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# YET ANOTHER INTRODUCTION TO ROUGH PATHS

ANTOINE LEJAY

ABSTRACT. This article provides another point of view on the theory of rough paths, which starts with simple considerations on ordinary integrals, and endows the importance of the Green-Riemann formula, as in the work of D. Feyel and A. de La Pradelle. This point of view allows us to introduce gently the required algebraic structures and provides alternative ways to understand why the construction provided by T. Lyons *et al.* is a natural generalization of the notion of integral of differential forms, in the sense it shares the same properties as integrals along smooth paths, when we use the “right notion” of path.

## 1. INTRODUCTION

The theory of rough paths [Lyo98, LQ02, Lej03a, LCL06] is now an active field of research, especially among the probabilistic community. Although this theory is motivated by stochastic analysis, it takes its roots in analysis and control theory, and is connected also to differential geometry and algebra.

Given a path  $x$  of finite  $p$ -variation with  $p \geq 2$  on  $[0, T]$  with values in  $\mathbb{R}^d$  or an  $\alpha$ -Hölder continuous path with  $\alpha \leq 1/2$ , this theory allows us to define the integral  $\int_x f$  of a differential form  $f$  along  $x$ , which is  $\int_x f = \int_0^T f(x_s) dx_s$ . Using a fixed point theorem, it is then possible to solve differential equations driven by  $x$  of type

$$y_t = y_0 + \int_0^t g(y_s) dx_s.$$

The case where  $p < 2$  (or  $\alpha > 1/2$ ) is covered by the Young integrals introduced by L.C. Young in [You36]. Some of the most common stochastic processes, including the Brownian motion, have trajectories that are of finite  $p$ -variation with  $p > 2$ . So, being able to define almost surely an integral along such irregular paths is of great practical interest, both theoretically and numerically. Yet we know it is not possible

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in general, and integrals of type Itô and Stratonovich are defined only as limits in probability of Riemann sums.

Introduced in the 50's by K.-T. Chen (see for example [Che58]), the notion of iterated integrals provides an algebraic tool to deal with a geometrical object which is a smooth path, and allows us to manipulate controlled differential equations using formal computations (see for example [Fli81, Isi95]).

The main feature of the rough paths theory is then to assert that, if it is possible to consider not only a path  $x$  but a path  $\mathbf{x}$  which encodes the iterated integrals (that cannot be canonically defined if  $x$  is of finite  $p$ -variation with  $p \geq 2$ ), then one may properly define the integral  $z_t = \int_0^t f(x_s) d\mathbf{x}_s$  and solve the differential equation  $y_t = y_0 + \int_0^t g(y_s) d\mathbf{x}_s$  provided that  $f$  and  $g$  are smooth enough. In addition, the maps  $\mathbf{x} \mapsto z$  and  $\mathbf{x} \mapsto y$  are continuous, with respect to the topology induced by the  $p$ -variation distance. The dimension of the path  $\mathbf{x}$ , or equivalently the number of “iterated integrals” to consider, depends on the regularity of  $x$ . For  $p \in [2, 3)$  (or  $\alpha \in (1/3, 1/2]$ ), then one has to consider only the iterated integrals of  $x$  along itself. This can be justified by the first order Taylor development of  $\int_s^t f(x_r) dx_r$ :

$$\begin{aligned} \sum_{i=1}^d \int_s^t f_i(x_r) dx_r^i \\ \approx \sum_{i=1}^d f_i(x_s)(x_t^i - x_s^i) + \sum_{i,j=1}^d \frac{\partial f_i}{\partial x_j}(x_s) \int_s^t (x_r^j - x_s^j) dx_r^i. \end{aligned}$$

If  $x$  is  $\alpha$ -Hölder continuous with  $\alpha \in (1/3, 1/2]$  and one has succeeded in constructing  $K_{s,t}^{i,j}(x) = \int_s^t (x_r^j - x_s^j) dx_r^i$ , then one can expect that  $|K_{s,t}^{i,j}(x)| \leq C|t-s|^{2\alpha}$ . Hence, we will use to approximate  $\int_0^T f(x_r) dx_r$  the sums

$$\begin{aligned} \sum_{k=0}^{n-1} \sum_{i=1}^d f_i(x_{kT/n})(x_{(k+1)T/n}^i - x_{kT/n}^i) \\ + \sum_{k=0}^{n-1} \sum_{i,j=1}^d \frac{\partial f_i}{\partial x_j}(x_{kT/n}) K_{kT/n, (k+1)T/n}^{i,j}(x) \end{aligned}$$

and show it converges as  $n \rightarrow \infty$ . Hence, the integral will be defined not along a path  $x$ , but along  $\mathbf{x}_{s,t}$  given by

$$\mathbf{x}_{s,t} = (1, x_t^1 - x_s^1, \dots, x_t^d - x_s^d, K_{s,t}^{1,1}(x), \dots, K_{s,t}^{d,d}(x)),$$

where the first component 1 is here for algebraic reasons. The element  $\mathbf{x}$  can be seen as an element of the truncated tensor space  $T(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ . By similarity with what happens for the power series constructed from the iterated integrals — sometimes called

the *signature* of the path —, one has that for all  $0 \leq s \leq r \leq t \leq T$ ,

$$\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t},$$

where  $\otimes$  is the tensor product on  $T(\mathbb{R})$  (where we keep only the tensor products of no more than 2 terms). In addition, it is possible to consider the formal logarithm of  $\mathbf{x}$ , and following also the properties of the Chen series, we look for paths  $\mathbf{x}$  such that  $\log(\mathbf{x}_{s,t})$  belongs to  $A(\mathbb{R}^d) = \mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d]$ , where  $[\mathbb{R}^d, \mathbb{R}^d]$  is the space generated by all the Lie brackets between two elements of  $\mathbb{R}^d$ . This algebraic property allows us to give proper definitions of rough paths and geometric rough paths from an algebraic point of view. The articles [Lyo98, Lej03a] and the books [LQ02, LCL06] use this point of view.

As noted first by N. Victoir, since  $(T_1(\mathbb{R}^d), \otimes)$  — the subset of  $T(\mathbb{R}^d)$  whose element have a first term equal to 1 — is a Lie group, one may describe  $\mathbf{x}_{s,t}$  by  $\mathbf{x}_{s,t} = (-\mathbf{x}_{0,s})^{-1} \otimes \mathbf{x}_{0,t}$ , and then, instead of considering the family  $(\mathbf{x}_{s,t})_{0 \leq s < t \leq T}$ , one may work with the path  $\mathbf{x}_t = \mathbf{x}_{0,t}$ , which lives in the non-commutative space  $(T_1(\mathbb{R}^d), \otimes)$ . This provides some simplifications on the statement of some theorems, but also opens the door to look for more connections with differential geometry.

Shortly after the publication of the article [Lyo98], other authors provided alternative constructions of the differential equations and integrals, still by using some of the ideas provided by the theory of rough paths. One of this work, from D. Feyel and A. de la Pradelle [FdLP05], uses a point of view from the differential geometry and endows the importance of the Gauss/Green-Riemann/Stokes formula to understand the need to “enhanced” the path with more information to get a rigorous definition. Another approach, by M. Gubinelli, rather relies on algebraic considerations [Gub04].

The idea of this article is then to justify the construction of the algebraic structures (tensor space, Lie groups) needed in the theory of rough paths from basic considerations on integrals of differential forms. To simplify, we consider that the dimension  $d$  of the state space is  $d = 2$  (for  $d = 1$ , there is no real problem, since any differential form is the differential of a function and the controlled differential equation  $y_t = g(y_t) dx_t$  is solved under reasonable assumptions on  $g$  by  $y_t = \Phi(x_t)$  with  $\Phi'(z) = \sigma(\Phi(z))$  if  $x$  is smooth and a density argument may then be used: On that topic, see for example the work of Doss and Sussmann [Dos77]). By considering all the pairs of components, it is easy to pass from  $d = 2$  to  $d > 2$ . In addition, we restrict ourselves to  $\alpha$ -Hölder continuous paths, which is not a stringent assumption at all, since a time change allows us to transform any path with  $p$ -finite variation into a path which is  $1/p$ -Hölder continuous.

Given a differential form, we wish to construct a map  $x \mapsto \int_x f$  which is continuous on the space  $C^\alpha$  of  $\alpha$ -Hölder continuous paths. If  $\alpha > 1/2$ , the existence of  $\int_x f$  is provided by the theory of Young integrals. We

also get that  $x \mapsto \int_x f$  is continuous on  $C^\alpha$  equipped with the  $\alpha$ -Hölder norm. Yet we construct some sequence  $(x^n)_{n \in \mathbb{N}}$  of functions in  $C^\alpha$  that converges to  $x$  in  $C^\beta$  with  $\beta < 1/2$ , and such that  $\int_0^T f(x_s^n) dx_s^n$  does not converges to  $\int_x f$ , but to  $\int_0^T f(x_s) dx_s + \int_0^T [f, f](x_s) d\varphi_s$  where  $[f, f]$  is the Lie bracket of  $f$  and  $\varphi$  is an arbitrary function. This counter-examples makes use of the Green-Riemann functions, and see that, if one consider not a path  $x$ , but a path  $(x, \varphi)$  with values in  $\mathbb{R}^3$ , then one can extend the notion of the integral to  $C^\alpha$  with  $\alpha \in (1/3, 1/2]$ . In some sense, the third component records the area enclosed between that path and its chord between times  $s$  and  $t$ . We can then provided an algebraic setting for describing such paths, still with a non-commutative operation. Then, we construct paths with values in  $A(\mathbb{R}^2)$ , a space of dimension 3, where the first two coordinates corresponds to an “ordinary” path in the Euclidean vector space  $\mathbb{R}^2$ . The non-commutativity comes from the fact that the area enclosed between  $x \cdot y$  — the concatenation of two paths  $x$  and  $y$  — and its chord is different from the area enclosed between  $y \cdot x$  and its chord. The degree of freedom we gain comes from the fact that small loops allows us to move in the third direction while staying at the same place. Any  $\alpha$ -Hölder continuous path with values in  $A(\mathbb{R}^2)$  (with the right distance) with  $\alpha > 1/3$  may be approximated by smooth paths lifted in  $A(\mathbb{R}^2)$  using their area. In addition, the convergence of paths with values in  $A(\mathbb{R}^2)$  in the  $\alpha$ -Hölder topology implies that the corresponding integrals form a Cauchy sequence in  $C^\beta$  for any  $\beta < \alpha$ . It is then possible to extend the notion of Young integrals to  $\alpha$ -Hölder continuous functions with values in  $A(\mathbb{R}^2)$ , and also to get the continuity result we need.

The basic idea to approximate some  $\alpha$ -Hölder continuous path  $\mathbf{x}$  taking its values in  $A(\mathbb{R}^2)$  with  $\alpha > 1/3$  consists in lifting paths  $x^n$  that take the same values as  $\mathbf{x}$  on the points of a partition of  $[0, T]$  and that links two successive times by a loop and a straight line. The loop is a way to “encode the area”. It may then be tempting to look for real geodesics. For this, we will interpret the space  $A(\mathbb{R}^2)$  as the subspace of the tangent space at any point of the tensor space  $T(\mathbb{R}^2)$ , and we will look for simple curves linking two points in  $T(\mathbb{R}^2)$ . There are several possibilities. One consists in using the tools from the sub-Riemannian geometry [FV06c, FV06a]. Another one consists in studying paths with values in a sub-manifold  $G(\mathbb{R}^2)$  of  $T(\mathbb{R}^2)$ , which is also a subgroup of  $(T(\mathbb{R}^2), \otimes)$ , and which is the Lie group corresponding to  $A(\mathbb{R}^2)$  seen as a Lie algebra. We give another way to define the integral by extending the differential form  $f$  to a differential form on  $G(\mathbb{R}^2)$  and construct geodesics that connect two points of  $G(\mathbb{R}^2)$ . Hence, instead of considering paths with values in  $A(\mathbb{R}^2)$ , we will consider paths with values in  $G(\mathbb{R}^2)$ , and the difference between two points in  $A(\mathbb{R}^2)$  corresponds then to a direction.

With this, we may redefine the integral as the limit of some Riemann sums, but where the addition has been replaced by some tensor product. Moreover, it becomes then possible to extend the notion of integrals to paths living in the bigger space  $T_1(\mathbb{R}^2)$ .

Consequently, using the concept of path living in a non-commutative space, the rough path theory provides a way to define an integral  $\int f(x_s) dx_s$  that shares the same properties as ordinary integrals:

- (a) It is a limit of expressions similar to Riemann sums.
- (b) It is a limit of integrals along approximations of the path construct from sampling the path at a finite set of points and connecting successive sample points by “simple” curves.

In addition, this map  $x \mapsto \int f(x_s) dx_s$  is continuous from  $C^\alpha([0, T]; T_1(\mathbb{R}^2))$  to  $C^\alpha([0, T]; T_1(\mathbb{R}^2))$  and may be used to solve differential equations driven by  $x$ , still with a continuity property.

The theory of rough paths turns out to be the natural extension of integrals on the space of  $\alpha$ -Hölder continuous paths with  $\alpha \in (1/3, 1/2]$ , in the same way Young integrals is the natural notion of integral against  $\alpha$ -Hölder continuous paths with  $\alpha \in (1/2, 1]$ .

**Outline.** In Section 2, we introduce our notations and recall some elementary facts about integrals of differential forms along smooth paths as well as about Hölder continuous paths. In section 3, we quickly present results about Young integrals, and thus show the properties of integrals along  $\alpha$ -Hölder continuous paths with  $\alpha > 1/2$ . In Section 4, we assume that one can integrate differential forms along  $\alpha$ -Hölder continuous path with  $\alpha \in (1/3, 1/2]$ , and we show how to transform this integral into a continuous one with respect to the path. In Section 5, we consider paths taking their values in  $A(\mathbb{R}^2)$ , and show how to define the integral  $\int_x f$  as limits of ordinary integrals. In Section 6, we continue our analysis of the space  $A(\mathbb{R}^2)$  and introduce the tensor space  $T(\mathbb{R}^2)$ . In Section 7, we give another way to define the integral of  $f$  along  $x$ , using an expression of Riemann sum type. This construction corresponds to the original one of T. Lyons [Lyo98, LQ02, LCL06]. In Section 8, we give some related results: case of the  $d$ -dimensional space, Chen series, other constructions for paths with quadratic variation, link with stochastic integrals. In Section 9, we solve differential equations. We end this article with appendix on the Heisenberg group and we recall a technical result about almost rough paths, on which the original construction of  $\int_x f$  is based.

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## 2. NOTATIONS

**2.1. Differential forms.** Let  $f_1, \dots, f_d$  be some functions from  $\mathbb{R}^d$  to  $\mathbb{R}^m$ . We consider the differential form

$$f(x) = f_1(x^1) dx^1 + \dots + f_d(x^d) dx^d$$

on  $\mathbb{R}^d$ .

**Definition 1.** For  $\gamma > 0$ ,  $f$  is said to be  $\gamma$ -Lipschitz if the  $f_i$  for  $i = 1, \dots, d$  are of class  $C^{\lfloor \gamma \rfloor}(\mathbb{R}^d; \mathbb{R}^m)$  with bounded derivative up to order  $\gamma$ , and the  $f_i^{\lfloor \gamma \rfloor}$  if  $(\gamma - \lfloor \gamma \rfloor)$ -Hölder continuous with a  $\gamma$ -Hölder constant  $H_i^\gamma$ . The class of such  $\gamma$ -Lipschitz differential forms is denoted by  $\text{Lip}(\gamma; \mathbb{R}^d \rightarrow \mathbb{R}^m)$ .

For  $f \in \text{Lip}(\gamma; \mathbb{R}^d \rightarrow \mathbb{R}^m)$ , define

$$\|f\|_{\text{Lip}} = \max_{i=1, \dots, d} \max\{\|f_i^{(0)}\|_\infty, \dots, \|f_i^{(\lfloor \gamma \rfloor)}\|_\infty, H_i^\gamma\},$$

which is a norm on  $\text{Lip}(\gamma; \mathbb{R}^d \rightarrow \mathbb{R}^m)$ .

*Remark 1.* If  $\gamma = 1$ , this definition is slightly different from the notion of Lipschitz functions, since this definition implies that  $f$  is of class  $C^1(\mathbb{R}^d; \mathbb{R}^m)$ , while with the definition that  $|f(x) - f(y)|/|x - y|$  is bounded as  $x \rightarrow y$  for all  $y \in \mathbb{R}^d$ , this means only that  $f$  is almost everywhere differentiable. Anyway, in our context, the case  $\gamma \in \mathbb{N}$  is never considered.

Given a path  $x \in C^1([0, T]; \mathbb{R}^d)$  and a continuous differential form  $f$ , we define the *integral of  $f$  along  $x$*  by

$$\int_x f = \int_0^T f(x_s) \frac{dx}{dt} \Big|_{t=s} ds = \sum_{i=1}^d \int_0^T f_i(x_s) \frac{dx^i}{dt} \Big|_{t=s} ds.$$

Let us recall a few facts on such integrals, that will be heavily used:

- (i) If  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is strictly increasing and continuous, then  $\int_{x \circ \varphi} f = \int_x f$ . In other words, the integral of  $f$  along  $x$  does not depend on the parametrization of  $x$ .
- (ii) If  $\varphi : [0, T] \rightarrow [0, T]$  is  $\varphi(t) = T - t$ , then  $\int_{x \circ \varphi} f = -\int_x f$ . In other words, reversing the time changes the sign of  $\int_x f$ .
- (iii) If  $x, y \in C_p^1([0, T]; \mathbb{R}^d)$  (the class of functions from  $[0, T]$  to  $\mathbb{R}^d$  which are piecewise in  $C^1$ ) and  $x \cdot y$  is the concatenation of  $x$  and  $y$ , then  $\int_{x \cdot y} f = \int_x f + \int_y f$ . This is the *Chasles relation*.
- (iv) If  $x \in C_p^1([0, T]; \mathbb{R}^2)$  is a closed loop in  $\mathbb{R}^2$ , that is  $x_T = x_0$ , then

$$(1) \quad \int_x f = \iint_{\text{Surface}(x)} [f, f](x^1, x^2) dx^1 dx^2,$$

where  $\text{Surface}(x)$  is the oriented surface surrounded by  $x$  and

$$[f, f] = \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}.$$

This is the *Green-Riemann/Stokes/Gauss formula*.

**2.2. Paths of finite  $p$ -variation.** Fix  $T > 0$ . Let  $x$  be a continuous path from  $[0, T]$  to  $\mathbb{R}^d$  and  $\Pi = \{t_i\}_{i=0, \dots, k}$  be a partition of  $[0, T]$  with  $k$  elements. For  $p \geq 1$ , define

$$\mathfrak{P}(x; \Pi, p) = \sum_{i=0}^{k-1} |x_{t_{i+1}} - x_{t_i}|^p.$$

The  $p$ -variation of  $x$  on  $[s, t] \subset [0, T]$  is defined by

$$\text{Var}_{p, [s, t]}(x) = \sup_{\Pi \text{ partition of } [0, T]} \mathfrak{P}(x|_{[s, t]}; \Pi \cap [s, t], p)^{1/p}.$$

**Definition 2.** A function  $x : [0, T] \rightarrow \mathbb{R}^d$  is said to be of *finite  $p$ -variation* if  $\text{Var}_{p, [0, T]}(x)$  is finite.

If  $x$  is of finite  $p$ -variation, then we get easily that

$$(2) \quad \text{Var}_{q, [0, T]}(x) \leq 2^{(q-p)/q} \|x\|_{\infty}^{(q-p)/q} (\text{Var}_{p, [0, T]}(x))^{p/q}$$

and then  $x$  is of finite  $q$ -variation for all  $q > p$ . Note that  $\text{Var}_{p, [0, T]}(x)$  defines a semi-norm on the space of functions of finite  $p$ -variation, but not a norm, since  $\text{Var}_{p, [0, T]}(x) = 0$  implies that  $x$  is constant. In addition, even restricted to the space of functions  $x$  with  $x_0 = 0$  and  $\text{Var}_{p, [0, T]}(x)$  on which  $\text{Var}_{p, [0, T]} < +\infty$  defines a norm, this norm is *not* equivalent to the uniform norm  $\|\cdot\|_{\infty}$ , and counter-example are easily constructed.

Following a recent remark due to P. Friz [Fri05], we may work with a more precise norm than the norm constructed from  $p$ -variation. Indeed, to simplify our approach, we work only with Hölder continuous paths.

If  $x$  is a path of finite  $p$ -variation and

$$\varphi(t) = \inf \{ s > 0 \mid \text{Var}_{p, [0, s]}(x)^p > t \},$$

then  $\varphi$  is increasing and  $x \circ \varphi$  is  $1/p$ -Hölder continuous. As the integral of a differential form keep the same value under a continuous, increasing time change, there is no difficulty in considering the  $1/p$ -Hölder norm, which is simpler to use than the  $p$ -variation norm (for some results on the relationship between  $p$ -variation and  $1/p$ -Hölder continuity, see for example [CG98]). Yet for convergence problems, this is not the most general framework, and dealing with the  $p$ -variation norm allows us to treat with more complete results (for example, in [Lej06, Lej03b], we prove the convergence only in  $p$ -variation although the path is  $\alpha$ -Hölder continuous, and this is due to a singularity at 0 of some term).

Let us denote by  $H_{\alpha}(x)$  the Hölder continuity modulus of a path  $x : [0, T] \rightarrow \mathbb{R}^d$  which is  $\alpha$ -Hölder continuous, that is

$$H_{\alpha}(x) = \sup_{0 \leq s < t \leq T} \frac{|x_t - x_s|}{|t - s|^{\alpha}}.$$



Of course, any  $\alpha$ -Hölder continuous path is also  $\beta$ -Hölder continuous for any  $\beta \leq \alpha$ . In addition, the equivalent of (2) is

$$(3) \quad \text{for } \beta \leq \alpha, \quad H_\beta(x) \leq 2^{1-\beta/\alpha} \|x\|_\infty^{1-\beta/\alpha} H_\alpha(x)^{\beta/\alpha}.$$

If  $H_\alpha(x) = 0$  then  $x$  is constant, and  $H_\alpha$  defines only a semi-norm.

**Notation 1.** If  $x : [0, T] \rightarrow \mathbb{R}^d$   $\alpha$ -Hölder continuous, then we set

$$\|x\|_\alpha = |x_0| + H_\alpha(x)$$

and by  $C^\alpha([0, T]; \mathbb{R}^d)$  the subset of functions  $x$  in  $C([0, T]; \mathbb{R}^d)$  such that  $\|x\|_\alpha$  is finite.

Equipped with  $\|\cdot\|_\alpha$ , this space  $C^\alpha([0, T], \mathbb{R}^d)$  is a Banach space. In addition, we get the following Lemma which is a consequence of the Ascoli Theorem and (3).

**Lemma 1.** *Let  $(x^n)_{n \in \mathbb{N}}$  such that  $x^n \in C^\alpha([0, T]; \mathbb{R}^d)$  and  $(\|x^n\|_\alpha)_{n \in \mathbb{N}}$  is bounded. Then there exists  $x$  in  $C^\alpha([0, T]; \mathbb{R}^d)$  and a subsequence of  $(x^n)_{n \in \mathbb{N}}$  that converges to  $x$  with respect to  $\|\cdot\|_\beta$  for each  $\beta < \alpha$ .*

*Remark 2.* It is important to note that here, we used the  $\|\cdot\|_\beta$  norm for the space  $C^\alpha([0, T]; \mathbb{R}^d)$  with  $\beta < \alpha$ . When equipped with this norm,  $(C^\alpha([0, T]; \mathbb{R}^d), \|\cdot\|_\beta)$  becomes a separable space, while the space  $(C^\alpha([0, T]; \mathbb{R}^d), \|\cdot\|_\alpha)$  is not separable: See [MS61] for example.

The next corollary follows easily.

**Corollary 1.** *Let  $\Pi$  be a partition of  $[0, T]$  and  $x^\Pi$  be the linear approximation of  $x \in C^\alpha([0, T]; \mathbb{R}^d)$  along  $\Pi$ . Then  $\|x^\Pi\|_\alpha \leq 3^{1-\alpha} \|x\|_\alpha$ .*

*If  $(\Pi^n)_{n \in \mathbb{N}}$  is a sequence of partitions of  $[0, T]$  whose meshes converge to 0, then  $(x^{\Pi^n})_{n \in \mathbb{N}}$  converges to  $x$  in  $(C^\alpha([0, T]; \mathbb{R}^d), \|\cdot\|_\beta)$  for all  $\beta < \alpha$ .*

*Proof.* Let  $\Pi = \{t_i\}_{i=1, \dots, J}$ . For  $0 \leq s < t \leq T$ , let  $s' = \min \Pi \cap [s, T]$  and  $t' = \max \Pi \cap [0, t]$ . If  $s \notin \Pi$  (resp.  $t \notin \Pi$ ), denote by  $s'' = \max \Pi \cap [0, s]$  (resp.  $t'' = \min \Pi \cap [t, T]$ ). As  $s', t' \in \Pi$ , if  $s, t \notin \Pi$ ,

$$\begin{aligned} |x_t^\Pi - x_s^\Pi| &\leq |x_t^\Pi - x_{t'}^\Pi| + |x_{t'}^\Pi - x_{s'}^\Pi| + |x_{s'}^\Pi - x_s^\Pi| \\ &\leq \frac{t - t'}{t'' - t'} |x_{t''} - x_{t'}| + |x_{t'} - x_{s'}| + \frac{s - s'}{s'' - s'} |x_{s''} - x_{s'}| \\ &\leq \|x\|_\alpha (t - t')^\alpha + \|x\|_\alpha (s' - s)^\alpha + \|x\|_\alpha (t' - s')^\alpha \\ &\leq 3^{1-\alpha} \|x\|_\alpha (t - s)^\alpha, \end{aligned}$$

the last inequality coming from the Jensen inequality applied by  $x \mapsto x^{1/\alpha}$ . The case where  $s$  or  $t$  belongs to  $\Pi$  is treated similarly. This proves that  $\|x^\Pi\|_\alpha \leq 3^{1-\alpha} \|x\|_\alpha$ .

The second part of this corollary is an immediate consequence of Lemma 1.  $\square$

*Remark 3.* One may wonder if it is possible to approximate a function  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  by piecewise linear functions that converge with

respect to  $\|\cdot\|_\alpha$ , and not with respect to  $\|\cdot\|_\beta$  for  $\beta < \alpha$ . As shown in [MS61] (see also [DN98, § 4.3]), this is only possible if  $x$  belongs to the class of functions such that

$$\lim_{\delta \rightarrow 0} \sup_{\substack{0 \leq s < t \leq T \\ |t-s| \leq \delta}} \frac{|x_t - x_s|}{(t-s)^\alpha} = 0,$$

or, in other words, if  $|x(t+h) - x(t)| = o(h^\alpha)$ . And of course, this class of functions is strictly included in  $C^\alpha([0, T]; \mathbb{R}^d)$ : the function  $f(x) = \sum_{k=0}^{+\infty} c^{-k\alpha} \sin(c^k x)$  for  $c$  large enough provides us with a counter-example, as it is easily proved using the results from [Cie60].

### 3. INTEGRALS ALONG $\alpha$ -HÖLDER CONTINUOUS PATHS, $\alpha \in (1/2, 1]$

For the sake of simplicity, consider  $d = 2$ . The construction of  $\mathfrak{I}$  on  $C^\alpha([0, T]; \mathbb{R}^2)$  for  $\alpha > 1/2$  is first deduced from the Young integral.

**3.1. Defining the integrals.** We recall here the construction of the integral of a  $\beta$ -Hölder continuous path driven by a  $\alpha$ -Hölder continuous path, provided that  $\alpha + \beta > 1$ . This theorem is due to L.C. Young [You36] (see also [DN98] for example).

**Theorem 1.** *Let  $\alpha, \beta \in (0, 1]$  with  $\alpha + \beta > 1$ . Then*

$$(x, y) \mapsto \left( t \mapsto \int_0^t y_s dx_s \right)$$

*is bilinear and continuous from  $C^\alpha([0, T]; \mathbb{R}) \times C^\beta([0, T]; \mathbb{R})$  to  $C^\alpha([0, T]; \mathbb{R})$ .*

*Sketch of the proof.* Fix  $n \in \mathbb{N}^*$ , and let us set, for  $t_k^n = Tk/2^n$ ,

$$J^n = \sum_{k=0}^{2^n-1} y_{t_k^n} (x_{t_{k+1}^n} - x_{t_k^n}).$$

Then

$$\begin{aligned} |J^{n+1} - J^n| &= \left| \sum_{k=0}^{2^{n+1}-1} (y_{t_{2k+1}^{n+1}} - y_{t_{2k}^{n+1}}) (x_{t_{2k+2}^{n+1}} - x_{t_{2k+1}^{n+1}}) \right| \\ &\leq \sum_{k=0}^{2^n-1} H_\beta(y) H_\alpha(x) T^{\alpha+\beta} 2^{-(n+1)(\alpha+\beta)} \\ &\leq 2^{-n(\alpha+\beta-1)} H_\beta(y) H_\alpha(x). \end{aligned}$$

As  $\alpha + \beta - 1 > 0$ , we deduce that the series  $\sum_{n \geq 0} (J_{n+1} - J_n)$  converges and thus that, if  $J \stackrel{\text{def}}{=} J_0 + \sum_{n \geq 0} (J_{n+1} - J_n)$ , then

$$(4) \quad |J - y_0(x_T - x_0)| \leq \zeta(\alpha + \beta - 1) T^{\alpha+\beta} H_\beta(y) H_\alpha(x),$$

where  $\zeta(\theta) = \sum_{n \geq 0} 1/n^\theta$ . Of course, we define  $\int_0^T y_s dx_s$  as  $J$ . From the last inequality in which  $t$  is substituted to  $T$  also proves that  $t \mapsto \int_0^t y_s dx_s$  is  $\alpha$ -Hölder continuous.

The other properties of the integral are easily, although tedious, deduced from this construction.  $\square$

*Remark 4.* Indeed, using the argument of Lemma 2.2.1, p. 244 [Lyo98], there is no need to consider dyadic partitions, but we keep them for simplicity. Note that however, especially when dealing with stochastic processes, some results in the rough paths theory are dependent from the choice of a dyadic partition (see for example [CL05]).

One may then define for  $0 \leq s \leq t \leq T$ ,

$$(5) \quad \mathfrak{I}(x; s, t) = \int_{x_{|[s,t]}} f = \int_s^t f_1(x_r) dx_r^1 + \int_s^t f_2(x_r) dx_r^2$$

as Young integrals with  $y_t = f(x_t)$ . Yet a global regularity condition is imposed on  $(x, y)$  which implies in particular that  $\alpha > 1/2$  and the minimal assumptions on the regularity of  $f$  also depends on  $\alpha$ .

**Notation 2.** For a path  $x$  defined on the time interval  $[S, T]$ , We will use  $\mathfrak{I}(x; s, t)$  to denote the integral  $\int_{x_{|[s,t]}} f$  when  $S \leq s < t \leq T$ , and  $\mathfrak{I}(x)$  to denote the function  $t \in [S, T] \mapsto \mathfrak{I}(x; S, t)$ .

The following corollaries follow from the construction of the Young integrals : see in particular [LZ94, Lej03a].

**Corollary 2.** Fix  $\alpha \in (1/2, 1]$  and  $f \in \text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R}^m)$  with  $\gamma > 1/\alpha - 1$ . Then  $\mathfrak{I}$  defined in (5) is well defined as a Young integral on  $C^\alpha([0, T]; \mathbb{R}^2)$  and is a locally Lipschitz map from  $(C^\alpha([0, T]; \mathbb{R}^2), \|\cdot\|_\alpha)$  to  $(C^\alpha([0, T]; \mathbb{R}^m), \|\cdot\|_\alpha)$ .

We also need the following Corollary.

**Corollary 3.** Fix  $\alpha \in (1/3, 1/2]$  and let  $f \in \text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R}^m)$  with  $\gamma > 1/\alpha - 1$ . Then

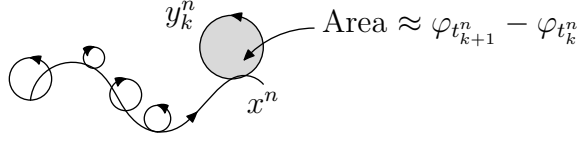
$$C^{2\alpha}([0, T]; \mathbb{R}) \times C^\alpha([0, T]; \mathbb{R}^2) \rightarrow C^{2\alpha}([0, T]; \mathbb{R}^m) \\ (\varphi, x) \mapsto \left( t \mapsto \int_0^t [f, f](x_s) d\varphi_s \right)$$

is well defined as a Young integral and is a locally Lipschitz map from  $(C^{2\alpha}([0, T]; \mathbb{R}^2), \|\cdot\|_{2\alpha}) \times (C^\alpha([0, T]; \mathbb{R}^2), \|\cdot\|_\alpha)$  to  $(C^{2\alpha}([0, T]; \mathbb{R}^2), \|\cdot\|_{2\alpha})$ .

**3.2. A problem of continuity.** We have to take great care of the meaning of the continuity result in Corollary 2: the norm  $\|\cdot\|_\alpha$  is *not* equivalent to the uniform norm. Convergence in  $C^\alpha$  implies uniform convergence but the converse is not true.

The following counter-example is the cornerstone to understand how  $\mathfrak{I}$  will be defined for dealing with irregular paths.

Let  $(x^n)_{n \in \mathbb{N}}$  and  $x$  be continuous paths such that  $x^n$  converges to  $x$  in  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha \in (1/2, 1]$ .


 FIGURE 1. The path  $x^n \bowtie \Phi^n$ .

Let  $\varphi$  be a function in  $C^\beta([0, T]; \mathbb{R})$  with  $\beta \in (2/3, 1]$ . Let us also assume that  $f$  belongs to  $\text{Lip}(\gamma; \mathbb{R}^d \rightarrow \mathbb{R})$  with

$$\gamma > 1, \quad \left(\frac{\gamma}{2} + 1\right) \beta > 1.$$

Let  $\Pi^n = \{t_k^n\}_{k=0, \dots, 2^n-1}$  be the dyadic partition of  $[0, T]$  at level  $n$ , that is  $t_k^n = k/2^n$ . For each  $n = 1, 2, \dots$ , we denote by  $\Phi^n = \{y_k^n\}_{k=0, \dots, 2^n-1}$  a set of functions piecewise of class  $C^1$  such that for a fixed  $\kappa > 1$ ,

$$(6a) \quad y_k^n : [t_k^n, t_{k+1}^n] \rightarrow \mathbb{R}^2 \text{ with } y_k^n(t_k^n) = y_k^n(t_{k+1}^n) = x^n(t_k^n),$$

$$(6b) \quad \sup_{n=1, 2, \dots, k=0, \dots, 2^n} \|y_k^n\|_{\beta/2} < +\infty,$$

$$(6c) \quad \text{uniformly in } n, k, |\text{Area}(y_k^n) - (\varphi(t_{k+1}^n) - \varphi(t_k^n))| \leq C 2^{-n\kappa},$$

where  $\text{Area}(y_k^n)$  is the algebraic area of the loop  $y_k^n$  defined by

$$\begin{aligned} \text{Area}(y_k^n) &= \frac{1}{2} \int_{t_k^n}^{t_{k+1}^n} (y_k^{1,n}(s) - y_k^{1,n}(t_k^n)) dy_k^{2,n}(s) \\ &\quad - \frac{1}{2} \int_{t_k^n}^{t_{k+1}^n} (y_k^{2,n}(s) - y_k^{2,n}(t_k^n)) dy_k^{1,n}(s). \end{aligned}$$

For such a sequence, we say that  $\varphi$  *encodes asymptotically the areas of*  $(\Phi^n)_{n \in \mathbb{N}}$ .

