the latter is equal to  $\text{darg}(w_n)/dt$  on  $I$ , we see that the rest of conditions on  $\mathcal{W}$  is satisfied.  $\blacksquare$

Before we prove Theorem 3.3 we need a simple observation that will be of further use later on.

**Lemma 3.9** *Let  $A$  be a compact set such that  $A \cap S = \emptyset$ . Further, let  $\{\mu_n\}$  and  $\{\lambda_n\}$  be two sequences of positive measures such that  $\text{supp}(\mu_n) \subset A$ ,  $\mu_n \xrightarrow{*} \mu$ ,  $\{\text{supp}(\lambda_n)\}$  is bounded, and  $\lambda_n \xrightarrow{*} \hat{\mu}$ , where  $\mu \in \Lambda(A)$  and  $\hat{\mu}$  is the balayage of  $\mu$  onto  $S$ . Then*

(a)

$$\limsup_{n \rightarrow \infty} \|\exp \{U^{\mu_n - \lambda_n}\}\|_S \leq \exp\{-c(\mu; \mathbb{C} \setminus S)\};$$

(b) *for any compact set  $K \subset \mathbb{C} \setminus S$  having a neighborhood  $\mathcal{V}$  such that  $\mathcal{V} \cap \text{supp}(\lambda_n) = \emptyset$  for  $n$  large enough, we have*

$$\limsup_{n \rightarrow \infty} \|\exp \{U^{\lambda_n - \mu_n}\}\|_K \leq \exp\{c(\mu; \mathbb{C} \setminus S)\} \|\exp \{-U_G^\mu(\cdot; \mathbb{C} \setminus S)\}\|_K.$$

where  $U_G^\mu(z; \mathbb{C} \setminus S)$  is the Green potential of  $\mu$  relative to  $\mathbb{C} \setminus S$ .

*Proof:* (a) By the principle of descent ([46, Thm. I.6.8])

$$\liminf_{n \rightarrow \infty} U^{\lambda_n}(z) \geq U^{\hat{\mu}}(z),$$

for any  $z \in \mathbb{C}$ . Further, for any  $z \in \mathbb{C} \setminus A$ , let  $\phi_z$  be a continuous function which is 1 on  $A$  and 0 outside a relatively compact neighborhood of  $A$  that does not contain  $z$  in its closure, we get by weak\* convergence since  $t \mapsto \phi_z(t) \log(1/|z - t|)$  is continuous with compact support that

$$\lim_{n \rightarrow \infty} \int \phi_z(t) \log \frac{1}{|z - t|} d\mu_n(t) = \int \phi_z(t) \log \frac{1}{|z - t|} d\mu(t)$$

which is equivalent to

$$\lim_{n \rightarrow \infty} U^{\mu_n}(z) = U^\mu(z).$$

Thus, for any  $x \in S$

$$\limsup_{n \rightarrow \infty} U^{\mu_n - \lambda_n}(x) \leq U^{\mu - \hat{\mu}}(x) = -c(\mu; \mathbb{C} \setminus S),$$

by the regularity of  $S$  and the properties of balayage. The assertion follows immediately from this.

(b) Because for  $n$  large  $\text{supp}(\lambda_n)$  is included in the compact set  $(\mathbb{C} \setminus \mathcal{V}) \cap \{|z| \leq R\}$  for some suitable  $R > 0$ , we may find a continuous function  $\phi_K$  with compact support which is 1 on this compact set and zero on  $K$ . Applying the principle of descent to  $\{U^{\mu_n}\}$  and the weak\* convergence argument to

$$\{U^{\lambda_n}\}(z) = \int \phi_K(t) \log \frac{1}{|z - t|} d\lambda_n(t), \quad z \in K,$$

we get in the same manner as before

$$\limsup_{n \rightarrow \infty} U^{\lambda_n - \mu_n}(z) \leq U^{\hat{\mu} - \mu}(z),$$

for any  $z \in K$ . This together with (2.18) implies the second assertion of the lemma. ■

*Proof of Theorem 3.3:* To prove the convergence in capacity we first establish an integral representation for the error  $r_n = (F - g_n)$ . As usual we denote by  $v_n = b_n \cdot w_n$  a singular vector associated to  $g_n$ , where  $b_n$  is a Blaschke product of degree  $n$  and  $w_n$  is an outer function. Since any critical point satisfies (see discussion before (3.4))

$$F - g_n = \frac{A_F(v_n)}{v_n},$$

equation (3.10) allows us to write for  $z \in \mathbb{D} \setminus \tilde{S}$

$$r_n(z) = \frac{1}{v_n(z)} \int \frac{v_n(t)}{z - t} d\tilde{\lambda}(t) = \frac{\tilde{q}_n(z)}{q_n(z)w_n(z)} \int \frac{q_n(t)}{z - t} \frac{w_n(t)}{\tilde{q}_n(t)} d\tilde{\lambda}(t). \quad (3.43)$$

In another connection, the orthogonality relations (3.7) yield

$$\int \frac{\tilde{q}_n(z) - \tilde{q}_n(t)}{z - t} q_n(t) \frac{w_n(t)}{\tilde{q}_n^2(t)} d\tilde{\lambda}(t) = 0.$$

This, in turn, implies that (3.43) can be rewritten as

$$r_n(z) = \frac{\tilde{q}_n^2(z)}{q_n(z)w_n(z)} \int \frac{q_n(t)}{z - t} \frac{w_n(t)}{\tilde{q}_n^2(t)} d\tilde{\lambda}(t), \quad z \in \mathbb{D} \setminus \tilde{S}. \quad (3.44)$$

Let  $K$  be a compact subset of  $\mathbb{D} \setminus S$  and let  $U$  an open neighborhood of  $K$  such that  $U \subset \mathbb{D}$  and  $U \cap S = \emptyset$ . Denote by  $\xi_{1,n}, \dots, \xi_{n,n}$  the zeros of  $q_n$  and define

$$q_{n,1}(z) = \prod_{\xi_{j,n} \in U} (z - \xi_{j,n}).$$

Corollary 3.7 implies that there exists  $m \in \mathbb{N}$  such that  $\deg q_{n,1} \leq m$  for all  $n \in \mathbb{N}$ . Let  $a < 1$  be such that  $S \subset \mathbb{D}_a$ . Then by Lemma 3.6 the number of zeros of  $q_n$  in  $D_a \setminus U$  tends to infinity with  $n$ . Thus, for  $n$  large enough and in any case bigger than  $s$ , we can pick  $s$  of them, say  $\xi_{1,n}, \dots, \xi_{s,n}$  and define

$$l_{s,n}(z) := \prod_{j=1}^s (z - \xi_{j,n}).$$

Since the polynomial  $q_{n,1}l_{s,n}$  has bounded degree, the asymptotic zero distribution of  $q_n^\# := q_n/(q_{n,1}l_{s,n})$  is the same as that of  $q_n$ . Thus, by Theorem 3.1 and Proposition A, points (b)-(e), the counting measures  $\nu_n^\#$  of  $q_n^\#$  converge weak\* to  $\mu_G^S$ , the Green equilibrium distribution on  $S$  relative to both  $\mathbb{D}$  and  $\mathbb{C} \setminus S^{-1}$ . Using the

orthogonality relations (3.7) and the fact that  $Q_s$  vanishes at the  $\eta_j$  with multiplicity  $m(\eta_j)$ , we can rewrite (3.44) as

$$r_n(z) = \frac{\tilde{q}_n^2(z)}{(q_n^\#)^2(z)(q_{n,1}Q_s)(z)(l_{s,n}w_n)(z)} \int \frac{(q_n^\#)^2(t)(q_{n,1}Q_s)(t)(l_{s,n}w_n)(t)}{\tilde{q}_n^2(t)} \frac{d\lambda(t)}{z-t}, \quad z \in \mathbb{D} \setminus \tilde{S}. \quad (3.45)$$

