ELAN ou la programmation par réécriture

Claude Kirchner

ELAN ou la programmation par réécriture

Claude Kirchner

LORIA & INRIA
Nancy
France

Toulouse, Mai 2000

Rules are ubiquitous in Mathematics, Logic and Computer Science

Production rules

Program transformation rules

Grammar rules

Transition rules

Logic programming rules

Constraint manipulation rules

Inference rules

Type checking rules

Derivative rules

and much more ...

Some examples

- the Unix mail system: look at the /etc/sendmail.cf file!
- tree automaton
- the so called *natural semantics*
- Mathematica
- Reve, Spass, Otter, Saturate
- ...

Some success

- re-engineering of Cobol programs [ASF+SDF, CWI Amsterdam]
- proof of the Robin's conjecture [New York Times, December 1996]
- ... ... and ELAN

fibonacci

$$
\begin{aligned}
fib(0) &\rightarrow 1 \\
fib(1) &\rightarrow 1 \\
fib(n) &\rightarrow fib(n-1) + fib(n-2)
\end{aligned}
$$

$$
\begin{aligned}
fib(3) &\rightarrow fib(2) + fib(1) \\
fib(2) + fib(1) &\rightarrow fib(2) + 1 \\
fib(2) + 1 &\rightarrow fib(1) + fib(0) + 1 \\
fib(1) + fib(0) + 1 &\rightarrow 1 + fib(0) + 1 \\
&\ldots
\end{aligned}
$$

/etc/sendmail.cf

```
#####  @(#)nullrelay.m4 8.19 (Berkeley) 5/19/1998  #####

#
#  This configuration applies only to relay-only hosts.  They send
#  all mail to a hub without consideration of the address syntax
#  or semantics, except for adding the hub qualification to the
#  addresses.
#
# This is based on a prototype done by Bryan Costales of ICSI.
#

##########################################################################
##########################################################################
#####
##### REWRITING RULES
#####
##########################################################################
##########################################################################
```

```
##########################################
###  Ruleset 3 -- Name Canonicalization  ###
##########################################
S3

# handle null input
R$@ $@ <@>

# strip group: syntax (not inside angle brackets!) and trailing semicolon
R$* $: $1 <@>mark addresses
R$* < $* > $* <@>$: $1 < $2 > $3 unmark <addr>
R$* :: $* <@>$: $1 :: $2 unmark node::addr
R:include: $* <@>$: :include: $1 unmark :include:...
R$* : $* <@>$: $2 strip colon if marked
.
.
.
.
.
.
.
```

```
.

##########################################################
###  Ruleset 4 -- Final Output Post-rewriting  ###
##########################################################
S4

R$* <@>$@ handle <> and list:;

# strip trailing dot off before passing to nullclient relay
R$* @ $+ . $1 @ $2
```

A simple game

The rules of the game:

●● → ○

○○ → ○

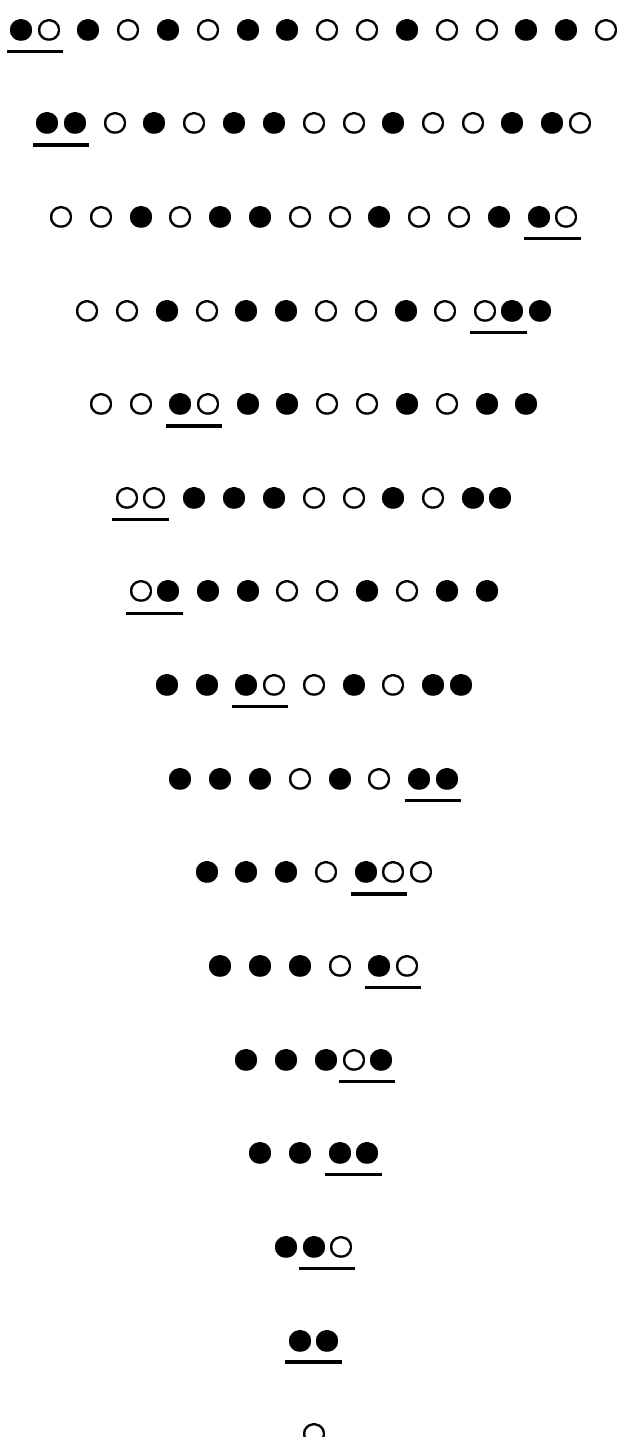
●○ → ●

○● → ●

A starting point:

●○●○●○●●●●○○●○○●●○

Who wins? (i.e. put the last white)



Rewrite description of a sorting algorithm

```
sorts NeList List ;   subsorts Nat < NeList < List ;
operators
  nil : List ;
  @ @ : (List List) List       [associative id: nil] ;
  @ @ : (NeList List) NeList   [associative] ;
  hd @ : (NeList) Nat ;
  tl @ : (NeList) List ;
  sort @ : (List) List ;
end
rules for List
  X, Y : Nat ; L L' L'' : List;
   hd (X L) => X ;                      tl (X L) => L ;
   sort nil => nil .
   sort (L X L' Y L'') =>  sort (L Y L' X L'')  if Y < X .
end

sort (6 5 4 3 2 1) => ...
```