We denote by  $x^n \bowtie \Phi^n$  the path from  $[0, 2T]$  to  $\mathbb{R}^2$  defined by

$$x^n \bowtie \Phi^n = y_0^n \cdot x_{|[t_0^n, t_1^n]}^n \cdot y_1^n \cdot x_{|[t_1^n, t_2^n]}^n \cdots y_{2^n-1}^n \cdot x_{|[t_{2^n-1}^n, t_{2^n}^n]}^n,$$

where  $x \cdot y$  is the concatenation between two path  $x$  and  $y$  (see Figure 1).

This path  $x^n \bowtie \Phi^n$  is defined on the time interval  $[0, 2T]$ .

Then, by the Chasles property of the integral,

$$\mathfrak{I}(x^n \bowtie \Phi^n; 0, 2T) = \mathfrak{I}(x^n; 0, T) + \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s)) dy_k^n(s).$$

By the Green-Riemann formula (1),

$$\int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s)) dy_k^n(s) = \iint_{\text{Surface}(y_k^n)} [f, f](x^1, x^2) dx^1 dx^2.$$

The idea is now the following,

$$\begin{aligned} \iint_{\text{Surface}(y_k^n)} [f, f](x^1, x^2) dx^1 dx^2 &\approx [f, f](x_{t_k^n}) \text{Area}(y_k^n) \\ &\approx [f, f](x_{t_k^n})(\varphi(t_{k+1}^n) - \varphi(t_k^n)). \end{aligned}$$

To be more precise, using our hypotheses on  $f$  and  $\Phi^n$ , with  $\Delta_n t = T2^{-n}$ ,

$$(7) \quad \left| \iint_{\text{Surface}(y_k^n)} [f, f](x^1, x^2) dx^1 dx^2 - [f, f](x_{t_k^n})(\varphi(t_{k+1}^n) - \varphi(t_k^n)) \right| \\ \leq 2\|\nabla f\|_\gamma \|y_k^n\|_{\beta/2}^\gamma (C\Delta_n t^\kappa + \|\varphi\|_\beta \Delta_n t^\beta) \Delta_n t^{\gamma\beta/2} + 2C\|\nabla f\|_\infty \Delta_n t^\kappa.$$

As there are now  $2^n$  terms to sum. By hypothesis,  $\beta + \gamma\beta/2 > 1$  and  $\kappa > 1$  so that, the sum of the right-hand side of (7) vanishes as  $n \rightarrow \infty$ . In addition, necessarily  $\beta + \gamma\alpha > 1$  so that one can consider  $\int [f, f](x_s) d\varphi_s$  as a Young integral. Thus, we easily get that

$$\sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s)) dy_k^n(s) \xrightarrow{n \rightarrow \infty} \int_0^T [f, f](x_s) d\varphi_s.$$

In other words,

$$\mathfrak{I}(x^n \rtimes \Phi^n; 0, 2T) \xrightarrow{n \rightarrow \infty} \mathfrak{I}(x; s, t) + \int_0^T [f, f](x_r) d\varphi_r.$$

It is important to note that here,  $(x^n \rtimes \Phi^n)_{n \in \mathbb{N}}$  is in general not bounded in  $C^\alpha([0, 2T]; \mathbb{R}^2)$ , but it is bounded in  $C^{\beta/2}([0, 2T]; \mathbb{R}^2)$ . Let us remark that for  $t \in [0, 2T]$ , if  $t/2 \in [t_k^{n+1}, t_{k+1}^{n+1}]$  and  $k$  is odd, then  $x^n \rtimes \Phi^n(t) = x^n(t/2)$ . If  $k$  is even, then  $x^n \rtimes \Phi^n(t) = y_k^n(t/2)$ . Thus,

$$|x^n \rtimes \Phi^n(t) - x^n \rtimes \Phi^n(s)| \leq \begin{cases} |x^n(t/2) - x^n(s/2)| \\ \quad \text{if } s/2 \in [t_{2k+1}^{n+1}, t_{2k+2}^{n+1}], t/2 \in [t_{2\ell+1}^{n+1}, t_{2\ell+2}^{n+1}], \\ |y_\ell^n(t/2) - y_\ell^n(t_\ell^n)| + |x^n(t_k^n) - x^n(s/2)| \\ \quad \text{if } s/2 \in [t_{2k+1}^{n+1}, t_{2k+2}^{n+1}], t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], \\ |y_\ell^n(t/2) - y_\ell^n(t_\ell^n)| + |y_\ell^n(t_\ell^n) - y_k^n(t_k^n)| + |y_k^n(t_k^n) - y_k^n(s/2)| \\ \quad \text{if } s/2 \in [t_{2k}^{n+1}, t_{2k+1}^{n+1}], t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], k \neq \ell, \\ |y_\ell^n(t/2) - y_\ell^n(s/2)| \\ \quad \text{if } s/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], \\ |x^n(t/2) - x^n(t_k^n)| + |y_k^n(t_k^n) - y_k^n(s/2)| \\ \quad \text{if } s/2 \in [t_{2k}^{n+1}, t_{2k+1}^{n+1}], t/2 \in [t_{2\ell+1}^{n+1}, t_{2\ell+2}^{n+1}]. \end{cases}$$

Using the convexity inequality, one gets that for some constant  $C$  that depends only on  $\alpha$  and  $\beta$ ,

$$\begin{aligned} & |x^n \rtimes \Phi^n(t) - x^n \rtimes \Phi^n(s)| \\ & \leq C \max\{\|x\|_\alpha, \sup_{k=0, \dots, 2^n-1} \|y_k^n\|_{\beta/2}\} \max\{(t-s)^{\beta/2}, (t-s)^\alpha\}. \end{aligned}$$

Since  $\beta/2 \leq \alpha$ , it follows that  $(x^n \rtimes \Phi^n)_{n \in \mathbb{N}}$  is bounded in  $C^{\beta/2}([0, 2T]; \mathbb{R}^2)$  assuming that of course, the  $y_k^n$  have a  $\beta/2$ -Hölder norm different from zero. As we required that  $\varphi$  is  $\beta$ -Hölder continuous, and if we choose for  $y_k^n$  some circles with area  $\varphi(t_{k+1}^n) - \varphi(t_k^n)$ , then their radius are  $\sqrt{|\varphi(t_{k+1}^n) - \varphi(t_k^n)|/\pi}$  and this is why we look for  $y_k^n$  that are  $\beta/2$ -Hölder continuous.

This also means that when one considers a sequence  $(x^n)_{n \in \mathbb{N}}$  of elements in  $C^\alpha([0, T]; \mathbb{R}^2)$  and a path  $x$  of  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$ , one has to consider the fact that  $(x^n)_{n \in \mathbb{N}}$  may converge to  $x$  with respect to some  $\beta$ -Hölder norm with  $\beta \leq 1/2$ . In addition, this counter-example ruins all hope to extend  $\mathfrak{I}$  naturally to  $C^\alpha([0, T]; \mathbb{R}^2)$  for  $\alpha < 1/2$ , since may construct at least two bounded sequences  $(x^n)_{n \in \mathbb{N}}$  and  $(z^n)_{n \in \mathbb{N}}$  in  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha < 1/2$  converging uniformly to  $x$  — hence that converge to  $x$  in  $C^\beta([0, T]; \mathbb{R}^2)$  for any  $\beta < \alpha$  — such that  $\mathfrak{I}(x^n; 0, T) \xrightarrow{n \rightarrow \infty} \mathfrak{I}(x; 0, T)$  and  $\mathfrak{I}(z^n; 0, T) \xrightarrow{n \rightarrow \infty} \mathfrak{I}(x; 0, T) + \int_0^T [f, f](x_s) d\varphi_s$ , which is different from  $\mathfrak{I}(x; 0, T)$  unless  $[f, f] = 0$  or  $\varphi$  is constant.

**3.3. A practical counter-example in the stochastic setting.** In [Lej02, LL06], we give a stochastic example of such a phenomenon coming from the homogenization theory. Let us consider some coefficients  $\sigma$  from  $\mathbb{R}^d$  to the space of  $d \times d$ -matrices and  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  smooth enough which are 1-periodic. We consider the SDE

$$X_t^\varepsilon = \int_0^t \sigma(X_s^\varepsilon/\varepsilon) dB_s + \frac{1}{\varepsilon} \int_0^t b(X_s^\varepsilon/\varepsilon) ds$$

for some Brownian motion  $B$ . It is well known from the homogenization theory (see [BLP78] for example) that  $X^\varepsilon$  converges as  $\varepsilon \rightarrow 0$  to  $\bar{\sigma}W$  for some Brownian motion  $W$  and a  $d \times d$ -matrix  $\bar{\sigma}$  which is constant, provided that the drift  $b$  satisfies some averaging property. One of the application of this theory is to provide a tool to replace (for modelling or numerical computations) a PDE of type  $\partial_t u^\varepsilon(t, x) + L^\varepsilon u^\varepsilon(t, x) = 0$ ,  $u^\varepsilon(T, x) = g(x)$  with  $L^\varepsilon = \sum_{i,j=1}^d \frac{1}{2} a_{i,j}(\cdot/\varepsilon) \partial_{x_i x_j}^2 + \sum_{i=1}^d \frac{1}{\varepsilon} b_i(\cdot/\varepsilon) \partial_{x_i}$  and  $a = \sigma \sigma^t$  by the simpler PDE  $\partial_t u(t, x) + \bar{L}u(t, x) = 0$  with  $\bar{L} = \sum_{i,j=1}^d \frac{1}{2} \bar{a}_{i,j} \partial_{x_i x_j}^2$  and  $\bar{a} = \bar{\sigma} \bar{\sigma}^t$ . From the probabilistic point of view, this means that  $X^\varepsilon$  behaves — thanks to a functional Central Limit Theorem and the ergodic behavior of its projection on the torus  $\mathbb{R}^d/[0, 1]^d$  — like a non-standard Brownian motion. However, one has to take care

when using  $X^\varepsilon$  as the driver of some SDE, since

$$i, j = 1, \dots, d, \quad \mathfrak{A}^{i,j}(X^\varepsilon; 0, t) \xrightarrow{\varepsilon \rightarrow 0} \mathfrak{A}^{i,j}(\bar{\sigma}W; 0, t) + t\bar{c}_{i,j}$$

uniformly and in  $p$ -variation for  $p > 2$ , where  $(\bar{c}_{i,j})_{i,j=1,\dots,d}$  is a  $d \times d$ -antisymmetric matrix that can be computed from  $a$  and  $b$ , and  $\mathfrak{A}^{i,j}$  is the Lévy area of  $(Y^i, Y^j)$ , *i.e.*,

$$\mathfrak{A}^{i,j}(Y; 0, t) = \frac{1}{2} \int_0^t (Y_s^i - Y_0^i) \circ dY_s^j - \frac{1}{2} \int_0^t (Y_s^i - Y_0^i) \circ dY_s^i$$

for a  $d$ -dimensional semi-martingale  $Y$ . If  $b = 0$ , then  $\bar{c} = 0$ , so that this effect comes from the presence of the drift.

From the Wong-Zakai theorem (see for example [IW89]), the Stratonovich integral appear as the natural extension of  $\mathfrak{J}$  on the subset  $\text{SM}([0, T]; \mathbb{R}^2)$  of  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha < 1/2$  that contains trajectories of semi-martingales. Let us note that  $\mathfrak{J}((Y^i, Y^j); 0, t) = \mathfrak{A}^{i,j}(Y; 0, t)$  for  $t \in [0, T]$  with  $(f_1, f_2) = \frac{1}{2}(x_i, x_j)$  and both  $X^\varepsilon$  and  $\bar{\sigma}W$  are  $\alpha$ -Hölder continuous with  $\alpha < 1/2$ . This proves that  $\mathfrak{J}$  cannot be continuous on  $\text{SM}([0, T]; \mathbb{R}^2) \subset C^\alpha([0, T]; \mathbb{R}^2)$ .

Counter-examples to the Wong-Zakai theorem (see [McS72, IW89]) also rely on the construction of approximations of the trajectories of the Brownian motion by using a “perturbation” of the piecewise linear approximation that gives rise, in the limit, to a non-vanishing supplementary area and then, for the SDE, to a drift term.

This problem of convergence may arise in natural setting and has then a practical interest.

#### 4. INTEGRALS ALONG $\alpha$ -HÖLDER CONTINUOUS PATHS, $\alpha \in (1/3, 1/2]$ : HEURISTIC CONSIDERATIONS

We present in this section a construction of the integral which is not the best possible one, but which allows us to understand the main ideas and problems.

The previous counter-example have endowed a few ideas: (1) We may use the Green-Riemann formula to deal with close loops. (2) For some  $\alpha > 1/2$ , we may add to our paths small loops whose radius is of order  $2^{-n\alpha/2}$  and thus whose area is of order  $2^{-n\alpha}$ . (3) As many loops are added, the sum of the areas does not vanish and gives rise to a supplementary term.

Our construction will now take these facts into account.

##### 4.1. Construction of the integral along a subset of $C^\alpha([0, T]; \mathbb{R}^2)$ .

As we wish our definition of the integral to be continuous, a naive construction is the following: Fix  $K > 0$ ,  $\alpha \in (1/3, 1/2]$  and  $f \in \text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R})$  with  $\gamma > 1/\alpha - 1$  (and then  $\gamma > 1$ ). Denote by  $\Pi^n$  the dyadic partition of  $[0, T]$  at level  $n$ , and by  $L^\alpha([0, T]; \mathbb{R}^2)$  the

set of functions  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  for which the linear approximations  $(x^{\Pi^n})_{n \in \mathbb{N}}$  satisfy

$$\mathfrak{J}(x) \stackrel{\text{def}}{=} \lim_{n \in \mathbb{N}} \mathfrak{J}(x^{\Pi^n}) \text{ exists in } C^\alpha([0, T]; \mathbb{R})$$

$$\text{and } |\mathfrak{J}(x|_{[s,t]}) - \mathfrak{J}(x^{\Pi^n}|_{[s,t]})| \leq K \|x - x^{\Pi^n}\|_\alpha |t - s|^\alpha, \quad 0 \leq s < t \leq T.$$

If  $K$  is large enough, it follows from Corollary 2 that  $L^\alpha([0, T]; \mathbb{R}^2)$  contains subsets of  $C^\beta([0, T]; \mathbb{R}^2)$  for all  $\beta > 1/2$  (this depends on  $f$  and the choice of  $K$ , since from Corollary 2,  $x \mapsto \mathfrak{J}(x)$  is locally Lipschitz) and it is also known (but for this, we need a more complete theory) that it contains paths that are not  $\beta$ -Hölder continuous for  $\beta > 1/2$ , such as Brownian motion's trajectories (see for example [Sip93, CL05]). Any element  $x$  of  $L^\alpha([0, T]; \mathbb{R}^2)$  may be identified with the sequence  $(x^{\Pi^n})_{n \in \mathbb{N}}$ .

Now, consider  $\varphi \in C^{2\alpha}([0, T]; \mathbb{R}^2)$  and  $(\Phi^n)_{n \in \mathbb{N}}$  a sequence of loops at each level  $n$  whose areas are asymptotically encoded by  $\varphi$ . Then, as previously,

$$\mathfrak{J}(x^{\Pi^n} \rtimes \Phi^n) \xrightarrow[n \rightarrow \infty]{C^\alpha} \mathfrak{J}(x, \varphi) \stackrel{\text{def}}{=} \mathfrak{J}(x) + \int [f, f](x_s) d\varphi_s.$$

For  $(x, \varphi) \in L^{\kappa, \alpha}([0, T]; \mathbb{R}^3) \stackrel{\text{def}}{=} L^\alpha([0, T]; \mathbb{R}^2) \times C^{2\alpha}([0, T]; \mathbb{R})$ , we may then define

$$\mathfrak{J}(x, \varphi) = \lim_{n \rightarrow \infty} \mathfrak{J}(x^{\Pi^n} \rtimes \Phi^n)$$

where  $\varphi$  encodes asymptotically the areas of  $(\Phi^n)_{n \in \mathbb{N}}$ . The space  $L^{\kappa, \alpha}([0, T]; \mathbb{R}^3)$  is naturally a Banach space when equipped with the norm  $\|(x, \varphi)\|_{\kappa, \alpha} = \|x\|_\alpha + \|\varphi\|_{2\alpha}$ .

The interesting point with this definition of the map  $(x, \varphi) \mapsto \mathfrak{J}(x, \varphi)$  is that its continuity follows naturally from its very construction.

**Proposition 1.** *For all  $\beta < \alpha$  with  $\alpha \in (1/3, 1/2]$ , the map  $\mathfrak{J}$  is continuous from  $(L^{\kappa, \alpha}([0, T]; \mathbb{R}^3), \|\cdot\|_{\kappa, \alpha})$  to  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$*

*Proof.* Let  $(x^n, \varphi^n)_{n \in \mathbb{N}}$  be a sequence of paths converging to  $(x, \varphi)$  in  $L^{\kappa, \alpha}([0, T]; \mathbb{R}^3)$ .

By definition,  $\mathfrak{J}(x^n, \varphi^n; s, t) = \mathfrak{J}(x^n; s, t) + \int_s^t [f, f](x_r^n) d\varphi_r^n$ . From Corollary 3, we know that  $\int_0^t [f, f](x^n) d\varphi^n$  converges to  $\int_0^t [f, f](x) d\varphi$  in  $C^{2\alpha}([0, T]; \mathbb{R})$ .

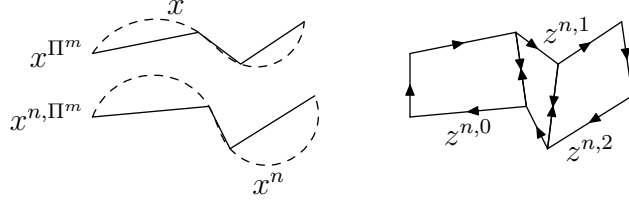
From the very definition of  $L^\alpha([0, T]; \mathbb{R}^2)$ ,

$$\|\mathfrak{J}(x^n, \Pi^n) - \mathfrak{J}(x^n)\|_\alpha \leq K \|x^n, \Pi^n - x^n\|_\alpha.$$

But it is easily shown with Corollary 1 that for all  $\beta < \alpha$  and some constant  $K_2$ ,  $\|x^n, \Pi^n - x^n\|_\beta \leq K_2 \|x^n\|_\alpha / 2^{m(\beta - \alpha)}$  and thus  $(\mathfrak{J}(x^n, \Pi^n))_{m \in \mathbb{N}}$  converges to  $\mathfrak{J}(x^n)$  in  $C^\beta([0, T]; \mathbb{R})$  at a rate which is uniform in  $n$  since  $(\|x^n\|_\alpha)_{n \in \mathbb{N}}$  is bounded.

It follows that for all  $\beta < \alpha$ ,  $\mathfrak{J}(x^n, \Pi^n)$  converges uniformly in  $n$  to  $\mathfrak{J}(x^n)$  in  $C^\beta([0, T]; \mathbb{R})$  as  $m \rightarrow \infty$ .



FIGURE 2. The paths  $z_k^{n,m}$ .

For  $s < t$  fixed, there exist some integers  $i_m$  and  $j_m$  such that  $t_{i_m-1}^m \leq s < t_{i_m}^m$  and  $t_{j_m}^m < t \leq t_{j_m+1}^m$ . To simplify the notations, we set  $t_{i_m-1}^m = s$  and  $t_{j_m+1}^m = t$ . For  $k = i_m - 1, \dots, j_m + 1$ , denote by  $z_k^{n,m}$  the following path (see Figure 2)

$$z_k^{n,m} = x_{[t_k^m, t_{k+1}^m]}^{\Pi^m} \cdot \overline{x_{t_{k+1}^m}^{\Pi^m} x_{t_k^m}^{n, \Pi^m}} \cdot x_{[t_{k+1}^m, t_k^m]}^{n, \Pi^m} \cdot \overline{x_{t_k^m}^{n, \Pi^m} x_{t_k^m}^{\Pi^m}}.$$

Hence, with the previous convention on  $t_{i_m-1}^m$  and  $t_{j_m}^m$ ,

$$(8) \quad \mathfrak{I}(x^{\Pi^m}; s, t) - \mathfrak{I}(x^{n, \Pi^m}; s, t) = \sum_{k=i_m-1}^{j_m} \int_{z_k^{n,m}} f + \int_{x_s^{\Pi^m} x_s^{n, \Pi^m}} f + \int_{x_t^{n, \Pi^m} x_t^{\Pi^m}} f$$

Let us note that

$$\begin{aligned} \left| \int_{z_k^{n,m}} f \right| &= \left| \iint_{\text{Surface}(z_k^{n,m})} [f, f](x^1, x^2) dx^1 dx^2 \right| \\ &\leq \frac{1}{2} \|f\|_{\text{Lip}} |x_{t_{k+1}^m}^m - x_{t_k^m}^m| \times |x_{t_k^m}^n - x_{t_k^m}^m| \\ &\leq \frac{(t_{k+1}^m - t_k^m)^\alpha}{2} \|x\|_\alpha \|f\|_{\text{Lip}} \|x^n - x\|_\infty. \end{aligned}$$

Using the convexity inequality with  $x \mapsto x^{1/\alpha}$ , since there are at most  $2^m$  terms in the series in the right-hand-side of (8), we get

$$|\mathfrak{I}(x^{\Pi^m}; s, t) - \mathfrak{I}(x^{n, \Pi^m}; s, t)| \leq \frac{2^{m(1-\alpha)}}{2} \|f\|_{\text{Lip}}^\alpha \|x\|_\alpha^\alpha \|x^n - x\|_\infty^\alpha (t-s)^\alpha.$$

This proves the convergence of  $\mathfrak{I}(x^{n, \Pi^m})$  to  $\mathfrak{I}(x^{\Pi^m})$  in  $C^\alpha([0, T]; \mathbb{R})$  as  $n \rightarrow \infty$  for any  $m$ .

It is now possible to complete the following diagram

$$\begin{array}{ccc} \mathfrak{I}(x^{n, \Pi^m}) & \xrightarrow[\substack{\|\cdot\|_\alpha \\ n \rightarrow \infty}]{} & \mathfrak{I}(x^{\Pi^m}) \\ \|\cdot\|_\beta \downarrow \substack{m \rightarrow \infty \\ \text{unif. in } n} & & \|\cdot\|_\beta \downarrow m \rightarrow \infty \\ \mathfrak{I}(x^n) & & \mathfrak{I}(x) \end{array}$$

to obtain that  $\mathfrak{I}(x^n, \varphi^n)$  converges in  $C^\beta([0, T]; \mathbb{R})$  to  $\mathfrak{I}(x, \varphi)$ .  $\square$



FIGURE 3. The paths  $x$ ,  $x^{\Pi^n}$ ,  $x^{\Pi^{n+1}}$  and the areas defined by  $\Phi^{n,n+1}$  (in gray).

Moreover, the following stability result is easily proved.

**Lemma 2.** *If  $\psi$  (resp.  $\varphi$ ) is given in  $C^{2\alpha}([0, T]; \mathbb{R})$  and that encodes asymptotically the areas of  $(\Psi^n)_{n \in \mathbb{N}}$  (resp.  $(\Phi^n)_{n \in \mathbb{N}}$ ), then*

$$\lim_{n \rightarrow +\infty} \mathfrak{J}(x^{\Pi^n} \bowtie \Phi^n \bowtie \Psi^n) = \mathfrak{J}(x, \varphi + \psi).$$

The function  $\varphi$  can be chosen arbitrary, so that we have gained a degree of freedom. In other words, to get a proper definition of  $\mathfrak{J}$  that respect the continuity, we have to consider not a path with values in  $\mathbb{R}^2$  but a path with values in  $\mathbb{R}^3$ . Indeed, this construction is far to be optimal, *i.e.*, the set  $L^{\bowtie, \alpha}([0, T]; \mathbb{R}^3)$  is not the biggest one that can be considered. Yet it gives a proper understanding of the problem.

**4.2. Is this construction natural?** Of course, the real question is to consider whether or not is it natural to extend  $\mathfrak{J}$  on (at least) a subset of  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha \in (1/3, 1/2]$  by considering paths with values in  $\mathbb{R}^2$  but with values in  $\mathbb{R}^3$ ?

Let us consider a path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$ . The piecewise linear path  $x^{\Pi^n}$  is an approximation of  $x$ , and for each  $m \geq n$ , we may define

$$\widehat{x}^{\Pi^m} \stackrel{\text{def}}{=} (x|_{[t_0^n, t_1^n]} \cdot x|_{[t_1^n, t_2^n]}) \cdot x|_{[t_2^n, t_3^n]} \cdots (x|_{[t_{2^n-1}^n, t_{2^n}^n]} \cdot x|_{[t_{2^n}^n, t_{2^n-1}^n]}) \cdot x|_{[t_{2^n-1}^n, t_{2^n}^n]}$$

on the time interval  $[0, 3T]$ . As we go back on forth on the segments composing  $x^{\Pi^n}$ , we get that  $\mathfrak{J}(\widehat{x}^{\Pi^m}; 0, 3T) = \mathfrak{J}(x^{\Pi^m}; 0, T)$ . We then define  $y_k^{n,m} = x|_{[t_k^n, t_{k+1}^n]} \cdot x|_{[t_{k+1}^n, t_k^n]}$ , that satisfies (6a)–(6b) and  $\Phi^{n,m} = \{y_k^{n,m}\}_{k=0, \dots, 2^n-1}$ . Since  $\widehat{x}^{\Pi^m} = x^n \bowtie \Phi^{n,m}$ ,

$$\mathfrak{J}(x^{\Pi^n}; 0, T) = \mathfrak{J}(\widehat{x}^{\Pi^m}; 0, 3T) = \mathfrak{J}(x^{\Pi^n} \bowtie \Phi^{n,m}; 0, 3T).$$

If we now set for example  $m = n^2$ , then *a priori* nothing ensures, unless  $x \in L^\alpha([0, T]; \mathbb{R}^2)$ , that the areas of  $(\Phi^{n,n^2})_{n \in \mathbb{N}}$  are asymptotically encoded by the function  $\varphi \equiv 0$ , not that there exists a function  $\varphi \in C^{2\alpha}([0, T]; \mathbb{R})$  that encodes the areas of  $(\Phi^{n,n^2})_{n \in \mathbb{N}}$ . In the last two cases, how then to consider the limit of  $\mathfrak{J}(x^{\Pi^{n^2}})$ , since it may differs from the limit of  $\mathfrak{J}(x^{\Pi^n} \bowtie \Phi^{n,n^2})$ ? Indeed,

$$\mathfrak{J}(x^{\Pi^n} \bowtie \Phi^{n,n^2}; 0, T) = \mathfrak{J}(x^{\Pi^n}; 0, T) + \sum_{k=0}^{2^n-1} \mathfrak{J}(y_k^{n,n^2}; t_k^n, t_{k+1}^n).$$

Yet with the Green-Riemann formula,

$$\mathfrak{J}(y_k^{n,n^2}; t_k^n, t_{k+1}^n) \approx [f, f](x_{t_k^n}) \text{Area}(y_k^{n,n^2}).$$

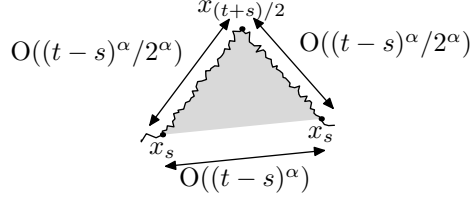


FIGURE 4. The area of some  $\alpha$ -Hölder continuous path between times  $s$  and  $t$  is of order  $(t-s)^{2\alpha}$ .

As we have seen it, the function  $\mathfrak{A}$  on  $C^\beta([0, T]; \mathbb{R}^2)$ ,  $\beta > 1/2$ , defined by

$$(9) \quad \mathfrak{A}(x; s, t) = \frac{1}{2} \int_s^t (x_r^1 - x_s^1) dx_r^2 - \frac{1}{2} \int_s^t (x_r^2 - x_s^2) dx_s^1$$

is *not* continuous with respect to the uniform norm: One has only to take  $f(x) = \frac{1}{2}x^1 dx^2 - \frac{1}{2}x^2 dx^1$  and to use the previous counter-examples. As  $\text{Area}(y_k^{n, n^2}) = \mathfrak{A}(x^{n^2}; t_k^n, t_{k+1}^n)$  and although  $y_k^{n, n^2}$  converges uniformly to 0, it may happens that  $\text{Area}(y_k^{n, n^2}; t_k^n, t_{k+1}^n)$  is of order  $2^{-2\alpha n}$  (this is possible since the distance between  $x_{t_k^n}$  and  $x_{t_{k+1}^n}$  is roughly of order  $2^{-\alpha n}$  if  $x$  is  $\alpha$ -Hölder continuous, see Figure 4). In this case,  $\sum_{k=0}^{2^n-1} \mathfrak{J}(y_k^{n, n^2})$  may have a limit different from 0 or have no limit at all.

In other words, the area contained between a path  $x$  and its chord for all couple of times  $(s, t)$  is “hidden” in  $x$  and has to be determined in an arbitrary manner<sup>1</sup>.

For some  $(x, \varphi) \in L^{\mathfrak{x}, \alpha}([0, T]; \mathbb{R}^3)$ , which is identified with a sequence converging uniformly to  $x$ , the element  $\varphi$  means in some sense that some area has been chosen and then that our integral is properly determined. Once this choice of  $\varphi$  has been done, Lemma 2 shows us how to deduce from it different integrals by choosing other areas.

**4.3. Justifications for a new setting.** The previous construction does not answer our main question: “How to construct an integral for paths in  $C^\alpha([0, T]; \mathbb{R}^2)$  for  $\alpha \in (1/3, 1]$ ?”. Yet it endows the fact one cannot define a map  $x \mapsto \mathfrak{J}$  which extend the map  $x \mapsto \int_x f$  on  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$  unless one adds some extra information. Here, this information corresponds to the choice of a function  $\varphi$ , so that we consider indeed a subset of  $C^\alpha([0, T]; \mathbb{R}^2) \times C^{2\alpha}([0, T]; \mathbb{R})$  (for  $\alpha \leq 1/2$ ) such that, when equipped with the norm  $\|(x, \varphi)\| = \|x\|_\alpha + \|\varphi\|_{2\alpha}$ , the map  $\mathfrak{J}$  is continuous.

<sup>1</sup>Consider the case of Brownian trajectories, where the Lévy area is a natural, but not the only one, choice and was the first example of stochastic integral [Lév65]. In addition, it is then defined as a limit in probability.

We have also seen that in Section 4.2 above that for considering integral along path in  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha \in (1/3, 1/2]$ , it is natural to consider the area contained between the path and its chord in view of defining some integral, although there is no way to define it canonically in general.

The drawback of our construction is that we assumed the convergence of the integrals along the piecewise linear approximations of  $x$ .

The idea is now to construct directly a path in  $\mathbb{R}^3$  in a way that such a path may be identified with a limit of converging sequence of piecewise smooth paths in  $\mathbb{R}^2$  whose integrals also converge. This allows us to get rid of the loops themselves, since the only information we need is the asymptotic limit of the area, while keeping enough information to construct the integral. Besides, this proves that the choice of a converging subsequence does not depend on the choice of the differential form which is integrated.

## 5. INTEGRALS ALONG $\alpha$ -HÖLDER CONTINUOUS PATHS, $\alpha \in (1/3, 1/2]$ : CONSTRUCTION BY APPROXIMATIONS

It is time to turn to the whole picture, now that the importance of knowing the area has been shown.

**5.1. Motivations.** The main idea in the previous approach was to replace an irregular path  $(x, \varphi) \in L^{\kappa, \alpha}([0, T]; \mathbb{R}^3)$  with a simpler path  $x^n \in C_p^1([0, T]; \mathbb{R}^2)$  which “approximates”  $x$  in the following sense:  $x_{t_k^n}^n = x_{t_k^n}$  for the dyadics point  $\{t_k^n\}_{k=0, \dots, 2^n}$  of  $[0, T]$ , and on  $[t_k^n, t_{k+1}^n]$  is composed of a loop  $y_k^n : [t_k^n, t_k^n + T2^{-n-1}] \rightarrow \mathbb{R}^2$  and then a segment joining  $x_{t_k^n}^n$  and  $x_{t_{k+1}^n}^n$ .

Once this family  $(x^n)_{n \in \mathbb{N}}$  has been constructed, one may study the convergence of the ordinary integrals  $\mathfrak{J}(x^n)$ , where the integrals of  $f$  on the loops have been transformed with the Green-Riemann formula into double integrals whose values are given approximatively by the areas of the loops times the Lie brackets of  $f$  at the starting points of the loop.

A simple approximation of  $\mathfrak{J}(x)$  is then given by, if  $x^n$  is then defined on  $[0, T]$  with loops on  $[t_k^n, t_{k+1}^n + T2^{-n-1}]$  and straight lines on  $[t_k^n + T2^{-n-1}, t_{k+1}^n]$ , is then given by

$$(10) \quad J^n = \sum_{k=0}^{2^n-1} \left( \int_{t_k^n + T2^{-n-1}}^{t_{k+1}^n} f(x_s^n) dx_s^n + [f, f](x_{t_k^n}^n) \mathfrak{A}(x^n; t_k^n, t_k^n + T2^{-n-1}) \right),$$

where  $\mathfrak{A}(x; s, t)$  has been defined by (9). Now, following the heuristic reasoning of Section 4.2, we replace the assumption

(H1) The path  $(x, \varphi)$  belongs to  $L^{\kappa, \alpha}([0, T]; \mathbb{R}^3)$ .

by the assumption

(H2) There exists some function  $\mathfrak{A}(x; s, t)$  which is the limit of  $\mathfrak{A}(x^n; s, t)$  for all  $0 \leq s \leq t \leq T$ .

Let us note that the assumption (H1) implies (H2) if  $f$  is the differential form  $f(x) = \frac{1}{2}(x^1 dx^2 - x^2 dx^1)$ . In (H2), there is no more reference to  $f$ , while the set  $L^{\kappa, \alpha}([0, T]; \mathbb{R}^3)$  depends *a priori* on  $f$ .