Lemma 3.8 says that  $\{w_n\}$  is bounded above and below on compact subsets of  $\mathbb{D}$ . This and the choice of  $l_{s,n}$  yield that

$$|r_n(z)| \leq \frac{c_1}{|q_{n,1}(z)Q_s(z)|} \left| \frac{\tilde{q}_n(z)}{q_n^\#(z)} \right|^2 \left\| \frac{q_n^\#}{\tilde{q}_n} \right\|_S^2 \quad (3.46)$$

for any  $z \in K$  and some positive  $c_1 < \infty$  which does not depend on  $n$ . We shall apply Lemma 3.9 with  $A = \mathbb{T}$ ,  $\lambda_n = \nu_n^\#$ , and  $\mu_n = \tilde{\nu}_n$ , where  $\tilde{\nu}_n$  is the balayage of  $\nu_n$  onto the unit circle and  $\nu_n$  is the counting measures of zeros of  $q_n$ . Because  $\nu_n(\mathbb{D} \setminus S) \rightarrow 0$  by Corollary 3.7 the weak\*-limit of the balayage is the balayage of the weak\* limit, hence Theorem 3.1 implies that  $\mu = \mu_G^{\tilde{S}}$  and Proposition A-(c) shows that  $\hat{\mu} = \mu_G^S$ . Also notice that in this case

$$\begin{aligned} U^{\lambda_n - \mu_n}(z) &= U^{\nu_n^\# - \tilde{\nu}_n}(z) = U^{\nu_n^\#}(z) - U^{\nu_n}(z) + U_G^{\nu_n}(z; \mathbb{D}) \\ &= \frac{1}{n} \log \left| \frac{1}{q_n^\#(z)} \right| - \frac{1}{n} \log \left| \frac{1}{q_n(z)} \right| + \int \log \left| \frac{1 - z\bar{t}}{z - t} \right| d\nu_n(t) = \frac{1}{n} \log \left| \frac{\tilde{q}_n(z)}{q_n^\#(z)} \right|, \quad z \in \mathbb{D} \setminus S, \end{aligned}$$

since  $c(\nu_n; \mathbb{D}) = 0$  by the boundedness of  $\mathbb{D}$ . As  $U_G^{\mu_G^{\tilde{S}}}(z; \mathbb{C} \setminus S)$  is a non-constant positive harmonic function in  $\mathbb{D} \setminus S$ , it is strictly positive there so there exists  $\beta > 0$  such that

$$0 < 2\beta \leq \inf_{z \in K} U_G^{\mu_G^{\tilde{S}}}(z; \mathbb{C} \setminus S).$$

Then Lemma 3.9, in particular, implies that for all  $n$  large enough

$$\left\| \frac{q_n^\#}{\tilde{q}_n} \right\|_S \leq \exp \left\{ -n \cdot c \left( \mu_G^{\tilde{S}}; \mathbb{C} \setminus S \right) + n\beta/2 \right\},$$

and

$$\left| \frac{\tilde{q}_n(z)}{q_n^\#(z)} \right| \leq \exp \left\{ n \cdot c \left( \mu_G^{\tilde{S}}; \mathbb{C} \setminus S \right) - 3n\beta/2 \right\}, \quad z \in K.$$

Combining the last two estimates and (3.46) we obtain that

$$|r_n(z)| \leq \frac{c_1 e^{-2n\beta}}{|q_{n,1}(z)Q_s(z)|}$$

for any  $z \in K$  and  $n$  large enough. Denote by  $K_n$  the following sets

$$K_n := \{z \in K : |q_{n,1}(z)Q_s(z)| \leq e^{-n\beta}\}.$$

Since each  $K_n$  is the lemniscate of a monic polynomial of degree at most  $s + m$  we get that ([45, Thm. 5.2.5])

$$\text{cap}(K_n) \leq \exp \left\{ -n \frac{\beta}{s + m} \right\}.$$

But for  $z \in K \setminus K_n$  we get

$$|r_n(z)| \leq c_1 e^{-n\beta}.$$

This establishes the main assertion of the theorem upon letting  $\gamma := \beta/(s + m)$ .



Let us finally fix  $p = 2$  and consider a compact set  $K \subset \mathbb{C} \setminus (\mathbb{D} \cup \mathcal{K})$ . In this case  $w_n \equiv 1$  for any  $n \in \mathbb{N}$ ,  $r_n(z)$  is defined everywhere outside of  $S$ , and has the representation

$$r_n(z) = \frac{\tilde{q}_n^2(z)}{(q_n^\#)^2(z)Q_s(z)l_{s,n}(z)} \int \frac{(q_n^\#)^2(t)Q_s(t)l_{s,n}(t)}{\tilde{q}_n^2(t)} \frac{d\lambda(t)}{z-t}, \quad z \in \mathbb{C} \setminus S,$$

where  $q_n^\# = q_n/l_{s,n}$  and we fixed  $s$  arbitrary zeros of  $q_n$  in  $\mathbb{D}_a$  in order to define  $l_{s,n}$ . Note that this time we do not introduce  $q_{n,1}$ , since zeros of  $q_n$  cannot accumulate on  $K$  by definition of  $\mathcal{K}$ . For any  $z \in K$ , the reasoning that led us to (3.46) now gives us

$$|r_n(z)| \leq c_2 \left| \frac{\tilde{q}_n(z)}{q_n^\#(z)} \right|^2 \left\| \frac{q_n^\#}{\tilde{q}_n} \right\|_S^2, \quad (3.47)$$

for some  $c_2$  independent of  $n$ . Since  $K$  does not contain any limit point of zeros of  $\{q_n^\#\}$ , applying Lemma 3.9 as before gives us in view of (3.47)

$$\limsup_{n \rightarrow \infty} |r_n(z)|^{1/2n} \leq \exp \left\{ -U_G^{\mu_S^\#}(z; \mathbb{C} \setminus S) \right\} = \exp \left\{ -\frac{1}{C(S, \mathbb{T})} \right\}$$

for any  $z \in K$ , where we used Proposition A-(e). This completes the proof of the theorem.  $\blacksquare$

*Proof of Theorem 3.4:* It was proved in Theorem 3.3 that  $\{g_n\}$  converges in capacity on compact subsets of  $\mathbb{D} \setminus S$  to  $F$ . Then the lower bound in (3.14) follows from the classical Gonchar's lemma ([21, Lemma 1]). Indeed, suppose to the contrary that there exist a number  $\delta > 0$  and a subsequence  $\mathbb{N}_1 \subset \mathbb{N}$  such that

$$\#\{S_n \cap B_\delta(\eta)\} < m(\eta)$$

for  $n \in \mathbb{N}_1$  and some fixed  $\eta \in S' \setminus S$ , where  $B_\delta(\eta) = \{z \in \mathbb{C} : |z - \eta| < \delta\}$ . Thus,  $\{g_n\}_{n \in \mathbb{N}_1}$  is a sequence of meromorphic functions in  $B_\delta(\eta)$  with at most  $m(\eta)$  poles, which converges in capacity on  $B_\delta(\eta)$  to the meromorphic function  $F|_{B_\delta(\eta)}$  with exactly one pole of multiplicity  $m(\eta)$  there. Then by Gonchar's lemma each  $g_n$  has exactly  $m(\eta)$  poles in  $B_\delta(\eta)$  and these poles converge to  $\eta$ . This contradiction proves the lower bound in (3.14).