Alternative Rings

$$
\begin{aligned}
0 + x &= x & \qquad 0 * x &= 0 \\
x * 0 &= 0 & \qquad i(x) + x &= 0 \\
i(x + y) &= i(x) + i(y) & \qquad i(i(x)) &= x \\
x * (y + z) &= (x * y) + (x * z) & \qquad (x + y) * z &= (x * z) + (y * z) \\
(x * y) * y &= x * (y * y) & \qquad (x * x) * y &= x * (x * y) \\
i(x) * y &= i(x * y) & \qquad x * i(y) &= i(x * y) \\
i(0) &= 0 & \qquad (x + y) + z &= x + (y + z) \\
x + y &= y + x
\end{aligned}
$$

Can we prove by rewriting the Moufang Identities?

$$
\begin{aligned}
(x * y) * x &= x * (y * x) \\
x * ((y * z) * x) &= (x * (y * z)) * x \\
x * (y * (x * z)) &= ((x * y) * x)) * z \\
((z * x) * y) * x) &= z * (x * (y * x)) \\
(x * y) * (z * x) &= (x * (y * z)) * x
\end{aligned}
$$

(R.Moufang, 1933)

(S.Anantharaman and J.Hsiang, JAR 6, 1990)

Needham-Schroeder public-key protocol

- proposed in '78 by Needham and Schroeder
- proved insecure in '95 by G. Lowe
- Needham-Schroeder public-key protocol has been already analyzed using several methodologies
  - model-checkers - FDR, Murφ
  - theorem proving - NRL
  - rewriting - Maude

The protocol

The Needham-Schroeder public-key protocol aims to establish a mutual authentication between an initiator and a responder that communicate via an insecure network.

		Initiator	Responder	Net
1.	$A \to B$: $\{N_A, A\}_{K(B)}$	A^{SLEEP}	B^{SLEEP}	$\emptyset$
		A^{WAIT}	B^{SLEEP}	$\{N_A, A\}_{K(B)}$
2.	$B \to A$: $\{N_A, N_B\}_{K(A)}$	A^{WAIT}	$B^{\{N_A, A\}_{K(B)}}$	$\emptyset$
		A^{WAIT}	B^{WAIT}	$\{N_A, N_B\}_{K(A)}$
3.	$A \to B$: $\{N_B\}_{K(B)}$	$A^{\{N_A, N_B\}_{K(A)}}$	B^{WAIT}	$\emptyset$
		A^{COMMIT}	B^{WAIT}	$\{N_B\}_{K(B)}$
4.	N_B *is the session key*	A^{COMMIT}	B^{COMMIT}	$\emptyset$

Methodology

Detect possible attacks of the protocol by the symbolic execution of all the possible scenarios.

Undeterministic strategies are needed to ease the description of the search space.

More details after introducing ELAN.

Two main kinds of rules

Computation rules

$2 + 3$

fib(33)

sort(data-base)

Compute as fast as possible the unique result

Deduction rules

Sequent calculus

Resolution method

Solve($2x + 4y - 3z - u = 0$ in $\mathbf{N}$)

The deduction has to be guided

Deduction rule application needs to be controlled

Lazy evaluation

Innermost/Outermost reduction

Search plans

Action plans

Tactics

User interaction

- This requires to search for a particular derivation corresponding to the desired control.

rewrite rewrite rewrite rewrite rewrite rewrite rewrite rewrite

Logic Programming, Theorem Proving, Constraint Solving, Expert Systems are instances of the same deduction schema:

Apply rewrite rules on formulas with some strategy, until getting specific forms

- Rewrite blindly: implements computations
- Rewrite wisely: implements deduction

Poincaré principle [BarendregtBarendsen97]

Deduction modulo [DowekHardinKirchner98]

What this talk is about

— Give a short presentation of ELAN with an emphasis on control of rewriting

— Present some examples on the use of ELAN

— The rewriting calculus as a semantical foundation for controled rule based systems

Presentation of ELAN

ELAN= computation rules + (deduction rules + strategies)

ELAN's General features

- ELAN has been designed to prototype, experiment and study deduction systems, in particular for constraint solving and theorem proving.
- ELAN is a declarative rewriting language.
- ELAN has two main originalities w.r.t. other algebraic languages:
  - non-deterministic computations
  - a user defined strategy language to control rewriting

Deductions and computations

- Rules for computations:
  - unique normal form required
  - leftmost innermost deterministic strategy fixed
- Rules for deductions:
  - no confluence nor termination required
  - application strategy required.
    - non-determinism handled in practice by backtracking
    - strategies are used to express choices
    - strategy may fail

Example 1: Very simple ...

```
module fib_builtin
import global builtinInt;
  end

operators global
        fib(@)  : (builtinInt) builtinInt ;
  end

rules for builtinInt
        n : builtinInt ;
global
    [] fib(0) => 1 end
    [] fib(1) => 1 end
    [] fib(n) => fib(n - 1) + fib(n - 2) if greater_builtinInt(n,1) end
  end
end
```

fib(33) = 5702887 11405773 rewrite steps in 0.695 s 16.411.184 rewrite/s

Digital 500/500, 128Mo

Example 2: Polynome derivative in ELAN

```
module poly1      import global int;    sort poly;
operators global
  X               : poly;
  @               : ( int ) poly;
  @ + @           : ( poly poly ) poly  assocRight;
  deriv(@)        : ( poly ) poly;
end
rules for poly
  p1, p2 : poly;  n : int;
global
  []  deriv(X)       => 1                        end
  []  deriv(n)       => 0                        end
  []  deriv(p1+p2)   => deriv(p1)+deriv(p2)      end
end
```

```
module poly3[Vars]
import global int Vars identifier list[identifier];
sort variable poly;
operators global
  FOR EACH Id:identifier SUCH THAT
    Id:=(listExtract) elem(Vars) : { Id : variable; }
  @           : ( variable )  poly;
  @           : ( int )       poly;
  @ + @       : ( poly poly ) poly  assocRight pri 1 (AC);
  (@ + @)     : ( poly poly ) poly  alias @ + @:;
  @ * @       : ( poly poly ) poly  assocRight pri 2 (AC);
  (@ * @)     : ( poly poly ) poly  alias @ * @:;
  deriv(@,@) : ( poly variable )   poly;
end

... ... ...
```