The assumption (H2) means that  $\mathfrak{A}(x^{n^2}; t_k^n, t_{k+1}^n)$  (which is equal to  $\mathfrak{A}(x^{n^2}; t_k^n, t_{k+1}^n + T2^{-n-1})$ ) is equivalent to  $\mathfrak{A}(x; t_k^n, t_{k+1}^n)$  as  $n \rightarrow \infty$ . Hence, one may replace (10) by

$$(11) \quad J^n = \sum_{k=0}^{2^n-1} \left( \int_{t_k^n + T2^{-n-1}}^{t_{k+1}^n} f(x_s^{\Pi^n}) dx_s^{\Pi^n} + [f, f](x_{t_k^n}^n) \mathfrak{A}(x; t_k^n, t_{k+1}^n) \right).$$

This form has the following advantage over the previous one: Under (H2), one can study, in the same way as for the proof of the Young integrals, the convergence of  $J^n$  by studying  $J^{n+1} - J^n$  in order to prove that  $\sum_{n \geq 0} (J^{n+1} - J^n)$  converges and to define the integral of  $x$  as the limit of this series plus  $J^0$ . This method is one of the core of the theory of rough paths.

Still using some approximation, we change (11) into

$$J^n = \sum_{k=0}^{2^n-1} \int_{t_k^n + T2^{-n-1}}^{t_{k+1}^n} f(x_s^{\Pi^n})(x_{t_{k+1}^n}^n - x_{t_k^n}^n) \frac{ds}{\Delta_n t} + \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} [f, f](x_s^{\Pi^n}) \mathfrak{A}(x; t_k^n, t_{k+1}^n) \frac{ds}{\Delta_n t},$$

with  $\Delta_n t = T2^{-n}$ . We use this expression to motivate our introduction of some algebraic structures. Our wish is then to interpret  $\mathfrak{A}(x; s, t)$  as some “vector”, in the same way as we can see, from the geometrical point of view,  $x_t - x_s$  as the vector that link the two points  $x_s$  and  $x_t$  and  $\mathbb{R}^2$  seen as some affine space. As we will see it below,  $\mathfrak{A}(x; s, t)$  is different in general from  $\mathfrak{A}(x; 0, t) - \mathfrak{A}(x; 0, s)$ . Hence, the Euclidean structure is not adapted.

We will now construct some space  $A(\mathbb{R}^2)$  of dimension 3, that will play the role both of an affine and a vector space, and the kind of vector we will consider will be  $(x_t^1 - x_s^1, x_t^2 - x_s^2, \mathfrak{A}(x; s, t))$ . Nevertheless, there will be constructed from the paths  $(x_t^1, x_t^2, \mathfrak{A}(x; 0, t))_{t \geq 0}$  living in  $A(\mathbb{R}^2)$  seen as some affine space.

In a first time, we define this space  $A(\mathbb{R}^2)$ , then we study the approximation of paths living in this space and finally, we define an integral as limit of ordinary integrals using the previously constructed approximations.

**5.2. What happens to the area?** For a continuous path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$ , let  $y_t = \mathfrak{A}(x; 0, t)$  be the area enclosed between the curve

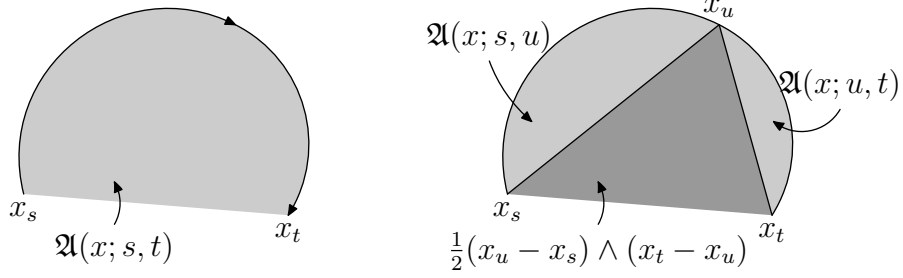


FIGURE 5. A geometrical illustration of (12).

$x|_{[0,t]}$  and its chord  $\overline{x_0x_t}$ , where  $\mathfrak{A}$  have been defined by (9). This path  $y$  is well defined by (9) and belongs to  $C^\alpha([0, T]; \mathbb{R})$ .

As we have seen that  $x \mapsto \mathfrak{A}(x; 0, \cdot)$  is not continuous in general on  $C^\alpha([0, T]; \mathbb{R}^2)$  for  $\alpha \leq 1/2$ , we are nonetheless willing to define the equivalent of a process  $y$  for an irregular path. This can be achieved using an algebraic setting. Remark first that if  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha \in (1/2, 1]$ ,

$$(12) \quad \mathfrak{A}(x; s, t) = \mathfrak{A}(x; s, u) + \mathfrak{A}(x; u, t) + \frac{1}{2}(x_u - x_s) \wedge (x_t - x_u)$$

for all  $0 \leq s < u < t \leq T$  (See Figure 5). Here,  $\wedge$  is the vector product between two vectors:  $a \wedge b = a^1b^2 - a^2b^1$ .

**5.3. Linking points.** In a first time, we consider, for a piecewise smooth paths  $x$ , the path  $(x^1, x^2, \mathfrak{A}(x))$  living in a three dimensional space. If  $u$  belongs to  $\mathbb{R}$ , then we set

$$(13) \quad \mathfrak{C}(x, u; t) = (x_t^1, x_t^2, u + \mathfrak{A}(x; 0, t))$$

for  $t \in [0, T]$ . In the following, we may think that  $x$  represents a 2-dimensional control trajectory of the position of a particle moving in  $\mathbb{R}^3$ .

Given two points  $a = (a^1, a^2, a^3)$  and  $b = (b^1, b^2, b^3)$ , we wish to construct a piecewise smooth path  $x$  from  $[0, 1]$  to  $\mathbb{R}^2$  such that the continuous path  $(x_t, a^3 + \mathfrak{A}(x; 0, t))$  from  $[0, 1]$  to  $\mathbb{R}^3$  goes from  $a$  to  $b$ .

Such a path is easily constructed. We give here a simple example, that serves as a prototype for our approach. Our choice, drawn in Figure 6, is

$$x_t = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} + \frac{\sqrt{|b^3 - a^3|}}{\sqrt{\pi}} \begin{bmatrix} \cos(4\pi t) - 1, \\ \operatorname{sgn}(b^3 - a^3) \sin(4\pi t) \end{bmatrix} \text{ if } t \in \left[0, \frac{1}{2}\right],$$

$$\text{and } x_t = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} + (2t - 1) \begin{bmatrix} b^1 - a^1 \\ b^2 - a^2 \end{bmatrix} \text{ if } t \in \left[\frac{1}{2}, 1\right].$$

Given two points  $a$  and  $b$  in  $\mathbb{R}^3$ , let us consider two paths  $x$  and  $y$  in  $C_p^1([0, T]; \mathbb{R}^2)$  such that  $x_0 = y_0 = 0$  and  $\mathfrak{C}(x, 0; T) = a$ ,  $\mathfrak{C}(y, 0; T) = b$ .

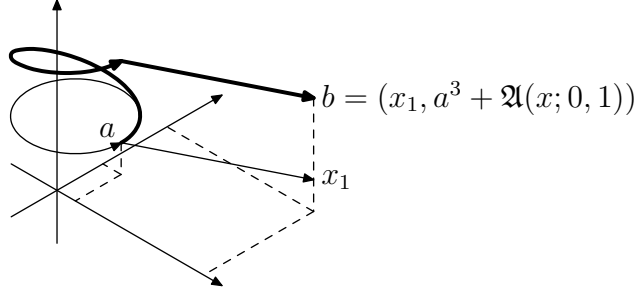
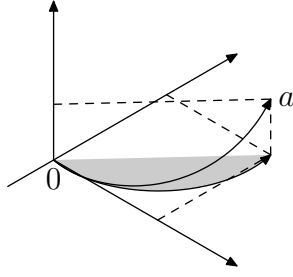
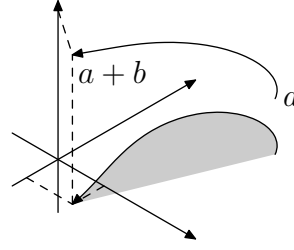


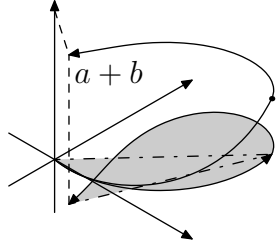
FIGURE 6. A simple path  $(x, y)$  from  $a$  to  $b$  controlled by a path  $x$  in  $\mathbb{R}^2$ .



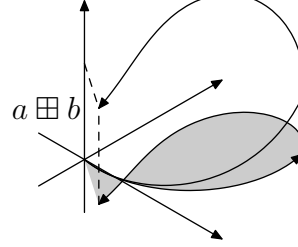
(a) The path  $\mathfrak{C}(x, 0)$ .



(b) The path  $\mathfrak{C}(\pi(a) + y, a^3)$ .



(c) The path  $\mathfrak{C}(x, 0) \cdot \mathfrak{C}(\pi(a) + y, a^3)$



(d) The path  $\mathfrak{C}(x \cdot y, 0)$

FIGURE 7. Reaching a point with the constraint of passing through another point

The concatenation  $x \cdot y$  of  $x$  and  $y$  gives rise to a path that goes from  $0$  to  $\pi(a + b)$  by passing through  $\pi(a)$ , where  $\pi$  is the projection  $\pi(a^1, a^2, a^3) = (a^1, a^2)$ . What can then be said on  $\mathfrak{C}(x \cdot y; 0, 2T)$ ? Due to (12), we get that  $\mathfrak{C}(x \cdot y)$  is a path that goes from  $0$  to the point we denote by  $a \boxplus b$  defined by

$$a \boxplus b = \left( a^1 + b^1, a^2 + b^2, a^3 + b^3 + \frac{1}{2} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \wedge \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} \right).$$

With this notation,  $\boxplus$  clearly defines an operation on  $\mathbb{R}^3$ , which is different from the usual addition (geometrically equivalent to some translation) in this space  $\mathbb{R}^3$ . In addition,  $\mathfrak{C}(x \cdot y, 0)$  passes through the point  $a$ .

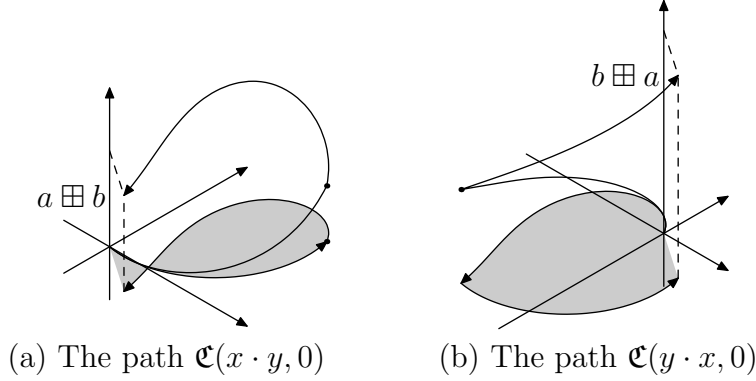


FIGURE 8. Illustration of the non-commutativity of  $A(\mathbb{R}^2)$ .

As illustrated in Figure 7, this gives rise to a different path as the one obtained by the concatenation of  $\mathfrak{C}(x, 0)$  and  $\mathfrak{C}(\pi(a) + y; a^3)$ , which ends at  $a + b$ .

**5.4. The space  $\mathbb{R}^3$  as a non-commutative group.** We have now equipped  $\mathbb{R}^3$  with an operation  $\boxplus$ , which is easily proved to be associative. When equipped with this operation  $\boxplus$ , we denote  $\mathbb{R}^3$  by  $A(\mathbb{R}^2)$ . We also set

$$[a, b] = a \boxplus b - b \boxplus a = \left( 0, 0, \frac{1}{2} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \wedge \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} \right).$$

This bracket  $[\cdot, \cdot]$  is our course linked to the fact that  $(A(\mathbb{R}^2), \boxplus)$  is a non-commutative group, called the *Heisenberg group* (see Appendix A.2).

**Lemma 3.** *The space  $(A(\mathbb{R}^2), \boxplus)$  is a non-commutative group with 0 as the neutral element. The inverse of any element  $a = (a^1, a^2, a^3)$  is  $-a = (-a^1, -a^2, -a^3)$ .*

*Proof.* That the inverse of  $a$  is  $-a$  is easily verified since

$$(-a^1, -a^2, -a^3) \boxplus (a^1, a^2, a^3) = -\frac{1}{2}[a, a] = 0.$$

The non-commutativity of  $\boxplus$  in general follows from  $b \boxplus a = a \boxplus b \boxplus [b, a]$ .  $\square$

The non-commutativity of  $\boxplus$  is illustrated in Figure 8. Of course, if  $a, b \in \mathbb{R}^3$  are of type  $a = (a^1, a^2, 0)$  and  $b = (b^1, b^2, 0)$ , then  $a \boxplus b = b \boxplus a$ : the non-commutativity concerns only the third component. If  $x : [0, 1] \rightarrow \mathbb{R}^2$  goes from  $a$  to  $b$  and  $y : [0, 1] \rightarrow \mathbb{R}^2$  goes from  $b$  to  $c$ , then  $x \cdot y$  goes from  $a$  to  $c$  and  $(y - b + a) \cdot (b - a + x)$  goes from also from  $a$  to  $c$ . Yet the area enclosed between these two paths and its chord is not the same.



It is now easy to remark that  $A(\mathbb{R}^2)$  is both a Lie algebra and a Lie group. For some introduction on these notion, see among many other books [SW93, War83, Var84, Hal03].

**Lemma 4.** *The space  $(A(\mathbb{R}^2), [\cdot, \cdot])$  is a Lie algebra.*

*Proof.* Clearly,  $(a, b) \mapsto [a, b]$  is bilinear,  $[a, b] = -[b, a]$  and the Jacobi identity is easily satisfied:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad \forall a, b, c \in A(\mathbb{R}^2).$$

This proves the Lemma.  $\square$

As for  $\mathbb{R}^3$ ,  $A(\mathbb{R}^2)$  may be equipped with the multiplication by a scalar, which is  $(\lambda, x) = \lambda \cdot x \stackrel{\text{def}}{=} (\lambda x^1, \lambda x^2, \lambda x^3)$  if  $x = (x^1, x^2, x^3) \in A(\mathbb{R}^2)$  and  $\lambda \in \mathbb{R}$ . But unlike  $\mathbb{R}^3$ , this operation is not distributive, since

$$\lambda \cdot (x \boxplus y) = (\lambda x) \boxplus (\lambda y) + \lambda(1 - \lambda)[x, y].$$

Thus,  $(A(\mathbb{R}^2), \boxplus, \cdot)$ , where  $\cdot$  denotes the multiplication by a scalar, is not a module.

Another natural external law equip naturally  $A(\mathbb{R}^2)$ , which is the *dilatation*. Given  $\lambda \in \mathbb{R}$ , we set

$$\delta_\lambda x = (\lambda x^1, \lambda x^2, \lambda^2 x^3) \text{ for } x = (x^1, x^2, x^3) \in A(\mathbb{R}^2).$$

Let us note that

$$\delta_\lambda(x \boxplus y) = (\delta_\lambda x) \boxplus \delta_\lambda y \text{ and } \delta_\lambda \delta_\mu x = \delta_{\lambda\mu} x$$

for  $\lambda, \mu \in \mathbb{R}$  and  $x \in A(\mathbb{R}^2)$ . However, we do not have that  $\delta_{\lambda+\mu} x = \delta_\lambda x \boxplus \delta_\mu x$ . Hence,  $(A(\mathbb{R}^2), \boxplus, \delta)$  is not a module.

This space  $A(\mathbb{R}^2)$  is equipped with a norm defined by

$$(14) \quad |a|_\star = \max\{|a^1|, |a^2|, |a^3|\}$$

and a *homogeneous norm* defined by

$$(15) \quad |a| = \max \left\{ |a^1|, |a^2|, \sqrt{\frac{1}{2}|a^3|} \right\},$$

which means that  $|a| = 0$  if and only if  $a = 0$ ,

$$|\delta_\lambda x| = |\lambda| \cdot |x| \text{ for } \lambda \in \mathbb{R} \text{ and } x \in A(\mathbb{R}^2),$$

and  $|-x| = |x|$  for all  $x \in A(\mathbb{R}^2)$  (see also Section A.1).

Remark that this choice ensures that  $|a \boxplus b| \leq 3/2(|a| + |b|)$ . We will see below in Sections 5.10 and A.1 that this homogeneous norm is equivalent to another homogeneous norm  $\|\cdot\|_{\text{CC}}$  which allows us to define a distance between two points  $a$  and  $b$  in  $A(\mathbb{R}^2)$  by  $\|(-a) \boxplus b\|_{\text{CC}}$  (with the  $\|\cdot\|_{\text{CC}}$ , the triangular inequality is satisfied, which is not the case with  $|\cdot|$ ). Because of the square root in the definition of  $|\cdot|$ , this distance is not equivalent to the one generated by  $|\cdot|_\star$ . Yet it generates the same topology.

*Remark 5.* Because  $|\cdot|$  does not satisfy the triangle inequality,  $d : (a, b) \mapsto |(-a) \boxplus b|$  does not define a distance. However, this may be called a *near-metric* because  $d(a, b) \leq C(d(a, c) + d(c, b))$  for some constant  $C > 0$  and all  $a, c, b \in A(\mathbb{R}^2)$ .

From this, we easily deduce that  $A(\mathbb{R}^2)$  is also a Lie group. We recall that a *Lie group*  $(G, \times)$  is a group with a differentiable manifold structure (and in particular a norm) such that  $(x, y) \mapsto x \times y$  and  $x \mapsto x^{-1}$  are continuous (see for example [SW93, War83, Var84, Hal03] and many other books).

**Lemma 5.** *The space  $(A(\mathbb{R}^2), \boxplus)$  is a Lie group.*

*Proof.* The continuity of  $(x, y) \mapsto x \boxplus y$  and  $x \mapsto -x$  is easily proved.  $\square$

**5.5. Enhanced paths and their approximations.** Of course, we have constructed the space  $A(\mathbb{R}^2)$  with the idea of considering paths living in  $A(\mathbb{R}^2)$ , the third component giving all the information we need.

Basically, a continuous path with values in  $A(\mathbb{R}^2)$  is a continuous path with values in the Euclidean space  $\mathbb{R}^3$  (recall that the norm  $|\cdot|_*$  we put on  $A(\mathbb{R}^2)$  is equivalent to the Euclidean norm). However, we will use the group operation  $\boxplus$  of  $A(\mathbb{R}^2)$  in replacement as the translation by a vector in  $\mathbb{R}^3$ , and thus the paths we consider will be seen differently from the usual paths.

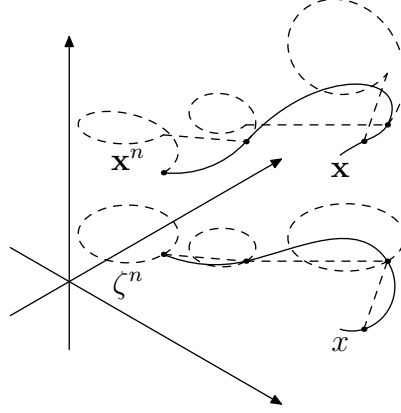
Let us recall that  $(\mathbb{R}^2, +)$  is in some sense contained in  $(A(\mathbb{R}^2), \boxplus)$ , and plays then a special role.

**Definition 3.** Given a continuous path  $x$  with values in  $\mathbb{R}^2$ , a continuous path  $\mathbf{x}$  with values in  $A(\mathbb{R}^2)$  with  $x = (\mathbf{x}^1, \mathbf{x}^2)$  may then be called an *enhanced path*, or a *path lying above  $x$* . Given a path  $x : [0, T] \rightarrow \mathbb{R}^2$ , a path  $\mathbf{x} : [0, T] \rightarrow A(\mathbb{R}^2)$  with lies above  $x$  is called a *lift of  $x$* .

Let  $x$  and  $y$  be two smooth paths lifted as  $\mathbf{x} = \mathfrak{C}(x, 0)$  and  $\mathbf{y} = \mathfrak{C}(y, 0)$ , where  $\mathfrak{C}$  has been defined by (13). We have seen that the usual concatenation  $\mathbf{x} \cdot \mathbf{y}$  of  $\mathbf{x}$  and  $\mathbf{y}$  seen as paths with values in  $\mathbb{R}^3$  is different from the path  $\mathfrak{C}(x \cdot y, 0)$ . We introduce then a new kind of concatenation of two paths  $\mathbf{x} : [0, T] \rightarrow A(\mathbb{R}^2)$  and  $\mathbf{y} : [0, S] \rightarrow A(\mathbb{R}^2)$ . This concatenation is defined by

$$(\mathbf{x} \boxminus \mathbf{y})_t = \begin{cases} \mathbf{x}_t & \text{if } t \in [0, T], \\ \mathbf{x}_T \boxplus ((-\mathbf{y}_0) \boxplus \mathbf{y}_{t-T}) & \text{if } t \in [T, T+S] \end{cases}$$

and gives rise to a continuous path from  $[0, T+S]$  to  $A(\mathbb{R}^2)$  when  $\mathbf{x}$  and  $\mathbf{y}$  are continuous. In addition,  $\mathbf{x} \boxminus \mathbf{y}$  lies above  $x \cdot y$  if  $\mathbf{x}$  (resp.  $\mathbf{y}$ ) lies above  $x$  (resp.  $y$ ). Yet we have to be warned of an important point: unlike the usual concatenation,  $(\mathbf{x} \boxminus \mathbf{y})_{T+S} \neq (\mathbf{x} \cdot \mathbf{y})_{T+S}$  in general, as we saw it in Section 5.3.

FIGURE 9. Approximation of a path  $\mathbf{x}$  in  $A(\mathbb{R}^2)$ .

If  $x : [0, T] \rightarrow \mathbb{R}^2$  and  $y : [0, S] \rightarrow \mathbb{R}^2$  are two piecewise smooth paths, then this concatenation satisfies

$$\mathfrak{C}(x \cdot y, 0) = \mathfrak{C}(x, 0) \square \mathfrak{C}(y, 0).$$

For two points  $a$  and  $b$  in  $A(\mathbb{R}^2)$ , let  $\psi_{a,b} \in C_p^1([0, 1]; \mathbb{R}^3)$  be a smooth path joining  $a$  and  $b$  lying above  $\zeta_{a,b} : [0, 1] \rightarrow \mathbb{R}^2$  (for example, we can use the one of Section 5.3). By definition of  $\zeta_{a,b}$  and  $\psi_{a,b}$ ,  $\psi_{a,b}(t) = \mathfrak{C}(\zeta_{a,b}, a^3; t)$ . Moreover, for  $a, b, c$  in  $A(\mathbb{R}^2)$ ,

$$\psi_{a,b} \square \psi_{b,c} = \mathfrak{C}(\zeta_{a,b} \cdot \zeta_{b,c}, a^3).$$

Thus,  $\psi_{a,b} \square \psi_{b,c}$  is a path that goes from  $a$  to  $c$  by passing through  $b$ .

Let  $\mathbf{x}$  be a continuous path from  $[0, T]$  living in  $A(\mathbb{R}^2)$ . It is then natural to look for an approximation of  $\mathbf{x}$  given by the sequence a paths

$$\mathbf{x}^n = \psi_{\mathbf{x}_0^n, \mathbf{x}_{t_1^n}} \square \psi_{\mathbf{x}_{t_1^n}, \mathbf{x}_{t_2^n}} \square \cdots \square \psi_{\mathbf{x}_{t_{n-1}^n}, \mathbf{x}_{t_n^n}}.$$

The path  $\mathbf{x}^n$  satisfies  $\mathbf{x}^n(t) = \mathbf{x}(t)$  for the dyadic times  $t$  at level  $n$ . In addition,

$$\mathbf{x}^n = \mathfrak{C}(\zeta^n, \mathbf{x}_0^3) \text{ with } \zeta^n = \zeta_{\mathbf{x}_0^n, \mathbf{x}_{t_1^n}} \cdot \zeta_{\mathbf{x}_{t_1^n}, \mathbf{x}_{t_2^n}} \cdot \cdots \cdot \zeta_{\mathbf{x}_{t_{n-1}^n}, \mathbf{x}_{t_n^n}},$$

and it is easily proved that  $\zeta^n$  converges uniformly to  $x$ , the path above which  $\mathbf{x}$  lives (See Figure 9).

Now, there are two natural questions: (1) Provided that  $\mathbf{x}$  is regular enough, does  $\mathbf{x}^n$  converge to  $\mathbf{x}$ , in which sense? (2) Is it possible to construct  $\mathfrak{I}(\mathbf{x})$  as the limit of the  $\mathfrak{I}(\zeta^n)$ 's, which are then ordinary integrals?

**5.6. Hölder continuous enhanced paths.** We have defined the space  $A(\mathbb{R}^2)$  as the space  $\mathbb{R}^3$  with a special non-commutative group structures, which is different from the translation.

Let  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$  and  $x_0 = 0$ . Set  $\mathbf{x} = (x^1, x^2, \mathfrak{A}(x))$ . With (12),

$$(-\mathbf{x}_s) \boxplus \mathbf{x}_t = (x_t^1 - x_s^1, x_t^2 - x_s^2, \mathfrak{A}(x; s, t)),$$

which means that  $(-\mathbf{x}_s) \boxplus \mathbf{x}_t$  can be constructed from the path  $x$  restricted to  $[s, t]$ . The same is true even if  $x_0 \neq 0$ .

For a path  $\mathbf{x}$  from  $[0, T]$  to  $A(\mathbb{R}^2)$ ,  $\mathbf{x}_{s,t} \stackrel{\text{def}}{=} (-\mathbf{x}_s) \boxplus \mathbf{x}_t$  may then be interpreted as an “increment” of  $\mathbf{x}$ , and we indeed get the following trivial identity  $\mathbf{x}_t = \mathbf{x}_s \boxplus \mathbf{x}_{s,t}$  for all  $0 \leq s \leq t \leq T$ , which is the equivalent to  $x_t = x_s + (x_t - x_s)$  in  $\mathbb{R}^2$ . Let us note that in general  $\mathbf{x}_{s,t}^3$  is different from  $\mathbf{x}_t^3 - \mathbf{x}_s^3$ , although  $\mathbf{x}_{s,t}^i = \mathbf{x}_t^i - \mathbf{x}_s^i$  for  $i = 1, 2$ .

Similarly, we may write the value of  $\mathbf{x}_t$  at time  $t$  in function the values of  $\mathbf{x}$  at times  $s \leq r \leq t$ :

$$(16) \quad \mathbf{x}_t = \mathbf{x}_s \boxplus \mathbf{x}_{s,r} \boxplus \mathbf{x}_{r,t}$$

for all  $0 \leq s \leq r \leq t \leq T$ . When one sees  $\mathbf{x}$  as a geometric object, (16) yields

$$(17) \quad \mathbf{x}_{[s,t]} = \mathbf{x}_{[s,r]} \boxplus \mathbf{x}_{[r,t]},$$

for all  $0 \leq s \leq r \leq t \leq T$ .

From now, to endow that fact that we work in  $A(\mathbb{R}^2)$ , we have to think to paths from  $[0, T]$  to  $A(\mathbb{R}^2)$  as continuous paths  $\mathbf{x}$  satisfying (17), although this relation is verified by any continuous path from  $[0, T]$  to  $\mathbb{R}^3$  (which means also that there are an infinite number of paths lying above a continuous path from  $[0, T]$  to  $\mathbb{R}^2$ ). But we will see below that if  $\mathbf{x}$  lies above a smooth path  $x$  and is also quite regular (in a sense to be defined), then (17) and the regularity condition will impose some “constraint” on the path  $\mathbf{x}$ .

**Lemma 6** ([Lyo98, Lemma 2.2.3, p. 250]). *Let  $\mathbf{x}$  and  $\mathbf{y}$  be two continuous paths from  $[0, T]$  to  $A(\mathbb{R}^2)$  such that  $(\mathbf{x}^1, \mathbf{x}^2) = (\mathbf{y}^1, \mathbf{y}^2)$ . Then there exists a continuous path  $\varphi : [0, T] \rightarrow \mathbb{R}$  such that  $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 + \varphi)$ , which means that*

$$(18) \quad ((-\mathbf{y}_s) \boxplus \mathbf{y}_t)^3 = ((-\mathbf{x}_s) \boxplus \mathbf{x}_t)^3 + \varphi_t - \varphi_s$$

for all  $0 \leq s \leq t \leq T$ .

*Proof.* It is sufficient to set  $\varphi_t = ((-\mathbf{y}_0) \boxplus \mathbf{y}_t)^3 - ((-\mathbf{x}_0) \boxplus \mathbf{x}_t)^3$ , which clearly satisfies (18).  $\square$

**Notation 3.** We denote by  $C^\alpha([0, T]; A(\mathbb{R}^2))$  the set of continuous paths  $\mathbf{x} : [0, T] \rightarrow A(\mathbb{R}^2)$  and such that

$$\|\mathbf{x}\|_\alpha = |\mathbf{x}_0| + \sup_{0 \leq s < t \leq T} \frac{|(-\mathbf{x}_s) \boxplus \mathbf{x}_t|}{|t - s|^\alpha}$$

is finite. If  $x = (x^1, x^2)$  is a path in  $C^\alpha([0, T]; \mathbb{R}^2)$  and  $\mathbf{x} = (x^1, x^2, y)$  a path in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ , then we say that  $\mathbf{x}$  lies above  $x$ .

**Lemma 7.** *Let  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$ . Then  $\mathbf{x} = (x, \mathfrak{A}(x; 0, \cdot))$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ . In addition the map  $x \mapsto \mathbf{x}$  is Lipschitz continuous from  $(C^\alpha([0, T]; \mathbb{R}^2), \|\cdot\|_\alpha)$  to  $(C^\alpha([0, T]; A(\mathbb{R}^2)), \|\cdot\|_\alpha)$ .*

*Proof.* By construction,  $\mathbf{x}$  is a path with value in  $A(\mathbb{R}^2)$ . Let us note that  $(-\mathbf{x}_s) \boxplus \mathbf{x}_t = (x_t^1 - x_s^1, x_t^2 - x_s^2, \mathfrak{A}(x; s, t))$ . From the construction of the Young integral (more specifically, from a variation of (4)),

$$(19) \quad |\mathfrak{A}(x; s, t)| \leq \zeta(2\alpha - 1)(t - s)^{2\alpha} \|x\|_\alpha^2$$

and then the result is proved.  $\square$

Let us note that in the previous proof, (19) does not mean that  $t \mapsto \mathfrak{A}(x; 0, t)$  is  $2\alpha$ -Hölder continuous (in which case  $2\alpha > 1$ ). Indeed,  $t \mapsto \mathfrak{A}(x; 0, t)$  is only  $\alpha$ -Hölder continuous, since  $x$  is  $\alpha$ -Hölder continuous.

On the other hand, any path in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha > 1/2$  can be expressed as a path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  lifted using its area  $\mathfrak{A}(x)$ .

**Lemma 8.** *Let  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha > 1/2$ . Then  $\mathbf{x} = \mathfrak{C}(x, \mathbf{x}_0^3) = (x, \mathbf{x}_0^3 + \mathfrak{A}(x))$  with  $x = (\mathbf{x}^1, \mathbf{x}^2)$ .*

*Remark 6.* If for some  $\alpha > 1/2$ ,  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$  is composed of paths of type  $\mathbf{x}^n = (x^n, \mathfrak{A}(x^n))$  with  $x^n \in C^\alpha([0, T]; \mathbb{R}^2)$  and  $\mathbf{x}^n$  converges in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  to some  $\mathbf{x}$ , then  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  is necessarily of type  $\mathbf{x} = (x, \mathfrak{A}(x))$  with  $x \in C^\alpha([0, T]; \mathbb{R}^2)$ . In Proposition 2 below, we will see how to construct a family of paths  $\mathbf{x}^n \in C^1([0, T]; A(\mathbb{R}^2))$  converging to  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha > 1/3$ . Thus, if one considers a path with values in  $A(\mathbb{R}^2)$  which is not of type  $(x, \mathfrak{A}(x))$  but which is piecewise smooth, one has to interpret it as a path in  $C^{1/2}([0, T]; A(\mathbb{R}^2))$  in order to identify it with a family of converging paths.

*Proof.* From Lemma 7,  $\mathbf{y} = \mathfrak{C}(x, \mathbf{x}_0^3)$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ , and from Lemma 6, there exists a function  $\varphi : [0, T] \rightarrow \mathbb{R}$  such that  $((-\mathbf{x}_s) \boxplus \mathbf{x}_t)^3 = ((-\mathbf{y}_s) \boxplus \mathbf{y}_t)^3 + \varphi_t - \varphi_s$  for all  $0 \leq s \leq t \leq T$ . Hence,  $\sqrt{|\varphi_t - \varphi_s|} \leq \|\mathbf{x}\|_\alpha |t - s|^\alpha$  and then  $|\varphi_t - \varphi_s| \leq \|\mathbf{x}\|_\alpha^2 |t - s|^{2\alpha}$ . As  $\alpha > 1/2$ , necessarily  $\varphi$  is constant.  $\square$

As we saw it, one can add a path with values in  $\mathbb{R}$  to the third component of a path with values in  $A(\mathbb{R}^2)$  to get a new path with values in  $A(\mathbb{R}^2)$ . Although a path with value in  $\mathbb{R}^2$  which is regular enough can be naturally lifted as a path with values in  $\mathbb{R}^3$ , we gain one degree of freedom: there are an infinite number of paths that lie above a path in  $\mathbb{R}^2$ . The next lemma, whose proof is immediate, precises the kind of paths we have to use to stay in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ .

**Lemma 9.** *For  $\alpha \leq 1/2$ , let  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  and  $\varphi \in C^{2\alpha}([0, T]; \mathbb{R})$ . Then  $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 + \varphi)$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ .*

Any path in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  can be seen as the limit of paths naturally constructed above path of finite variation. Before proving this,

we state a lemma on relative compactness, which is just an adaptation of Lemma 2.

**Lemma 10.** *Let  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  such that  $\mathbf{x}^n \in C^\alpha([0, T]; A(\mathbb{R}^2))$  and is bounded. Then there exists  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  and a subsequence of  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  that converges to  $\mathbf{x}$  in  $(C^\alpha([0, T]; A(\mathbb{R}^2)), \|\cdot\|_\beta)$  for each  $\beta < \alpha$ .*

We shall now prove the main result of this section: any path  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2)$  may be identified as the limit of  $\mathfrak{C}(x^n, \mathbf{x}_0^3)$ , where  $x^n$  are paths in  $C_p^\infty([0, T]; \mathbb{R}^2)$ . Paths taking their values in  $A(\mathbb{R}^2)$  are then objects that are easier to deal with than sequences of paths with loops as we did previously.