Now, for each  $\eta_i \in S'$  define  $D_{\eta_i} := B_{\delta_i}(\eta_i)$ ,  $\delta_i$ 's to be chosen later. Moreover, assume that  $D_{\eta_i} \cap S' = \{\eta_i\}$  and for each  $\eta_i \in S' \setminus S$  assume further that  $D_{\eta_i} \cap S = \emptyset$ , which is always possible by the appropriate choice of  $\delta_i$ 's.

Fix any  $\eta \in S' \setminus S$ . In general, the existence of  $C_\eta$  in (3.14) follows from Corollary 3.7 applied to  $\mathbb{D} \setminus D_\eta$ , but it is possible to say more. Let  $I_{S,m} = \cup_{j=1}^m [a_j, b_j]$  cover  $S$  and let  $S_n$  stand for the set of all poles of  $g_n$ ,  $S_n = \{\xi_{1,n}, \dots, \xi_{n,n}\}$ . Then by Lemma 3.6 we have

$$\sum_{j=1}^n (\pi - \theta(\xi_{j,n})) \leq V(\varphi) + V_{\mathcal{W}} + (m + s - 1)\pi + \sum_{\eta_j \in S'} m(\eta_j)\theta(\eta_j), \quad (3.48)$$

where  $\theta(\cdot)$  is the angle function defined in (3.13) for the system of intervals  $I_{S,m}$  and  $V_{\mathcal{W}}$  was defined in (3.15). The finiteness of  $V_{\mathcal{W}}$  was obtained in the proof of Theorem 3.2. Then for  $n$  large enough (3.48) yields

$$\#\{S_n \cap D_\eta\} - m(\eta) \leq \frac{V(\varphi) + V_{\mathcal{W}} + (m - 1)\pi + \sum_{\eta_i \in S'} m(\eta_i)(\theta(\eta_i) + \max_{\xi \in D_{\eta_i}} \theta(\xi))}{\pi - \max_{\xi \in D_\eta} \theta(\xi)}.$$

Denote by  $C$  the right hand-side of the previous inequality. By a suitable choice of  $\delta_i$ 's and the continuity of  $\theta(\cdot)$  in  $\mathbb{D} \setminus [-1, 1]$  we can make  $C$  such that

$$\lfloor C \rfloor \leq \frac{V(\varphi) + V_{\mathcal{W}} + (m - 1)\pi + 2 \sum_{\eta_i \in S'} m(\eta_i)\theta(\eta_i)}{\pi - \theta(\eta)} < C,$$

where  $\lfloor C \rfloor$  is the greatest integer smaller than  $C$ . Since  $C$  is a bound for an integer, we get that

$$\#\{S_n \cap D_\eta\} \leq m(\eta) + \frac{V(\varphi) + V_W + (m-1)\pi + 2 \sum_{\eta_i \in S'} m(\eta_i) \theta(\eta_i)}{\pi - \theta(\eta)}.$$

■

## 4 Multipoint Padé Approximation

In this section we investigate the behavior of the poles of multipoint Padé approximants to functions of type (2.4). These are analogs of rational approximants solving **MA**(2) except that the interpolation points not implicitly determined by an optimality property, but rather have to be chosen *a priori*. In what follows we no longer assume that  $S' \subset \mathbb{D}$  and  $S \subset (-1, 1)$ , i.e.  $S'$  is a set of  $s$  elements from  $\mathbb{C}$ , counting multiplicities, while  $S$  may be any regular compact subset of  $\mathbb{R}$ .

Let

$$\mathbf{A} = \{A_k\}_{k \in \mathbb{N}}, \quad A_k \subset \overline{\mathbb{C}} \setminus (S \cup S'), \quad \overline{A}_k = A_k,$$

be a conjugate-symmetric interpolation scheme, as defined in Section 2. Thus,  $A_k$  consists of  $2k$  points  $\zeta_{1,k}, \dots, \zeta_{2k,k}$ , not necessarily finite nor distinct, none of which lies in  $\tilde{S} = S \cup S'$ , and such that  $\bar{\zeta}_{j,k} \in A_k$  for each  $j \in \{1, \dots, 2k\}$ . If we define the polynomial  $w_{2k}$  as in (2.11), the conjugate-symmetry implies that it has real coefficients.

Given  $F$  of type (2.4), we put  $\Pi_k = p_k/q_k$  for the multipoint Padé approximant associated with  $A_k$ . We denote, as usual, the convex hull of  $S$  by  $I$ .

The interpolation scheme  $\mathbf{A}$  is said to be *admissible* if the set of limit points of  $\mathbf{A}$ , indicated by  $\mathcal{K}(\mathbf{A})$ , is disjoint from  $S' \cup I$ , and if the counting measures of the points in  $A_k$  converge in the weak\* topology, as  $k \rightarrow \infty$ , to some probability measure with finite logarithmic energy  $\sigma = \sigma(\mathbf{A}) \in \Lambda(\mathcal{K}(\mathbf{A}))$ , i.e.

$$\sigma_k := \frac{1}{2k} \sum_{j=1}^{2k} \delta_{\zeta_{j,k}} \xrightarrow{*} \sigma.$$

We call such  $\sigma$  an *asymptotic distribution for  $\mathbf{A}$* . Note that  $\mathcal{K}(\mathbf{A})$  is not necessarily compact. If it is not compact, the finiteness of the logarithmic energy of  $\sigma$  is understood as follows. Since  $\mathcal{K}(\mathbf{A})$  is closed and does not intersect  $\tilde{S}$ , there exists  $z_0 \in \mathbb{C} \setminus \cup_k A_k$  such that  $z_0 \notin \mathcal{K}(\mathbf{A})$ . Pick such a  $z_0$  and set  $M_{z_0}(z) := 1/(z - z_0)$ . Then, all  $M_{z_0}(A_k)$  are contained in some compact set and their counting measures converge weak\* to  $\sigma^\sharp$  such that  $\sigma^\sharp(B) := \sigma(M_{z_0}^{-1}(B))$  for any Borel set  $B \subset \mathbb{C}$ . We say that  $\mathbf{A}$  is admissible if  $\sigma^\sharp$  has finite logarithmic energy. Obviously this definition does not depend on a particular choice of  $z_0$ . If all the interpolation points are assigned to infinity, i.e.  $w_{2n} \equiv 1$  for all  $n \in \mathbb{N}$ , then  $\Pi_n$  is just the classical Padé approximant to  $F$ .

We shall follow the same exposition as in the previous section on meromorphic approximation. We start with an analog to Theorem 3.1 which is the basis of our approach.