A simple strategy

```
[]  P+0                              => P                    end
[]  P*0                              => 0                    end
[]  deriv(x,x)                       => 1                    end
[]  deriv(n,x)                       => 0                    end
[]  deriv(p1+p2,x)       => deriv(p1,x) + deriv(p2,x)        end

[factorize]  P + n1*p1 + n2*p1       => P + n3*p1
                                        where n3:=()n1+n2 end
[factorize]  P + (p1*p2) + (p1*p3) => P + p1*(p2+p3)         end
[expand]     P + (p1+p2)*p3          => P + p2*p3 + p1*p3 end
end
strategies for poly
  [] simplify  =>
    normalize( one(expand) ) ; normalize( one(factorize) )
  end
```

Elementary strategies

Built from a few built-in constructors:

- Sequential composition

  $S_1; S_2$

- Iteration

  repeat*(S) or iterate*(S)

- Choice points

  dk$(S_1, \ldots, S_n)$: all results of all S_i

  dc$(S_1, \ldots, S_n)$: all results of one S_i

  first$(S_1, \ldots, S_n)$: all results of the first S_i

- Cut points

  dc one$(S_1, \ldots, S_n)$: one result of one S_i

  first one$(S_1, \ldots, S_n)$: one result of the first S_i

- Failure **fail** and Identity **id**

Another example: propositional sequent calculus

$$\frac{H, P \vdash Q}{H \vdash \neg P, Q}\text{neg-r}$$

```
rules for Seq
        P, Q, R : Pred;    H : Pred;    S1, S2 : Seq;
global
  [neg-r]     H |- ^P : Q    => S1
                                   where S1 := (dedstrat) H : P |- Q
  end

...

  [axio]    P : H |- P : R  =>  $
                                if neq_Pred(P, EmptyP)
```

The true code

Built (for later use) the proof term:

```
[negr]    H |- ^P : Q    =>  [#negd,H |- ^P : Q] <> S1
                                 where S1 := (dedstrat) H : P |- Q
end
```

Strategies

```
strategies  for Seq
implicit

        [] SetRules  =>  first one(
                                 axio
                                 ,negd ,disjd
                                 ,impd ,negg ,conjg
                                 ,disjg ,conjd ,impg)
        end
end

strategies  for Seq
implicit
        [] dedstrat  =>  first one( Start );
                         repeat*( SetRules )
        end
end
```

The resulting proof term

```
[dedstrat](A |=> B |- ^(B) |=> ^(A))
```

evaluates to:

```
#infer[#impd]<(A#to B)#vdash(#neg(B)#to#neg(A))>
<#infer[#negd]<(A#to B),#neg(B)#vdash#neg(A)>
<#infer[#negg]<A,(A#to B),#neg(B)#vdash EmptyP>
<#infer[#impg]<A,(A#to B)#vdash B>
<#infer[#axiom]<A,B#vdash B><#mbox<>>&
#infer[#axiom]<A#vdash A,B><#mbox<>>>>>>
end
```

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\ Axiom \quad \dfrac{}{A \vdash A, B}\ Axiom}{A, (A \to B) \vdash B}\ \to_G}{A, (A \to B), \neg(B) \vdash}\ \neg_G}{(A \to B), \neg(B) \vdash \neg(A)}\ \neg_D}{(A \to B) \vdash (\neg(B) \to \neg(A))}\ \to_D$$

To summarize: Rules in ELAN

Rules are possibly labeled conditional rewrite rules with local variable affectations

$$[\mathsf{lab}] : l(x) \Rightarrow r(y) \quad \textbf{if } v \quad \textbf{where } y := [S]u(x)$$

— lab is the rule label,
— l and r the respective left and right-hand sides,
— v the condition and
— $y := [S]u$ a local affectation, assigning to the local variable y one of the results of the strategy S applied to the term u.