Let  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2)$  lying above  $x$ . Denote by  $x^{\Pi^n}$  the linear interpolation of  $x$  along the dyadic partition  $\Pi^n = \{t_k^n\}_{k=0, \dots, 2^n}$  at level  $n$ , with  $t_k^n = Tk/2^n$ . Also define

$$(20a) \quad \theta_k^n = ((-\mathbf{x}_{t_{k+1}^n}) \boxplus \mathbf{x}_{t_k^n})^3.$$

Set  $\Phi^n = \{y_k^n\}_{k=0, \dots, 2^n-1}$  with  $y_k^n : [t_k^n, t_{k+1}^n] \rightarrow \mathbb{R}^2$  and

$$(20b) \quad y_k^n(t) = \sqrt{\frac{|\theta_k^n|}{\pi}} \begin{bmatrix} \cos\left(2\pi \frac{t-t_k^n}{t_{k+1}^n-t_k^n}\right) - 1 \\ \operatorname{sgn}(\theta_k^n) \sin\left(2\pi \frac{t-t_k^n}{t_{k+1}^n-t_k^n}\right) \end{bmatrix}.$$

Finally, set

$$(20c) \quad x_t^n = x^{\Pi^n} \boxtimes \Phi^n(t/2) \text{ for } t \in [0, T] \text{ and } \mathbf{x}^n = (x^n, \mathbf{x}_0^3 + \mathfrak{A}(x^n; 0, \cdot)).$$

This corresponds to join the points of  $\{\mathbf{x}_{t_k^n}\}_{k=0, \dots, 2^n}$  by the simple paths constructed in Section 5.3 (see Figure 6).

**Proposition 2.** *With the previous notations (20a)-(20c),  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  is uniformly bounded in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  and converges to  $\mathbf{x}$  with respect to  $\|\cdot\|_\beta$  for all  $\beta < \alpha$ .*

*Remark 7.* We have considered a path  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  above a path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$ , but we have not shown how to construct such a path, except of  $\alpha > 1/2$ . For that, we may either use the results in [LV05], that assert it is always possible to do so, or to deal with particular cases. For example, many trajectories of stochastic processes have been dealt with (Brownian motion [Sip93], semi-martingales [CL05], fractional Brownian motion [CQ02, Cou06], Wiener process [LLQ02], ... see also the book [FV06b]). All these results are done in general in connection with an approximation results.

Choosing a path  $\mathbf{x}$  above  $x$  corresponds to a determination of the limit of  $\mathfrak{A}(x^n; s, t)$  where  $x^n$  converges to  $x$ , and is then a slightly weaker hypothesis than (H2).

*Proof of Proposition 2.* Let us note first that for  $t = t_k^n$ ,  $\mathbf{x}_{t_k^n}^n = \mathbf{x}_{t_k^n}$ . For  $t \in [0, T)$ , let  $\underline{M}(t, n)$  be the biggest integer such that  $t_{\underline{M}(t, n)}^n \leq t$ .

Then, for  $0 \leq t < T$ ,

$$\begin{aligned} |\mathbf{x}_t^n - \mathbf{x}_t| &\leq |\mathbf{x}_t^n - \mathbf{x}_{t_{\underline{M}(t,n)}^n}^n| + |\mathbf{x}_t - \mathbf{x}_{t_{\underline{M}(t,n)}^n}^n| \\ &\leq \max\left\{\sqrt{|\theta_k^n|/\pi}, |\mathbf{x}_{t_{\underline{M}(t,n)+1}^n} - \mathbf{x}_{t_{\underline{M}(t,n)}^n}^n|\right\} + \|\mathbf{x}\|_\alpha (t - t_{\underline{M}(t,n)}^n)^\alpha \\ &\leq 2\|\mathbf{x}\|_\alpha T^\alpha 2^{-\alpha n}. \end{aligned}$$

This prove that  $\mathbf{x}^n$  converges uniformly to  $\mathbf{x}$ .

The convergence in  $C^\beta([0, T]; A(\mathbb{R}^2))$  follows from the uniform boundedness of the  $\alpha$ -Hölder norm of  $\mathbf{x}^n$  and Lemma 10.

So, it remains to estimate the  $\alpha$ -Hölder norm of  $\mathbf{x}^n$  in  $A(\mathbb{R}^2)$ . For  $0 \leq s < t \leq T$ , let  $\overline{M}(s, n)$  be the smallest integer such that  $s \leq t_{\overline{M}(s,n)}^n$ . Then, unless  $s, t$  belongs to the same dyadic interval  $[t_k^n, t_{k+1}^n]$  for some  $k = 0, \dots, 2^n - 1$ ,

$$\mathbf{x}_{s,t}^n = \mathbf{x}_{s, t_{\overline{M}(s,n)}^n}^n \boxplus \mathbf{x}_{t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n}^n \boxplus \mathbf{x}_{t_{\underline{M}(t,n)}^n, t}^n.$$

for all  $0 \leq s < t \leq T$ . In addition,  $\mathbf{x}_{t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n}^n = \mathbf{x}_{t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n}^n$  for any integer  $n$ . Since  $|\cdot|$  is a homogeneous norm on  $A(\mathbb{R}^2)$ , it follows that for some universal constant  $C_0$ ,

$$\begin{aligned} |\mathbf{x}_{s,t}^n| &\leq C_0 |\mathbf{x}_{s, t_{\overline{M}(s,n)}^n}^n| + C_0 |\mathbf{x}_{t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n}^n| + C_0 |\mathbf{x}_{t_{\underline{M}(t,n)}^n, t}^n| \\ &\leq C_0 |\mathbf{x}_{s, t_{\overline{M}(s,n)}^n}^n| + C_0 \|\mathbf{x}\|_\alpha (t_{\underline{M}(t,n)}^n - t_{\overline{M}(s,n)}^n)^\alpha + C_0 |\mathbf{x}_{t_{\underline{M}(t,n)}^n, t}^n|. \end{aligned}$$

Let us assume that we have proved that for some constant  $K$ ,

$$(21) \quad \|\mathbf{x}_{s,t}^n\| \leq K(t-s)^\alpha \text{ for all } t_k^n \leq s \leq t \leq t_{k+1}^n, \quad k = 0, \dots, 2^n - 1,$$

then the boundedness of  $(\|\mathbf{x}^n\|_\alpha)_{n \in \mathbb{N}}$  follows easily as in the proof of Corollary 1 by applying (21) to  $s, t$  in the same dyadic interval, and to  $|\mathbf{x}_{s, t_{\overline{M}(s,n)}^n}^n|$  and to  $|\mathbf{x}_{t_{\underline{M}(t,n)}^n, t}^n|$ .

We now turn to the proof of (21). First, let us consider that for some  $k \in \{0, \dots, 2^n - 1\}$ , either  $s, t \in [t_k^n, t_{k+1}^n - T2^{-n-1}]$  or  $s, t \in [t_k^n + T2^{-n-1}, t_{k+1}^n]$ . In the latter case,

$$\mathbf{x}_{s,t}^n \stackrel{\text{def}}{=} (-\mathbf{x}_s^n) \boxplus \mathbf{x}_t^n = \begin{bmatrix} T^{-1}2^{n+1}(t-s)(x_{t_{k+1}^n}^1 - x_{t_k^n}^2) \\ T^{-1}2^{n+1}(t-s)(x_{t_{k+1}^n}^1 - x_{t_k^n}^2) \\ 0 \end{bmatrix}$$

and then  $|\mathbf{x}_{s,t}^n| \leq \|\mathbf{x}\|_\alpha |t-s|^\alpha$ . In the former case, setting  $\Delta_n t = T2^{-n}$ ,

$$\mathbf{x}_{s,t}^n = \begin{bmatrix} \sqrt{\frac{|\theta_k^n|}{\pi}} \left( \cos\left(\frac{\pi}{\Delta_{n+1}t}(t-t_k^n)\right) - \cos\left(\frac{\pi}{\Delta_{n+1}t}(s-t_k^n)\right) \right) \\ \text{sgn}(\theta_k^n) \sqrt{\frac{|\theta_k^n|}{\pi}} \left( \sin\left(\frac{\pi}{\Delta_{n+1}t}(t-t_k^n)\right) - \sin\left(\frac{\pi}{\Delta_{n+1}t}(s-t_k^n)\right) \right) \\ \theta_k^n \frac{t-s}{\Delta_{n+1}t} \end{bmatrix}.$$

Thus, for some universal constant  $C_1$ ,

$$|\mathbf{x}_{s,t}^n| \leq C_1 2^{n+1} \sqrt{|\theta_k^n|} \frac{t-s}{T} \leq 2C_1 2^{n(1-\alpha)} \|\mathbf{x}\|_\alpha \frac{t-s}{T} \leq C_2 \|\mathbf{x}\|_\alpha (t-s)^\alpha,$$

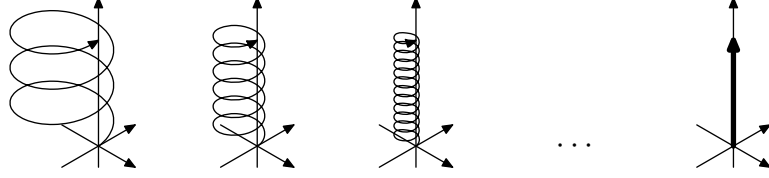


FIGURE 10. Moving freely in the third direction.

where  $C_2$  depends only on  $C_1$  and  $T$ .

Now, if  $t_k^n \leq s \leq t_k^n + T2^{-n-1} \leq t \leq t_{k+1}^n$ , we get by combining the previous estimates that

$$\begin{aligned} |\mathbf{x}_{s,t}^n| &\leq C_0 C_2 \|\mathbf{x}\|_\alpha ((t - T2^{-n-1})^\alpha + (T2^{-n-1} - s)^\alpha) \\ &\leq 2^{\alpha-1} C_0 C_2 \|\mathbf{x}\|_\alpha (t - s)^\alpha. \end{aligned}$$

We have then proved (21) with a constant which is in addition proportional to  $\|\mathbf{x}\|_\alpha$ .  $\square$

Let us come back to the Remark 6 following Lemma 8. For  $\alpha \in (1/3, 1/2]$ , let us consider  $\mathbf{x}_t = (0, 0, \varphi_t)$  where  $\varphi \in C^{2\alpha}([0, T]; \mathbb{R})$ , then one can find  $x^n \in C_p^1([0, T]; \mathbb{R})$  such that  $x^n$  converges uniformly to 0,  $\mathbf{x}^n = (x^n, \mathfrak{A}(x^n; 0, \cdot))$  is uniformly bounded in  $C^\alpha([0, T]; \mathbb{A}(\mathbb{R}^2))$  and converges in  $C^\beta([0, T]; \mathbb{A}(\mathbb{R}^2))$  to  $\mathbf{x}$  for any  $\beta < \alpha$ . For this, one may simply consider (see Figure 10)

$$z_t^n = \frac{1}{n\sqrt{\pi}} (\cos(2\pi t n^2) - 1, \sin(2\pi t n^2)),$$

and then set  $x_t^n = z_{\varphi_t}^n$ .

Thus, moving freely in the “third direction” is equivalent to accumulate the areas of small loops. Using the language of differential geometry, which we develop below, this new degree of freedom comes from the lack of commutativity of  $(\mathbb{A}(\mathbb{R}^2), \boxplus)$ : a small loop of radius  $\sqrt{\varepsilon}$  around the origin in the plane  $\mathbb{R}^2$  is equivalent to a small displacement of length  $\varepsilon$  in the third direction. To rephrase Remark 6, even if  $\varphi \in C^1([0, T]; \mathbb{R})$ , then one has to see  $\mathbf{x}$  as a path in  $C^{1/2}([0, T]; \mathbb{A}(\mathbb{R}^2))$  that may be approximated by paths in  $C_p^1([0, T]; \mathbb{A}(\mathbb{R}^2))$  which converge to  $\mathbf{x}$  only in  $\|\cdot\|_\beta$  for any  $\beta < 1/2$ . Hence, we recover the problem underlined in Section 3.2.

**5.7. Construction of the integral.** Let  $f$  be a differential form in  $\text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R})$  with  $\gamma > 1/\alpha - 1$ .

If  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{A}(\mathbb{R}^2))$  with  $\alpha > 1/2$ , then from Lemma 8,  $\mathbf{x} = (x, \mathbf{x}_0^3 + \mathfrak{A}(x))$  with  $x = (\mathbf{x}^1, \mathbf{x}^2)$ . Hence we set  $\mathfrak{J}(\mathbf{x}) \stackrel{\text{def}}{=} \mathfrak{J}(x) = \int_{x|_{[0, \cdot]}} f$  which is well defined as a Young integral.

The next proposition will be refined later.



**Proposition 3.** *Let  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2]$ . Let  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  be constructed by (20a)–(20c). Then  $(\mathfrak{J}(x^n))_{n \in \mathbb{N}}$  has a unique limit in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$  for all  $\beta < \alpha$ , which we denote by  $\mathfrak{J}(\mathbf{x})$  (of course, the limit does not depend on  $\beta$ ). Both the  $\alpha$ -Hölder continuity modulus of  $\mathfrak{J}(\mathbf{x})$  and the rate of convergence with respect to  $\|\cdot\|_\beta$  depends only on  $T$ ,  $\alpha$ ,  $\gamma$ ,  $\beta$ ,  $\|\mathbf{x}\|_\alpha$  and  $\|f\|_{\text{Lip}}$ .*

Other properties of this map  $x \mapsto \mathfrak{J}(\mathbf{x})$  will be proved below. Indeed, this map is obviously an extension of the one we have constructed beforehand on  $L^{\alpha, \alpha}([0, T]; \mathbb{R}^3)$ , with a more convenient way to encode the loops.

*Proof.* Let us fix a dyadic level  $n$ . We remark first that for  $k \in \{0, \dots, 2^n - 1\}$ ,  $t_k^n \leq s < t \leq t_{k+1}^n$ ,

$$\begin{aligned} & \mathfrak{J}(x^n; s, t) \\ &= \begin{cases} \iint_{\text{Part}^n(s, t)} [f, f](z^1, z^2) dz^1 dz^2 & \text{if } t_k^n \leq s \leq t \leq t_k^n + T2^{-n-1}, \\ \iint_{\text{Part}^n(s, t_k^n + T2^{-n-1})} [f, f](z^1, z^2) dz^1 dz^2 \\ \quad + \int_{t_k^n}^{t_k^n + 2(t - t_k^n - 2^{n+1}T)} f(x_r^{\Pi^n}) dx_r^{\Pi^n} & \text{if } t_k^n \leq s \leq t_k^n + T2^{-n-1} \leq t \leq t_{k+1}^n, \\ \int_{t_k^n + 2(s - t_k^n - 2^{n+1}T)}^{t_k^n + 2(t - t_k^n - 2^{n+1}T)} f(x_r^{\Pi^n}) dx_r^{\Pi^n} & \text{if } t_k^n + T2^{-n-1} \leq s < t \leq t_{k+1}^n, \end{cases} \end{aligned}$$

where  $\text{Part}^n(s, t)$  is the portion of the loop enclosed between the loop  $x_{[t_k^n, t_k^n + T2^{-n-1}]}$  and the segment  $\overline{x_s^n x_t^n}$ . Of course, the integral of  $f$  over  $\text{Part}^n(t_k^n, t_k^n + T2^{-n-1})$  is the integral of  $[f, f]$  over the surface of the loop  $x_{[t_k^n, t_k^n + T2^{-n-1}]}$ .

If  $t_k^n \leq s < t \leq t_k^n + T2^{-n-1}$ , then the algebraic area of  $\text{Part}^n(s, t)$  is  $\theta_k^n(t - s)2^{n+1}/T$ . In addition, the maximal distance between two points in  $\text{Part}^n(s, t)$  smaller than  $(t - s)\sqrt{|\theta_k^n|}2(t - s)2^{n+1}/T$ . Hence, we deduce that for  $r \in [s, t]$ , there exists a constant  $C$  that depends only on  $T$  such that

$$(22) \quad \left| \iint_{\text{Part}^n(s, t)} [f, f](z^1, z^2) dz^1 dz^2 - [f, f](x_r) \theta_k^n \frac{t - s}{T2^{-n-1}} \right| \leq C \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha^{2+\gamma} (t - s)^{\alpha(2+\gamma)}$$

since  $|\theta_k^n| \leq \|\mathbf{x}\|_\alpha^2 2^{-2n\alpha}$ . We also deduce that for some constant  $C'$  that depends only on  $T$ ,  $\|\mathbf{x}\|_\alpha$  and  $\|f\|_{\text{Lip}}$ ,

$$(23) \quad \left| \iint_{\text{Part}^n(s, t)} [f, f](z^1, z^2) dz^1 dz^2 \right| \leq C' (t - s)^{2\alpha}.$$

If  $t_k^n + T2^{-n-1} \leq s < t \leq t_{k+1}^n$ , then

$$(24) \quad \left| \int_{t_k^n + 2(s - t_k^n - 2^{n+1}T)}^{t_k^n + 2(t - t_k^n - 2^{n+1}T)} f(x_r^{\Pi^n}) dx_r^{\Pi^n} \right| \leq \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha (T2^n)^{1-\alpha} (t-s) \leq \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha (t-s)^\alpha.$$

It follows from (23) and (24) that for some constant  $C_1$ ,

$$(25) \quad |\mathfrak{J}(x^n; s, t)| \leq C_1 (t-s)^\alpha.$$

for all  $t_k^n \leq s \leq t \leq t_{k+1}^n$ ,  $k = 0, \dots, 2^n - 1$ .

Yet this is not sufficient to bound  $|\mathfrak{J}(x^n; s, t)|$  by  $C(t-s)^\alpha$  for all  $0 \leq s < t \leq T$ . We then use another computation.

Let us remark first that  $t_{2k}^{n+1} = t_k^n$ ,  $t_{2k+2}^{n+1} = t_{k+1}^n$ , and that

$$\begin{aligned} \mathfrak{J}(x^{\Pi^{n+1}}; t_{2k}^{n+1}, t_{2k+1}^{n+1}) + \mathfrak{J}(x^{\Pi^{n+1}}; t_{2k+1}^{n+1}, t_{2k+2}^{n+1}) - \mathfrak{J}(x^{\Pi^n}; t_{2k}^{n+1}, t_{2k+2}^{n+1}) \\ = \iint_{T_k^n} [f, f](z) dz, \end{aligned}$$

where  $T_k^n$  is  $\text{Triangle}(x_{t_{2k}^n}, x_{t_{2k+1}^n}, x_{t_{2k+2}^n})$  with area

$$\text{Area}(T_k^n) = -\frac{1}{2}(x_{t_{2k+1}^{n+1}} - x_{t_{2k}^{n+1}}) \wedge (x_{t_{2k+2}^{n+1}} - x_{t_{2k+1}^{n+1}}).$$

In addition,

$$\begin{aligned} \mathfrak{J}(x^n; t_k^n, t_k^n + T2^{-n-1}) &= \iint_{\text{Part}^n(t_k^n, t_{k+1}^n)} [f, f](z^1, z^2) dz^1 dz^2 \\ &= [f, f](x_{t_k^n}) \theta_k^n + \zeta_k^n, \end{aligned}$$

where, from (22),  $|\zeta_k^n| \leq C_2 2^{-n\alpha(2+\gamma)}$  for some constant  $C_2$  that depends only on  $\|\mathbf{x}\|_\alpha$ ,  $\|f\|_{\text{Lip}}$  and  $T$ .

Let us recall that from (12),

$$\theta_{2k}^{n+1} + \theta_{2k+1}^{n+1} + \frac{1}{2}(x_{t_{2k+1}^{n+1}} - x_{t_{2k}^{n+1}}) \wedge (x_{t_{2k+2}^{n+1}} - x_{t_{2k+1}^{n+1}}) = \theta_k^n.$$

Hence, we get easily that

$$\begin{aligned} \mathfrak{J}(x^{n+1}; t_{2k}^{n+1}, t_{2k+1}^{n+1}) + \mathfrak{J}(x^{n+1}; t_{2k+1}^{n+1}, t_{2k+2}^{n+1}) - \mathfrak{J}(x^n; t_{2k}^{n+1}, t_{2k+2}^{n+1}) \\ = \zeta_{2k}^{n+1} + \zeta_{2k+1}^{n+1} - \zeta_k^n + ([f, f](x_{t_{2k+1}^{n+1}}) - [f, f](x_{t_{2k}^{n+1}})) \theta_{2k+1}^{n+1} + \xi_k^n, \end{aligned}$$

where

$$\xi_k^n = \iint_{T_k^n} [f, f](z^1, z^2) dz^1 dz^2 - [f, f](x_{t_{2k}^{n+1}}) \text{Area}(T_k^n).$$

As in (22),

$$|\xi_k^n| \leq \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha^{2+\gamma} \Delta_n t^{\alpha(\gamma+2)},$$

where  $\Delta_n t = T2^{-n}$ . Thus, for some constant  $C_3$  that depends only on  $\|f\|_{\text{Lip}}$ ,  $\|\mathbf{x}\|_\alpha$ ,

$$(26) \quad |\mathfrak{J}(x^{n+1}; t_{2k}^{n+1}, t_{2k+1}^{n+1}) + \mathfrak{J}(x^{n+1}; t_{2k+1}^{n+1}, t_{2k+2}^{n+1}) - \mathfrak{J}(x^n; t_{2k}^{n+1}, t_{2k+2}^{n+1})| \leq C_3 2^{-n\alpha(\gamma+2)}.$$

For  $m \leq n$  and  $k \in \{0, \dots, 2^m - 1\}$ ,

$$\mathfrak{J}(x^n; t_k^m, t_{k+1}^m) - \mathfrak{J}(x^m; t_k^m, t_{k+1}^m) = \sum_{\ell=m}^{n-1} (\mathfrak{J}(x^{\ell+1}; t_k^\ell, t_{k+1}^\ell) - \mathfrak{J}(x^\ell; t_k^\ell, t_{k+1}^\ell)).$$

As there are exactly  $2^{\ell-m}$  dyadics intervals of the form  $[t_i^\ell, t_{i+1}^\ell]$  contained in  $[t_k^m, t_{k+1}^m]$  for all  $\ell \geq m$ , we deduce from the Chasles relation and (26) that

$$(27) \quad |\mathfrak{J}(x^n; t_k^m, t_{k+1}^m) - \mathfrak{J}(x^m; t_k^m, t_{k+1}^m)| \leq C_3 \sum_{\ell=m}^{n-1} \frac{2^{\ell-m}}{2^{\ell\alpha(\gamma+2)}} \leq \frac{C_4}{2^{m\alpha(\gamma+2)}},$$

where  $C_4$  depends on  $C_3$  and the choice of  $\alpha$  and  $\gamma$  (note that our choice of  $\alpha$  and  $\gamma$  ensures that the involved series converges as  $n \rightarrow \infty$ ).

We now choose for  $m(0)$  the largest integer such that there exists some  $k \in \{0, \dots, 2^{m(0)} - 1\}$  for which  $[t_k^{m(0)}, t_{k+1}^{m(0)}] \subset [t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n]$ , where  $\overline{M}(s,n)$  (resp.  $\underline{M}(t,n)$ ) is the smallest (resp. the largest) integer such that  $s \leq t_{\overline{M}(s,n)}^n$  (resp.  $t \geq t_{\underline{M}(t,n)}^n$ ).

From the Chasles relation,

$$\begin{aligned} & \mathfrak{J}(x^n; t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n) \\ &= \mathfrak{J}(x^n; t_{\overline{M}(s,n)}^n, t_k^{m(0)}) + \mathfrak{J}(x^n; t_k^{m(0)}, t_{k+1}^{m(0)}) + \mathfrak{J}(x^n; t_{k+1}^{m(0)}, t_{\underline{M}(t,n)}^n). \end{aligned}$$

By combining (25) and (27), we get that  $|\mathfrak{J}(x^n; t_k^{m(0)}, t_{k+1}^{m(0)})| \leq C_5 2^{-m(0)\alpha}$  for some constant  $C_5$  that depends only on  $T$ ,  $\alpha$ ,  $\gamma$ ,  $\|f\|_{\text{Lip}}$  and  $\|\mathbf{x}\|_\alpha$ .

We may now find some integers  $m(1)$  and  $k(1)$  such that  $[t_{k(1)}^{m(1)}, t_{k(1)+1}^{m(1)}]$  is the biggest interval of this type contained in  $[t_{\overline{M}(s,n)}^n, t_k^{m(0)}]$ , in order to estimate  $\mathfrak{J}(x^n; t_{\overline{M}(s,n)}^n, t_k^{m(0)})$ . Similarly, we can find some integers  $m'(1)$  and  $k'(1)$  such that  $[t_{k'(1)}^{m'(1)}, t_{k'(1)+1}^{m'(1)}]$  is the biggest interval of this type contained in  $[t_{k+1}^{m(0)}, t_{\underline{M}(t,n)}^n]$ , in order to estimate  $\mathfrak{J}(x^n; t_{k+1}^{m(0)}, t_{\underline{M}(t,n)}^n)$ . Note that necessarily,  $m(1)$  and  $m'(1)$  are strictly greater than  $m(0)$ .

Hence, proceeding recursively, we obtain with (22) and (27) that

$$|\mathfrak{J}(x^n; t_{\overline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n)| \leq \frac{C_5}{2^{m(0)\alpha}} + \sum_{j \in J} \frac{C_5}{2^{m(j)\alpha}} + \sum_{j \in J'} \frac{C_5}{2^{m'(j)\alpha}},$$

where  $(m(j))_{j \in J}$  and  $(m'(j))_{j \in J'}$  are two finite increasing families of integers, that are bounded by  $n$  and greater than  $m(0)$ . This kind of computation is the core of the proof of the Kolmogorov Lemma (see for

example Corollary of Theorem 4.5 in [IW89]) and is also an important tool in the theory of rough paths.

For some constant  $C_6$ , we then obtain that

$$|\mathfrak{J}(x^n; t_{\underline{M}(s,n)}^n, t_{\underline{M}(t,n)}^n)| \leq \frac{C_6}{2^{m(0)\alpha}}.$$

Let us note that  $T2^{-m(0)} \leq t_{\underline{M}(t,n)}^n - t_{\underline{M}(s,n)}^n < T2^{-m(0)+1}$ . With (25) and the Chasles relation, we then obtain that

$$(28) \quad |\mathfrak{J}(x^n; s, t)| \leq C_1(t_{\underline{M}(s,n)}^n - s)^\alpha + \frac{C_6}{2^{m\alpha}} + C_1(t - t_{\underline{M}(t,n)}^n)^\alpha \\ \leq \max\{C_1, C_6/T^\alpha\}(t - s)^\alpha.$$

Since  $\mathfrak{J}(x^n; 0) = 0$ , this proves that  $\mathfrak{J}(x^n; s, t)$  is uniformly bounded in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\alpha)$ . It follows that there exists a convergent subsequence in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$ , whose limit is denoted by  $\mathfrak{J}(x)$ , which is also a  $\alpha$ -Hölder continuous function.

We may however give more information on the limit. With (27) and (28), for some constant  $C_7$  and any integers  $0 \leq m \leq n$  and any  $0 \leq s \leq t \leq T$  with  $t - s > T2^{-m}$ ,

$$|\mathfrak{J}(x^n; s, t) - \mathfrak{J}(x^m; s, t)| \leq C_7(t_{\underline{M}(s,m)}^m - s)^\alpha + C_7(t - t_{\underline{M}(t,m)}^m)^\alpha \\ + \frac{C_4(\underline{M}(t, m) - \overline{M}(s, m))}{2^{m\alpha(\gamma+2)}}.$$

As  $\underline{M}(t, m) - \overline{M}(s, m) \leq 2^m$  and  $\alpha(\gamma + 2) > 1 + \alpha$ , it follows that

$$(29) \quad |\mathfrak{J}(x^n; s, t) - \mathfrak{J}(x^m; s, t)| \\ \leq C_7(t_{\underline{M}(s,m)}^m - s)^\alpha + C_7(t - t_{\underline{M}(t,m)}^m)^\alpha + \frac{C_4}{2^{m\alpha}}.$$

Set

$$R_m(s, t; \alpha) = \max \left\{ C_7(t_{\underline{M}(s,m)}^m - s)^\alpha, C_7(t - t_{\underline{M}(t,m)}^m)^\alpha, \frac{C_4}{2^{m\alpha}} \right\}.$$

As  $R_m(s, t; \alpha)$  converges to 0 when  $m \rightarrow \infty$ , the sequence  $(\mathfrak{J}(x^n; s, t))_{n \in \mathbb{N}}$  is a Cauchy sequence for any  $0 \leq s \leq t \leq T$ , which has a unique limit, which is necessarily  $\mathfrak{J}(x; s, t)$ . Besides, we get from (29) that for some constant  $C_8$  and any  $\beta < \alpha$ ,

$$|\mathfrak{J}(x; s, t) - \mathfrak{J}(x^m; s, t)| \leq C_8(t - s)^\beta R_m(s, t; \alpha - \beta),$$

when  $m$  is large enough so that  $T2^{-m} < t - s$ . If  $T2^{-m} > t - s$ , then there is at most one point  $t_k^m$  such that  $s \leq t_k^m \leq t$  and then for some constant  $C_9$ ,

$$|\mathfrak{J}(x; s, t) - \mathfrak{J}(x^m; s, t)| \leq |\mathfrak{J}(x; s, t)| + |\mathfrak{J}(x^m; s, t)| \\ \leq C_9(t - s)^\alpha \leq \frac{C_9 T^{\alpha-\beta}}{2^{-m(\alpha-\beta)}}(t - s)^\beta.$$

We get that the whole sequence  $(\mathfrak{J}(x^n))_{n \in \mathbb{N}}$  converges to  $\mathfrak{J}(x)$  in the space  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$  for any  $\beta < \alpha$ .

The proposition is then proved.  $\square$

**Corollary 4.** *Let  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  be a sequence of paths converging to  $\mathbf{x}$  in  $(C^\alpha([0, T]; A(\mathbb{R}^2)), \|\cdot\|_\alpha)$ . Then for all  $\beta < \alpha$ ,  $\mathfrak{J}(\mathbf{x}^n; 0, \cdot)$  converges to  $\mathfrak{J}(\mathbf{x}; 0, \cdot)$  in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$ .*

*Proof.* The proof follows the same line as the proof of Proposition 1.

To simplify the notation, we denote  $\mathbf{x}$  by  $\mathbf{x}^\infty$ .

Since  $\mathbf{x}^n$  is convergent in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ , the sequence  $(\|\mathbf{x}^n\|_\alpha)_{n \in \mathbb{N}}$  is bounded and then, from Proposition 3,  $(\mathfrak{J}(\mathbf{x}^n))_{n \in \mathbb{N}}$  is bounded in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\alpha)$ .

For  $n \in \mathbb{N} \cup \{\infty\}$ , let  $(\mathbf{x}^{n,m})_{m \in \mathbb{N}}$  be the sequence of paths converging to  $\mathbf{x}^n$  given by Proposition 2. We have seen in Proposition 3 for any  $\beta < \alpha$ , there exists some constant  $K^n$  that depends on  $\|\mathbf{x}^n\|_\alpha$  such that  $\|\mathfrak{J}(\mathbf{x}^{n,m}) - \mathfrak{J}(\mathbf{x}^n)\|_\beta \leq K^n 2^{m(\beta-\alpha)}$ . In addition, the sequence  $(K^n)_{n \in \mathbb{N}}$  is bounded if  $(\|\mathbf{x}^n\|_\alpha)_{n \in \mathbb{N}}$  is bounded. As  $\mathfrak{J}(\mathbf{x}^{n,m})$  is a Young integral, it follows from Corollary 2 that  $\mathfrak{J}(\mathbf{x}^{n,m})$  converges to  $\mathfrak{J}(\mathbf{x}^\infty, m)$  in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$ . Hence, this is sufficient to prove that  $\mathfrak{J}(\mathbf{x}^n)$  converges to  $\mathfrak{J}(\mathbf{x})$  in  $(C^\alpha([0, T]; \mathbb{R}), \|\cdot\|_\beta)$ , as in the proof of Proposition 1.  $\square$

*Remark 8.* Let us consider the following equivalence relation  $\sim$  between two sequences  $(x^n)_{n \in \mathbb{N}}$  and  $(y^n)_{n \in \mathbb{N}}$  of sequences of path converging in  $(C^\alpha([0, T]; \mathbb{R}^2), \|\cdot\|_\beta)$  with  $\alpha > 1/2$  and  $\beta \in (1/3, 1]$ :  $(x^n)_{n \in \mathbb{N}} \sim (y^n)_{n \in \mathbb{N}}$  if  $\mathbf{x} \stackrel{\text{def}}{=} \lim_{n \in \mathbb{N}} \mathfrak{C}(x^n, 0) = \lim_{n \in \mathbb{N}} \mathfrak{C}(y^n, 0)$  in  $(C^\gamma([0, T]; A(\mathbb{R}^2)), \|\cdot\|_\beta)$  for some  $\gamma > \beta$ . This implies that  $\mathfrak{J}(x^n; s, t)$  and  $\mathfrak{J}(y^n; s, t)$  converge to the same limit  $\mathfrak{J}(\mathbf{x}; s, t)$ . Hence, one may identify  $C^\gamma([0, T]; A(\mathbb{R}^2), \|\cdot\|_\gamma)$  with the quotient space  $(C^\alpha([0, T]; \mathbb{R}^2), \|\cdot\|_\beta)^{\mathbb{N}} / \sim$ , and two elements in the same class of equivalence give rise to the same integral.

Here, we have used the dyadics partitions<sup>2</sup>, so that one may ask whether  $\mathfrak{J}(\mathbf{x}; s, t)$  is equal to  $\mathfrak{J}(\mathbf{x}|_{[s,t]})$ ? As this is true for ordinary integrals, we easily get the following result.

**Lemma 11.** *Let  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ . Then, for all  $0 \leq s \leq t \leq T$ ,  $\mathfrak{J}(\mathbf{x}; s, t) = \mathfrak{J}(\mathbf{x}|_{[s,t]})$ .*

From this lemma, we deduce that if  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  and  $\mathbf{y} \in C^\alpha([0, S]; A(\mathbb{R}^2))$ , then

$$\mathfrak{J}(\mathbf{x} \square \mathbf{y}; 0, t) = \begin{cases} \mathfrak{J}(\mathbf{x}; 0, t) & \text{if } t \in [0, T], \\ \mathfrak{J}(\mathbf{x}; 0, T) + \mathfrak{J}(\mathbf{y}; 0, t - T) & \text{if } t \in [S, T]. \end{cases}$$

<sup>2</sup>We will give below another construction of  $\mathfrak{J}$  for which there family of partitions different from the dyadics ones can be used.

*Proof.* This lemma means that the integral constructed using the dyadics on  $[0, T]$  but restricted to  $[s, t]$  corresponds to the integral constructed using the dyadics on  $[s, t]$ . One knows that such a relation holds for ordinary integrals, since the integral does not depend on the choice of the family of partitions on which approximations of the integrals are defined.

Let  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  be the approximation of  $\mathbf{x}$  given by Proposition 2. Then  $\mathfrak{I}(\mathbf{x}^n)$  is an ordinary integral. Hence  $\mathfrak{I}(\mathbf{x}^n; s, t) = \mathfrak{I}(\mathbf{x}^n_{|[s,t]}; 0, t-s)$  (the last integral means that  $T$  is replaced by  $t-s$  and thus that we consider the dyadic partitions of  $[0, t-s]$ ). The result follows from passing to the limit.  $\square$

Let us end this section with an important remark. Consider  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2)$  and  $\varphi$  in  $C^{2\alpha}([0, T]; \mathbb{R})$ . We saw in Lemma 9 that  $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 + \varphi)$  also belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ .

Hence, we set  $y_t^n = x_t^{\Pi^n} \times \Phi^n \times \Psi^n(t/3)$  for  $t \in [0, 3T]$  where  $\Psi^n = \{z_k^n\}_{k=0, \dots, 2^n-1}$  with  $z_k^n : [t_k^n, t_{k+1}^n] \rightarrow \mathbb{R}^2$  defined by

$$z_k^n(t) = \frac{\varphi_{t_{k+1}^n} - \varphi_{t_k^n}}{\sqrt{\pi}} \begin{bmatrix} \cos\left(2\pi \frac{t-t_k^n}{t_{k+1}^n - t_k^n}\right) - 1 \\ \sin\left(2\pi \frac{t-t_k^n}{t_{k+1}^n - t_k^n}\right) \end{bmatrix},$$

so that  $\varphi$  asymptotically encodes the area of  $(\Psi^n)_{n \in \mathbb{N}}$ .

Similarly as in Section 3.2, it is then easily shown that

$$\mathfrak{I}(y^n; 0, t) \xrightarrow{n \rightarrow \infty} \mathfrak{I}(\mathbf{y}; 0, t) = \mathfrak{I}(\mathbf{x}; 0, t) + \int_0^t [f, f](x_s) d\varphi_s.$$

Hence, adding a path  $\varphi$  to the third component of  $\mathbf{x}$  consists in adding a term  $\int_0^t [f, f](x_s) d\varphi_s$  to  $\mathfrak{I}(\mathbf{x})$ .

**5.8. Construction of an enhanced path lying above an integral.** We have seen the importance to take care of the area. We now assume that the functions  $f_1$  and  $f_2$  takes their values in  $\mathbb{R}^2$  — or equivalently that  $f = (f_1, f_2) \in \text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R}^2)$ . so that  $\mathfrak{I}(\mathbf{x})$  for  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  is a path with values in  $C^\alpha([0, T]; \mathbb{R}^2)$ . It is possible to construct  $\mathfrak{I}(\mathbf{x})$  as an element of  $C^\alpha([0, T]; A(\mathbb{R}^2))$ ? If yes, then one can use  $\mathfrak{I}(\mathbf{x})$  as a path which is integrated along another differential form, ... By iterations, this leads to the construction of solutions of differential equations using a Picard scheme.

We set  $\mathfrak{I}(\mathbf{x}) = (\mathfrak{I}^1(\mathbf{x}), \mathfrak{I}^2(\mathbf{x}))$  and  $\mathfrak{I}(\mathbf{x}; s, t) = \mathfrak{I}(\mathbf{x}_{|[s,t]})$ . If  $\alpha > 1/2$ , define

$$\mathfrak{I}^3(\mathbf{x}; 0, t) = \frac{1}{2} \int_0^t \mathfrak{I}^1(\mathbf{x}; 0, s) d\mathfrak{I}^2(\mathbf{x}; 0, s) - \frac{1}{2} \int_0^t \mathfrak{I}^2(\mathbf{x}; 0, s) d\mathfrak{I}^1(\mathbf{x}; 0, s),$$

where

$$\int_0^t \mathfrak{I}^i(\mathbf{x}; 0, s) d\mathfrak{I}^j(\mathbf{x}; 0, s) = \sum_{k, \ell=1}^2 \int_0^t \left( \int_0^s f_k^i(x_r) dx_r^k \right) f_\ell^j(x_s) dx_s^\ell.$$

Naturally, we set  $\mathfrak{J}^3(\mathbf{x}; s, t) = (-\mathfrak{J}^3(\mathbf{x}; 0, s)) \boxplus \mathfrak{J}^3(\mathbf{x}; 0, t)$ .

If  $\mathbf{x}$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2]$ , then  $\mathfrak{J}^i(\mathbf{x})$  is itself  $\alpha$ -Hölder continuous for  $i = 1, 2$ . Thus, can one consider the integral of  $\mathfrak{J}^i(\mathbf{x})$  against  $\mathfrak{J}^j(\mathbf{x})$  for  $(i, j) = (1, 2)$  or  $(i, j) = (2, 1)$ ? We will see in Section 7.4 that we may construct also the area of  $(\mathfrak{J}^1(\mathbf{x}), \mathfrak{J}^2(\mathbf{x}))$ , which proves some kind of stability result: the integral of a path in  $A(\mathbb{R}^2)$  is then a path in  $A(\mathbb{R}^2)$  if  $f$  takes its values in  $\mathbb{R}^2$ .

Here, we prove only a weak statement: a stronger one, similar to Proposition 3 and Corollary 4 will be given in Section 7.4.

**Proposition 4.** *Let  $\mathbf{x}$  be a path in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  and  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  be the sequence of paths constructed in Proposition 2.*

*For  $\alpha \in (1/3, 1/2]$ , there exists a subsequence of  $(\mathfrak{J}^1(\mathbf{x}^n), \mathfrak{J}^2(\mathbf{x}^n), \mathfrak{J}^3(\mathbf{x}^n))_{n \in \mathbb{N}}$  which converges in  $(C^\alpha([0, T]; A(\mathbb{R}^2)), \|\cdot\|_\beta)$  for any  $\beta < \alpha$ .*

*Proof.* It is clear from the very construction of  $\mathfrak{J}^3$  that  $(\mathfrak{J}^1(\mathbf{x}^n), \mathfrak{J}^2(\mathbf{x}^n), \mathfrak{J}^3(\mathbf{x}^n))_{n \in \mathbb{N}}$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ . The proposition will be proved if one shows that for some constant  $C$ ,  $|\mathfrak{J}(\mathbf{x}^n; s, t)| \leq C(t-s)^{2\alpha}$  for any  $0 \leq s \leq t \leq T$ .

Fix  $n \in \mathbb{N}$  and let us estimate first  $\mathfrak{J}^3(\mathbf{x}^n; t_k^n, t_{k+1}^n)$  for  $k = 0, \dots, 2^n - 1$ . Let us remark that for some constant  $K$ ,

$$\begin{aligned} & \left| \int_{t_k^n}^{t_k^n + T2^{-n-1}} \mathfrak{J}^1(\mathbf{x}^n; t_k^n, r) d\mathfrak{J}^2(\mathbf{x}^n; t_k^n, s) \right| \\ &= \left| \frac{2^{n+2}}{T} \sqrt{\pi|\theta_k^n|} \int_{t_k^n}^{t_k^n + T2^{-n-1}} \mathfrak{J}^1(\mathbf{x}^n; t_k^n, r) f(x_r^n) \sin(2^{n+2}\pi(r - t_k^n)/T) dr \right| \\ & \leq \frac{K}{2^{2n\alpha}} \|f\|_{\text{Lip}} \|\mathfrak{J}(\mathbf{x}^n)\|_\alpha \|\mathbf{x}^n\|_\alpha. \end{aligned}$$

A similar inequality holds for when one swaps the role of  $\mathfrak{J}^1$  and  $\mathfrak{J}^2$ . Besides,

$$\begin{aligned} \mathfrak{J}^3(\mathbf{x}^n; t_k^n, t_{k+1}^n) &= \mathfrak{J}^3(\mathbf{x}^n; t_k^n, t_k^n + T2^{-n-1}) + \mathfrak{J}^3(\mathbf{x}^n; t_k^n + T2^{-n-1}, t_{k+1}^n) \\ & \quad + \frac{1}{2} \mathfrak{J}(\mathbf{x}^n; t_k^n, t_k^n + T2^{-n-1}) \wedge \mathfrak{J}(\mathbf{x}^n; t_k^n + T2^{-n-1}, t_{k+1}^n) \end{aligned}$$

and one gets also easily that for  $(i, j) = (1, 2)$  and  $(i, j) = (2, 1)$ ,

$$\left| \int_{t_k^n + T2^{-n-1}}^{t_{k+1}^n} \mathfrak{J}^i(\mathbf{x}^n; t_k^n, r) d\mathfrak{J}^j(\mathbf{x}^n; t_k^n, s) \right| \leq \frac{K}{2^{2n\alpha}} \|f\|_{\text{Lip}} \|\mathfrak{J}(\mathbf{x}^n)\|_\alpha \|\mathbf{x}^n\|_\alpha$$

for some constant  $K$ .

Since  $\|\mathbf{x}^n\|_\alpha$  and  $\|\mathfrak{J}(\mathbf{x}^n)\|_\alpha$  are bounded and  $|a \wedge b| \leq 2|a| \times |b|$  for all  $a, b \in \mathbb{R}^2$ , we get by combining the previous facts that for some constant  $C_1$ ,

$$(30) \quad |\mathfrak{J}^3(\mathbf{x}^n; t_k^n, t_{k+1}^n)| \leq C_1(t-s)^{2\alpha}.$$

Now, let us estimate  $\mathfrak{J}^3(\mathbf{x}^n; 0, T)$  the following way:

$$\begin{aligned} |\mathfrak{J}^3(\mathbf{x}^n; 0, T)| &\leq |\mathfrak{J}^3(\mathbf{x}^n; 0, T/2)| + |\mathfrak{J}^3(\mathbf{x}^n; T/2, T)| \\ &\quad + |\mathfrak{J}(\mathbf{x}^n; 0, T/2)| \times |\mathfrak{J}(\mathbf{x}^n; T/2, T)|. \end{aligned}$$

We can also break  $\mathfrak{J}^3(\mathbf{x}^n; 0, T/2)$  and  $\mathfrak{J}^3(\mathbf{x}^n; T/2, T)$  in two parts, and so on. We stop after  $n$  steps. In this case,

$$\begin{aligned} |\mathfrak{J}^3(\mathbf{x}^n; 0, T)| &\leq \sum_{k=0}^{2^n-1} |\mathfrak{J}^3(\mathbf{x}^n; t_k^n, t_{k+1}^n)| \\ &\quad + \sum_{k=1}^n \sum_{\ell=0}^{2^k-1} |\mathfrak{J}(\mathbf{x}^n; t_{2^\ell}^k, t_{2^{\ell+1}}^k)| \times |\mathfrak{J}(\mathbf{x}^n; t_{2^{\ell+1}}^k, t_{2^{\ell+2}}^k)|. \end{aligned}$$

As  $(\mathfrak{J}^1(\mathbf{x}^n), \mathfrak{J}^2(\mathbf{x}^n))$  is  $\alpha$ -Hölder continuous with a constant  $K$  which is independent from  $n$ , we get with (30) that

$$|\mathfrak{J}^3(\mathbf{x}^n; 0, T)| \leq \sum_{k=0}^{2^n-1} \frac{C_1 T^{2\alpha}}{2^{2n\alpha}} + \frac{1}{2} \sum_{k=1}^n K T^{2\alpha} \frac{2^{k-1}}{2^{2k\alpha}}.$$

Since  $2\alpha > 1$ , it follows that the series in the right hand side of the previous expression converges as  $n \rightarrow \infty$ . Hence, for some constant  $C_2$  that does not depend on  $n$ ,

$$|\mathfrak{J}^3(\mathbf{x}^n; 0, T)| \leq C_2 T^{2\alpha}.$$

Now, if  $\mathbf{y} = \mathbf{x}_{|[t_k^m, t_{k+1}^m]}$  for some integer  $m$  and  $k \in \{0, \dots, 2^k - 1\}$ , then  $|\mathfrak{J}^3(\mathbf{y}^n; 0, T/2^m)| \leq C_2 T^{2\alpha} / 2^{2\alpha m}$ , where  $\mathbf{y}^n$  is the approximation of  $\mathbf{y}$  constructed by Proposition 2. Let us remark that for all  $0 \leq s < t \leq T$ ,  $\mathfrak{J}^3(\mathbf{x}_{|[s, t]}^n; 0, t - s) = \mathfrak{J}^3(\mathbf{x}^n; s, t)$  and that  $\mathbf{y}_{|[0, T/2^m]}^n = \mathbf{x}_{|[t_k^m, t_{k+1}^m]}^{n+m}$ . Hence, we obtain that for any integers  $0 \leq n \leq m$ ,

$$|\mathfrak{J}^3(\mathbf{x}^n; t_k^m, t_{k+1}^m)| \leq \frac{C_2 T^{2\alpha}}{2^{2\alpha m}}.$$

We now proceed as in the proof of the Kolmogorov Lemma, using the same kind of computations we already used in the proof of Proposition 3.

Fix  $0 \leq s < t \leq T$  and choose  $m(0)$  and  $k(0)$  such that  $[t_{k(0)}^{m(0)}, t_{k(0)+1}^{m(0)}] \subset [s, t]$  and  $m(0)$  is as large as possible. Then

$$\begin{aligned} (31) \quad \mathfrak{J}^3(\mathbf{x}^n; s, t) &= \mathfrak{J}^3(\mathbf{x}^n; s, t_{k(0)}^{m(0)}) + \mathfrak{J}^3(\mathbf{x}^n; t_{k(0)}^{m(0)}, t_{k(0)+1}^{m(0)}) \\ &\quad + \mathfrak{J}^3(\mathbf{x}^n; t_{k(0)+1}^{m(0)}, t) + \frac{1}{2} \mathfrak{J}(\mathbf{x}^n; s, t_{k(0)}^{m(0)}) \wedge \mathfrak{J}(\mathbf{x}^n; t_{k(0)}^{m(0)}, t) \\ &\quad + \frac{1}{2} \mathfrak{J}(\mathbf{x}^n; t_{k(0)}^{m(0)}, t_{k(0)+1}^{m(0)}) \wedge \mathfrak{J}(\mathbf{x}^n; t_{k(0)+1}^{m(0)}, t). \end{aligned}$$



Hence, for some constants  $C_3$  and  $C_4 = 2^{\alpha-1}C_3$ ,

$$(32) \quad \left| \mathfrak{J}(\mathbf{x}^n; s, t_{k(0)}^{m(0)}) \wedge \mathfrak{J}(\mathbf{x}^n; t_{k(0)}^{m(0)}, t) \right| \leq C_3 (t_{k(0)}^{m(0)} - s)^\alpha (t - t_{k(0)}^{m(0)})^\alpha \\ \leq 2^{\alpha-1} C_3 (t_{k(0)}^{m(0)} - s)^\alpha \left( (t - t_{k(0)+1}^{m(0)})^\alpha + (t_{k(0)+1}^{m(0)} - t_{k(0)}^{m(0)})^\alpha \right) \\ \leq T^{2\alpha} \frac{C_4}{2^{2m(0)\alpha}},$$

since from our choice of  $m(0)$ ,  $0 \leq t - t_{k(0)+1}^{m(0)} < T/2^{m(0)}$  and  $0 \leq t_{k(0)}^{m(0)} - s < T/2^{m(0)}$ . A similar inequality holds for  $|\mathfrak{J}(\mathbf{x}^n; t_{k(0)}^{m(0)}, t_{k(0)+1}^{m(0)}) \wedge \mathfrak{J}(\mathbf{x}^n; t_{k(0)+1}^{m(0)}, t)|$ .

We may now construct recursively some integers  $m(i)$  (resp.  $m'(i)$ ) and  $k(i)$  (resp.  $k'(i)$ ) in  $\{0, \dots, 2^{m(i)} - 1\}$  such that  $[t_{k(i)}^{m(i)}, t_{k(i)+1}^{m(i)}] \subset [s, t_{k(i-1)}^{m(i-1)}]$  (resp.  $[t_{k'(i)}^{m'(i)}, t_{k'(i)+1}^{m'(i)}] \subset [t_{k(i-1)+1}^{m(i-1)}, t]$ ) and  $m(i)$  (res.  $m'(i)$ ) is as large as possible. Note that the sequences  $(m(i))_{i \geq 0}$  and  $(m'(i))_{i \geq 0}$  are strictly increasing, and that

$$(33) \quad [s, t] = \dots \cup [t_{k(1)}^{m(1)}, t_{k(1)+1}^{m(1)}] \cup [t_{k(0)}^{m(0)}, t_{k(0)+1}^{m(0)}] \cup [t_{k'(1)}^{m'(1)}, t_{k'(1)+1}^{m'(1)}] \cup \dots$$

Moreover, (32) holds when  $m(0)$  and  $k(0)$  are replaced by  $m(i)$  (resp.  $m'(i)$ ) and  $k(i)$  (resp.  $k'(i)$ ). Using recursively a decomposition similar to (31) to  $\mathfrak{J}^3(\mathbf{x}^n; s, t_{k(i)}^{m(i)})$  and  $\mathfrak{J}^3(\mathbf{x}^n; t_{k'(i)+1}^{m'(i)})$ , we thus obtain that

$$|\mathfrak{J}^3(\mathbf{x}^n; s, t)| \leq T^{2\alpha} C_4 \left( \frac{1}{2^{2\alpha m(0)}} + \sum_{i \geq 1} \frac{1}{2^{2\alpha m(i)}} + \sum_{i \geq 1} \frac{1}{2^{2\alpha m'(i)}} \right).$$

Since  $2\alpha > 1$ , it follows from (33) that  $|\mathfrak{J}^3(\mathbf{x}^n; s, t)| \leq C_4 (t - s)^{2\alpha}$ . Hence, from Lemma 10, the sequence of functions  $(\mathfrak{K}(\mathbf{x}^n))_{n \in \mathbb{N}}$  with  $\mathfrak{K}(\mathbf{x}^n) = (\mathfrak{J}^1(\mathbf{x}^n), \mathfrak{J}^2(\mathbf{x}^n), \mathfrak{J}^3(\mathbf{x}^n))$  has a convergent subsequence to some  $\mathfrak{K}(\mathbf{x})$  in  $C^\alpha([0, T]; \mathbb{A}(\mathbb{R}^2))$  in  $(C^\alpha([0, T]; \mathbb{A}(\mathbb{R}^2)), \|\cdot\|_\beta)$  for any  $\beta < \alpha$ . Moreover, from Proposition 3,  $\mathfrak{K}(\mathbf{x})$  lies above  $(\mathfrak{J}^1(\mathbf{x}), \mathfrak{J}^2(\mathbf{x}))$ .  $\square$

**5.9. A sub-Riemannian point of view.** Our definition of  $\mathfrak{J}$  consists in approximating a path  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{A}(\mathbb{R}^2))$  by a family of paths  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  in  $C^1([0, T]; \mathbb{A}(\mathbb{R}^2))$  such that  $\mathfrak{J}(\mathbf{x}^n)$  converges with respect to the  $\beta$ -Hölder norm in  $C^\alpha([0, T]; \mathbb{R})$  as  $n \rightarrow \infty$  for all  $\beta < \alpha$ . The integral  $\mathfrak{J}(\mathbf{x})$  is then defined as the limit if  $\mathfrak{J}(\mathbf{x}^n)$ . In addition, necessarily, it follows from Lemma 8 that  $\mathbf{x}^n = (x^{1,n}, x^{2,n}, \mathbf{x}_0^3 + \mathfrak{A}(x^n))$ , where  $x^n$  is a family of functions in  $C_p^1([0, T]; \mathbb{R}^2)$ .

The paths  $\mathbf{x}^n$  were constructed by replacing  $\mathbf{x}|_{[t_k^n, t_{k+1}^n]}$  by some paths obtained by combining loops and segment. Of course, other choice are possible, and a natural one consists in using geodesics.

Let  $a$  be a point in  $\mathbb{A}(\mathbb{R}^2)$ . How to find a path  $\mathbf{x} : [0, 1] \rightarrow \mathbb{A}(\mathbb{R}^2)$  with  $\mathbf{x}_0 = 0$ ,  $\mathbf{x}_1 = a$  and whose length (or whose energy) is minimal? Of course, one can use the segment  $\mathbf{y} = (ta^1, ta^2, ta^3)_{t \in [0, 1]}$  that goes from 0 to  $a$ , which is the natural geodesic in  $\mathbb{R}^3$ . But  $\mathfrak{A}(\mathbf{y}^1, \mathbf{y}^2; t) = 0$  and thus

$\mathbf{y}$  is not of type  $(y, \mathfrak{A}(y))$  and does not belong to  $C^1([0, T]; A(\mathbb{R}^2))$ . We will use this point of view in Section 7.2, and this will help us to bridge our construction with another one of Riemann sum type. So, we may reformulate our question by imposing the condition that  $\mathbf{y}$  is of type  $\mathbf{y} = (y, \mathfrak{A}(y))$ , which means that  $\mathbf{y}_t^3 = \mathfrak{A}(\mathbf{y}^1, \mathbf{y}^2; 0, t)$  for  $t \in [0, 1]$ . This kind of problem is related to sub-Riemannian geometry: see [Gro96, BBI01, Mon02, Bau04] for example.

The notion of length we use is then the length of the path  $(\mathbf{y}^1, \mathbf{y}^2)$ :

$$\text{Length}(\mathbf{y}) = \int_0^1 \sqrt{\dot{\mathbf{y}}_s^1 + \dot{\mathbf{y}}_s^2} ds.$$

Such a path — which will be characterized from the differentiable point of view in the next section —, is called *horizontal*. It is then possible to introduce a distance between two points of  $A(\mathbb{R}^2)$  by

$$d(a, b) = \inf_{\substack{\mathbf{y}: [0, 1] \rightarrow A(\mathbb{R}^2) \text{ horizontal} \\ \mathbf{y}_0 = a, \mathbf{y}_1 = b}} \text{Length}(\mathbf{y}),$$

which is called the *Carnot-Carathéodory* distance. We may then define  $\|x\|_{CC} = d(0, x)$ , which becomes a homogeneous sub-additive norm on  $A(\mathbb{R}^2)$  (see Section A.1) *i.e.*,  $\|x\|_{CC} = 0$  if and only  $x = 0$  and for all  $x, y \in A(\mathbb{R}^2)$  and  $\lambda \in \mathbb{R}$ ,  $\|\delta_\lambda x\|_{CC} = |\lambda| \cdot \|x\|_{CC}$ ,  $\|x^{-1}\|_{CC} = \|x\|_{CC}$  and  $\|x \boxplus y\|_{CC} \leq \|x\|_{CC} + \|y\|_{CC}$ , which is the *sub-additive property*.

For any  $a \in A(\mathbb{R}^2)$ , we succeed in constructing in Section 5.3 a path that goes from 0 to  $a$ , so that  $\|a\|$  is finite. Of course,  $d(a, b) = \|a^{-1} \boxplus b\|$  for all  $a, b \in A(\mathbb{R}^2)$ . If  $a^3 = 0$ , then the shortest horizontal path from 0 to  $a$  is the segment that goes from 0 to  $a$ . If  $a = (0, 0, a^3)$  with  $a^3 \neq 0$ , then this problem is equivalent to the isoperimetric problem, whose solution is known to be circle.

In the general case, this problem is called the *Dido problem*, and the solutions are known to be arcs of circle (see for example [Str87, Mon02]), but they are less practical to use than our construction with circles and loops (see below in the proof of Proposition 5).

These solutions are not real geodesics in  $A(\mathbb{R}^2)$ , but they are called *sub-Riemannian geodesics*. The sub-Riemannian geodesic that links  $a$  to  $b$  is then denoted by  $\psi_{a,b}$  and belongs to  $C^1([0, T]; A(\mathbb{R}^2))$ .

If we define the energy of a path by  $\text{Energy}(\mathbf{y}) = \frac{1}{2} \int_0^1 (\dot{\mathbf{y}}_s^1 + \dot{\mathbf{y}}_s^2) ds$ , then  $\psi_{a,b}$  is also a minimizer for the energy among all the paths with constant speed  $\text{Length}(\psi_{a,b})$ .

To a path  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ , we associate

$$(34) \quad \mathbf{x}_t^n = \psi_{\mathbf{x}_{t_k^n}, \mathbf{x}_{t_{k+1}^n}} \left( \frac{t - t_k^n}{t_{k+1}^n - t_k^n} \right) \text{ for } t \in [t_k^n, t_{k+1}^n],$$

for  $n = 0, 1, 2, \dots$

**Proposition 5.** *The sequence of paths  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  constructed by (34) is a family of paths in  $C^1([0, T]; A(\mathbb{R}^2))$  which converges to  $\mathbf{x}$  in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with respect to  $\|\cdot\|_\beta$  for any  $\beta < \alpha$ .*

*Proof.* The proof is similar to the one of Corollary 1 or of Proposition 2.

Obviously,  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  converges uniformly to  $\mathbf{x}$ . Let us remark that  $\mathbf{x}_{s,t}^n = \mathbf{x}_{s,t_k}^n \boxplus \mathbf{x}_{t_k,t_{k+1}}^n \boxplus \mathbf{x}_{t_{k+1},t}^n$  and that  $\mathbf{x}_{t_k,t_{k+1}}^n = \mathbf{x}_{t_k,t_{k+1}}^n$ . Using the same argument as in Corollary 1, the  $\alpha$ -Hölder norm of  $\mathbf{x}^n$  is then deduced from estimates on  $\mathbf{x}_{s,t_k}^n$  and  $\mathbf{x}_{t_{k+1},t}^n$  for  $t \in [t_k, t_{k+1}]$  for  $k = 0, \dots, 2^n - 1$ .

After a translation, we are looking for establishing an estimate of type  $|\psi_{0,x}(t)| \leq Ct|x|$  for  $t \in [0, 1]$  for some constant  $C$ . If this holds, then for  $t \in [t_k, t_{k+1}]$ ,

$$|\psi_{\mathbf{x}_{t_k}^n, \mathbf{x}_{t_{k+1}}^n}(t/\Delta_n t)| \leq C \frac{t}{\Delta_n t} |\mathbf{x}_{t_k, t_{k+1}}^n| \leq C \frac{t \Delta_n t^\alpha}{\Delta_n t} \|\mathbf{x}\|_\alpha \leq Ct^\alpha \|\mathbf{x}\|_\alpha.$$

We now give two proofs: one is done “by hand”, and the second one uses the properties of the Carnot-Carathéodory distance.

◦ If  $x^3 = 0$ , then  $\psi_{0,x}(t)$  is a segment and for  $t \in [0, 1]$ ,

$$|\psi_{0,x}(t)| \leq |x|t.$$

which gives the desired result.

Now, if  $x^3 \neq 0$ , let us note first that for some constants  $a \neq 0$  and  $r, \varphi \in [0, 2\pi)$ ,

$$\begin{cases} \psi_{0,x}^1(t) &= a(\cos(rt + \varphi) - \cos(\varphi)), \\ \psi_{0,x}^2(t) &= a(\sin(rt + \varphi) - \sin(\varphi)), \\ \psi_{0,x}^3(t) &= a^2 rt \end{cases}$$

since the minimizers lies above arcs of circles. Hence,  $a^2 r = x^3$  and

$$(x^1)^2 + (x^2)^2 = \psi_{0,x}^1(1)^2 + \psi_{0,x}^2(1)^2 = 2a^2(1 - \cos(r)).$$

It is easily seen that one may find  $a$  and  $r$  in order to satisfy  $\psi_{0,x}(1) = x$ .

If  $r \in [\pi/2, 3\pi/2]$ , then  $1 \leq 1 - \cos(r) \leq 2$ ,  $a^2 \leq \max\{|x^1|^2, |x^2|^2\}$  and

$$\max\{|\psi_{0,x}^1(t)|, |\psi_{0,x}^2(t)|\} \leq \sqrt{2}\pi t \max\{|x^1|, |x^2|\},$$

and  $|\psi_{0,x}^3(t)| \leq 4\pi^{-1}t \max\{|x^1|, |x^2|\}^2$ . This is sufficient to conclude.

In the other case, since  $\cos$  and  $\sin$  are Lipschitz continuous and  $|a^2 r| \leq |x^3|$ , we get that

$$\psi_{0,x}^1(t)^2 + \psi_{0,x}^2(t)^2 = 2a^2(1 - \cos(rt)) \leq 2|x^3|t \leq 2|x|^2 t.$$

Hence,  $|\psi_{0,x}(t)| \leq \sqrt{2}|x|t$ .

It follows that  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  is bounded in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  and this is sufficient to conclude.

◦ (*Alternative proof*). As the Carnot-Carathéodory norm is equivalent to any homogeneous norm (see Proposition 10 in Section A.1), it follows

that for some universal constants  $C$  and  $C'$ ,

$$(35) \quad \forall t \in [0, 1], |\psi_{0,x}(t)| \leq C \|\psi_{0,x}(t)\|_{\text{CC}} = Ct \|x\|_{\text{CC}} \leq CC' t |x|,$$

since  $\psi_{0,x}(t)$  is a sub-Riemannian geodesic and then  $\psi_{0,x}(t) = td(0, x)$ . The inequalities (35) yields the result.  $\square$

The point of view of the sub-Riemannian geometry, which is natural in the context of Heisenberg groups, have been used by P. Friz and N. Victoir in [FV06c] and [FV06a].

**5.10. A sub-Riemannian point of view: differentiable paths in  $A(\mathbb{R}^2)$ .** We have introduced the set of paths  $C^\alpha([0, T]; A(\mathbb{R}^2))$  for  $\alpha \in [1/2, 1/3)$ , but we have that the value of  $\alpha$  does not really refer to the regularity of the path  $\mathbf{x}$  in such a set, but to the norms to be used to approximate  $\mathbf{x}$  by a family of paths  $x^n$  that are naturally lifted as  $\mathbf{x}^n = (x^n, \mathfrak{A}(x^n))$ . It is then possible to consider some path  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha < 1/2$  that are differentiable: for example, if  $\mathbf{x}$  in  $C^1([0, T]; A(\mathbb{R}^2))$  and  $\varphi$  in  $C^1([0, T]; \mathbb{R})$ , then  $\mathbf{y}_t = (\mathbf{x}_t^1, \mathbf{x}_t^2, \mathbf{x}_t^3 + \varphi_t)$  is almost everywhere differentiable, in the sense that

$$(36) \quad i = 1, 2, 3, \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{y}_{t+\varepsilon}^i - \mathbf{y}_t^i}{\varepsilon} = \alpha^i(t)$$

exists for almost every  $t$ . Another natural way of thinking the derivative of  $\mathbf{y}$  consists in setting

$$(37) \quad i = 1, 2, 3, \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (-\mathbf{y}_t) \boxplus \mathbf{y}_{t+\varepsilon}^i = \beta^i(t)$$

when this limit exists. If  $t \in [0, T]$  is such that (36) holds, then  $\beta^i(t)$  exists and

$$\beta(t) = \alpha(t) - \frac{1}{2}[\mathbf{y}_t, \alpha(t)].$$

Reciprocally, if (37) holds, then (36) holds and

$$\alpha(t) = \beta(t) + \frac{1}{2}[\mathbf{y}_t, \beta(t)].$$

Of course,  $(\alpha^1(t), \alpha^2(t)) = (\beta^1(t), \beta^2(t))$  for all  $t$  at which  $\mathbf{y}_t$  is differentiable.

If the path  $\mathbf{y}$  is of type  $(y, \mathfrak{A}(y))$ , then

$$\alpha^1(t) = \frac{dy_t^1}{dt}, \quad \alpha^2(t) = \frac{dy_t^2}{dt} \quad \text{and} \quad \alpha^3(t) = \frac{1}{2}y_t \wedge \frac{dy_t}{dt} = \frac{1}{2}y_t \wedge \begin{bmatrix} \alpha^1(t) \\ \alpha^2(t) \end{bmatrix}.$$

At each point  $a$  of  $A(\mathbb{R}^2)$ , the associate the 2-dimensional vector space

$$\Theta(a) = \left\{ (v^1, v^2, v^3) \in \mathbb{R}^3 \mid v^3 = \frac{1}{2} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \wedge \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \right\}$$

as well as the space  $\Xi(a)$  orthogonal to  $\Theta(a)$  with respect to the usual scalar product in  $\mathbb{R}^3$ . The one-dimensional space  $\Xi(a)$  is generated

by the vector  $(-a^2/2, a^1/2, 1)^T$ . It is easily seen that  $a \mapsto \Xi(a)$  and  $a \mapsto \Theta(a)$  forms two subbundles of the tangent bundle of  $A(\mathbb{R}^2)$ .

We then obtain the next result:

**Lemma 12.** *A differentiable curve  $\mathbf{y}$  is the natural lift  $(y, \mathfrak{A}(y))$  of a differentiable curve  $y$  if and only if  $\dot{\mathbf{y}}_t$  belongs to  $\Theta(\mathbf{y}_t)$  for any  $t \in [0, T]$ .*

For a differentiable path  $\mathbf{y} : [0, T] \rightarrow A(\mathbb{R}^2)$ , let  $\beta(t)$  be given by (37). The condition that  $\dot{\mathbf{y}}_t$  belongs to  $\Theta(\mathbf{y}_t)$  is equivalent to  $\beta^i(t) = (\dot{\mathbf{y}}_t^1, \dot{\mathbf{y}}_t^2, 0)$ , and more generally, if  $\pi_{\Xi(a)}$  is the projection from  $\mathbb{R}^3$  identified with the tangent plane of  $A(\mathbb{R}^2)$  at  $a$  onto  $\Xi(a)$ , then for  $t \in [0, T]$ ,

$$\beta(t) = (\dot{\mathbf{y}}_t^1, \dot{\mathbf{y}}_t^2, \pi_{\Xi(\mathbf{y}_t)}(\dot{\mathbf{y}}_t)).$$

Thus, a differentiable path  $\mathbf{y}$  from  $[0, T]$  to  $(A(\mathbb{R}^2), \boxplus)$  is necessarily of type  $(y, \mathfrak{A}(y) + \varphi)$  where  $y = (\mathbf{y}^1, \mathbf{y}^2)$  and  $\varphi$  is differentiable, and  $\beta(t) = (\dot{y}_t^1, \dot{y}_t^2, \dot{\varphi}_t)$  for  $t \in [0, T]$ .

## 6. GEOMETRIC AND ALGEBRAIC STRUCTURES

**6.1. Motivations.** Up to now, we have introduced a space  $A(\mathbb{R}^2)$  and considered paths in  $C^\alpha([0, T]; A(\mathbb{R}^2))$ . For a path  $\mathbf{x} \in C^\alpha([0, T]; A(\mathbb{R}^2))$ , we have seen how to construct a sequence  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  of paths converging to  $C^\beta([0, T]; A(\mathbb{R}^2))$  with  $\beta < \alpha$  such that  $x^n = (\mathbf{x}^{1,n}, \mathbf{x}^{2,n})$  is piecewise smooth and  $\mathbf{x}^{3,n} = \mathbf{x}_0^3 + \mathfrak{A}(x^n)$ . As  $\mathbf{x}^n$  lies above a piecewise smooth path  $x^n$ ,  $\mathfrak{J}(\mathbf{x}^n)$  is well defined as a Young integral, and we have shown in Proposition 3 that the sequence  $(\mathfrak{J}(\mathbf{x}^n))_{n \in \mathbb{N}}$  converges and its limit defines  $\mathfrak{J}(\mathbf{x})$ .