**Theorem 4.1** *Let  $\{q_n\}_{n \in \mathbb{N}}$  be a sequence of polynomials of degree at most  $n$  satisfying the weighted orthogonality relations*

$$\int t^j q_n(t) \frac{d\tilde{\lambda}(t)}{w_{2n}(t)} = 0, \quad j = 0, \dots, n-1, \quad (4.1)$$

*where  $\{w_{2n}\}_{n \in \mathbb{N}}$  is the sequence of monic polynomials associated via (2.11) to some admissible interpolation scheme  $\mathbf{A}$  with asymptotic distribution  $\sigma$ , and where  $\tilde{\lambda} = \lambda + \lambda'$  is given by (2.2)-(2.6) for some  $\lambda \in \mathbf{BVT}$ . Then the counting measures  $\nu_n$  of the zeros of  $q_n$  converge in the weak\* sense to  $\mu_w^S$ , the weighted equilibrium distribution on  $S = \text{supp}(\lambda)$  with external field given by  $-U^\sigma$ .*

Note that, by Proposition B, the measure  $\mu_w^S$  is nothing but the balayage  $\hat{\sigma}$  of  $\sigma$  onto  $S$  out of  $\mathbb{C} \setminus S$ . Let us stress also that, in general, the polynomials  $q_n$  satisfying (4.1) need not be unique up to a multiplicative constant nor have exact degree  $n$ . In the above theorem, it is understood that  $q_n$  is *any* sequence of such

polynomials, and that  $\nu_n$  is normalized by  $1/n$  so that it may no longer be a probability measure. This is of no importance since, as will be shown later, the defect  $n - \deg q_n$  is uniformly bounded.

Let  $\Gamma$  be any closed Jordan curve that separates  $\tilde{S}$  and  $\mathcal{K}(\mathbf{A})$  and contains  $\tilde{S}$  in the bounded component of its complement, say  $D$ . Then, the fact that  $(q_n F - p_n)(z)/w_{2n}(z) = O(1/z^{n+1})$  as  $z \rightarrow \infty$ , and that it is analytic in  $\mathbb{C} \setminus \tilde{S}$ , yields that

$$\int_{\Gamma} z^j q_n(z) F(z) \frac{dz}{w_{2n}(z)} = 0, \quad j = 0, \dots, n-1, \quad z \in D,$$

which is equivalent to (4.1) by definition (2.4) of  $F$ , the Fubini-Tonelli theorem, and the residue formula. Hence the denominators of the multipoint Padé approximants to  $F$  are polynomials satisfying a non-Hermitian orthogonality relation with varying complex measures as in the previous theorem. The asymptotic distribution of their zeros is described in the following theorem which gives one answer, for the case of multipoint Padé approximants, to question (a) raised in Section 2.

**Theorem 4.2** *Let  $\{\Pi_n\}_{n \in \mathbb{N}}$  be a sequence of multipoint Padé approximants, corresponding to some admissible interpolation scheme  $\mathbf{A}$ , to  $F$  given by (2.4) with  $\lambda \in \mathbf{BVT}$ . Then, the counting measures of the poles of  $\Pi_n$  converge to  $\mu_w^S$  in the weak\* sense.*

For classical Padé approximants, this theorem reduces to the following result.

**Corollary 4.3** *Let  $\{\Pi_n\}_{n \in \mathbb{N}}$  be a sequence of classical Padé approximants to  $F$  given by (2.4) with  $\lambda \in \mathbf{BVT}$ . Then the counting measures of the poles of  $\Pi_n$  converge in the weak\* sense to  $\mu_S$ , the equilibrium distribution on  $S$ .*

As before, taking into account the special structure of  $\tilde{\lambda}$ , we deduce from (4.1) that

$$\int P_{n-s-1}(t) Q_s(t) q_n(t) \frac{d\lambda(t)}{w_{2n}(t)} = 0, \quad P_{n-s-1} \in \mathcal{P}_{n-s-1} \quad (4.2)$$

which is the equation we shall really work with when proving Theorem 4.2.

The forthcoming theorem is of classical scope and stands as an analog to Theorem 3.3 for multipoint Padé approximants. It deals with their convergence in capacity to  $F$  and offers one answer to question (c) raised in Section 2. In a way, it generalizes Theorem 6.1.6(a) in [49].

**Theorem 4.4** *Let  $\{\Pi_n\}_{n \in \mathbb{N}}$ ,  $F$ , and  $\lambda$  be as in Theorem 4.2 for some admissible interpolation scheme  $\mathbf{A}$ . Further, let  $\sigma$  stand for the asymptotic distribution of  $\mathbf{A}$ . Then  $\{\Pi_n\}$  converges to  $F$  in capacity on  $\mathbb{C} \setminus S$ , and for any compact set  $K \subset \mathbb{C} \setminus S$  there exist positive constants  $\beta$  and  $\gamma$  such that*

$$\text{cap} \{z \in K : |r_n(z)| \geq e^{-\beta n}\} \leq e^{-\gamma n},$$

where  $r_n := (F - \Pi_n)$ . Further, for each compact set  $K \subset \mathbb{C} \setminus (\tilde{S} \cup \mathcal{K})$  we have

$$\limsup_{n \rightarrow \infty} \|r_n\|_K^{1/2n} \leq \|\exp\{-U_G^\sigma(\cdot; \mathbb{C} \setminus S)\}\|_K,$$

where  $U_G^\sigma(z; \mathbb{C} \setminus S)$  is the Green potential of  $\sigma$  relative to  $\mathbb{C} \setminus S$  and  $\mathcal{K}$  is the set of limit points of the poles of  $\Pi_n$ .

Our last theorem asserts the convergence of some of the poles of the multipoint Padé approximants of  $F$  to its polar singularities. It parallels Theorem 3.4 and yields one answer to question (b) of Section 2.

**Theorem 4.5** *Let  $\{\Pi_n\}_{n \in \mathbb{N}}$ ,  $F$ , and  $\lambda$  be as in Theorem 4.2. Moreover, let  $S_n$  denote the set of poles of  $\Pi_n$ . Then for each  $\eta \in S' \setminus S$  there exists  $D_\eta$ , an open neighborhood of  $\eta$ , a constant  $C_\eta = C_\eta(F)$ , and a number  $N = N(\eta) \in \mathbb{N}$  such that, for any  $n \geq N$ ,*

$$m(\eta) \leq \# \{S_n \cap D_\eta\} \leq m(\eta) + C_\eta,$$

where  $m(\eta)$  is the multiplicity of  $\eta$ .

As in Theorem 3.4, we shall obtain in the course of the proof of the above result an upper bound for  $C_\eta$ . Namely if  $I_{S,m} := \{[a_j, b_j]\}_{j=1}^m$  is a system of intervals covering  $S$ , then

$$C_\eta \leq \frac{V(\varphi) + (m-1)\pi + 2 \sum_{\eta_i \in S'} m(\eta_i) \theta(\eta_i)}{\pi - \theta(\eta)},$$

where  $\theta(\cdot)$  is the angle function, defined in (3.13), applied to  $I_{S,m}$ .

The proof of Theorem 4.1 runs parallel to the proof of Theorem 3.1. It relies on the following lemma.

**Lemma 4.6** *Let  $q_n(z) = \prod_{j=1}^{d_n} (z - \xi_{j,n})$  be a  $n$ -th orthogonal polynomial in the sense of (4.2). Then*

$$\sum_{j=1}^{d_n} (\pi - \theta(\xi_{j,n})) + (n - d_n)\pi \leq V(\varphi) + \sum_{\eta_j \in S'} m(\eta_j) \theta(\eta_j) + (m + s - 1)\pi,$$

where  $\theta(\cdot)$  is the angle function defined in (3.13) for a system of intervals  $I_{S,m} = \cup_{j=1}^m [a_j, b_j]$  which covers  $S$ , with  $I = [a_1, b_m]$  the convex hull of  $S$ .