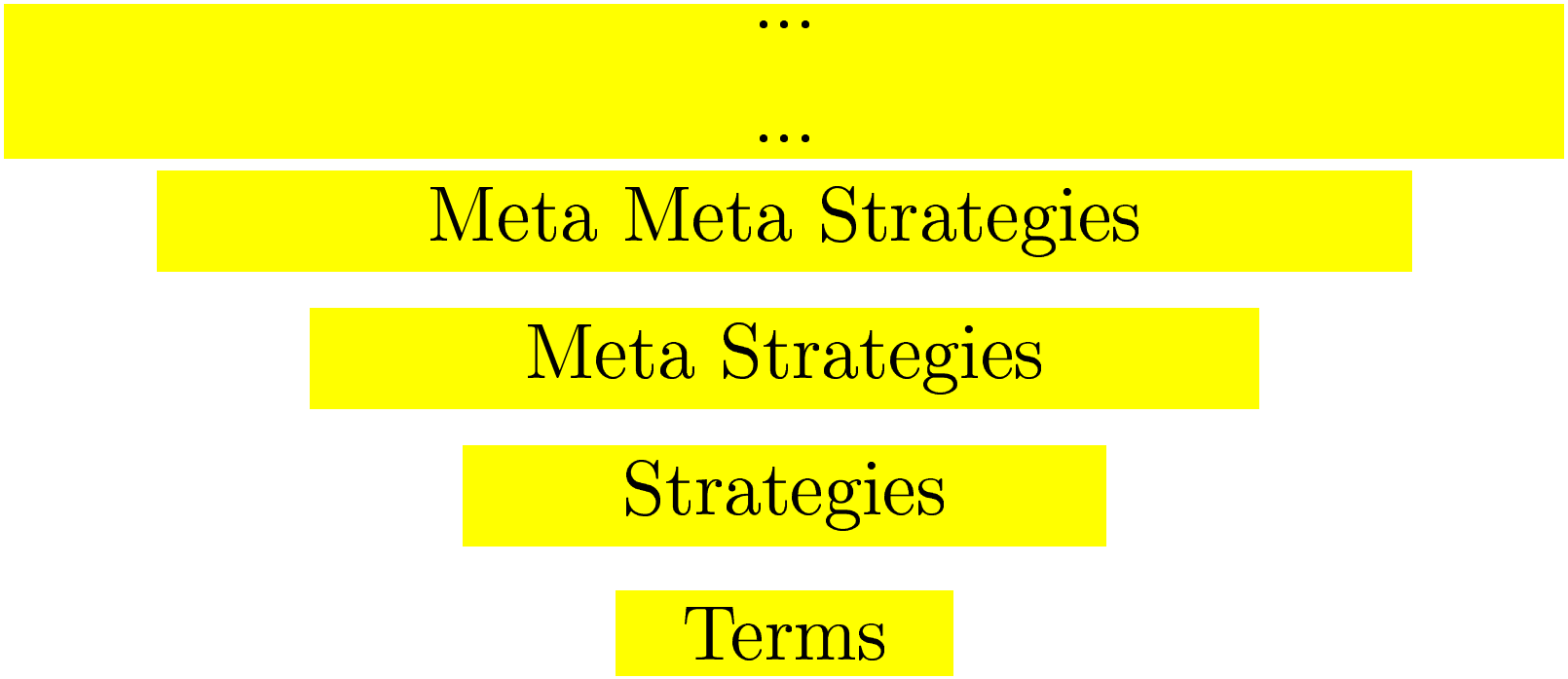
Computation and Deduction in ELAN

Computations are described with unlabeled rules

Deductions are described by labeled rules

Strategies for deductions are themselves described using labeled or unlabeled rewrite rules.

The Computational System Tower



Rewriting with strategies IS now a realistic programming paradigm

On typical applications, ELAN applies per second:

- 10 millions of non-labeled rules
- 1 million of labeled rules
- 100 000 of AC rules

On real size applications:

- Jobs shop 10x10: 2 billions rewrites applied in 2 hours

Some Applications developed in ELAN

- Theorem provers (KBs, Spike, B)
- Automata computations
- CP on finite domains
- Unifications
- Combination of deduction systems
- ...

N-queens

$$queens(n, size) \rightarrow x.ql \text{ if } n > 0$$
$$\text{where } ql := (queens\_strat)queens(n-1, size)$$
$$\text{where } x := (iterate * (dc(range\_rule)))size$$
$$\text{if } noattack(1, x, ql)$$
$$\ldots$$

	Interpreter	Compiler	Compiler
Query	queens(8)	queens(8)	queens(11)
Applied rules	128,949	128,949	19,286,638
Speed (inf/sec)	6,570	2,387,000	2,743,000

- **Note:** local affectations and non-deterministic strategies
- **Note:** efficient management of non-deterministic strategies
- **Speedup:** 360

Completion à la Knuth Bendix

- 225 nonamed rules
- 105 named rules
- 43 strategies (8 dont know, 58 dont care, 6 iterate, 14 repeat)

	Interpreter	Compiler	Compiler
Query	group	group	p8
Applied rules	181,752	181,752	21,607,279
Speed (inf/sec)	1,452	826,145	919,458

- **Note:** conditional rules, local affectations and non-deterministic strategies,...
- **Speedup:** 135

Nat10

Arithmetic rules introduced by Marché et all

- 56 rules rooted by the AC symbol +
- 11 rules rooted by the AC symbol *
- 82 syntactic rules

	Interpreter	Compiler	Compiler
Query	Fib(16)	Fib(16)	Fib(25)
Applied rules	15,384	15,384	1,171,043
Speed (inf/sec)	44	49,625	53,717

Encoding and analysing the Needham-Schroeder public-key protocol in ELAN

Thanks to Horatiu Cirstea

- user-definable number of initiators and responders
- user-definable size of the network
- the strategies guiding the rewrite rules describe a form of model-checking in which all the possible behaviors are explored

Data structures

- The agent

```
@ + @ + @  : ( AgentId SWC Nonce ) Agent;
@          : ( Agent ) listAgent;
@ || @     : ( listAgent  listAgent ) listAgent (AC);
```

- The messages

```
@-->@:@[@,@,@] : ( AgentId AgentId Key Nonce Nonce Address ) message;
@         : ( message ) network;
@ & @     : ( network  network ) network (AC);
```

- The intruders

```
@ # @ # @ : ( AgentId listNonce network ) intruder;
```

- The global state

```
@ <> @ <> @ <> @ : ( listAgent listAgent intruder network ) state;
```

Rewrite rules for the agents

- initiator starts the communication with a responder

```
[initiator-1]
   x+SLEEP+resp  || E <> D <> w#l#ll <> ls                                  =>
   x+WAIT+N(x,y) || E <> D <> w#l#ll <> x-->y:K(y)[N(x,y),DN,A(x)] & ls
                 where (Agent)y+std+init :=(extAgent) elemIA(D || w+SLEEP+DN)
end
```

- responder reads the message and sends the acknowledgement

```
[responder-1]
   E <> y+SLEEP+init  || D <> I <> w-->y:K(y)[N(n1,n3),N(n2,n4),A(z)] & ls =>
   E <> y+WAIT+N(y,z) || D <> I <> y-->z:K(z)[N(n1,n3),N(y,z),A(y)] & ls
end
```

Rewrite rules for the agents

- initiator receives the acknowledgement and checks its validity

```
[initiator-2]
 x+WAIT+N(x,v) || E <> D <> I <> w-->x:K(x)[N(n1,n3),N(n2,n4),A(z)] & ls =>
 S
  choose
    try
      if x==n1 and v==n3
      where S:=() x+COMMIT+N(x,v) || E <> D <> I <> x-->v:K(v)[N(n2,n4),DN,DA] & ls
    try
      if x!=n1 or v!=n3
      where S:=() ERROR
  end
end
```

Rewrite rules for the intruder

- the intruder intercepts all the messages in the network but the messages generated by himself and stores or decrypts them.

```
[intruder-1]
   E <> D <> w#l#ll <> z-->x:K(w)[N(n1,n3),N(n2,n4),A(v)] & ls  =>
   E <> D <> w#N(n1,n3) | N(n2,n4) | l#ll <> ls
                if w!=z
end
```

- the nonces obtained previously by the intruder are used in order to generate fake messages that are sent to all the agents.

```
[intruder-4]
   E <> D <> w # resp | l # ll <> ls                          =>
   E <> D <> w # l # ll <> w-->y:K(y)[resp,DN,A(xadd)] & ls
               where (Agent)y+std+dn :=(extAgent) elemIA(D || E)
               where (Agent)xadd+std1+dn1 :=(extAgent) elemIA(D || E)
end
```