On the other hand, we may rewrite

$$\mathfrak{J}(\mathbf{x}^n; 0, T) = \sum_{k=0}^{2^n-1} \mathfrak{J}(\mathbf{x}_{[t_k^n, t_{k+1}^n]}^n) \text{ and } \mathfrak{J}(\mathbf{x}; 0, T) = \sum_{k=0}^{2^n-1} \mathfrak{J}(\mathbf{x}_{[t_k^n, t_{k+1}^n]}).$$

The path  $\mathbf{x}_{[t_k^n, t_{k+1}^n]}^n$  was constructed in Section 5.3 from the values of  $\mathbf{x}_{t_{k+1}^n}$  and  $\mathbf{x}_{t_k^n}$ . Hence,  $\int_{t_k^n}^{t_{k+1}^n} f(x_s^n) dx_s^n$  is an approximation of  $\mathfrak{J}(\mathbf{x}_{[t_k^n, t_{k+1}^n]}^n)$ , and  $\mathfrak{J}(\mathbf{x}^n)$  is constructed only from the values of  $\{\mathbf{x}_{t_k^n}\}_{k=0, \dots, 2^n-1}$ .

We have proposed two constructions of integrals that rely on approximation of the path. We are now looking for a Riemann sum like expression, which consists in finding approximations of  $\mathfrak{J}(\mathbf{x}; 0, T)$  and to sum them over the dyadic partitions of  $[0, T]$ .

Let us note first that if  $\mathbf{x}$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha > 1/2$  and  $x^{\Pi^n}$  is the piecewise linear approximation of  $x$  along the dyadic partition  $\Pi^n$ , then

$$|\mathfrak{J}(x^{\Pi^n}; t_k^n, t_{k+1}^n) - \mathfrak{J}(\mathbf{x}; t_k^n, t_{k+1}^n)| \leq \|f\|_{\text{Lip}} |\mathfrak{A}(x; t_k^n, t_{k+1}^n)| \leq \frac{T^{2\alpha} \|f\|_{\text{Lip}} \|x\|_\alpha^2}{2^{2n\alpha}}$$

and thus, since  $\alpha > 1/2$ ,

$$\begin{aligned}
 \mathfrak{J}(\mathbf{x}; 0, T) &= \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n}) \frac{x_{t_{k+1}^n} - x_{t_k^n}}{t_{k+1}^n - t_k^n} ds \\
 (38) \qquad &= \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n}) \frac{dx_s^{\Pi^n}}{ds} ds
 \end{aligned}$$

where  $x$  is the path above which  $\mathbf{x}$  lies.

The first idea is then to find a formulation similar to (38), by looking for another way to draw a piecewise differentiable path  $\mathbf{y}^n$  on lying above a path  $y^n : [0, T] \rightarrow \mathbb{R}^2$  with  $y^n(t_k^n) = x_{t_k^n}$  for  $k = 0, \dots, 2^n$  and for which the expression  $\xi_k^n = \int_{t_k^n}^{t_{k+1}^n} f(y_s^n) \frac{dy_s^n}{ds} ds$  provides a good approximation of  $\mathfrak{J}(\mathbf{x}; t_k^n, t_{k+1}^n)$ , in the sense that for some  $\theta > 1$  and  $C > 0$ ,

$$|\xi_k^n - \mathfrak{J}(\mathbf{x}; t_k^n, t_{k+1}^n)| \leq \frac{C}{2^{n\theta}}.$$

The space in which  $\mathbf{y}$  lives has to be precised, but it is natural to assume that  $\frac{dy_k^n(s)}{ds}$  belongs to  $A(\mathbb{R}^2)$ , and then one has to extend the definition of  $f$  into a differential form on  $A(\mathbb{R}^2)$  accordingly.

The second idea would then to get an expression of type  $\sum_{k=0}^{2^n-1} f(x_{t_k^n}) \Delta_k^n \mathbf{x}$  where  $\Delta_k^n \mathbf{x}$  depends only on  $\mathbf{x}_{t_{k+1}^n}$  and  $\mathbf{x}_{t_k^n}$ . As we deal with second-order calculus, the things are not too simple: think to the difference between the Stratonovich and the Itô integrals for the Brownian motion.

**6.2. Another formulation for the integral.** We rewrite  $\mathfrak{J}(\mathbf{x}^n; t_k^n, t_{k+1}^n)$  as

$$\mathfrak{J}(\mathbf{x}^n; t_k^n, t_{k+1}^n) = \int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n}) dx_s^{\Pi^n} + \iint_{\text{Surface}(y_k^n)} [f, f](z) dz$$

where  $y_k^n$  has been defined by (20b). Setting  $\mathbf{x}_{s,t} = (-\mathbf{x}_s) \boxplus \mathbf{x}_t$  and  $\Delta_n t = T2^{-n}$ , we have already seen that

$$\left| \iint_{\text{Surface}(y_k^n)} [f, f](z) dz - \mathbf{x}_{t_k^n, t_{k+1}^n}^3 [f, f](x_{t_k^n}) \right| \leq \Delta_n t^{\alpha(\gamma+2)} \|f\|_{\text{Lip}} \|\mathbf{x}\|_{\alpha}^{2+\gamma}.$$

On the other hand,

$$\left| \mathbf{x}_{t_k^n, t_{k+1}^n}^3 [f, f](x_{t_k^n}) - \mathbf{x}_{t_k^n, t_{k+1}^n}^3 \int_{t_k^n}^{t_{k+1}^n} [f, f](x_s^{\Pi^n}) \frac{ds}{\Delta_n t} \right| \leq \Delta_n t^{\alpha(\gamma+2)} \|f\|_{\text{Lip}} \|\mathbf{x}\|_{\alpha}^{2+\gamma}.$$

Hence, this means that one can replace  $\mathfrak{J}(\mathbf{x}^n; t_k^n, t_{k+1}^n)$  by

$$\begin{aligned} \xi_k^n &= \mathbf{x}_{t_k^n, t_{k+1}^n}^1 \int_{t_k^n}^{t_{k+1}^n} f_1(x_s^{\Pi^n}) \frac{ds}{\Delta_n t} \\ &\quad + \mathbf{x}_{t_k^n, t_{k+1}^n}^2 \int_{t_k^n}^{t_{k+1}^n} f_2(x_s^{\Pi^n}) \frac{ds}{\Delta_n t} + \mathbf{x}_{t_k^n, t_{k+1}^n}^3 \int_{t_k^n}^{t_{k+1}^n} [f, f](x_s^{\Pi^n}) \frac{ds}{\Delta_n t}, \end{aligned}$$

in the sense that  $\mathfrak{J}(\mathbf{x}; 0, T) = \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \xi_k^n$ .

Let us denote by  $\{e_1, e_2, [e_1, e_2]\}$  the canonical basis of  $A(\mathbb{R}^2)$ , and by  $\{e^1, e^2, [e^1, e^2]\}$  its dual basis. For  $z = (z^1, z^2, z^3) \in A(\mathbb{R}^2)$ , let us define the differential form

$$(39) \quad \mathfrak{E}_{A(\mathbb{R}^2)}(f)(z) = f_1(z^1, z^2)e^1 + f_2(z^1, z^2)e^2 + [f, f](z^1, z^2)[e^1, e^2].$$

With  $\mathbf{x}^{\Pi^n} = (x^{\Pi^n}, 0)$ , the term  $\xi_k^n$  may be put in a more synthetic form

$$\xi_k^n = \int_{t_k^n}^{t_{k+1}^n} \mathfrak{E}_{A(\mathbb{R}^2)}(f)(\mathbf{x}_s^{\Pi^n}) \mathbf{x}_{t_k^n, t_{k+1}^n} \frac{ds}{\Delta_n t}.$$

We are now looking for a curve  $\mathbf{y}^n(t)$  on  $[0, T]$  which is piecewise differentiable and such that

$$(40) \quad \frac{d\mathbf{y}^n(t)}{dt} = \frac{1}{\Delta_n t} \mathbf{x}_{t_k^n, t_{k+1}^n}, \quad t \in (t_k^n, t_{k+1}^n).$$

Of course, from (40), such a path lies above  $x^{\Pi^n}$ . The problem is now to find the space in which  $\mathbf{y}^n$  lives.

Let us recall the results from Section 5.4: The space  $(A(\mathbb{R}^2))$  is a non-commutative group when equipped with  $\boxplus$ , and it is also a *Lie algebra* when equipped with the brackets  $[\cdot, \cdot]$ .

We have already denoted the basis of  $A(\mathbb{R}^2)$  by  $\{e_1, e_2, [e_1, e_2]\}$ . The choice of  $[e_1, e_2]$  to denote the third component follows naturally from the bilinearity of  $[\cdot, \cdot]$ .

The Lie algebra structure is particularly important here, since one knows that  $A(\mathbb{R}^2)$  may be identified with the tangent space at any point of a Lie group. We will now construct such a Lie group.

**6.3. Lie groups.** Let  $(G, \times)$  be a Lie group (see Section 5.4). We recall here some general results about  $G$ .

For  $x \in G$ , let us denote by  $T_x(G)$  the tangent plane at  $x$ . A *vector field*  $X$  is a differentiable application  $X : x \in G \mapsto X_x \in T_x(G)$ . If  $\Psi$  is a differentiable map from  $G$  to  $G$ , then its differential  $d\Psi$  maps  $T_x(G)$  to  $T_{\Psi(x)}$ . For a one-to-one map  $\Psi$ , two linear operators on vector fields may then be defined:

$$\Psi_* X = d\Psi(X \circ \Psi^{-1}) \quad \text{and} \quad \Psi^* X = d\Psi^{-1}(X \circ \Psi).$$

A *left-invariant* vector field  $X$  is a vector field such that  $L_x^* X = X$  for all  $x \in G$ , where  $L_x(y) = x \times y$ . It is easily shown that for such a

vector field,

$$X_x = d_1 L_x X, \quad \forall x \in G,$$

where 1 is the neutral element of the Lie group  $G$ . In other words, a left-invariant vector field is fully characterized by the tangent vector  $X_1$  in the tangent plane  $T_1(G)$  at the identity of  $G$ .

An *integral curve* of  $X$  is a differentiable curve  $\gamma : \mathbb{R}_+ \rightarrow G$  such that

$$\frac{d\gamma(t)}{dt} = X_{\gamma(t)}.$$

A *one-parameter subgroup* of  $G$  is a differentiable curve  $\gamma : \mathbb{R} \rightarrow G$  such that  $\gamma(t+s) = \gamma(t) \times \gamma(s)$  for all  $s, t \in \mathbb{R}$  (note that  $\gamma(-t) = \gamma(t)^{-1}$  for all  $t \in \mathbb{R}$ ). This implies in particular that  $\gamma(0) = 1$ . If  $\gamma$  is an integral curve of a left-invariant vector field  $X$ , then  $\gamma$  is deduced from the tangent vector  $X_1$  at the identity 1 of  $G$ . This vector  $X_1$  is then called the *generator* of  $\gamma$ . Given a vector  $v$  in  $T_1(G)$ , it is usual to denote by  $(\exp(tv))_{t \in \mathbb{R}}$  the one-parameter subgroup of  $G$  generated by  $v$ .

One may define a map  $\text{Ad}$  on  $G$  such that  $\text{Ad}(x) : y \mapsto x \times y \times x^{-1}$ . This map  $\text{Ad}(x) : G \rightarrow G$  is linear. Let us denote by  $\text{ad}(x)$  its differential, which is then a linear map from  $T_1(G)$  to  $T_1(G)$ , since  $\text{Ad}(1) = 1$ . For  $X, Y$  in  $T_1(G)$ ,  $(X, Y) \mapsto \text{ad}(X)Y$  is a bilinear map and we set  $[X, Y] = \text{ad}(X)Y$ . With this bracket  $[\cdot, \cdot]$ ,  $T_1(G)$  is then a Lie algebra.

**6.4. Tensor algebra.** We consider now the following *tensor algebra*  $T(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2)$  where  $\mathbb{R}^2 \otimes \mathbb{R}^2$  is the tensor product of  $\mathbb{R}^2$  (on this notion see for example [DP91]). If  $\{e_1, e_2\}$  is the canonical basis of  $\mathbb{R}^2$ , then  $\mathbb{R}^2 \otimes \mathbb{R}^2$  is the vector space of dimension 4 with basis  $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ . For  $x, y \in \mathbb{R}^2$ ,

$$x \otimes y = (x^1 e_1 + x^2 e_2) \otimes (y^1 e_1 + y^2 e_2) = \sum_{i,j=1,2} x^i y^j e_i \otimes e_j,$$

$$\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y), \quad \forall \lambda \in \mathbb{R}.$$

An element  $x \in T(\mathbb{R}^2)$  may be decomposed as  $x = (x^0, x^1, x^2)$  where  $x^0 \in \mathbb{R}$ ,  $x^1 \in \mathbb{R}^2$  and  $x^2 \in \mathbb{R}^2 \otimes \mathbb{R}^2$ . This space is equipped with the addition term-wise addition  $+$ , and the multiplication  $\otimes$  defined by

$$x \otimes y = xy \text{ if } x \in \mathbb{R}, y \in T(\mathbb{R}^2),$$

$$x \otimes y \otimes z = 0 \text{ if } x, y, z \in \mathbb{R}^2.$$

The element  $e_0 = 1 = (1, 0, 0)$  is the neutral element of  $T(\mathbb{R}^2)$  for  $\otimes$ , while  $0 = (0, 0, 0)$  is the neutral element of  $+$ . The space  $(T(\mathbb{R}^2), +, \otimes)$  is an associative algebra, which is obtained by quotienting the tensor algebra  $\mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \otimes \mathbb{R}^2 \oplus \dots$  by the ideal formed by all the elements which belongs to  $(\mathbb{R}^2)^{\otimes 3} \oplus (\mathbb{R}^2)^{\otimes 4} \oplus \dots$ .



*Remark 9.* Let us consider the space  $\mathbb{R}\langle X_1, X_2 \rangle$  of polynomials with two non-commutative variables  $X_1$  and  $X_2$ , as well as the equivalence relation  $\sim$  on  $\mathbb{R}\langle X_1, X_2 \rangle$  defined by  $P \sim Q$  if  $P - Q$  is the sum of terms of total degree at least 3. Then there exists an associative algebra isomorphism  $\Phi$  from  $(T(\mathbb{R}^2), +, \otimes)$  to  $(\mathbb{R}\langle X_1, X_2 \rangle / \sim, +, \times)$  such that  $\Phi(e_i) = X_i$  for  $i = 1, 2$ . In other words, the elements of  $T(\mathbb{R}^2)$  are manipulated as polynomials where the terms of total degree at most 2 are kept.

For  $\xi \in \{0, 1\}$ , we denote by  $T_\xi(\mathbb{R}^2)$  the subset of  $T(\mathbb{R}^2)$  defined by

$$T_\xi(\mathbb{R}^2) = \{ (\xi, x^1, x^2) \mid x^1 \in \mathbb{R}^2, x^2 \in \mathbb{R}^2 \otimes \mathbb{R}^2 \}.$$

**Lemma 13.** *The space  $(T_1(\mathbb{R}^2), \otimes)$  is a non-commutative group.*

*Proof.* Clearly, if  $x, y \in T_1(\mathbb{R}^2)$ , then  $x \otimes y \in T_1(\mathbb{R}^2)$ . That  $(T_1(\mathbb{R}^2), \otimes)$  is non-commutative follows from the very definition of  $\otimes$ . To show it is a group, it remains to compute the inverse of each element. If  $x = (1, x^1, x^2)$ , then  $x^{-1} = (1, -x^1, -x^2 + x^1 \otimes x^1)$  is the inverse of  $x$ .  $\square$

For  $x, y \in T(\mathbb{R}^2)$ , we define the bracket of  $x$  and  $y$  by

$$[x, y] = x \otimes y - y \otimes x.$$

If  $x = (x^0, x^1, x^2)$  and  $y = (y^0, y^1, y^2)$  belong to  $T(\mathbb{R}^2)$ , then

$$[x, y] = [x^1, y^1] = (x^1 \wedge y^1)[e_1, e_2].$$

Let us also note that  $[x, y] = -[y, x]$ .

A natural sub-vector space of  $(T_0(\mathbb{R}^2), +) \subset (T(\mathbb{R}^2), +)$  is then

$$\mathfrak{g}(\mathbb{R}^2) = \{ x \in T_0(\mathbb{R}^2) \mid x = x^1 + x^a[e_1, e_2], x^1 \in \mathbb{R}^2, x^a \in \mathbb{R} \}.$$

Although not stable under  $\otimes$ ,  $\mathfrak{g}(\mathbb{R}^2)$  is stable under  $[\cdot, \cdot]$ : if  $x = (x^1, x^a), y = (y^1, y^a) \in \mathfrak{g}(\mathbb{R}^2)$ , then

$$[x, y] = x^1 \wedge y^1[e_1, e_2] \in \mathfrak{g}(\mathbb{R}^2).$$

This space  $\mathfrak{g}(\mathbb{R}^2)$  is of dimension 3. For  $x = x^1 + x^a[e_1, e_2]$  and  $y = y^1 + y^a[e_1, e_2]$ , we set

$$\begin{aligned} x \boxplus y &= x^1 + y^1 + (x^a + y^a)[e_1, e_2] + \frac{1}{2}[x^1, y^1] \\ &= x^1 + y^1 + (x^a + y^a + \frac{1}{2}x^1 \wedge y^1)[e_1, e_2]. \end{aligned}$$

Finally, we define  $i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}$  by

$$i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}(x) = (x^{1,1}, x^{1,2}, x^a) \text{ if } x = x^{1,1}e_1 + x^{1,2}e_2 + x^a[e_1, e_2].$$

It is clear that  $i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}$  is one-to-one from  $\mathfrak{g}(\mathbb{R}^2)$  to  $A(\mathbb{R}^2)$ , and an additive group homomorphism from  $(\mathfrak{g}(\mathbb{R}^2), \boxplus)$  to  $(A(\mathbb{R}^2), \boxplus)$ . In addition,  $[i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}(x), i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}(y)] = i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}[x, y]$  for all  $x, y \in \mathfrak{g}(\mathbb{R}^2)$ , which means that  $i_{\mathfrak{g}(\mathbb{R}^2), A(\mathbb{R}^2)}$  is also a Lie homomorphism. Hence, we identify the spaces  $\mathfrak{g}(\mathbb{R}^2)$  and  $A(\mathbb{R}^2)$ . Lemmas 4 and 5 are then rewritten in the following way.

**Lemma 14.** *The space  $(\mathfrak{g}(\mathbb{R}^2), [\cdot, \cdot])$  is a Lie algebra, and  $(\mathfrak{g}(\mathbb{R}^2), \boxplus)$  is a Lie group with the neutral element 0.*

On  $T_0(\mathbb{R}^2)$ , let us define

$$\exp(x) = 1 + x^1 + x^2 + \frac{1}{2}x^1 \otimes x^1 \text{ for } x = (0, x^1, x^2).$$

This map  $\exp$  is given by the first terms of the formal expansion of the exponential, since we are working in a truncated tensor algebra.

Similarly, let us define on  $T_1(\mathbb{R}^2)$ ,

$$\log(x) = x^1 + x^2 - \frac{1}{2}x^1 \otimes x^1 \text{ for } x = (1, x^1, x^2) \in T_1(\mathbb{R}^2).$$

It is easily seen that for  $\exp \circ \log$  and  $\log \circ \exp$  are equal to the identity respectively on  $T_1(\mathbb{R}^2)$  and on  $T_0(\mathbb{R}^2)$ .

If  $x, y \in T_0(\mathbb{R}^2)$ , then

$$\begin{aligned} \exp(x) \otimes \exp(y) &= 1 + x^1 + y^1 + x^2 + y^2 \\ &\quad + \frac{1}{2}x^1 \otimes x^1 + \frac{1}{2}y^1 \otimes y^1 + x^1 \otimes y^1 + x^2 \otimes y^2 \end{aligned}$$

and then

$$(41) \quad \log(\exp(x) \otimes \exp(y)) = x \boxplus y.$$

This is the truncated version of the *Baker-Campbell-Hausdorff-Dynkin formula* (see for example [Hal03, Reu93]).

**Lemma 15.** *If  $G(\mathbb{R}^2) = \exp(\mathfrak{g}(\mathbb{R}^2))$ , then  $G(\mathbb{R}^2)$  is a subgroup of  $(T_1(\mathbb{R}^2), \otimes)$  and  $\exp$  is a group isomorphism from  $(\mathfrak{g}(\mathbb{R}^2), \boxplus)$  to  $(G(\mathbb{R}^2), \otimes)$ .*

Let us note that  $\exp(-x)$  is the inverse of  $\exp(x)$  in  $G(\mathbb{R}^2)$ , for all  $x \in \mathfrak{g}(\mathbb{R}^2)$ .

For a sub-vector space  $V$  of  $T(\mathbb{R}^2)$ ,  $\pi_V(x)$  denotes the projection onto  $V$ . For  $e, x \in T(\mathbb{R}^2)$ , let  $\pi_e(x)$  be the real  $\lambda$  such that  $\lambda e = \pi_{\text{Vect}(e)}(x)$ . For  $x \in T(\mathbb{R}^2)$ , set

$$\begin{aligned} \mathfrak{s}(x) &= \sum_{i,j=1,2} \frac{1}{2}(\pi_{e_i \otimes e_j}(x) + \pi_{e_j \otimes e_i}(x))e_i \otimes e_j, \\ \mathfrak{a}(x) &= \frac{1}{2}(\pi_{e_1 \otimes e_2}(x) - \pi_{e_2 \otimes e_1}(x))[e_1, e_2]. \end{aligned}$$

If  $x$  belongs to  $\mathbb{R}^2 \otimes \mathbb{R}^2$ , then

$$(42) \quad x = \mathfrak{s}(x) + \mathfrak{a}(x),$$

and  $\mathfrak{s}(x)$  (resp.  $\mathfrak{a}(x)$ ) corresponds to the symmetric (resp. anti-symmetric) part of  $x$ . Finally, let us note that for  $x \in T(\mathbb{R}^2)$ ,

$$(43) \quad \mathfrak{s}(x \otimes x) = \pi_{\mathbb{R}^2 \otimes \mathbb{R}^2}(x \otimes x),$$

For  $z = \exp(x) \in G(\mathbb{R}^2)$ , we have

$$(44) \quad \mathfrak{s}(z) = \frac{1}{2}\mathfrak{s}(x \otimes x) = \frac{1}{2}x \otimes x$$

and

$$\mathbf{a}(z) = \pi_{[e_1, e_2]}(x)[e_1, e_2].$$

Hence, for  $x \in \mathfrak{g}(\mathbb{R}^2)$ , one may rewrite

$$(45) \quad \exp(x) = 1 + \pi_{\mathbb{R}^2}(x) + \frac{1}{2}x \otimes x + \mathbf{a}(x) \text{ and } x = \pi_{\mathbb{R}^2}(x) + \mathbf{a}(x).$$

In particular, for  $z \in G(\mathbb{R}^2)$ ,  $\mathbf{a}(\log(z)) = \mathbf{a}(z)$ .

**6.5. The tensor space as a Lie group.** It is possible to find a norm  $|\cdot|$  on  $\mathbb{R}^2 \otimes \mathbb{R}^2$  such that  $|x \otimes y| \leq |x| \cdot |y|$  for all  $x, y \in \mathbb{R}^2$  (there are indeed several possibilities).

For  $x = (1, x^1, x^2) \in T_1(\mathbb{R}^2)$  or for  $x = (0, x^1, x^2) \in T_0(\mathbb{R}^2)$ , we set

$$\|x\|_{\star} = \max\{|x^1|, |x^2|\}$$

and

$$\|x\| = \max\left\{|x^1|, \sqrt{\frac{1}{2}|x^2|}\right\}.$$

Then  $\|\cdot\|$  is an *homogeneous gauge* for the dilatation operator  $\delta_t$  defined by  $\delta_t x = (1, tx^1, t^2 x^2)$ ,  $t \in \mathbb{R}$ , since  $\|\delta_t x\| = |t| \cdot \|x\|$  (see Section A.1). Besides,  $\|x \otimes y\| \leq 3/2(\|x\| + \|y\|)$  for all  $x, y \in T_1(\mathbb{R}^2)$ . We have introduced in Section 5.4 a dilatation operator, also denoted by  $\delta$ , in a similar way. Note that for  $x \in A(\mathbb{R}^2)$  and  $t \in \mathbb{R}$ ,  $\exp(\delta_t x) = \delta_t \exp(x)$ .

The the next lemma is easily proved.

**Lemma 16.** *With the norm  $\|\cdot\|_{\star}$ , the spaces  $(T_1(\mathbb{R}^2), \otimes)$  and  $(G(\mathbb{R}^2), \otimes)$  are Lie groups, and  $G(\mathbb{R}^2)$  is a closed subgroup of  $T_1(\mathbb{R}^2)$ .*

For  $x \in \mathfrak{g}(\mathbb{R}^2)$ ,  $t \in \mathbb{R} \mapsto \gamma_x(t) \stackrel{\text{def}}{=} \exp(tx) \in G(\mathbb{R}^2)$  is a one-parameter subgroup of  $(G(\mathbb{R}^2), \otimes)$ . The point  $x$  is the tangent vector to  $\gamma_x(t)$  for  $t = 0$ :

$$\frac{d\gamma_x(0)}{dt} = x.$$

Hence,  $\mathfrak{g}(\mathbb{R}^2)$  may be identified with the tangent plane of  $G(\mathbb{R}^2)$  at the point 1, and indeed at any point  $y \in G(\mathbb{R}^2)$ .

The bracket allows us to characterize the lack of commutativity of  $G(\mathbb{R}^2)$ , as it follows from the next result, which is classical in the theory of Lie group (see Figure 11): For  $x, y \in \mathfrak{g}(\mathbb{R}^2)$  and for  $t \geq 0$ , set

$$\theta_{x,y}(t) = \gamma_x(\sqrt{t}) \otimes \gamma_y(\sqrt{t}) \otimes (\gamma_x(-\sqrt{t})) \otimes \gamma_y(-\sqrt{t}).$$

Then  $\theta_{x,y}(0) = 1$  and

$$\left. \frac{d\theta_{x,y}}{dt} \right|_{t=0} = [x, y].$$

In our case, it follows from the truncated version the Baker-Campbell-Hausdorff-Dynkin formula (41) that  $\theta_{x,y}(t) = \exp(t[x, y])$  for all  $t \geq 0$ .

To any Lie group corresponds a Lie algebra, which is identified to the tangent plane at the neutral elements, and then at any point. Of

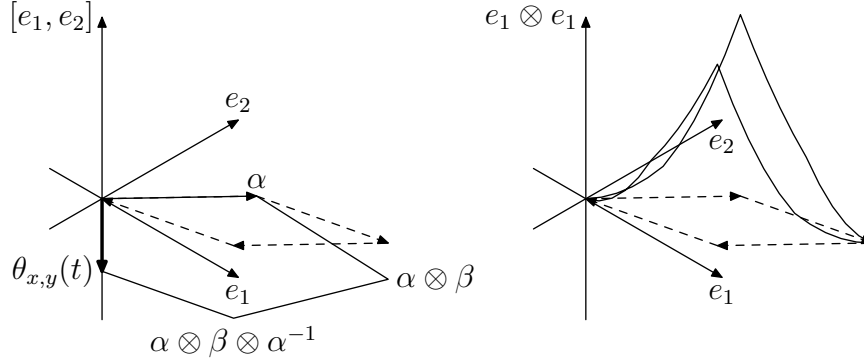


FIGURE 11. Illustration of the non-commutativity with  $\alpha = \gamma_x(\sqrt{t})$  and  $\beta = \gamma_y(\sqrt{t})$ .

course,  $\mathfrak{g}(\mathbb{R}^2) \cong \mathbb{A}(\mathbb{R}^2)$  has been constructed to be the tangent plane of  $G(\mathbb{R}^2)$  at any point.

**Lemma 17.** *The tangent plane of  $G(\mathbb{R}^2)$  at any point may be identified with  $\mathbb{A}(\mathbb{R}^2)$ .*

*Remark 10.* In Section A.2, we will see how to see the elements of  $\mathbb{A}(\mathbb{R}^d)$  and  $G(\mathbb{R}^2)$  as matrices.

We end with a very useful lemma, whose proof is straightforward. The notion of Lipschitz functions on spaces with homogeneous gauges is similar to the notion of Lipschitz functions (See Definition 9 in Section A.1).

**Lemma 18.** *The applications  $\exp$  is Lipschitz continuous from  $(\mathbb{A}(\mathbb{R}^2), |\cdot|)$  to  $(G(\mathbb{R}^2), \|\cdot\|)$ , and  $\log$  is Lipschitz continuous from  $(G(\mathbb{R}^2), \|\cdot\|)$  to  $(\mathbb{A}(\mathbb{R}^2), |\cdot|)$ .*

*The maps  $\exp$  and  $\log$  are differentiable, so that  $\exp$  is a diffeomorphism between  $\mathbb{A}(\mathbb{R}^2)$  and  $G(\mathbb{R}^2)$ . Besides,  $d_x \exp : T_x(\mathbb{A}(\mathbb{R}^2)) \cong \mathbb{A}(\mathbb{R}^2) \rightarrow \mathbb{A}(\mathbb{R}^2)$  is the identity map for any  $x \in \mathbb{A}(\mathbb{R}^2)$ .*

In particular, it follows from this lemma that  $\exp$  provides a change of variable between  $\mathbb{A}(\mathbb{R}^2)$  and  $G(\mathbb{R}^2)$ . For a point  $x$  in  $G(\mathbb{R}^2)$ ,  $\log(x)$  is called the *normal coordinates* of  $x$ .

**6.6. A transverse decomposition of the tensor space.** We have introduced a subgroup  $G(\mathbb{R}^2)$  of  $T_1(\mathbb{R}^2)$ . Is this subgroup strict or not?

The tangent space of  $T_1(\mathbb{R}^2)$  at any point may be identified with the vector space  $(T_0(\mathbb{R}^2), +)$ , which is dimension 6. We have also seen that the tangent space of  $G(\mathbb{R}^2)$  at any point may be identified with  $\mathbb{A}(\mathbb{R}^2)$ , and thus is of dimension 3. Then, of course,  $G(\mathbb{R}^2) \neq T_1(\mathbb{R}^2)$ . Indeed, we may be more precise on the decomposition of  $T_1(\mathbb{R}^2)$ .

We denote by  $S(\mathbb{R}^2)$  the subset of  $T_0(\mathbb{R}^2)$  defined by

$$S(\mathbb{R}^2) = \left\{ x = (0, 0, x^2) \in T_0(\mathbb{R}^2) \left| \begin{array}{l} x^2 = \lambda e_1 \otimes e_1 + \mu e_2 \otimes e_2 \\ + \nu(e_1 \otimes e_2 + e_2 \otimes e_1), \\ \lambda, \mu, \nu \in \mathbb{R} \end{array} \right. \right\}$$

In other words, an element of  $S(\mathbb{R}^2)$  belongs to  $\mathbb{R}^2 \otimes \mathbb{R}^2$  and is symmetric. Of course,  $S(\mathbb{R}^2)$  is linear, stable under  $\otimes$  and  $+$  (indeed, if  $x, y \in S(\mathbb{R}^2)$ , then  $x \otimes y = x + y$ ), and is a vector space of dimension 3.

To an element  $e$  of the basis of  $T(\mathbb{R}^2)$ , we denote by  $\pi_e$  the projection from  $T(\mathbb{R}^2)$  to  $T(\mathbb{R}^2)$ , such that  $x = \pi_1(x) + \sum_{i=1,2} \pi_{e_i}(x)e_i + \sum_{i,j=1,2} \pi_{e_i \otimes e_j}(x)e_i \otimes e_j$ .

The next result follows easily from the construction of the projection operator  $\widehat{\Upsilon}_s : T_0(\mathbb{R}^2) \rightarrow S(\mathbb{R}^2)$  and  $\widehat{\Upsilon}_a : T_0(\mathbb{R}^2) \rightarrow A(\mathbb{R}^2)$  defined by

$$\widehat{\Upsilon}_s(x) = \mathfrak{s}(x) \text{ and } \widehat{\Upsilon}_a(x) = \pi_{\mathbb{R}^2}(x) + \mathfrak{a}(x).$$

**Proposition 6.** *The space  $T_0(\mathbb{R}^2)$  is the direct sum of  $A(\mathbb{R}^2)$  and  $S(\mathbb{R}^2)$ .*

This decomposition holds at the level of the tangent spaces at any point of  $T_1(\mathbb{R}^2)$ .

**Proposition 7.** *Any element  $x$  of  $T_1(\mathbb{R}^2)$  may be written as the sum  $x = y + z$  with  $y \in G(\mathbb{R}^2)$  and  $z \in S(\mathbb{R}^2)$ .*

*Proof.* For  $x \in T(\mathbb{R}^2)$ , let us set

$$\begin{aligned} \Upsilon_s(x) &= \mathfrak{s}(x) - \frac{1}{2}x \otimes x, \\ \Upsilon_a(x) &= 1 + \pi_{\mathbb{R}^2}(x) + \mathfrak{a}(x) + \frac{1}{2}x \otimes x. \end{aligned}$$

With (42),  $\Upsilon_a(x) + \Upsilon_s(x) = x$  for all  $x \in T(\mathbb{R}^2)$ . Also, thanks to (43) and (45),  $\Upsilon_s(T_1(\mathbb{R}^2)) \subset S(\mathbb{R}^2)$  and  $\Upsilon_a(T_1(\mathbb{R}^2)) \subset G(\mathbb{R}^2)$ .  $\square$

We have to note that with the previous decomposition,  $G(\mathbb{R}^2)$  is not a linear subspace of  $T_1(\mathbb{R}^2)$ , and  $\Upsilon_a$  and  $\Upsilon_s$  are not linear projections, since they involve quadratic terms. This is why we do not write  $T_1(\mathbb{R}^2)$  as the direct sum of  $G(\mathbb{R}^2)$  and  $S(\mathbb{R}^2)$ . However, as the tangent plane of  $S(\mathbb{R}^2)$  is  $S(\mathbb{R}^2)$  itself, if  $G(\mathbb{R}^2)$  and  $\exp(S(\mathbb{R}^2)) = \{1 + x \mid x \in S(\mathbb{R}^2)\}$  are submanifolds of  $T_1(\mathbb{R}^2)$ , we get that  $G(\mathbb{R}^2)$  and  $\exp(S(\mathbb{R}^2))$  provides a *transverse decomposition* of  $T_1(\mathbb{R}^2)$ .

We define a homogeneous norm  $\|\cdot\|_{G(\mathbb{R}^2) \times S(\mathbb{R}^2)}$  by

$$(46) \quad \|x\|_{G(\mathbb{R}^2) \times S(\mathbb{R}^2)} = \max \left\{ \|\Upsilon_a(x)\|, \sqrt{\frac{1}{2}\|\Upsilon_s(x)\|} \right\}.$$

It is easily shown that this homogeneous norm is equivalent to the homogeneous gauge  $\|\cdot\|$  on  $T^1(\mathbb{R}^2)$ .