*Proof:* The proof is the same as that of Lemma 3.6, observing that

$$V(\text{Arg}(w_{2n}), I_{S,m}) = 0,$$

since  $w_{2n}$  is real on  $S$ . ■

Note that Lemma 4.6, applied *e.g.* with  $m = 1$  and  $I_{S,1} = I$ , implies that the defect  $n - d_n$  is bounded above independently of  $n$ .

**Corollary 4.7** *Let  $U$  be a neighborhood of  $S$ . Then there exists a constant  $k_U \in \mathbb{N}$  such that each  $q_n$  has at most  $k_U$  zeros outside of  $U$  for  $n$  large enough.*

*Proof of Theorem 4.1:* Recall the notation  $\mathbf{A} = \{A_n\}$  for the admissible interpolation scheme, and let  $\sigma_n$  be the counting measure of the points in  $A_n$ . Set  $q_n(z) = \prod_{j=1}^{d_n} (z - \xi_{j,n})$  and recall that it satisfies (4.2) (the more general relation (4.1) is not used in this proof).

We claim that we may suppose  $\mathbf{A}$  is contained in a compact set. Indeed, if this is not the case, we can pick a real number  $x_0 \notin \mathcal{K}(\mathbf{A}) \cup S' \cup I$ , where  $\mathcal{K}(\mathbf{A})$  is, as usual, the set of limit points of  $\mathbf{A}$ . Consider the analytic automorphism of  $\mathbb{C}$  given by  $M_{x_0}(z) := 1/(z - x_0)$ , with inverse  $M_{x_0}^{-1}(\tau) = x_0 + 1/\tau$ . If we put  $A_n^\sharp := M_{x_0}(A_n)$ , then  $\mathbf{A}^\sharp = \{A_n^\sharp\}$  is an admissible interpolation scheme having asymptotic distribution  $\sigma^\sharp$ , with  $\sigma^\sharp(B) = \sigma(M_{x_0}^{-1}(B))$  for any Borel set  $B \subset \mathbb{C}$ . Moreover, the choice of  $x_0$  yields that  $\mathcal{K}(\mathbf{A}^\sharp)$  is compact. Now, if we let

$$\begin{aligned} \ell_n(\tau) &= \tau^n q_n(M_{x_0}^{-1}(\tau)), \\ L_s(\tau) &= \tau^s Q_s(M_{x_0}^{-1}(\tau)), \\ P_{n-s-1}^\sharp(\tau) &= \tau^{n-s-1} P_{n-s-1}(M_{x_0}^{-1}(\tau)), \\ w_{2n}^\sharp(\tau) &= \tau^{2n} w_{2n}(M_{x_0}^{-1}(\tau)), \end{aligned}$$

then  $\ell_n$  is a polynomial of degree  $n$  with zeros at  $M_{x_0}(\xi_{j,n})$ ,  $j = 1, \dots, d_n$ , and a zero at the origin with multiplicity  $n - d_n$ . In addition,  $w_{2n}^\sharp$  is a polynomial with a zero at each point of  $A_n^\sharp$ , counting multiplicity. Thus, up to a multiplicative constant,  $w_{2n}^\sharp$  is the polynomial associated with  $A_n^\sharp$  via (2.11). Analogously,  $L_s$  is a polynomial of degree  $s$  with a zero of multiplicity  $m(\eta_j)$  at  $M_{x_0}(\eta_j)$ ,  $\eta_j \in S'$ , and  $P_{n-s-1}^\sharp$  is an arbitrary polynomial of degree at most  $n - s - 1$ . Making the substitution  $t = M_{x_0}^{-1}(\tau)$  in (4.2), we get

$$\int_{M_{x_0}(S)} P_{n-s-1}^\sharp(\tau) L_s(\tau) \ell_n(\tau) \frac{d\lambda^\sharp(\tau)}{w_{2n}^\sharp(\tau)} = 0, \quad P_{n-s-1}^\sharp \in \mathcal{P}_{n-s-1},$$

where  $d\lambda^\sharp(\tau) = \tau d\lambda(M_{x_0}^{-1}(\tau))$  is a complex measure with compact support  $M_{x_0}(S) \subset \mathbb{R}$ , having an argument of bounded variation and total variation measure  $|\lambda^\sharp| \in \mathbf{BVT}$ . Note that  $\tau$  is bounded away from zero on  $\text{supp}(\lambda^\sharp)$ , since  $S$  is compact and therefore bounded away from infinity. Now, since Lemma 4.6 implies that  $n - d_n$  is uniformly bounded above, the asymptotic distribution of the counting measures of zeros of  $\ell_n$  is the same as the asymptotic distribution of the images of the counting measures of zeros of  $q_n$  under the map  $M_{x_0}$ . As the counting measures of the points in  $A_n^\sharp$  converge weak $^\sharp$  to  $\sigma^\sharp$ , the claim follows from the fact that  $\mu_w^{M_{x_0}(S)} = \widehat{\sigma}^\sharp$  since  $\mu_w^S = \widehat{\sigma}$  by Proposition B and since balayage is preserved under  $M_{x_0}$  (e.g. because harmonic functions are, cf. equation (2.16)). Hence we assume in the rest of the proof that  $\mathbf{A}$  is contained in a compact set, say  $K_0$ , which is disjoint from  $\widetilde{S}$  by the definition of admissibility.

Let  $\Gamma$  be a closed Jordan arc such that the bounded component of  $\mathbb{C} \setminus \Gamma$ , say  $D$ , contains  $\widetilde{S}$  while the unbounded component, say  $D'$ , contains  $K_0$ . Then  $q_n = q_{n,1} \cdot q_{n,2}$ , where

$$q_{n,1}(z) = \prod_{\xi_{j,n} \in D} (z - \xi_{j,n})$$

and

$$q_{n,2}(z) = \prod_{\xi_{j,n} \notin D} (z - \xi_{j,n}).$$

Corollary 4.7 assures that degrees of polynomials  $q_{n,2}$  are uniformly bounded with respect to  $n$ , therefore the asymptotic distribution of the zeros of  $q_{n,1}$  coincides with that of  $q_n$ . Denote by  $\nu_{n,1}$  the zero counting measure of  $q_{n,1}$  normalized with  $1/n$ . Since all  $\nu_{n,1}$  are supported on a fixed compact set, Helly's selection theorem and Lemma 4.6 yield the existence of a subsequence  $\mathbb{N}_1$  such that  $\nu_{n,1} \xrightarrow{*} \nu$  for  $n \in \mathbb{N}_1$  and some measure  $\nu \in \Lambda(S)$ ; remember the defect is bounded which is why  $\nu$  is a probability measure in spite of the normalization of  $q_{n,1}$  with  $1/n$ .

Next, we observe it is enough to show that the logarithmic potential of  $\nu - \sigma$  is constant q.e. on  $S$ . Indeed, the same argument as in the proof of Theorem 3.1 shows in this case that  $\nu$  has finite logarithmic energy, and then we get from the characterization (2.14) of balayage that  $\nu = \widehat{\sigma} (= \mu_w^S)$ , as desired.