The invariants

- authenticity of the responder: if an initiator `x` committed with a responder `y`, then `y` has really been involved in the protocol.

```
[attack-1] x+COMMIT+N(x,y) || E <> D <> I <> ls =>
           ATTACK
                if y!=i
                if not(existAgent(y+WAIT+N(y,x),D)) and
                   not(existAgent(y+COMMIT+N(y,x),D))
end
```

- authenticity of the initiator: if a responder `y` committed with an initiator `x` then the initiator have committed as well with `y`.

```
[attack-2] E <> y+COMMIT+N(y,x) || D <> I <> ls =>
           ATTACK
                if x!=i
                if not(existAgent(x+COMMIT+N(x,y),E))
end
```

The strategy

We apply repeatedly all the rewrite rules in any order and in all the possible ways until one of the attack rules can be applied.

```
[]attStrat  =>  repeat*(
                   dk(
                      attack-1, attack-2,
                      intruder-1, intruder-2, intruder-3, intruder-4,
                      initiator-1, initiator-2, responder-1, responder-2)
                );
                attackFound
end
```

where

```
[attackFound]   ATTACK   =>   ATTACK              end
```

The attack

I.1.	$A \to I$	$: \{N_A, A\}_{K(I)}$
II.1.	$I(A) \to B$	$: \{N_A, A\}_{K(B)}$
II.2.	$B \to I(A)$	$: \{N_A, N_B\}_{K(A)}$
I.2.	$I \to A$	$: \{N_A, N_B\}_{K(A)}$
I.3.	$A \to I$	$: \{N_B\}_{K(I)}$
II.3.	$I(A) \to B$	$: \{N_B\}_{K(B)}$

A second strategy

```
[]attStrat  =>  repeat*(
                        dk(
                           attack-1, attack-2,
                           initiator-1, initiator-2, responder-1, responder-2,
                           intruder-1, intruder-2, intruder-3, intruder-4)
                );
                attackFound
end
```

I.1.	$A \to I$	: $\{N_A, A\}_{K(I)}$
II.1.	$I(A) \to B$	: $\{N_A, A\}_{K(B)}$
I+II.2.	$B \to A$	: $\{N_A, N_B\}_{K(A)}$
I.3.	$A \to I$	: $\{N_B\}_{K(I)}$
II.3.	$I(A) \to B$	: $\{N_B\}_{K(B)}$

The corrected protocol

1. $A \rightarrow B$: $\{N_A, A\}_{K(B)}$
2. $B \rightarrow A$: $\{N_A, N_B, B\}_{K(A)}$
3. $A \rightarrow B$: $\{N_B\}_{K(B)}$

Modified rule: `initiator-2`

```
[initiator-2] x+WAIT+N(x,v) || E <> D <> I <> w-->x:K(x)[N(n1,n3),N(n2,n4),A(z)] & ls  =>
S
  choose
    try
      if v==z                 // expected responder
      if x==n1 and v==n3
      where S:=() x+COMMIT+N(x,v) || E <> D <> I <> x-->v:K(v)[N(n2,n4),DN,DA] & ls
    try
      if v!=z                 // not expected responder
      if x!=n1 or v!=n3
      where S:=() ERROR
  end                                                                    end
```

Optimizations of the current implementation

- each message is given a type according to its relevant fields (nonce and address),
- initiator and responder listen only to messages coming from the intruder,
- the intruder sends fake messages only to those agents that are able to handle it

Comparison with Murφ

ini	res	net	Murφ	ELAN simple	ELAN(Murφ like)	ELAN optimized
1	1	1	2578 rules 0.56s	7442 rules 0.167s	4308 rules 0.125s	711 rules 0.064s
1	1	2	136273 rules 18.40s	453514 rules 7.007s	333552 rules 4.980s	6700 rules 0.162s
2	1	1	25701 rules 6.81s	575101 rules 8.571s	257087 rules 3.946s	26011 rules 0.483s
2	2	1	557430 rules 303.46s	118214389 rules 1658.296s	22985807 rules 333.096s	753785 rules 12.392s

Concluding remarks on this example

- computational systems can be used as a logical framework for representing the Needham-Schroeder public-key protocol,
- this approach can be easily extended to other authentication protocols (TMN),
- the strategy used for guiding the application of the rewrite rules is important when an attack on the protocol exists,
- the behavior of intruders can be easily modified by changing the corresponding rules,
- the state space is finite and thus, some other methods should be used in order to show that properties proved for the finite model can be lifted to an unbounded model,

How does ELAN compilation work?

- leftmost-innermost normalization
- many-to-one matching
- re-using left-hand-side parts
- shared terms allowed
- particular choice point management

→ What about the semantics?

The rewriting calculus as a semantics of ELAN

Towards a new calculus

That gives a first class status to:

- rewrite rules

and

- strategies

The main ideas (1)

apply a rule at the top level of a term

$$[l \to r](t)$$

The main ideas (2)

The application operator [ ]( ) returns results that should be handled explicitely.

For example, if + is commutative, what is the result of the application of the rule

$x + y \to x$

on

$a + b$

should it be a?
or b?
or $a \diamond b$?

The main ideas (3)

— rule application

— results object

should be explicit objects of the calculus

The calculus ingredients

Five components:

1. The syntax of terms and substitutions,
2. The description of the substitution application on terms,
3. The matching algorithm used to bind variables to their actual arguments,
4. The evaluation rules describing the way the calculus operates locally.
5. The strategy describing how the evaluation rules operate globally.