7. THE ROUGH PATHS AND THEIR INTEGRALS

**7.1. What is rough paths?** If  $x \in G(\mathbb{R}^2)$ , then it is easily seen that for some universal constants  $c$  and  $c'$ ,  $c\|x\| \leq |\log(x)| \leq c'\|x\|$ , where  $|\cdot|$  is the homogeneous norm we have defined on  $A(\mathbb{R}^2)$  by (15).

**Definition 4.** A *rough path* is a continuous path  $\mathbf{x}$  with values in  $T_1(\mathbb{R}^2)$ .

Denote by  $C^\alpha([0, T], T_1(\mathbb{R}^2))$  the set of rough paths  $\mathbf{x} : [0, T] \rightarrow T_1(\mathbb{R}^2)$  such that

$$\|\mathbf{x}\|_\alpha \stackrel{\text{def}}{=} \sup_{0 \leq s < t \leq T} \frac{\|\mathbf{x}_s^{-1} \otimes \mathbf{x}_t\|}{|t - s|^\alpha}$$

is finite.

A particular class of path is the notion of geometric rough paths. The next definition follows from [FV06c], which correct some result of [Lyo98].

**Definition 5.** A *geometric rough path* is a continuous path with values in  $G(\mathbb{R}^2)$ .

A *smooth rough path* is an element of the set

$$C^\infty([0, T]; G(\mathbb{R}^2)) = \left\{ \exp(\mathbf{x}) \left| \begin{array}{l} \mathbf{x} = x + \mathfrak{A}(x)[e_1, e_2] \\ \text{with } x \in C_p^\infty([0, T]; \mathbb{R}^2) \end{array} \right. \right\}.$$

A *weak geometric  $p$ -rough path with Hölder control* is a path with values in  $G(\mathbb{R}^2)$  which is  $1/p$ -Hölder continuous.

A *geometric  $p$ -rough path* is the closure of the set of smooth rough paths with respect to the  $\|\cdot\|_{1/p}$ -norm.

*Remark 11.* As discussed in Section 5.10, a rough path which is smooth is not necessarily a smooth rough path.

The space of weak geometry  $1/\alpha$ -rough paths with Hölder control is denoted by  $C^\alpha([0, T]; G(\mathbb{R}^2))$ , while the space of  $1/\alpha$ -rough path with Hölder control is denoted by  $C^{0,\alpha}([0, T]; G(\mathbb{R}^2))$ . This latter space is strictly included in  $C^\alpha([0, T]; G(\mathbb{R}^2))$ . In addition  $C^{0,\alpha}([0, T]; G(\mathbb{R}^2))$  is a Polish space, while  $C^\alpha([0, T]; G(\mathbb{R}^2))$  is not a polish space. The difference between weak geometric  $p$ -rough paths and a geometric  $p$ -rough paths comes from an extension of the properties of Hölder continuous paths given in Remarks 2 and 3. For practical applications, the difference between weak geometric rough paths and geometric rough path is not that important, if we are ready to weaken the Hölder norm (this is in general cost free).

Of course, there is a one-to-one correspondence between the paths with values in  $G(\mathbb{R}^2)$  and the one with values in  $A(\mathbb{R}^2)$ : Since  $\exp$  and  $\log$  are Lipschitz continuous (see Lemma 18), we get easily the following lemma, which is illustrated by Figure 12.

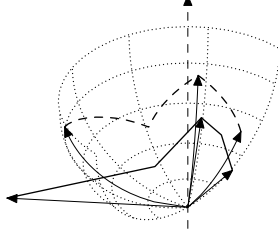


FIGURE 12. From the tangent plane  $A(\mathbb{R}^2)$  at the point 1 (perpendicular to the vertical axis) to the manifold  $G(\mathbb{R}^2)$ : the paths  $\mathbf{x}$  (dashed) and  $\log(\mathbf{x})$  (plain).

**Lemma 19.** *A path  $\mathbf{x}$  belongs to  $C^\alpha([0, T]; G(\mathbb{R}^2))$  if and only if  $\log(\mathbf{x})$  belongs to  $C^\alpha([0, T]; A(\mathbb{R}^2))$ .*

A path  $\mathbf{y} = (a(t), b(t), c(t))_{t \in [0, T]}$  with value in  $A(\mathbb{R}^2)$  is then transformed into a path  $\mathbf{x}_t = \exp(\mathbf{y}_t)$  with value in  $G(\mathbb{R}^2)$  by the relation

$$\begin{aligned} \mathbf{x}_t = & a(t)e_1 + b(t)e_2 + \frac{1}{2}a(t)^2e_1 \otimes e_1 + \frac{1}{2}b(t)^2e_2 \otimes e_2 \\ & + (a(t)b(t) + c(t))e_1 \otimes e_2 + (a(t)b(t) - c(t))e_2 \otimes e_1. \end{aligned}$$

Similarly, a path  $\mathbf{x}$  with values in  $G(\mathbb{R}^2)$  is transformed into a path  $\mathbf{y}$  with values in  $A(\mathbb{R}^2)$  by setting  $\mathbf{y}_t = \log(\mathbf{x}_t)$ .

In addition, let us note that  $\mathbf{x}_{s,t} \stackrel{\text{def}}{=} \mathbf{x}_s^{-1} \otimes \mathbf{x}_t = \exp((-y_s) \boxplus y_t)$  and then

$$\mathfrak{s}(\mathbf{x}_{s,t}) = \frac{1}{2}(x_t - x_s) \otimes (x_t - x_s)$$

where  $x_t = a(t)e_1 + b(t)e_2$  is the path above which  $\mathbf{x}$  lies.

Let us assume now that  $\mathbf{y}$  belongs  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha > 1/2$ . We have seen in Lemma 7 that necessarily,  $c(t) = \mathfrak{A}(x; 0, t)$ . Hence, from (45),

$$(47) \quad \mathbf{x}_t = 1 + x_t + \mathfrak{A}(x; 0, t)[e_1, e_2] + \frac{1}{2}(x_t - x_0) \otimes (x_t - x_0).$$

As for  $0 \leq s \leq t \leq T$ ,

$$\begin{aligned} \mathfrak{A}(x; s, t) = & \frac{1}{2} \int_s^t (x_r^1 - x_s^1) dx_r^2 - \frac{1}{2} \int_s^t (x_r^2 - x_s^2) dx_r^1 \\ \text{and } \frac{1}{2}(x_t^i - x_s^i)^2 = & \int_s^t (x_r^i - x_s^i) dx_r^i \text{ for } i = 1, 2, \end{aligned}$$

we may rewrite (47) as

$$(48) \quad \mathbf{x}_t = 1 + x_t + \sum_{i,j=1,2} \left( \int_0^t (x_r^i - x_0^i) dx_r^j \right) e_i \otimes e_j.$$

Let us note also that

$$\mathbf{x}_{s,t} \stackrel{\text{def}}{=} (-\mathbf{x}_s) \otimes \mathbf{x}_t = 1 + x_t - x_s + \sum_{i,j=1,2} \left( \int_s^t (x_r^i - x_s^i) dx_r^j \right) e_i \otimes e_j.$$

This means that the terms of  $\mathbf{x}_t$  in  $\mathbb{R}^2 \otimes \mathbb{R}^2$  are the *iterated integrals of  $x$* . When  $\alpha < 1/2$ , the difficulty comes from the fact that these iterated integrals are not canonically constructed. As the iterated integrals have some nice algebraic properties (see Section 8.2), we replace them by an object — the rough paths — which shares the same algebraic properties, whose existence is not discussed in this article.

Let us end this Section with a result on path that are not geometric. If  $\mathbf{x}$  belongs to  $C^\alpha([0, T]; T_1(\mathbb{R}^2))$  (with  $\alpha \in (0, 1]$ ) and  $\mathbf{x}_t - 1 \in S(\mathbb{R}^2)$  for all  $t$ , then

$$\|\mathbf{x}_s^{-1} \otimes \mathbf{x}_t\| = \sqrt{\frac{1}{2}|\mathbf{x}_t^2 - \mathbf{x}_s^2|} \leq \|\mathbf{x}\|_\alpha |t - s|^\alpha$$

with  $\mathbf{x}_t = (1, \mathbf{x}_t^1, \mathbf{x}_t^2)$ . This implies that  $\mathbf{x}_t$  can be identified with a path in  $C^{2\alpha}([0, T]; \mathbb{R}^3)$  (note that if  $\alpha > 1/2$ , then  $\mathbf{x}$  is constant).

### 7.2. Joining two points by staying in $G(\mathbb{R}^2)$ : geodesics in $G(\mathbb{R}^2)$ .

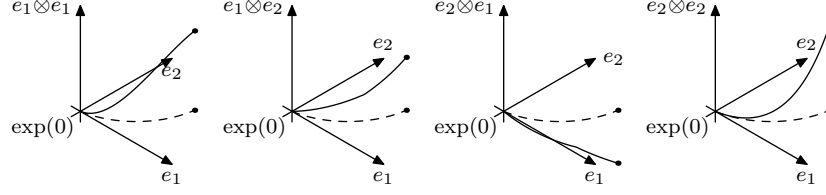
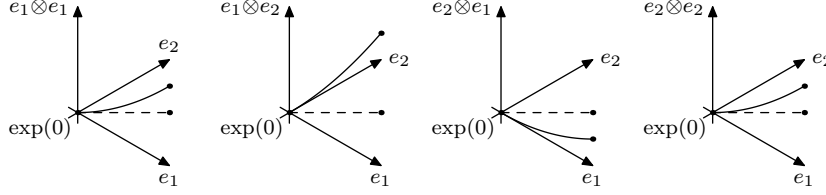
Given two points  $x$  and  $y$  in  $A(\mathbb{R}^2)$ , we have constructed a path  $z : [0, 1] \rightarrow \mathbb{R}^2$  such that  $(z, \mathfrak{A}(z, 0, \cdot))$  goes from  $x$  to  $y$ . We have constructed the spaces  $T_1(\mathbb{R}^2)$  and  $G(\mathbb{R}^2)$  such that  $A(\mathbb{R}^2)$  may be interpreted as the tangent plane of  $G(\mathbb{R}^2)$  at any point. Thus, a point in  $A(\mathbb{R}^2)$  is identified with the initial direction of a curve in  $G(\mathbb{R}^2)$ . We have also seen that it is natural that, in order to construct integrals, to study the limit of integral along a family of piecewise smooth paths whose values at the dyadics points of  $[0, T]$  corresponds to the values of the paths. The problem is how to link two points in  $G(\mathbb{R}^2)$ . In the case  $\alpha > 1/2$ , it is sufficient to use a linear approximation. As straight lines correspond to geodesics in  $\mathbb{R}^2$ , we will now construct geodesics in  $G(\mathbb{R}^2)$ . These geodesics aim to replace our approximation of paths constructed using a loop and a straight line.

We consider  $x, y \in G(\mathbb{R}^2)$ . There are several possibilities to construct what can be seen as the “shortest path” to join  $x$  and  $y$ : the path may stay in  $G(\mathbb{R}^2)$  or join directly  $x$  and  $y$  by going through  $T_1(\mathbb{R}^2)$ . The latter case requires a supplementary assumption on  $\mathbf{x}$  (see Section 8.3 below). To stay in  $G(\mathbb{R}^2)$ , we have already considered in Section 5.10 sub-Riemannian geodesics, which means that we use some smooth rough path. We will now consider the construction of a real geodesic in  $G(\mathbb{R}^2)$ , which is indeed a segment in  $A(\mathbb{R}^2)$ , but which is no longer a smooth rough path.

For this, let us define for  $t \in [0, 1]$  (see Figure 13 (b)),

$$\varphi_{x,y}(t) = x \otimes \exp(t \log(x^{-1} \otimes y))$$



(a) A sub-Riemannian geodesic in  $G(\mathbb{R}^2)$  as constructed from Section 5.10.FIGURE 13. (b) The path  $\varphi_{x,y}$  with  $x = \exp(0)$  and  $y = \exp((1, 1, 1))$ .

with

$$\log(x^{-1} \otimes y) = y - x + \frac{1}{2}x^1 \otimes x^1 - \frac{1}{2}y^1 \otimes y^1 - \frac{1}{2}x^1 \otimes y^1 + \frac{1}{2}y^1 \otimes x^1.$$

It is clear that  $\varphi_{x,y}$  is continuous and that  $\varphi_{x,y}(0) = x$  and  $\varphi_{x,y}(1) = y$ . In addition,  $\varphi_{x,y}(t) \in G(\mathbb{R}^2)$  for all  $t \in [0, 1]$ .

There are several ways to compute the derivative. Let us note that for  $x, y, z \in \mathfrak{g}(\mathbb{R}^2)$  and  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} \lambda(x \boxplus y) &= (\lambda x) \boxplus y + (\lambda - 1)y, \\ x \boxplus (y + z) &= x \boxplus y + x \boxplus z - x. \end{aligned}$$

After a short computation, we get

$$\begin{aligned} \varphi_{x,y}(t) &= (1-t)x + ty \\ &\quad + \frac{t(1-t)}{2} (x^1 \otimes x^1 - y^1 \otimes y^1 + y^1 \otimes x^1 + x^1 \otimes y^1). \end{aligned}$$

We can now compute the derivative of  $\varphi_{x,y}$  by identifying  $\varphi_{x,y}$  with a path with values in the tangent bundle  $T(T_1(\mathbb{R}^2)) = T_0(\mathbb{R}^2) \cong T_1(\mathbb{R}^2)$ . The spaces  $T_\xi(\mathbb{R}^2)$ ,  $\xi = 0, 1$  are naturally equipped with a scalar product and  $\{e_k, e_i \otimes e_j\}_{k,i,j=1,2}$  forms an orthonormal frame at any point of  $T_1(\mathbb{R}^2)$ . Hence,

$$(49) \quad \begin{aligned} \frac{d_{T_0(\mathbb{R}^2)}\varphi_{x,y}(t)}{dt} &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (\varphi_{x,y}(t + \varepsilon) - \varphi_{x,y}(t)) \\ &\stackrel{\text{def}}{=} \log(x^{-1} \otimes y) + t \log(x^{-1} \otimes y) \otimes \log(x^{-1} \otimes y). \end{aligned}$$

Indeed, the derivative of  $\varphi_{x,y}$  does not belong to  $\mathfrak{g}(\mathbb{R}^2)$ . Yet, the second term in the right-hand side of (49) belongs to  $S(\mathbb{R}^2)$ , which is orthogonal to  $\mathfrak{g}(\mathbb{R}^2)$ . As we are interested in paths only in  $G(\mathbb{R}^2)$ , the variation of  $\varphi_{x,y}$  seen by the path is the covariant derivative of

$\varphi_{x,y} : [0, 1] \rightarrow T_0(\mathbb{R}^2) \cong T_1(\mathbb{R}^2)$ , — here denoted by  $\frac{d\varphi_{x,y}}{dt}$  —, which is the projection of  $\frac{d_{T_0(\mathbb{R}^2)}\varphi_{x,y}}{dt}$  on  $\mathfrak{g}(\mathbb{R}^2)$ . Hence

$$(50) \quad \frac{d\varphi_{s,y}(t)}{dt} = \log(x^{-1} \otimes y)$$

and the covariant derivative of  $\varphi_{x,y}$  is constant. This means that  $\varphi_{x,y}$  is a *geodesic*.

A simpler way is the following: Using Lemma 18 and the normal coordinates  $\zeta_{x,y}(t) = \log(\varphi_{x,y}(t))$ , since the differential of  $\exp$  is the identify, we directly get that

$$\frac{d\varphi_{x,y}}{dt}(t) = \frac{d\zeta_{x,y}}{dt}(t) = \log(x^{-1} \otimes y).$$

To a path  $\mathbf{x} : [0, T] \rightarrow G(\mathbb{R}^2)$  and which is piecewise differentiable, we set

$$\mathfrak{J}(\mathbf{x}; 0, T) = \int_0^T \mathfrak{E}_{A(\mathbb{R}^2)}(f)(\mathbf{x}_s) \frac{d\mathbf{x}_s}{ds} ds,$$

where  $\mathfrak{E}_{A(\mathbb{R}^2)}(f)$  is the extension of the differential form  $f$  defined by (39), and  $\frac{d\mathbf{x}_s}{ds}$  belongs to  $\mathfrak{g}(\mathbb{R}^2) \cong A(\mathbb{R}^2)$ .

**Proposition 8.** *For  $\mathbf{x} \in C^\alpha([0, T]; G(\mathbb{R}^2))$  with  $\alpha > 1/3$ , let  $\mathbf{x}^n$  be the path in  $C^\alpha([0, T]; G(\mathbb{R}^2))$  defined by*

$$\mathbf{x}_t^n = \varphi_{\mathbf{x}_{t_k^n}, \mathbf{x}_{t_{k+1}^n}} \left( \frac{t - t_k^n}{t_{k+1}^n - t_k^n} \right) \text{ for } t \in [t_k^n, t_{k+1}^n],$$

for  $n \in \mathbb{N}^*$  and  $t_k^n = Tk/2^n$ ,  $k = 0, \dots, 2^n$ . Then

$$\mathfrak{J}(\mathbf{x}) = \lim_{n \rightarrow \infty} \mathfrak{J}(\mathbf{x}^n) \text{ in } C^\beta([0, T]; \mathbb{R})$$

for all  $\beta < \alpha$ . Besides,  $\mathfrak{J}(\mathbf{x}) \in C^\alpha([0, T]; \mathbb{R})$ .

*Proof.* We have only to prove that  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  is bounded in  $C^\alpha([0, T]; G(\mathbb{R}^2))$ . The results will then follow immediately from Corollary 4 applied to  $\log(\mathbf{x}^n)$ , since  $\log$  is continuous.

It can also be justified in the same way as Proposition 3 with the computations of Section 6.2.

To prove that  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  is bounded, we may use the same computations as in the proof of Corollary 1. Hence, we have only to estimate  $\mathbf{x}_{s,t}^n = (\mathbf{x}_s^n)^{-1} \otimes \mathbf{x}_t^n$  for  $s, t$  in the same interval  $[t_k^n, t_{k+1}^n]$ . Since  $\log$  and  $\exp$  are Lipschitz, for some constant  $C$ ,

$$|\mathbf{x}_{s,t}^n| = \exp \left( 2^n \frac{t-s}{T} \log(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) \right) \leq C(t-s)^\alpha,$$

which allows us to prove that  $\|\mathbf{x}^n\|_\alpha \leq C \cdot 3^{1-\alpha} \|\mathbf{x}\|_\alpha$ .  $\square$

**7.3. A Riemann sum like definition.** We are now willing to give another definition of the integral in the spirit of Riemann sums, to get rid of the integrals between the successive times  $t_k^n$  and  $t_{k+1}^n$  for  $k = 0, \dots, 2^n - 1$ . For this, we use the Taylor development of  $f$ : For  $x, y \in \mathbb{R}^2$  and  $i = 1, 2$ ,

$$f_i(y^1, y^2) = f_i(x^1, x^2) + \sum_{j=1,2} \frac{\partial f_i}{\partial x_j}(x^1, x^2) z^j + \kappa_1^i(z)$$

with  $|\kappa_1^i(z)| \leq \|f\|_{\text{Lip}} |z|^{2+\gamma}$  and  $z = y - x$ . In addition,

$$[f, f](y^1, y^2) = [f, f](x^1, x^2) + \kappa_2(z) \text{ with } |\kappa_2(z)| \leq \|f\|_{\text{Lip}} |z|^\gamma.$$

Let us set  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$  with  $\alpha > 1/3$  and  $\mathbf{x}^n$  constructed as in Proposition 8. In addition, we define  $x$  and  $x^n$  by  $x = \pi_{\mathbb{R}^2}(\mathbf{x})$  and  $x^n = \pi_{\mathbb{R}^d}(\mathbf{x})$ . Let us remark that  $x^n = x^{\Pi^n}$ , the piecewise linear interpolation of  $x$ . For  $\Delta_n t = T2^{-n}$ ,

$$\left| \int_{t_k^n}^{t_{k+1}^n} [f, f](x_s^n) \mathbf{a}(\mathbf{x}_{t_k^n, t_{k+1}^n}) \frac{ds}{\Delta_n t} - [f, f](x_{t_k^n}) \mathbf{a}(\mathbf{x}_{t_k^n, t_{k+1}^n}) \right| \leq \Delta_n t^{\alpha(\gamma+2)} \|\mathbf{x}\|_\alpha^{\alpha(2+\gamma)} \|f\|_{\text{Lip}}.$$

In addition, with the Taylor formula,

$$(51) \quad \left| \int_{t_k^n}^{t_{k+1}^n} f(x_s^n) (x_{t_{k+1}^n}^n - x_{t_k^n}^n) \frac{ds}{\Delta_n t} - f(x_{t_k^n}^n) (x_{t_{k+1}^n}^n - x_{t_k^n}^n) - \sum_{i,j=1,2} \frac{1}{2} \frac{\partial f_i}{\partial x_j}(x_{t_k^n}^n) \pi_{e_j}(x_{t_{k+1}^n}^n - x_{t_k^n}^n) \pi_{e_i}(x_{t_{k+1}^n}^n - x_{t_k^n}^n) \right| \leq \Delta_n t^{\alpha(\gamma+2)} \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha^{2+\gamma}.$$

If  $e^i \otimes e^j$  is the dual element of  $e_i \otimes e_j$  for  $i, j = 1, 2$ , we denote by  $\nabla f$  the operator

$$\nabla f(x) = \sum_{i,j=1,2} \frac{\partial f_i}{\partial x_j}(x) e^j \otimes e^i$$

so that with (44),

$$\frac{1}{2} \sum_{i,j=1,2} \frac{\partial f_i}{\partial x_j}(x_{t_k^n}^n) \pi_{e_j}(x_{t_{k+1}^n}^n - x_{t_k^n}^n) \pi_{e_i}(x_{t_{k+1}^n}^n - x_{t_k^n}^n) = \nabla f(x_{t_k^n}^n) \mathbf{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n).$$

Hence, with (42), we deduce that

$$(52) \quad \int_{t_k^n}^{t_{k+1}^n} \mathbf{e}_{\Lambda(\mathbb{R}^2)}(f)(\mathbf{x}_s^n) \frac{d\mathbf{x}_s^n}{ds} ds = f(x_{t_k^n}^n) \pi_{\mathbb{R}^2}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) + \nabla f(x_{t_k^n}^n) \pi_{\mathbb{R}^2 \otimes \mathbb{R}^2}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) + \theta_k^n$$

with  $|\theta_k^n| \leq \|f\|_{\text{Lip}} \|\mathbf{x}\|_\alpha^{2+\gamma} \Delta_n t^{\alpha(\gamma+2)}$ . Since  $\alpha(\gamma+1) > 1$ ,  $\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} |\theta_k^n| = 0$ . We then define a differential form  $\mathfrak{E}_{T_1(\mathbb{R}^2)}(f)$  on  $T_1(\mathbb{R}^2)$  by

$$\mathfrak{E}_{T_1(\mathbb{R}^2)}(f)(x) = 1 \cdot e^0 + \sum_{i=1,2} f_i(\pi_{\mathbb{R}^2}(x)) de^i + \sum_{i,j=1,2} \frac{\partial f_i}{\partial x_j}(\pi_{\mathbb{R}^2}(x)) de^i \otimes e^j,$$

where  $e^0$  is the dual element of  $e_0 = 1$ .

With (52) and the property of the  $\theta_k^n$ 's, we get that

$$(53) \quad \mathfrak{J}(\mathbf{x}; 0, T) = \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \mathfrak{E}_{T_1(\mathbb{R}^2)}(f)(x_{t_k^n}) \mathbf{x}_{t_k^n, t_{k+1}^n},$$

which is a Riemann sum like expression.

This means also that  $\mathfrak{E}_{T_1(\mathbb{R}^2)}(f)(x_{t_k^n}) \mathbf{x}_{t_k^n, t_{k+1}^n}$  is a “good” approximation of  $\mathfrak{J}(\mathbf{x}; t_k^n, t_{k+1}^n)$ .

**7.4. Another construction of the integral.** Let us assume that the functions  $f_1, f_2$  take their values in the space  $\mathbb{R}^m$  with  $m > 1$ . For the sake of simplicity, we assume that  $m = 2$ . The integral  $\mathfrak{J}(\mathbf{x}) = (\mathfrak{J}^1(\mathbf{x}), \mathfrak{J}^2(\mathbf{x}))$  becomes then a path in  $\mathbb{R}^2$ , and we are interested in construction its iterated integrals.

If  $x$  belongs to  $C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 1/2$ , then  $\mathfrak{J}(\mathbf{x})$  also corresponds to a Young integral and belongs to  $C^\alpha([0, T]; \mathbb{R}^2)$ . Hence, we use the natural lift in (48), which means that we have to define only  $t \mapsto \mathfrak{A}(\mathfrak{J}(\mathbf{x}; 0, t); 0, t)$ , or equivalently,  $\int_s^t \mathfrak{J}^i(\mathbf{x}; s, r) d\mathfrak{J}^j(\mathbf{x}; s, r)$  for  $i, j = 1, 2$ .

We remark that, if  $x_r = x_s + (r - s)(t - s)^{-1}(x_t - x_s)$ ,

$$\begin{aligned} \int_s^t \left( \int_s^r f_i^k(x_u) dx^i \right) f_j^\ell(x_r) dx_r^j &= f_i^j(x_s) f_j^\ell(x_s) \int_s^t (x_r^i - x_s^i) dx_r^j \\ &+ \int_s^t \left( \int_s^r (f_i^k(x_r) - f_i^k(x_s)) dx_r^i \right) f_j^\ell(x_r) dx_r^j \\ &+ \int_s^t f_i^k(x_s) (x_r^i - x_s^i) (f_j^\ell(x_r) - f_j^\ell(x_s)) dx_r^j. \end{aligned}$$

This suggests to take for an approximation of  $\int_{t_k^n}^{t_{k+1}^n} \mathfrak{J}^i(\mathbf{x}; t_k^n, s) d\mathfrak{J}^j(\mathbf{x}; t_k^n, s)$  the quantity

$$\mathbf{y}_{t_k^n, t_{k+1}^n}^{k, \ell} = \sum_{i, j=1, 2} f_i^k(x_{t_k^n}) f_j^\ell(x_{t_k^n}) \mathbf{x}_{t_k^n, t_{k+1}^n}^{2, i, j}, \quad k, \ell = 1, 2.$$

With (53), we also set

$$\mathbf{y}_{t_k^n, t_{k+1}^n}^i = \mathfrak{E}_{T_1(\mathbb{R}^2)}(f^i)(x_{t_k^n}) \mathbf{x}_{t_k^n, t_{k+1}^n}, \quad i = 1, 2.$$

Let  $\{\check{e}_1, \check{e}_2\}$  be the canonical basis of  $\mathbb{R}^2$  and  $\{\check{e}^1, \check{e}^2\}$ , which we distinguish from  $\{e_1, e_2\}$  to refer to the space in which  $f$  takes its values.

Then we introduce the differential form  $\mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)$  with value in  $\mathbb{T}_1(\mathbb{R}^2)$  by, for  $z \in \mathbb{T}_1(\mathbb{R}^2)$ ,

$$\begin{aligned} \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(z) &= \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2)}(f^1)(z)\check{e}^1 + \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2)}(f^2)(z)\check{e}^2 \\ &\quad + \sum_{i,j=1,2} f^i(\pi_{\mathbb{R}^2}(z))f^j(\pi_{\mathbb{R}^2}(z))\check{e}^1 \otimes \check{e}^2, \end{aligned}$$

or in the more compact form,

$$\mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2)}(f)(z) = \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2)}(f)(z) + f(\pi_{\mathbb{R}^2}(z)) \otimes f(\pi_{\mathbb{R}^2}(z))$$

with  $f = f^1\check{e}^1 + f^2\check{e}^2$ . Hence, in order to approximate  $\mathfrak{J}(\mathbf{x}; s, t)$  and its iterated integral, we may then set

$$(54) \quad \mathbf{y}_{s,t} = \mathfrak{F}(f, \mathbf{x}; s, t) \stackrel{\text{def}}{=} \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)\mathbf{x}_{s,t}$$

and set, for  $t \in (t_{\underline{M}(t,n)-1}^n, t_{\underline{M}(t,n)}^n]$  and  $s \in [t_{\overline{M}(s,n)}^n, t_{\overline{M}(s,n)}^n)$ ,

$$(55) \quad \mathfrak{J}^n(\mathbf{x}; s, t)$$

$$\stackrel{\text{def}}{=} \mathfrak{F}(\mathbf{x}; s, t_{\overline{M}(s,n)}^n) \otimes \left( \bigotimes_{k=\overline{M}(s,n)}^{\underline{M}(t,n)-1} \mathfrak{F}(\mathbf{x}; t_k^n, t_{k+1}^n) \right) \otimes \mathfrak{F}(\mathbf{x}; t_{\underline{M}(t,n)}^n, t).$$

Finally, we set

$$(56) \quad \mathfrak{J}(\mathbf{x}; s, t) = \lim_{n \rightarrow \infty} \mathfrak{J}^n(\mathbf{x}; s, t)$$

when this limit exists.

In the definition of  $\mathfrak{F}(\mathbf{x})$ , we have assumed that  $\mathbf{x}$  is a path with values in  $\mathbb{G}(\mathbb{R}^2)$ . Indeed, this definition may be extended to paths with values in  $\mathbb{T}_1(\mathbb{R}^2)$ . In addition, let us note that if  $\mathbf{x}$  takes its values in  $\mathbb{G}(\mathbb{R}^2)$ , then  $\mathfrak{F}(\mathbf{x}; s, t) \in \mathbb{G}(\mathbb{R}^2)$ . The analysis of  $\mathfrak{F}(\mathbf{x}; s, t)$  for  $\mathbf{x}$  in  $\mathbb{S}(\mathbb{R}^2)$  is performed in Section 7.5.

We will see below that the integral defined by (55)-(56) satisfies the relation

$$(57) \quad \mathfrak{J}(\mathbf{x}; s, t) = \mathfrak{J}(\mathbf{x}; s, r) \otimes \mathfrak{J}(\mathbf{x}; r, t), \quad \forall 0 \leq s \leq r \leq t \leq T,$$

which means that  $t \in [0, T] \mapsto \mathfrak{J}(\mathbf{x}; 0, t)$  is a path with values in  $\mathbb{T}_1(\mathbb{R}^2)$  and  $\mathfrak{J}(\mathbf{x}; s, t)$  represents its increments.

But  $\mathfrak{J}^n(\mathbf{x})$  does not satisfy (57), unless  $s, r, t$  belong to  $\{t_k^n\}_{k=0, \dots, 2^n-1}$ .

The next results are borrowed from [LQ02, Section 3.2, p. 40] or from [Lyo98, Section 3.1, p. 273].

**Definition 6.** A function  $\mathbf{y}_{s,t}$  from  $\Delta_+ = \{(s, t) \in [0, T]^2 \mid 0 \leq s \leq t \leq T\}$  to  $\mathbb{T}_1(\mathbb{R}^2)$  is an *almost rough path* if there exists some constants  $C > 0$  and  $\theta > 1$  such that

$$\|\mathbf{y}_{s,t} - \mathbf{y}_{s,r} \otimes \mathbf{y}_{r,t}\|_* \leq C|t - s|^\theta, \quad \forall 0 \leq s \leq r \leq t \leq T.$$

where  $\|\cdot\|_*$  is the norm defined by  $\|x\|_* = \max\{|x^1|, |x^2|\}$ .

An almost rough path is the “basic brick” for constructing a rough path. We give a proof of the next theorem in Section B in the appendix.

**Theorem 2.** *Let  $\mathbf{y} : \Delta_+ \rightarrow \mathbb{T}_1(\mathbb{R}^2)$  be an almost rough path such that  $\|\mathbf{y}_{s,t}\| \leq C|t-s|^\alpha$  for  $\alpha \in (1/3, 1]$  and  $C > 0$ . Set*

$$\mathbf{y}_{s,t}^n \stackrel{\text{def}}{=} \mathbf{y}_{s, \overline{M}(s,n)} \otimes \left( \bigotimes_{k=\overline{M}(s,n)}^{M(t,n)-1} \mathbf{y}_{t_k^n, t_{k+1}^n} \right) \otimes \mathbf{y}_{\overline{M}(t,n), t}, \quad \forall (s, t) \in \Delta_+.$$

*Then there exists a unique path  $\mathbf{z}$  in  $C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$  and a sequence  $(K_n)_{n \in \mathbb{N}}$  decreasing to 0 such that*

$$\|\mathbf{z}_{s,t} - \mathbf{y}_{s,t}^n\|_* \leq K_n |t-s|^\theta.$$

*If  $\mathbf{y}$  is an almost rough path in  $\mathbb{G}(\mathbb{R}^2)$ , then  $\mathbf{z}$  is a weak geometric rough path with  $\alpha$ -Hölder control.*

*In addition, if  $\mathbf{y}$  and  $\mathbf{y}'$  are both almost rough paths with*

$$|\pi_{\mathbb{R}^2}(\mathbf{y}_{s,t} - \mathbf{y}'_{s,t})| \leq \varepsilon |t-s|^\alpha, \quad |\pi_{\mathbb{R}^2 \otimes \mathbb{R}^d}(\mathbf{y}_{s,t} - \mathbf{y}'_{s,t})| \leq \varepsilon |t-s|^{2\alpha}$$

*for all  $(s, t) \in \Delta_+$ , then the corresponding rough paths  $\mathbf{z}$  and  $\mathbf{z}'$  satisfy*

$$|\pi_{\mathbb{R}^d}(\mathbf{z}_{s,t} - \mathbf{z}'_{s,t})| \leq \varepsilon K |t-s|^\alpha, \quad |\pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{z}_{s,t} - \mathbf{z}'_{s,t})| \leq \varepsilon K |t-s|^{2\alpha}$$

*for some constant  $K$  that depends only on  $T$ ,  $\alpha$ , and  $\theta$ .*

The existence of  $\mathfrak{J}(\mathbf{x})$  in (56) as a (weak geometric) rough path when  $\mathbf{x}$  is a (weak geometric) rough path is then justified by the next proposition and the application of Theorem 2.

Roughly speaking, the proof follows the same line as the one of the Young integral: the reader is referred to [Lyo98, Section 3.2.2, p. 289], [LQ02, Section 5.2, p. 117], [Lej03a, Section 3] or [LV06].

**Proposition 9.** *For  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1]$ , the function  $(s, t) \in \Delta_+ \mapsto \mathfrak{F}(\mathbf{x}; s, t)$  is an almost rough path. In addition, if  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$ , then  $\mathfrak{F}(\mathbf{x}; s, t)$  belongs to  $\mathbb{G}(\mathbb{R}^2)$ . Hence,  $\mathfrak{J}(\mathbf{x})$  given by (56) exists and belongs to  $C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$  (resp.  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$ ) if  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$  (resp.  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$ ).*

We have already seen that the integral  $\mathfrak{J}(\mathbf{x})$  lies above the integral we constructed in Section 6 using some approximation of  $\mathbf{x}$ . With Theorem 2, we have not only the continuity of  $\mathbf{x} \mapsto \mathfrak{J}(\mathbf{x})$ , but we also get that it is a locally Lipschitz map and we are not bound in using the  $\|\cdot\|_\beta$  norm with  $\beta < \alpha$  while working with  $\alpha$ -Hölder paths. In addition, we may consider any family of partitions whose mesh decreases to zero (see Remark 4 or the proof of Theorem 5 in Appendix B).