Now, suppose that  $U^{\nu-\sigma}$  is not a constant q.e. on  $S$ . Then there exist nonpolar Borel subsets of  $S$ , say  $E_-$  and  $E_+$ , and two constants  $d$  and  $\tau > 0$  such that

$$U^{\nu-\sigma}(x) \geq d + \tau, \quad x \in E_+, \quad U^{\nu-\sigma}(x) \leq d - 2\tau, \quad x \in E_-.$$

Exactly as in Theorem 3.1, this leads to the existence of  $y_0 \in \text{supp}(\nu)$  and  $\rho > 0$  such that

$$\left| \frac{q_{n,1}^2(y)}{w_{2n}(y)} \right| \leq e^{-2nd},$$

for any  $y \in [y_0 - 2\rho, y_0 + 2\rho]$  and  $n \in \mathbb{N}_1$  large enough. We also remark that the same bound holds if  $\{q_n\}$  is replaced by a sequence of monic polynomials, say  $\{v_n\}$ , of degree  $n + o(n)$ , whose counting measures normalized with  $1/n$  have asymptotic distribution  $\nu$ . Moreover, in this case

$$\left| \frac{q_{n,1}(y)v_n(y)}{w_{2n}(y)} \right| \leq e^{-2nd} \quad (4.3)$$

for any  $y \in [y_0 - 2\rho, y_0 + 2\rho]$  and  $n \in \mathbb{N}_1$  large enough.

In another connection, arguing still as in the proof of Theorem 3.1, there exists  $x \in E_- \cap S_0$  (where we will choose  $S_0$  in exactly the same way as in that proof) such that

$$\left| \frac{q_{n,1}^2(x)}{w_{2n}(x)} \right| \geq e^{-2n(d-\tau)} \quad (4.4)$$

for  $n \in \mathbb{N}_2 \subset \mathbb{N}_1$ . Let  $x_n$  be a maximum place for  $|q_{n,1}^2/w_{2n}|$  on  $S_0$ :

$$M_n := \max_{x \in S_0} \left| \frac{q_{n,1}^2(x)}{w_{2n}(x)} \right| = \left| \frac{q_{n,1}^2(x_n)}{w_{2n}(x_n)} \right|.$$

Since  $w_{2n}$  has no zeros on  $D$ , the function  $\log |q_{n,1}^2/w_{2n}|$  is subharmonic there. Thus, by (4.4) and an obvious majorization, the two-constants theorem on  $D \setminus S_0$  (compare (3.33)) yields

$$\left| \frac{q_{n,1}^2(z)}{w_{2n}(z)} \right| \leq M_n \exp \left\{ 2n \left( d - \tau + \log \frac{\text{diam}(D)}{\text{dist}(\Gamma, K_0)} \right) (1 - \omega_{D \setminus S_0}(z, S_0)) \right\}, \quad z \in D,$$

where  $\omega_{D \setminus S_0}(\cdot, S_0)$  is the harmonic measure on  $D \setminus S_0$ . Note that  $d - \tau + \log(\text{diam}(D)/\text{dist}(\Gamma, K_0))$  is necessarily positive otherwise  $q_{n,1}^2/w_{2n}$  would be constant in  $D$  by the maximum principle, which is absurd. Then, following the arguments after (3.33), we get from the regularity of  $S_0$  that for any  $\delta > 0$  there is  $r(\delta) < \text{dist}(S_0, \Gamma)$  such that

$$\left| \frac{q_{n,1}^2(x)}{w_{2n}(x)} \right| \geq \frac{1}{2} e^{-2n(d-\tau)} \quad (4.5)$$

for  $x \in I_n$ , where

$$I_n := \left[ x_n - \frac{r(\delta)}{8e^{n\delta}}, x_n + \frac{r(\delta)}{8e^{n\delta}} \right].$$

Now, Lemma 3.5(c) guarantees that there exist a polynomial  $T$  of degree, say,  $k$ , and a number  $\beta \in (0, \pi/32)$  such that

$$\left| \text{Arg} \left( e^{i\varphi(t)} Q_s(t) T(t) \right) \right| \leq \frac{\pi}{2} - 2\beta,$$

for all  $t \in I$  such that  $(TQ_s)(t) \neq 0$ , where  $\varphi$  is as in (2.2). Denote by  $2\alpha$  the smallest even integer strictly greater than  $k + s + 1$ . Then, as in the proof of Theorem 3.1, there exist  $\beta_{1,n}, \dots, \beta_{2\alpha,n}$ , zeros of  $q_{n,1}$ , so close to  $[y_0 - \rho, y_0 + \rho]$  that

$$\left| \sum_{j=1}^{2\alpha} \text{Arg} \left( \frac{1}{x - \bar{\beta}_{j,n}} \right) \right| = \left| \sum_{j=1}^{2\alpha} \text{Arg} (x - \bar{\beta}_{j,n}) \right| \leq \beta, \quad x \in \mathbb{R} \setminus [y_0 - 2\rho, y_0 + 2\rho].$$

Define for  $n \in \mathbb{N}_2$  sufficiently large

$$P_n^*(z) = \frac{\overline{q_n(\bar{z})} T(z) (z - \zeta^*)}{\prod_{j=1}^{2\alpha} (z - \beta_{j,n})},$$

where the factor  $(z - \zeta^*)$  is present only if  $\text{Arg}(w_{2n}(t)) \equiv \pi$  on  $I$  in which case  $\zeta^*$  is a real zero of  $w_{2n}$  lying to the right of  $I$  (such a  $\zeta^*$  exists for  $w_{2n}$  is positive at infinity and real on  $\mathbb{R}$  by the conjugate-symmetry of  $A_n$ , while it does not vanish on  $I$  when  $n$  is large enough since  $\mathcal{K}(\mathbf{A}) \cap I = \emptyset$  by assumption). Then

$$\begin{aligned} \left| \text{Arg} \left( (P_n^* Q_s q_n)(x) \frac{e^{i\varphi(x)}}{w_{2n}(x)} \right) \right| &= \left| \text{Arg} \left( |q_n(x)|^2 \cdot \prod_{j=1}^{2\alpha} \frac{1}{(x - \bar{\beta}_{j,n})} \cdot \frac{(x - \zeta^*)(TQ_s)(x) e^{i\varphi(x)}}{w_{2n}(x)} \right) \right| \\ &\leq \pi/2 - \delta, \end{aligned} \quad (4.6)$$

for all  $x \in I \setminus [y_0 - 2\rho, y_0 + 2\rho]$  except if  $T(x)Q_s(x) = 0$ , where  $\delta$  was chosen so small that  $\delta < \beta$  and where  $n \in \mathbb{N}_2$  is sufficiently large. Just like in the proof of Theorem 3.1, we fix  $S_0$  to be at positive distance from the zeros of  $TQ_s$ , which is a polynomial independent of  $n$ .