1 ρ-Calculus Syntax

- elements in $\mathcal{X}$ (variables) and in $\mathcal{T}(\mathcal{F},\mathcal{X})$ are ρ-terms,

If $t, u, t_1, \ldots, t_m$ are ρ-terms and $f \in \mathcal{F}_m$ then the following are ρ-terms:

- $f(t_1, \ldots, t_m)$
- $\{t_1, \ldots, t_m\}$ (if $m = 0$ we have the ρ-term $\emptyset$),
- $[t](u)$ (application of the ρ-term t to the ρ-term u),
- $t \to u$ (rewrite rule).

$$t ::= x \mid \{t, \ldots, t\} \mid f(t, \ldots, t) \mid [t](t) \mid t \to t \qquad (x \in \mathcal{X},\ f \in \mathcal{F})$$

Examples of ρ-terms

• $f(x, y)$	a first order term
• $f(x, x) \to x$	a "standard" rewrite rule
• $[a \to b](a)$	the application of the rule $a \to b$ to the term a
• $[f(x, y) \to g(x, y)](f(a, b))$	a classical rule application
• $[x \to x](a)$	the result is $\{a\}$
• $[x \to y](a)$	the result is $\{y\}$
• $[x \to (x \to x)]([x \to y](a))$	similar to the λ-term $((\lambda x.\lambda x.x)\ ((\lambda x.y)\ a))$
• $[x \to [x](x)](x \to [x](x))$	the well-known $(\Delta\ \Delta)$ λ-term
• $[(a \to b) \to c](a)$	a more complicated ρ-term
• $[\{a \to b, a \to c\}](a)$	apply several rules

Handling results

Handling results via the set datastructure:

- has consequences on the calculus evaluation rules and their properties
- could be done in a different ways using e.g. list, multisets, ...

2 Substitution application on ρ-terms

Takes care of variable capture.

Grafting	Substitution
$x \mapsto u$	x/u
$\{x \mapsto a + y\}(y \to y + x) = y \to y + (a + y)$	$\{x/a + y\}(y \to y + x) = y' \to y' + (a + y)$

4 ρ-Calculus: Evaluation Rules

The main rule in general:

$$\textbf{Fire} \quad [l \to r](t) \quad \mapsto\!\!\!\!\!\rightarrow \quad r\langle\!\langle Solution(l \ll^{?}_{T} t)\rangle\!\rangle$$

For example

$$\textbf{Fire} \quad [f(x) \to x](f(a)) \quad \mapsto\!\!\!\!\!\rightarrow \quad x\langle\!\langle Solution(f(x) \ll^{?}_{T} f(a))\rangle\!\rangle$$

In case of syntactic matching:

$$\begin{array}{lll} \textbf{Fire} \quad [l \to r](\sigma(l)) & \mapsto\!\!\!\!\!\rightarrow & \{\sigma(r)\} \\ \textbf{Fire} \quad [l \to r](t) & \mapsto\!\!\!\!\!\rightarrow & \emptyset \\ & & \textsf{if } l \text{ does not match } t \end{array}$$

Congruence rules on operators

Congruence	$[f(u_1,\ldots,u_n)](f(v_1,\ldots,v_n))$	$\mapsto\!\!\!\!\rightarrow$	$\{f([u_1](v_1),\ldots,[u_n](v_n))\}$
Congruence_fail	$[f(u_1,\ldots,u_n)](g(v_1,\ldots,v_m))$	$\mapsto\!\!\!\!\rightarrow$	$\emptyset$

Equivalent to the reduction of

$[f(x_1,\ldots,x_n) \rightarrow f([u_1](x_1),\ldots,[u_n](x_n))](t)$

Congruence rules on sets

Distrib	$[\{u_1, \ldots, u_n\}](v)$	$\mapsto\!\!\!\!\rightarrow$	$\{[u_1](v), \ldots, [u_n](v)\}$
Batch	$[v](\{u_1, \ldots, u_n\})$	$\mapsto\!\!\!\!\rightarrow$	$\{[v](u_1), \ldots, [v](u_n)\}$
SwitchL	$\{u_1, \ldots, u_n\} \to v$	$\mapsto\!\!\!\!\rightarrow$	$\{u_1 \to v, \ldots, u_n \to v\}$
SwitchR	$u \to \{v_1, \ldots, v_n\}$	$\mapsto\!\!\!\!\rightarrow$	$\{u \to v_n, \ldots, u \to v_n\}$
OpOnSet	$f(v_1, \ldots, \{u_1, \ldots, u_m\}, \ldots, v_n)$	$\mapsto\!\!\!\!\rightarrow$	$\{f(v_1, \ldots, u_1, \ldots, v_n),$ $\ldots,$ $f(v_1, \ldots, u_m, \ldots, v_m)\}$

Handling sets of sets

Flat $\{u_1, \ldots, \{v_1, \ldots, v_n\}, \ldots, u_m\} \quad \twoheadrightarrow \quad \{u_1, \ldots, v_1, \ldots, v_n, \ldots, u_m\}$

Applying substitutions

Meta-rules in this version

$$\begin{array}{lll} \textbf{Propagate} & r\langle\langle\{s_1,\ldots,s_n\}\rangle\rangle & \rightsquigarrow \quad \{s_1(r),\ldots,s_n(r)\} \\ \textbf{Clash} & r\langle\langle\emptyset\rangle\rangle & \rightsquigarrow \quad \emptyset \end{array}$$

Could be made explicit: $\rho\sigma$-calculus

Remarks

- ρ-calculus describes rewrite rules application at the top level of a term

$$[f(x) \to g(x)](f(b)) \underset{Fire}{\mapsto\!\!\!\!\!\!\rightarrow} \{g(b)\}$$

but

$$[x + 0 \to x](f(3 + 0)) \underset{Fire}{\mapsto\!\!\!\!\!\!\rightarrow} \emptyset$$

and

$$[f(x + 0 \to x)](f(3 + 0)) \underset{Congruence}{\mapsto\!\!\!\!\!\!\rightarrow} f([x + 0 \to x](3 + 0)) \underset{Fire}{\mapsto\!\!\!\!\!\!\rightarrow} f(\{3\})$$

- The evaluation rules expressing the behavior of ρ-calculus are applied *everywhere* in the term (e.g. like β-reduction).

3 Matching

In general

For a theory T a *T-match-equation* is a formula of the form $t \ll^?_T t'$, where t and t' are ρ-terms.

A substitution σ is a T-solution of $t \ll^?_T t'$ when

$$\sigma(t) =_T t'$$

T-matching is in general undecidable.

Decidable cases:

- higher-order pattern matching [Miller-89]
- higher-order matching (up to the fourth order [PadovaniThese-96])
- many first-order equational theories and their combinations

Solutions set

We define the function $\mathcal{S}olution$ on a T-matching system $\mathcal{S}$ as returning the set of:

— all T-matches of $\mathcal{S}$ when $\mathcal{S}$ is not trivial

— $\{Id\}$ (the identity substitution) when $\mathcal{S}$ is trivial

— $\emptyset$ when the matching algorithm fails.

Specificities of the ρ_T-calculus

- calculus of explicit rule application
- get the "matching power"
- explicit handling of result sets,
- one of the main parameter of the calculus is the matching algorithm,

- allows the full control of the rewrite rule application
- allows a uniform combination of higher-order and first-order features,
- allows expressing strategies (e.g. search strategies)

- the ρ-calculus is quite different from the rewrite relation generated by a trs.

The $\rho_\emptyset$-calculus

An instance of ρ_T-calculus where:

— $T = \emptyset$

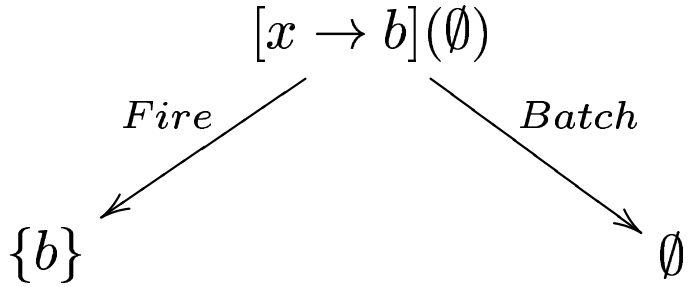
— the left members of matching equations are composed only of first-order terms (i.e. not containing arrows or applications).

From now on we only consider $\rho_\emptyset$-calculus.

No difficulty to extend to the case of first-order equational matching (e.g. AC-matching)

Technical specificities

- A rule application always fires ... but the result may be the empty set (when the rule does not match an under-evaluated term),
- Thus the wild $\rho_\emptyset$-calculus is trivialy non-confluent:



ConfStratStrict

The term $[l \to r](t)$ is reduced using the evaluation rule **Fire** only if $t \in \mathcal{T}(\mathcal{F})$.

ConfStrat

The term $[l \to r](t)$ is reduced using the evaluation rule **Fire** only if:

- the term l is linear,
- the term l weakly subsumes the term t ($\mathcal{FP}os(l) \subseteq \mathcal{FP}os(t)$),
- the term t contains no empty set and no set with more than one element,
- the term t contains no sub-term of the form $[u](v)$ where u is not a rewriting rule,
- for all sub-term $[u \to w](v)$ of t, u subsumes v,

  or

- $t \in \mathcal{T}(\mathcal{F})$ is a closed first order term.

ConfStratOper

The term $[l \to r](t)$ is reduced using the evaluation rule **Fire** only if $t \in \mathcal{T}(\mathcal{F})$ or:

- the term l is linear and
- the term l weakly subsumes the term t

  and

  - the rewrite rule $l \to r$ is conservative ($FV(l) \subseteq FV(r)$)

    or

  - the term t contains no empty set
  - the term t contains no sub-term of the form $[u](v)$ where u is not a rewriting rule,
  - for all sub-term $[u \to w](v)$ of t, u subsumes v,

  and

  - the rewrite rule $l \to r$ is right linear

    or

  - the term t contains no set with more than one element.

When using one of the strategies *ConfStratStrict*, *ConfStrat* or *ConfStratOper*, the $\rho_{\emptyset}$-calculus in confluent.

About expressiveness of the ρ-calculus

Encoding the λ-calculus

$t ::= x \mid \{t\} \mid f(t, \ldots, t) \mid [t](t) \mid x \to t$

FirePropagate $[x \to r](t) \quad \mapsto\!\!\!\!\!\rightarrow \quad \{\{x/t\}r\}$

$\varphi(\lambda x.t) = x \to t$

$\varphi((s\ t)) = [s](t)$

Given t and t' two λ-terms. Then, $t \longrightarrow_\beta t'$ iff $\varphi(t) \longrightarrow_\rho \{\varphi(t')\}$.

Proof: For any λ-term $(\lambda x.t\ u)$ that reduces to $\{x/u\}t$ the corresponding ρ-reduction is $[x \to t](u) \longrightarrow^*_{FirePropagate} \{\{x/u\}t\}$. $\square$

Encoding rewriting

$t ::= x \mid \{t\} \mid f(t, \ldots, t) \mid [t](l) \mid l \to l$

Given t and t' two terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\mathcal{R}$ a first order term rewrite system.

If $t \longrightarrow^*_{\mathcal{R}} t'$ then there exists a ρ-term u constructed using the rules in $\mathcal{R}$ such that $[u](t) \longrightarrow^*_{\rho} \{t'\}$.

Proof: Let us consider a rewrite rule $(l \to r)$ that applies on the term t at position p and thus, $t[\theta l]_p \longrightarrow_{\mathcal{R}} t[\theta r]_p = t'$.

The ρ-term u to be applied in the ρ-calculus is $t[l \to r]_p$ and we get:

$$[t[l \to r]_p](t[\theta l]_p) \longrightarrow^*_{\rho} \{t[\theta r]_p\}$$

□

(Normalization) Strategies

Given a rewrite theory $\mathcal{R}$ does it exist a ρ-term $\xi_{\mathcal{R}}$ such that for any term u, if u reduces to the term v in the rewrite theory $\mathcal{R}$ then $[\xi_{\mathcal{R}}](u)$ ρ-reduces to $\{\ldots, v, \ldots\}$?

Given a rewrite theory $\mathcal{R}$ does it exist a ρ-term $\xi_{\mathcal{R}}$ such that for any term u, if u normalizes to the term v in the rewrite theory $\mathcal{R}$ then $[\xi_{\mathcal{R}}](u)$ ρ-normalizes to $\{\ldots, v, \ldots\}$?