Denote by  $m_{n,1}$  the number of zeros of  $q_{n,2}$  of modulus at least  $2 \max_{x \in S} |x|$ , and put  $\alpha_n$  for the inverse of their product. Let  $m_{n,2} := \deg q_{n,2} - m_{n,1}$ . Then, for  $t \in S$ , we have

$$(\text{dist}(S, \Gamma))^{m_{n,2}} (1/2)^{m_{n,1}} \leq |\alpha_n q_{n,2}(t)| \leq \left( 3 \max_{x \in S} |x| \right)^{m_{n,2}} (3/2)^{m_{n,2}},$$

and since  $m_{n,1} + m_{n,2} = \deg q_{n,2}$  is uniformly bounded with  $n$  so is  $\{|\alpha_n q_{n,2}|\}$  bounded above and below on  $S$ . Therefore, using (4.6), (4.5), (4.3) with  $v_n(z) = P_n^*(z)/\overline{q_{n,2}(\bar{z})}$ , and reasoning as in the proof of Theorem 3.1, we obtain

$$\text{Re} \left( \int_{S \setminus [y_0 - 2\rho, y_0 + 2\rho]} |\alpha_n|^2 P_n^*(t) Q_s(t) q_n(t) \frac{e^{i\varphi(t)}}{w_{2n}(t)} d|\lambda|(t) \right) \geq c_1 e^{-2nd + n(2\tau - L\delta)} \quad (4.7)$$

and

$$\left| \int_{[y_0-2\rho, y_0+2\rho]} |\alpha_n|^2 P_n^*(t) Q_s(t) q_n(t) \frac{e^{i\varphi(t)}}{w_{2n}(t)} d|\lambda|(t) \right| \leq c_2 e^{-2nd}. \quad (4.8)$$

This completes the proof, since  $\delta$  can be taken such that  $2\tau - L\delta > 0$  which would contradict the orthogonality relation (4.2) because, for  $n$  large enough, the integral in (4.7) is much bigger than in (4.8).  $\blacksquare$

*Proof of Theorem 4.4:* Exactly as in the proof of Theorem 4.1 we can suppose that all the interpolation points are contained in some compact set  $K_0$  disjoint from  $\tilde{S}$ . By the Hermite interpolation formula, (cf. [49, Lemma 6.1.2, (1.23)]) the error  $r_n$  has the following representation

$$r_n(z) = \frac{w_{2n}(z)}{q_n(z)} \int \frac{q_n(t)}{w_{2n}(t)} \frac{d\tilde{\lambda}(t)}{z-t}. \quad (4.9)$$

Let  $K \subset \mathbb{C} \setminus S$  be compact and let  $U$  be a bounded open neighborhood of  $K$  not intersecting  $S$ . Further, let  $D$  be an open disk around the origin such that  $(S \cup \overline{U}) \subset D$ . Denote by  $\xi_{1,n}, \dots, \xi_{d_n,n}$  zeros of  $q_n$  and define

$$\begin{aligned} q_{n,1}(z) &= \prod_{\xi_{j,n} \in U} (z - \xi_{j,n}) \\ q_{n,2}(z) &= \prod_{\xi_{j,n} \in \mathbb{C} \setminus \overline{D}} (z - \xi_{j,n}). \end{aligned}$$

Corollary 4.7 implies that there exist  $m_1, m_2 \in \mathbb{N}$  independent of  $n$  such that  $\deg q_{n,j} \leq m_j$  for any  $n \in \mathbb{N}$  and  $j = 1, 2$ . Since at least  $d_n - m_1 - m_2$  zeros lie in  $D \setminus U$  we can fix  $s$  of them there, say  $\xi_{1,n}, \dots, \xi_{s,n}$ , for  $n$  large enough. Denote by

$$l_{s,n}(z) := \prod_{j=1}^s (z - \xi_{j,n}).$$

By Theorem 4.1, the counting measures of  $q_n^\# := q_n / (q_{n,1} q_{n,2} l_{s,n})$ , denoted by  $\nu_n^\#$ , converge weak\* to  $\hat{\sigma}$ , the balayage of  $\sigma$  onto  $S$ . Using the orthogonality relations (4.1) in the same way as we did to obtain (3.45), we can rewrite (4.9) as

$$r_n(z) = \frac{w_{2n}}{(q_n^\#)^2 (q_{n,1} Q_s) (|\alpha_n| q_{n,2}) l_{s,n}}(z) \int \frac{(q_n^\#)^2 (q_{n,1} Q_s) (|\alpha_n| q_{n,2}) l_{s,n}}{w_{2n}}(t) \frac{d\lambda(t)}{z-t},$$

where  $\alpha_n$  is the inverse of the product of the zeros of  $q_{n,2}$ . Then, the choice of  $l_{s,n}$  and  $\alpha_n$  yields that

$$|r_n(z)| \leq \frac{c_1}{|q_{n,1}(z) Q_s(z)|} \left| \frac{w_{2n}(z)}{(q_n^\#)^2(z)} \right| \left\| \frac{(q_n^\#)^2}{w_{2n}} \right\|_S \quad (4.10)$$

for all  $z \in K$  and some  $c_1 < \infty$  which does not depend on  $n$ . Now we may apply Lemma 3.9 with  $A = K_0$ ,  $\lambda_n = \nu_n^\#$ , and  $\mu_n = \sigma_n$ , where  $\sigma_n$  is the counting measure of the zeros of  $w_{2n}$ . Notice that in this case

$$U^{\lambda_n - \mu_n}(z) = \frac{1}{2n} \log \left| \frac{w_{2n}(z)}{(q_n^\#)^2(z)} \right|.$$

By the regularity of  $S$ , the Green potential  $U_G^\sigma(z; \mathbb{C} \setminus S)$  tends to zero whenever  $z$  tends to  $\xi \in S$ . Moreover, it is strictly positive and superharmonic in  $\mathbb{C} \setminus S$ . Thus, by the generalized minimum principle ([46, Thm. I.2.4]), there exists  $\beta > 0$  such that

$$0 < 5\beta/2 \leq \inf_{z \in K} U_G^\sigma(z; \mathbb{C} \setminus S).$$

Then, Lemma 3.9 implies that for all  $n$  large enough

$$\left\| \frac{(q_n^\#)^2}{w_{2n}} \right\|_S \leq \exp \{-2n \cdot c(\sigma; \mathbb{C} \setminus S) + 2n\beta\}$$

and

$$\left| \frac{w_{2n}(z)}{(q_n^\sharp)^2(z)} \right| \leq \exp \{2n \cdot c(\sigma; \mathbb{C} \setminus S) - 4n\beta\}, \quad z \in K.$$

Combining the last two estimates and (4.10) we obtain for any  $z \in K$  and  $n$  large that

$$|r_n(z)| \leq \frac{c_1 e^{-2n\beta}}{|q_{n,1}(z)Q_s(z)|}.$$

This yields the first assertion of the theorem exactly as in the proof of Theorem 3.3.

Now, let  $K \subset \mathbb{C} \setminus (\tilde{S} \cup \mathcal{K})$  be a compact set. Then for any  $z \in K$

$$|r_n(z)| \leq c_3 \left| \frac{w_{2n}(z)}{(q_n^\sharp)^2(z)} \right| \left\| \frac{(q_n^\sharp)^2}{w_{2n}} \right\|_S$$

for some constant  $c_3$  which is independent of  $n$ . The second assertion then follows upon applying Lemma 3.9.  $\blacksquare$

*Proof of Theorem 4.5:* The proof is exactly the same as the proof of Theorem 3.4. The only difference is that Lemma 4.6 states that

$$\sum_{j=1}^{d_n} (\pi - \theta(\xi_{j,n})) + (n - d_n)\pi \leq V(\varphi) + (m + s - 1)\pi + \sum_{\eta_j \in S'} m(\eta_j)\theta(\eta_j),$$

and therefore

$$C_\eta \leq \frac{V(\varphi) + (m - 1)\pi + 2 \sum_{\eta_i \in S'} m(\eta_i)\theta(\eta_i)}{\pi - \theta(\eta)}.$$

## 5 Numerical Experiments

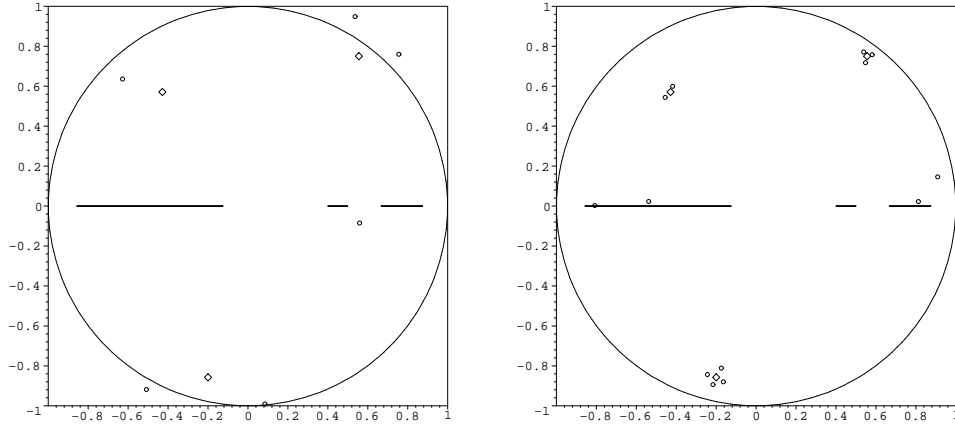
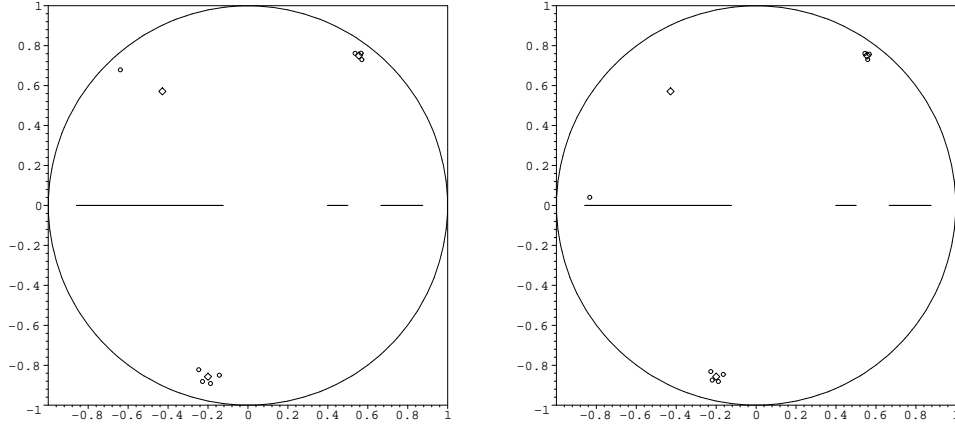
The Hankel operator  $A_F$  with symbol  $F \in H^\infty + C(\mathbb{T})$  is of finite rank if and only if  $F$  is a rational function [36, Thm. 3.11]. In practice one can only compute with finite rank operators, due to the necessity of ordering the singular values, so a preliminary rational approximation to  $F$  is needed when the latter is not rational. One way to handle this problem is to truncate the Fourier series of  $F$  at some high order  $N$ . This provides us with a rational function  $F_N$  that approximates  $F$  in the Wiener norm which, in particular, dominates any  $L^p$  norm on the unit circle,  $p \in [1, \infty]$ . It was proved in [26] that the best approximation operator from  $H_n^\infty$  (mapping  $F$  to  $g_n$  according to (3.3)) is continuous in the Wiener norm provided  $(n + 1)$ -st singular value of the Hankel operator is simple. It was shown in [9, Cor. 2] that the last assertion is satisfied for Hankel operators with symbols in some open dense subset of  $H^\infty + C(\mathbb{T})$ , and the same technique can be used to prove that it is also the case for the particular subclass (2.4). Thus, even though the simplicity of singular values cannot be asserted beforehand, it is generically true. When it prevails, one can approximate  $F_N$  instead of  $F$  and get a close approximation to  $g_n$  when  $N$  is large enough. This amounts to perform the singular value decomposition of  $A_{F_N}$  (see [51, Ch. 16]). When  $2 \leq p < \infty$  there is no difficulty with continuity issues, but the computation of  $g_n$  has to rely on a numerical search. To numerically construct rational approximants when  $p = 2$ , we used the above truncation technique together with the *Hyperion* software described in [25].

As to Padé approximants, we restricted ourselves to the classical case and we constructed their denominators by solving the orthogonality relations (4.1) with  $w_{2n} \equiv 1$ . Thus, finding these denominators amounts to solving a system of linear equations whose coefficients are obtained from the moments of the measure  $\lambda$ .

In the numerical experiments below we approximate function  $F$  given by the formula

$$\begin{aligned} F(z) &= 7 \int_{[-6/7, -1/8]} \frac{e^{it} dt}{z - t} - (3 + i) \int_{[2/5, 1/2]} \frac{1}{t - 2i} \frac{dt}{z - t} + (2 - 4i) \int_{[2/3, 7/8]} \frac{\ln(t) dt}{z - t} \\ &+ \frac{2}{(z + 3/7 - 4i/7)^2} + \frac{6}{(z - 5/9 - 3i/4)^3} + \frac{24}{(z + 1/5 + 6i/7)^4}. \end{aligned}$$

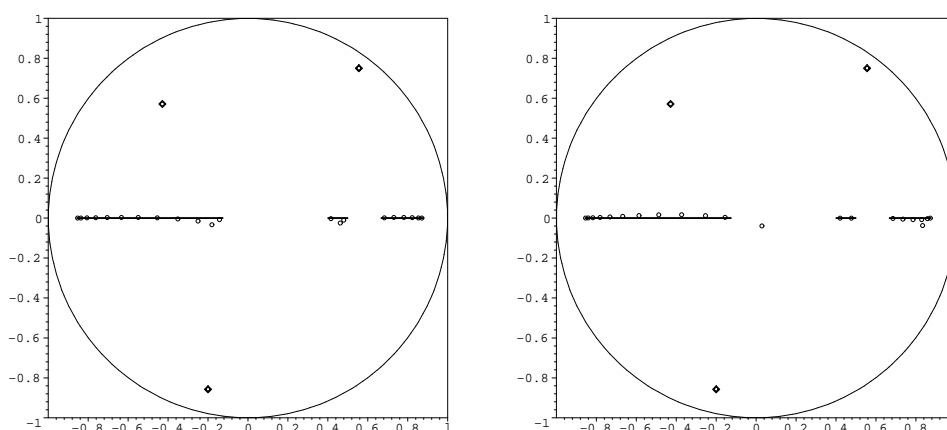



 Figure 1: Padé approximants to  $F$  of degree 8 and 13

 Figure 2: AAK (left) and rational (right) approximants to  $F$  of degree 8

On the figures the solid lines stand for the support of the measure, diamonds depict the polar singularities of  $F$ , and circles denote the poles of the correspondent approximants. Note that the poles of  $F$  seem to attract the singularities first.

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 Figure 3: Padé (left) and AAK (right) approximants to  $F$  of degree 30

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