Extending the calculus: with syntactic sugar

$$t ::= id \mid fail \mid t;t \mid dk(t,t)$$

Evaluation rules

Identity	id	$\mapsto\!\!\!\!\rightarrow$	$x \rightarrow x$
Fail	$fail$	$\mapsto\!\!\!\!\rightarrow$	$x \rightarrow \emptyset$
Compose	$[s_1; s_2](t)$	$\mapsto\!\!\!\!\rightarrow$	$[s_2]([s_1](t))$
DK	$[dk(s_1, s_2)](t)$	$\mapsto\!\!\!\!\rightarrow$	$\{[s_1](t), [s_2](t)\}$

The *first* operator

First	$[first(s_1, \ldots, s_n)](t)$	$\mapsto\!\!\!\!\rightarrow$	$\langle [s_1](t), \ldots, [s_n](t) \rangle$
First_fail	$\langle \emptyset, t_1, \ldots, t_n \rangle$	$\mapsto\!\!\!\!\rightarrow$	$\langle t_1, \ldots, t_n \rangle$
First_success	$\langle t, t_1, \ldots, t_n \rangle$	$\mapsto\!\!\!\!\rightarrow$	$\{t\}$
			if $t \neq \emptyset$ *is closed and contains no redex*
First_single	$\langle\rangle$	$\mapsto\!\!\!\!\rightarrow$	$\{\}$

Term traversal

Tseq	$[\Phi(r)](f(u_1,\ldots,u_n))$	$\mapsto\!\!\!\!\gg$	$\langle\{f([r](u_1),\ldots,u_n)\},\ldots,\{f(u_1,\ldots,[r](u_n))\}\rangle$
Tseq_ct	$[\Phi(r)](c)$	$\mapsto\!\!\!\!\gg$	$\emptyset$
Tpar	$[\Psi(r)](f(u_1,\ldots,u_n))$	$\mapsto\!\!\!\!\gg$	$\{f([r](u_1),\ldots,[r](u_n))\}$
Tpar_ct	$[\Psi(r)](c)$	$\mapsto\!\!\!\!\gg$	$\{c\}$

Fix point operators

$$\Theta = [A](A) \quad with \quad A = x \rightarrow (y \rightarrow [y]([[x](x)](y)))$$

Thus

$$[\Theta](G) \xrightarrow{*}_{\rho} \{[G]([\Theta](G))\}$$

Bottom-up application

$$Once_{bu}(r) \equiv [\Theta](H_{bu}(r))$$

where

$$H_{bu}(r) \equiv f \rightarrow (x \rightarrow [first(\Phi(f), r)](x))$$

Ex: $[Once_{bu}(a \rightarrow b)](f(a, g(a))) \Longrightarrow_{\rho} \{f(b, g(a))\}$

Repeat operator

$$repeat*\,(r) \equiv [\Theta](J(r))$$

where

$$J(r) \equiv f \rightarrow (x \rightarrow [first(r; f, id)](x))$$

Normalization strategies

$$im(r) \equiv repeat*\,(Once_{bu}(r))$$

$$om(r) \equiv repeat*\,(Once_{td}(r)).$$

Encoding Conditional Rewriting

The conditional rewrite rule:

$$l \to r \text{ if } c$$

is simply encoded by

$$l \to [\{\mathbf{T} \to r, \mathbf{F} \to \emptyset\}]([lmim](c))$$

Or even simpler

$$l \to [\mathbf{T} \to r]([lmim](c))$$

ELAN's semantics

Using rewriting logic [Meseguer 92]

Using ρ-calculus

The ELAN basic object: rewrite rules with where

Rules are labelled conditional rewrite rules with local variable affectations

$$[\mathsf{lab}] \quad l \Rightarrow \quad r(x,y) \quad \textbf{where} \;\; x := [S_1]u_1 \quad \textbf{if} \;\; c_1(x) \quad \textbf{where} \;\; y := [S_2]u_2 \quad \textbf{if} \;\; c_2(x,y)$$

— lab is the rule label,

— l and r are the respective left and right-hand sides,

— c_1, c_2 are the conditions and

— $z := [S]u$ are local affectations, assigning to the local variable z one of the result of the strategy S applied to the term u.

Example of ELAN's rule

$$[\text{deriveSum}] \quad p_1 + p_2 \quad \Rightarrow \quad p_1' + p_2'$$
$$\text{where } p_1' := (derive)p_1$$
$$\text{where } p_2' := (derive)p_2$$

can be represented by the following two ρ-terms

$$p_1 + p_2 \to [derive](p_1) + [derive](p_2)$$

$$p_1 + p_2 \to [p_1' \to [p_2' \to p_1' + p_2'][derive](p_2))][derive](p_1))$$

An expression of ELAN's rules in ρ-calculus

$$[\mathsf{lab}] \quad l \Rightarrow \quad r(x) \quad \textbf{where} \;\; x := [S_1]u_1 \quad \textbf{if} \;\; c_1(x)$$

The rule is expressed as the ρ-term:

$$l \;\rightarrow\; [x \rightarrow [True \rightarrow r(x)]([lmim](c_1(x)))]([S_1](u_1))$$

With two `where if`

$$[\mathsf{lab}] \quad l \Rightarrow \quad r(x) \ \textbf{where}\ x := [S_1]u_1 \ \textbf{if}\ c_1(x) \ \textbf{where}\ y := [S_2]u_2 \ \textbf{if}\ c_2(x,y)$$

The rule is expressed as the ρ-term:

$$\begin{array}{lll} l \rightarrow & [x \rightarrow [& True \rightarrow [\quad y \rightarrow [True \rightarrow r(x,y)]([lmim](c_2(x,y))) \\ & & \quad]([S_2](u_2)) \\ & \quad]([lmim](c_1(x))) & \\ \quad]([S_1](u_1)) & & \end{array}$$

To sum-up

- Rule based programming a la ELAN allows to integrate the functional, transitional and nondeterministic programmation paradigms
- ELAN makes exactly the *deduction modulo* concept at work
- It is now practical to use thanks to the compilation techniques developped (Moreau, Vittek)
- A relatively large number of programs have been developped in ELAN (Needham-Schroeder, CSP, TP) but also in other rule based programming environments (ASF+SDF, Maude, OBJ, Stratego, ...)

→ The programming methodology is still to be better understood

→ Connection with other systems at work (COQ-ELAN, CoFI, ASF+SDF parsing and editing advanced technology)

- The works on ELAN's semantics give rise to a new appealing calculus
- ρ_T-calculus is a simple, elegant and powerful new calculus
  - explicit rule application (and thus "matching power")
  - explicit handling of result sets
  - paramaterized by a matching theory T
- It makes a clear distinction between the rewrite *relation* and the rewrite *calculus*
- It allows uniform combination of first and higher-order computations
- ρ-calculus gives a simple semantics to ELAN programs
- The simply typed ρ-calculus

→ Models of the ρ-calculus

→ The ρ-typed ρ-calculus

More on 

`http://www.loria.fr/ELAN`

Current release: V3.4

The ELAN team:
Peter Borovanský, Horatiu Cirstea, Hubert Dubois,
Claude Kirchner, Hélène Kirchner, Pierre-Etienne Moreau,
Huy Nguyen, Christophe Ringeissen, Marian Vittek.