We introduce a new norm  $\|\cdot\|_{*,\alpha}$  on  $C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^d))$ , which is not equivalent to  $\|\cdot\|_\alpha$  but which generates the same topology: for  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^d))$ ,

$$\|\mathbf{x}\|_{*,\alpha} = \sup_{0 \leq s < t \leq T} \max \left\{ \frac{|\pi_{\mathbb{R}^d}(\mathbf{x}_{s,t})|}{(t-s)^\alpha}, \frac{|\pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x}_{s,t})|}{(t-s)^{2\alpha}} \right\}.$$

In the next theorem, we summarize Proposition 9 as well as some continuity results.

**Theorem 3.** *If  $f \in \text{Lip}(\gamma; \mathbb{R}^2 \rightarrow \mathbb{R}^2)$  with  $\gamma > 1/\alpha - 1$  and  $\alpha > 1/3$ , then the limit of  $(\mathfrak{J}^n(\mathbf{x}; s, t))_{n \in \mathbb{N}}$  in (56) exists and is unique. Besides,  $\mathfrak{J}$  maps  $(C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2)), \|\cdot\|_{*,\alpha})$  to  $(C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2)), \|\cdot\|_{*,\alpha})$  and is locally Lipschitz continuous.*

*In addition, if  $\mathbf{x}$  is a smooth rough path, then*

$$\mathfrak{J}(\mathbf{x}; 0, t) = \exp \left( \int_0^t f(x_s) dx_s + \mathfrak{A}(\mathfrak{J}(\mathbf{x}; 0, t); 0, t)[\check{e}_1, \check{e}_2] \right)$$

*and  $\mathfrak{J}(\mathbf{x})$  is also a smooth rough path.*

Hence, for  $\mathbf{x} \in C^{0,\alpha}([0, T]; \mathbb{G}(\mathbb{R}^2))$ , then there exists a sequence of paths  $\mathbf{x}^n \in C_p^\infty([0, T]; \mathbb{G}(\mathbb{R}^2))$  converges to  $\mathbf{x}$  in  $\|\cdot\|_\alpha$ , then  $\mathfrak{J}(\mathbf{x}) = \lim_{n \rightarrow \infty} \mathfrak{J}(\mathbf{x}^n)$  and  $\mathfrak{J}(\mathbf{x}^n)$  is a smooth rough path and  $\mathfrak{J}(\mathbf{x})$  belongs to  $C^{0,\alpha}([0, T]; \mathbb{G}(\mathbb{R}^2))$ .

Now, if  $\mathbf{x}$  is only a weak geometric  $1/\alpha$ -rough path with Hölder control, then we have seen that  $\mathbf{x}$  may be approximated by some smooth rough paths  $\mathbf{x}^n$  in the  $\beta$ -Hölder norm  $\|\cdot\|_\beta$  with  $\beta < \alpha$ . Hence,  $\mathfrak{J}(\mathbf{x}^n)$  converges to  $\mathfrak{J}(\mathbf{x})$  in  $\|\cdot\|_\beta$  with  $\beta < \alpha$ . Anyway,  $\mathfrak{J}(\mathbf{x})$  belongs to  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$ .

We then deduce the following stability result.

**Corollary 5.** *If  $\mathfrak{J}$  is defined by Theorem 3, then  $\mathfrak{J}$  maps  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$  into  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$  and  $C^{0,\alpha}([0, T]; \mathbb{G}(\mathbb{R}^2))$  into  $C^{0,\alpha}([0, T]; \mathbb{G}(\mathbb{R}^2))$ .*

We end this section with a lemma similar to Lemma 11.

**Lemma 20.** *For any  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$ ,  $\mathfrak{J}(\mathbf{x}; s, t) = \mathfrak{J}(\mathbf{x}|_{[s,t]})$  for all  $0 \leq s < t \leq T$ .*

*Proof.* If  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2))$ , then the proof of this Lemma is similar to the one of Lemma 11. If  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$ , then the results at the end in Section 7.5 allows us to conclude in the same way.  $\square$

**7.5. Integral along a path living in the tensor space.** Now, we consider  $\mathbf{x} \in C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2)$ . What can be said about  $\mathfrak{J}(\mathbf{x})$ ? From Proposition 7, one may decompose  $\mathbf{x}_t$  as the sum  $\mathbf{x}_t = \mathbf{y}_t + \mathbf{z}_t$  with  $\mathbf{y} = \Upsilon_a(\mathbf{x})$  and  $\mathbf{z} = \Upsilon_s(\mathbf{x})$ . In addition,  $(\mathbf{y}, \mathbf{z})$  belongs to  $C^\alpha([0, T]; \mathbb{G}(\mathbb{R}^2) \times \mathbb{S}(\mathbb{R}^2))$ , where the homogeneous norm on  $\mathbb{G}(\mathbb{R}^2) \times \mathbb{S}(\mathbb{R}^2)$  has been defined by (46). In particular, this implies that  $\pi_{e_i \otimes e_j}(\mathbf{z})$  belongs to  $C^{2\alpha}([0, T]; \mathbb{R})$ , *i.e.*, each of its component is  $2\alpha$ -Hölder continuous. In addition, for  $(s, t) \in \Delta_+$ ,

$\mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{y}_t - \mathbf{y}_s)$  belongs to  $G(\mathbb{R}^2)$ , while

$$\begin{aligned} \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{z}_t - \mathbf{z}_s) &= \sum_{k,i,j=1,2} \frac{\partial f_i^k}{\partial x_j}(x_s) \pi_{e_i \otimes e_j}(\mathbf{z}_t - \mathbf{z}_s) \check{e}_k \\ &+ \sum_{k,\ell,i,j=1,2} f_i^k(x_s) f_j^\ell(x_s) \pi_{e_i \otimes e_j}(\mathbf{z}_t - \mathbf{z}_s) \check{e}_k \otimes \check{e}_\ell. \end{aligned}$$

Since  $\pi_{e_i \otimes e_j}(\mathbf{z}_t - \mathbf{z}_s) = \pi_{e_j \otimes e_i}(\mathbf{z}_t - \mathbf{z}_s)$ , we get that  $\mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{z}_t - \mathbf{z}_s)$  belongs to  $\mathbb{R}^2 \oplus S(\mathbb{R}^2)$ . Besides, for  $t \in [0, T]$ ,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{k=0}^{\underline{M}(t,n)} \pi_{\mathbb{R}^2}(\mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2), \mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_{t_k^n})(\mathbf{z}_{t_{k+1}^n} - \mathbf{z}_{t_k^n})) \\ &= \sum_{k=1,2} \check{e}^k \int_0^t \frac{\partial f_1^k}{\partial x_1}(x_s) d\pi_{e_1 \otimes e_1}(z_s) + \check{e}^k \int_0^t \frac{\partial f_2^k}{\partial x_2}(x_s) d\pi_{e_2 \otimes e_2}(z_s) \\ &\quad + \check{e}^k \int_0^t \left( \frac{\partial f_1^k}{\partial x_2}(x_s) + \frac{\partial f_2^k}{\partial x_1}(x_s) \right) d\pi_{e_1 \otimes e_2}(z_s) \end{aligned}$$

which we can write in the more compact form  $\int_0^t \nabla f(x_s) dz_s$ .

In addition, if  $\{\alpha_k\}_{k=0,\dots,m}$  and  $\{\beta_k\}_{k=0,\dots,m}$  belongs to  $\mathbb{T}_0(\mathbb{R}^2)$ , then

$$\begin{aligned} \bigotimes_{k=0}^m (1 + \alpha_k + \beta_k) &= \bigotimes_{k=1}^m (1 + \alpha_k) + \sum_{k=0,\dots,m} \alpha_k^1 \otimes \left( \sum_{\ell=k+1,\dots,m} \beta_\ell^1 \right) \\ &+ \sum_{k=0,\dots,m} \beta_k^1 \otimes \left( \sum_{\ell=k+1,\dots,m} \alpha_\ell^1 \right) + \sum_{k=0,\dots,m} \beta_k^1 \otimes \left( \sum_{\ell=k+1,\dots,m} \beta_\ell^1 \right) \end{aligned}$$

with  $\alpha_k^1 = \pi_{\mathbb{R}^2}(\alpha_k)$  and  $\beta_k^1 = \pi_{\mathbb{R}^2}(\beta_k)$ . We remark that

$$\sum_{k=0,\dots,m} \beta_k^1 \otimes \left( \sum_{\ell=k+1,\dots,m} \alpha_\ell^1 \right) = \sum_{\ell=1}^m \sum_{k=0}^{\ell} \alpha_k \otimes \beta_\ell,$$

and

$$\sum_{k=0,\dots,m} \beta_k^1 \otimes \left( \sum_{\ell=k+1,\dots,m} \beta_\ell^1 \right) = \frac{1}{2} \left( \sum_{k=0}^m \beta_k \right) \otimes \left( \sum_{k=0}^m \beta_k \right).$$

We set

$$1 + \alpha_k = \mathfrak{F}(f, \mathbf{y}; t_k^n, t_{k+1}^n) \text{ and } \beta_k = \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^d)}(f)(\mathbf{x}_{t_k^n})(\mathbf{z}_{t_{k+1}^n} - \mathbf{z}_{t_k^n}).$$

In addition,  $\sum_{k=0,\dots,\underline{M}(t,n)} \alpha_k^1$  converges to  $\pi_{\mathbb{R}^2}(\mathfrak{J}(\mathbf{x}; 0, t))$ , while  $\sum_{k=0,\dots,\underline{M}(t,n)} \beta_k^1$  converges to  $\int_0^t \nabla f(x_s) dz_s$ . We also remark that if  $\beta_k^2 = \pi_{\mathbb{R}^2 \otimes \mathbb{R}^d}(\beta_k)$ , then

$$\sum_{k=0}^{\underline{M}(t,n)} \beta_k^2 \xrightarrow{n \rightarrow \infty} \int_0^t f(x_s) \otimes f(x_s) dz_s.$$



By combining all the facts and using techniques similar to the one in [LQ02, Section 3.3.3, p .56] or in [LV06], since the components of  $\int_0^t \nabla f(x_s) d\mathbf{z}_s$  are  $2\alpha$ -Hölder continuous, we can get that

$$\bigotimes_{k=0}^{M(t,n)-1} \mathfrak{F}(f, \mathbf{x}; t_k^n, t_{k+1}^n) \xrightarrow{n \rightarrow \infty} \int_0^t f(x_s) d\mathbf{y}_s + \mathfrak{K}(\mathbf{y}, \mathbf{z}; 0, t)$$

with

$$\begin{aligned} (58) \quad \mathfrak{K}(\mathbf{y}, \mathbf{z}; 0, t) &= \int_0^t \nabla f(y_s) d\mathbf{z}_s + \int_0^t f(y_s) \otimes f(y_s) d\mathbf{z}_s \\ &+ \sum_{k,\ell=1,2} \check{e}_k \otimes \check{e}_\ell \int_0^t \nabla f^k(y_s) \left( \int_0^s f^\ell(y_s) d\mathbf{y}_s \right) d\mathbf{z}_s \\ &+ \sum_{k,\ell=1,2} \check{e}_k \otimes \check{e}_\ell \int_0^t f^k(y_s) \left( \int_0^s \nabla f^\ell(y_s) d\mathbf{z}_s \right) d\mathbf{y}_s \\ &+ \frac{1}{2} \left( \int_0^t \nabla f(y_s) d\mathbf{z}_s \right) \otimes \left( \int_0^t \nabla f(y_s) d\mathbf{z}_s \right). \end{aligned}$$

In the previous expression, we have to remember that  $\mathbf{x}$  and  $\mathbf{y}$  lives above the same path  $x = y$ .

Thus, if for each  $n \in \mathbb{N}$ ,  $\mathbf{z}^n$  belongs to  $C_p^\infty([0, T]; S(\mathbb{R}^2))$  and converges to  $\mathbf{z}$ , while  $\mathbf{y}^n \in C_p^\infty([0, T]; G(\mathbb{R}^2))$  converges to  $\mathbf{y}$ , one get that  $\mathbf{x}^n = \mathbf{y}^n + \mathbf{z}^n$  converges to  $C^\alpha([0, T]; T_1(\mathbb{R}^2))$  and

$$\mathfrak{J}(\mathbf{x}) = \lim_{n \rightarrow \infty} (\mathfrak{J}(\mathbf{y}^n) + \mathfrak{K}(\mathbf{y}^n, \mathbf{z}^n)),$$

where the limit is in  $C^\beta([0, T]; T_1(\mathbb{R}^2))$  for all  $\beta < \alpha$ . Of course, both  $\mathfrak{K}(\mathbf{y}^n, \mathbf{z}^n)$  and  $\mathfrak{J}(\mathbf{y}^n)$  correspond to integrals of differential forms along piecewise smooth paths, and hence to ordinary integrals.

Yet it has to be note the following fact: If  $\mathbf{x} \in C^\alpha([0, T]; T_1(\mathbb{R}^2))$  but  $\mathbf{x} \notin C^\alpha([0, T]; G(\mathbb{R}^2))$ , then it is not possible to find a family  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  of smooth rough paths such  $\mathfrak{J}(\mathbf{x}^n)$  converges to  $\mathfrak{J}(\mathbf{x})$ . This means that  $\mathfrak{J}(\mathbf{x})$  cannot be approximated by the ordinary integrals  $\mathfrak{J}(\mathbf{x}^n)$ . This motivates our definition of geometric rough paths. However, using the decomposition of  $T_1(\mathbb{R}^2)$  as  $G(\mathbb{R}^2) \times S(\mathbb{R}^2)$ , it is then possible to interpret any  $\alpha^{-1}$ -rough path as a geometric  $(\alpha^{-1}, (2\alpha)^{-1})$ -rough path in the sense defined in [LV06].

**7.6. On geometric rough paths lying above the same path.** We have seen in Lemma 6 that if  $\mathbf{x}$  and  $\mathbf{y}$  are two paths in  $C^\alpha([0, T]; A(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2)$  and lying above the same path taking its values in  $\mathbb{R}^2$  (i.e.,  $\pi_{\mathbb{R}^2}(\mathbf{x}) = \pi_{\mathbb{R}^2}(\mathbf{y})$ ), then there exists a path  $\varphi \in C^{2\alpha}([0, T]; \mathbb{R})$  such that  $\mathbf{x} = \mathbf{y} + \varphi[e_1, e_2]$ . In addition,  $(-\mathbf{x}_s) \boxplus \mathbf{x}_t = (-\mathbf{y}_s) \boxplus \mathbf{y}_t + (\varphi_t - \varphi_s)[e_1, e_2]$ . Now, if we lift  $\mathbf{x}$  and  $\mathbf{y}$  as paths in  $C^\alpha([0, T]; G(\mathbb{R}^2))$  by  $\widehat{\mathbf{x}}_t = \exp(\mathbf{x}_t)$  and  $\widehat{\mathbf{y}}_t = \exp(\mathbf{y}_t)$ , we deduce that there exists  $\psi \in$

$C^\alpha([0, T]; T_0(\mathbb{R}^2))$  such that  $\widehat{\mathbf{x}}_t = \widehat{\mathbf{y}}_t + \psi_t$  and in addition,  $\widehat{\mathbf{x}}_s^{-1} \otimes \widehat{\mathbf{x}}_t = \widehat{\mathbf{y}}_s^{-1} \otimes \widehat{\mathbf{y}}_t + \psi_t - \psi_s$ . This path is given by

$$\psi_t = \varphi_t e_1 \otimes e_2 - \varphi_t e_2 \otimes e_1 = \varphi_t [e_1, e_2].$$

Each component of  $\psi$  is  $2\alpha$ -Hölder continuous. Using the map  $\mathfrak{K}$  previously defined by (58), we get that

$$\mathfrak{J}(\mathbf{x}) = \mathfrak{J}(\mathbf{y}) + \mathfrak{K}(\mathbf{y}, \psi).$$

Finally, using the fact that  $\psi$  is anti-symmetric, setting  $[f, f] = \check{e}_1[f^1, f^1] + \check{e}_2[f^2, f^2]$ , we get that

$$\begin{aligned} \mathfrak{K}(\mathbf{y}, \psi; s, t) &= \int_s^t [f, f](y_s) d\varphi_s \\ &+ \sum_{k, \ell=1,2} \check{e}_k \otimes \check{e}_\ell \int_0^t [f^k, f^k](y_s) \int_0^s f^\ell(y_s) d\mathbf{y}_s d\varphi_s \\ &+ \sum_{k, \ell=1,2} \check{e}_k \otimes \check{e}_\ell \int_0^t f^\ell(y_s) \int_0^s [f^k, f^k](y_s) d\varphi_s d\mathbf{y}_s \\ &+ \frac{1}{2} \left( \int_0^t [f, f](y_s) d\varphi_s \right) \otimes \left( \int_0^t [f, f](y_s) d\varphi_s \right) \end{aligned}$$

for all  $0 \leq s \leq t \leq T$ .

If  $[f, f] = 0$ , we deduce that  $\mathfrak{K}(\mathbf{y}, \psi) = 0$  and then that  $\mathfrak{J}(\mathbf{y}) = \mathfrak{J}(\mathbf{x})$ . In other words, any rough path lying above the same path  $x$  gives rise to the same integral.

With the results in [LV05], which asserts that it is always possible to lift a path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  to a path  $\mathbf{x} \in C^\alpha([0, T]; G(\mathbb{R}^2))$  when  $\alpha \in (1/3, 1/2]$ , this means that if  $[f, f] = 0$ , one may define  $\mathfrak{J}$  only on  $C^\alpha([0, T]; \mathbb{R}^2)$  for  $\alpha \in (1/3, 1/2]$ , but the continuity of  $\mathfrak{J}$  remains an open question.

## 8. VARIATIONS IN THE CONSTRUCTION OF THE INTEGRAL

**8.1. Case of a path living in a  $d$ -dimensional space.** The case of a space of dimension  $d$  is not harder to treat than the case of  $d = 2$ . For this, one has only to consider the area between the components grouped by pairs.

The tensor space  $T(\mathbb{R}^d)$  becomes then the space  $T(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$  whose basis is, if  $\{e_1, \dots, e_d\}$  is a basis of  $\mathbb{R}^d$ ,

$$1, e_1, \dots, e_d, e_1 \otimes e_1, e_1 \otimes e_2, \dots, e_d \otimes e_d,$$

Hence,  $T(\mathbb{R}^d)$  is a space of dimension  $1 + d + d^2$ .

The space  $A(\mathbb{R}^2)$  is a space of dimension  $d + d(d-1)/2$ , whose basis is given by

$$\{e_i \mid i = 1, \dots, d\} \cup \{[e_i, e_j] \mid i \neq j, i, j = 1, \dots, d\}$$

with  $[e_i, e_j] = e_i \otimes e_j - e_j \otimes e_i$ . The space  $A(\mathbb{R}^d)$  is then  $\mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d]$ , where  $[\mathbb{R}^d, \mathbb{R}^d] = \{ [x, y] \mid x, y \in \mathbb{R}^d \}$ .

The applications  $\exp$  and  $\log$  are defined as previously:

$$\exp(x) = 1 + x + \frac{1}{2}x \otimes x \text{ for } x \in A(\mathbb{R}^d)$$

$$\text{and } \log(1 + x) = x - \frac{1}{2}x \otimes x \text{ for } x \in T(\mathbb{R}^d), \pi_1(x) = 0.$$

The space  $G(\mathbb{R}^d) = \exp(A(\mathbb{R}^d))$  is a subgroup of  $(T_1(\mathbb{R}^d), \otimes)$ , where  $T_1(\mathbb{R}^d) = \{ x \in T(\mathbb{R}^d) \mid \pi_1(x) = 1 \}$ , and  $(A(\mathbb{R}^d), [\cdot, \cdot])$  is the Lie algebra of  $(G(\mathbb{R}^d), \otimes)$ . It may also be identified with its tangent plane at any point.

A smooth path  $x$  in  $\mathbb{R}^d$  is then lifted into a path  $\widehat{\mathbf{x}}$  in  $A(\mathbb{R}^d)$  by

$$\widehat{\mathbf{x}}_t = x_t + \sum_{i,j=1,\dots,d, i<j} \mathfrak{A}((x^i, x^j); 0, t)[e_i, e_j],$$

where  $(x^i, x^j)$  is the two dimensional path composed of the  $i$ -th and  $j$ -th component of  $x$ . Let us remark that  $\mathfrak{A}((x^i, x^j)) = -\mathfrak{A}((x^j, x^i))$  and  $\mathfrak{A}((x^i, x^i)) = 0$ .

The path  $\widehat{\mathbf{x}}$  is then lifted into a path  $\mathbf{x}$  in  $\mathbb{R}^d$  by  $\mathbf{x} = \exp(\widehat{\mathbf{x}})$ , and thus

$$\mathbf{x}_t = 1 + x_t + \sum_{i,j=1}^d \int_0^t (x_s^j - x_0^j) dx_s^i.$$

The symmetric part  $\mathfrak{s}(\mathbf{x})$  of  $\pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x})$  is

$$\mathfrak{s}(\mathbf{x}_t) = \frac{1}{2}(x_t - x_0) \otimes (x_t - x_0)$$

while the anti-symmetric part  $\mathfrak{a}(\mathbf{x})$  of  $\pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x})$  is

$$\mathfrak{a}(\mathbf{x}_t) = \sum_{i,j=1}^d \mathfrak{A}((x^i, x^j); 0, t)e_i \otimes e_j = \sum_{\substack{i=1,\dots,d \\ i<j}} \mathfrak{A}((x^i, x^j); 0, t)[e_i, e_j].$$

Hence, all the previous notions and results are easily extended to this case.

Finally, note that the theory of rough paths may also be applied to the infinite dimensional case (see [LLQ02] for example).

**8.2. Using iterated integrals.** We saw in Sections 7.1 and 8.1 that a path  $x \in C^\alpha([0, T]; \mathbb{R}^d)$  with  $\alpha > 1/2$  may be naturally lifted as a path  $\mathbf{x}$  in  $G(\mathbb{R}^d)$  with

$$\mathbf{x}_t = 1 + x_t + \sum_{i,j=1}^d \left( \int_0^t (x_r^i - x_0^i) dx_r^j \right) e_i \otimes e_j.$$

The term  $K^{i,j}(x; 0, t) = \pi_{e_i \otimes e_j}(\mathbf{x}_t)$  is called an *iterated integral* of  $x$ .

Fix  $d \geq 1$  and consider the tensor space  $T^\infty(\mathbb{R}^d)$  defined by

$$T^\infty(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus (\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d) \oplus \dots,$$

and, for a smooth path  $x : [0, T] \rightarrow \mathbb{R}^d$ , the iterated integrals

$$K^{i_1, \dots, i_\ell}(x; 0, t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{\ell-1}} dx_{i_\ell}^{i_1} \dots dx_{t_1}^{i_\ell}$$

for each integer  $\ell$  and each  $(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell$ . It was noted first by K.T. Chen in the 50's [Che58, Che57] that the formal power series

$$\Psi(x; 0, t) = \sum_{\ell \geq 0} \sum_{(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell} K^{i_1, \dots, i_\ell}(x; 0, t) e_{i_1} \otimes \dots \otimes e_{i_\ell}$$

in  $T^\infty(\mathbb{R}^d)$  provides an algebraic way to encode the geometric object which is the path  $x$ . For that,  $\Psi(x; 0, t)$  is sometimes called the *signature* of the path.

With the tensor product  $\otimes$ ,  $T^\infty(\mathbb{R}^d)$  remains a group, and thus if  $x : [0, T] \rightarrow \mathbb{R}^d$  and  $y : [0, S] \rightarrow \mathbb{R}^d$  are two smooth paths,  $\Psi(x \cdot y; 0, T + S) = \Psi(x; 0, T) \otimes \Psi(y; 0, S)$ . In addition, if  $\bar{x}$  is the path  $\bar{x}_t = x_{T-t}$ , then  $\Psi(\bar{x}; T) = \Psi(x; T)^{-1}$ . The signature characterizes  $x$  in the sense that there is a one-to-one equivalence<sup>3</sup> between the algebraic object  $\Psi(x)$  and the geometric object  $x$  in  $C_p^\infty([0, T]; \mathbb{R}^d)$  (see also [HL06] for some extension).

Let  $[x, y]$  be the Lie bracket  $[x, y] = x \otimes y - y \otimes x$ . Let us denote by  $A^\infty(\mathbb{R}^d)$  the subset of  $T^\infty(\mathbb{R}^d)$  defined by

$$A^\infty(\mathbb{R}^d) = \mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d] \oplus [\mathbb{R}^d, [\mathbb{R}^d, \mathbb{R}^d]] \oplus \dots$$

This subset is stable under the application of the Lie bracket  $[\cdot, \cdot]$ . The tensor space  $T^\infty(\mathbb{R}^d)$  is the universal Lie algebra of  $A^\infty(\mathbb{R}^d)$  (see [Reu93] for example). One may then define two maps  $\exp : A^\infty(\mathbb{R}^d) \rightarrow T_1^\infty(\mathbb{R}^d)$  and  $\log : T_1^\infty(\mathbb{R}^d) \rightarrow A^\infty(\mathbb{R}^d)$ , where  $T_1^\infty(\mathbb{R}^d)$  is the subset of  $T^\infty(\mathbb{R}^d)$  such that  $\pi_{\mathbb{R}}(x) = 1$  which are given by

$$\begin{aligned} \exp(x) &= 1 + x + \frac{1}{2}x \otimes x + \frac{1}{6}x \otimes x \otimes x + \dots, \\ \log(1 + x) &= x - \frac{1}{2}x \otimes x + \frac{1}{3}x \otimes x \otimes x - \dots. \end{aligned}$$

In particular, if  $G^\infty(\mathbb{R}^d) = \exp(A^\infty(\mathbb{R}^d))$ , then  $(G^\infty(\mathbb{R}^d), \otimes)$  is a closed subgroup of  $(T_1^\infty(\mathbb{R}^d), \otimes)$ . In addition,  $\exp$  is one-to-one from  $A^\infty(\mathbb{R}^d)$  to  $G^\infty(\mathbb{R}^d)$ , and  $\log$  is its inverse.

One of the striking result from K.T. Chen, which uses some properties of the iterated integrals, is that  $\Psi(x; 0, t)$  belongs to  $G^\infty(\mathbb{R}^d)$ , or equivalently, with the Baker-Campbell-Hausdorff-Dynkin formula, that  $\log(\Psi(x; 0, t))$  belongs to  $A^\infty(\mathbb{R}^d)$ .

<sup>3</sup>In fact, this equivalence is not exactly one-to-one, unless one eliminates paths such that, on some time interval,  $x$  goes from a point  $a$  to a point  $b$  and then back to  $a$  by reversing the path.

This approach proved to be very useful, since it allows us to consider equation driven by smooth paths or differential equations in an algebraic setting, and allows us formal computations. Numerous topics in control theory uses this point of view (see for example [Fli81, Isi95, Kaw98]). It way also used in the stochastic context to deal with flow of Stochastic Differential Equations (see for example [Yam79, FNC82, BA89, Cas93]... or the book [Bau04]).

For some integer  $k$ , we may truncate  $\Psi$  by considering that  $e_{i_1} \otimes \cdots \otimes e_{i_\ell} = 0$  for all  $\ell > k$ . For such a truncated power series  $\Psi_k(x)$  we still get the relationship  $\Psi_k(x \cdot y; 0, T + S) = \Psi_k(x; 0, T) \otimes \Psi_k(y; 0, S)$ . In particular, we deduce that

$$\Psi_k(x_{|[s,t]}; s, t) = \Psi_k(x_{|[s,r]}; s, r) \otimes \Psi_k(x_{|[r,t]}; r, t)$$

for all  $0 \leq s \leq r \leq t \leq T$ . With  $k = 2$ , we get exactly that our natural lift of  $\mathbf{x}_t = \Psi_2(x; 0, t)$  satisfies the relationship  $\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}$ .

Thus, given a path  $\mathbf{x}$  in  $T_1(\mathbb{R}^2)$ , one can think of  $\pi_{e_i \otimes e_j}(\mathbf{x}_t)$  as the iterated integrals of  $x^j$  against  $x^i$ . Of course, one knows that for irregular paths, there is no canonical way to define them (think of Brownian motion trajectories). Anyway, for weak geometric rough paths, this iterated integrals are approximated by iterated integrals of some smooth paths.

We may now present another heuristic argument to derive the expression of  $\mathfrak{F}(f, \mathbf{x}; s, t)$  and then (55). This argument is the historical one (see [Lyo98, Lej03a, LQ02, LCL06]). Consider a smooth path  $x : [0, T] \rightarrow \mathbb{R}^d$  and a smooth function  $f = (f_1, \dots, f_d)$ . Then using a Taylor development, one gets that

$$\begin{aligned} \sum_{i=1}^d \int_s^t f_i(x_r) dx_r^i &= \sum_{i=1}^d f_i(x_s)(x_t^i - x_s^i) \\ &+ \sum_{\ell \geq 1} \sum_{(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell} \frac{\partial^\ell f_i}{\partial x_{i_1} \cdots \partial x_{i_\ell}}(x_0) K^{i_\ell, \dots, i_1, i}(x; s, t) \\ &= \mathfrak{E}_{T^\infty(\mathbb{R}^d)}(f)(x_s) \Psi(x; s, t) \end{aligned}$$

with, for  $z \in \mathbb{R}^d$ ,

$$\mathfrak{E}_{T^\infty(\mathbb{R}^d)}(f)(z) = \sum_{\ell \geq 0} \sum_{(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell} \frac{\partial^\ell f_i}{\partial x_{i_1} \cdots \partial x_{i_\ell}}(z) e^{i_1} \otimes \cdots \otimes e^{i_\ell}.$$

In the usual case, we keep only the first term  $\sum_{i=1}^d f_i(x_s)(x_t^i - x_s^i)$  as an approximation of  $\sum_{i=1}^d \int_s^t f_i(x_r) dx_r^i$ , and we use it as the term in a Riemann sum. Keeping higher order terms has no influence, since  $K^{i_1, \dots, i_\ell}(x; s, t) \leq (1/\ell!) \|x'\|_\infty^\ell (t - s)^\ell$ .

The idea is then to keep enough terms, if  $x$  is  $\alpha$ -Hölder continuous and we get an object  $\mathbf{x}^{(k)}$  having the same algebraic properties as  $\Psi_k(x; s, t)$  for some integer  $k$ , to get a Riemann sum that converges. In

[Lyo98, LQ02], T. Lyons and his co-authors proved that the number of terms shall be  $k = \lfloor 1/\alpha \rfloor$ . In particular, from  $\mathbf{x}^{(k)}$ , it is possible to reconstruct an object living in  $T^\infty(\mathbb{R}^d)$  and equal to  $\Psi(x)$  when  $x$  is smooth and possessing the same algebraic properties as  $\Psi(x)$ .

For  $k = 2$  and using the path  $\mathbf{x}$  as the object  $\mathbf{x}^{(2)}$ , we get the expression (55).

**8.3. Paths with quadratic variation.** For the Brownian motion or a semi-martingale, one knows how to construct several integrals — the major ones are the Itô and the Stratonovich integrals — whose difference depends on the fact that their trajectories have finite quadratic variation. We will see how to construct the equivalent of these integrals, just by choosing different interpolations.

**Definition 7.** Given  $\alpha \in (1/3, 1/2]$ , a path  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  has a *quadratic variation* if there exists a process  $\mathfrak{Q}(x) \in C^\alpha([0, T]; \mathbb{S}(\mathbb{R}^2))$  such that  $\xi_0 = 0$  and, if  $z^{\otimes 2} = z \otimes z$  for  $z \in \mathbb{R}^2$ ,

$$\begin{aligned} \mathfrak{Q}_n(x; t) = & \frac{t - t_{\underline{M}(t,n)}^n}{t_{\underline{M}(t,n)+1}^n - t_{\underline{M}(t,n)}^n} (x_{t_{\underline{M}(t,n)+1}^n} - x_{t_{\underline{M}(t,n)}^n})^{\otimes 2} \\ & + \sum_{k=0}^{\underline{M}(t,n)-1} (x_{t_{k+1}^n} - x_{t_k^n})^{\otimes 2} \end{aligned}$$

and  $\mathfrak{Q}(x) = \lim_{n \rightarrow \infty} \mathfrak{Q}_n(x)$  where the limit holds in  $C^\alpha([0, T]; \mathbb{S}(\mathbb{R}^2))$ .

*Remark 12.* Note that with the norm we use, this means that the components of  $\mathfrak{Q}(x)$  are  $2\alpha$ -Hölder continuous.

*Remark 13.* If  $x \in C^\alpha([0, T]; \mathbb{R}^2)$  with  $\alpha > 2$ , then it is easily seen that necessarily,  $\mathfrak{Q}(x; t) = 0$  for  $t \in [0, T]$ .

Trajectories of the Brownian motion and of Hölder continuous martingales present this feature (see [Sip93, CL05]).

Given a path  $x$  with values in  $\mathbb{R}^2$  and provided we succeed in constructing a path  $\mathbf{x}$  with value in  $T_1(\mathbb{R}^2)$  lying above  $x$ , that is  $\pi_{\mathbb{R}^2}(\mathbf{x}) = x$ , then we have seen how  $\mathbf{x}$  can be modified to construct several distinct integrals, by the use of  $\mathfrak{K}$ .

Given a path  $\mathbf{x} \in C^\alpha([0, T]; G(\mathbb{R}^2))$  with  $\alpha \in (1/3, 1/2]$ , we may also think to construct different integrals. The original construction of T. Lyons is related to the choice of a family  $(\mathbf{x}^n)_{n \in \mathbb{N}}$  of approximations of  $\mathbf{x}$  such that  $\mathbf{x}_{t_k^n}^n = \mathbf{x}_t$  with  $t_k^n = Tk2^{-n}$ , for  $k = 0, \dots, 2^n$ . We have constructed our approximations in order that they stay in the submanifold  $G(\mathbb{R}^2)$ . We did so because we have extended the differential form  $f$  to a differential form  $\mathfrak{E}_{A(\mathbb{R}^2)}(f)$  on  $A(\mathbb{R}^2)$ , identified with the tangent plane at any point of  $G(\mathbb{R}^2)$ . Later, we also extended  $f$  to a differential form on  $T_1(\mathbb{R}^2)$ .

One can then think to use straight paths in  $T_1(\mathbb{R}^2)$  to join two successive points. Thus, let us define  $\mathbf{x}^n$  by

$$\mathbf{x}_t^n = \psi_{\mathbf{x}_{t_k^n}, \mathbf{x}_{t_{k+1}^n}} \left( \frac{t - t_k^n}{t_{k+1}^n - t_k^n} \right) \text{ for } t \in [t_k^n, t_{k+1}^n],$$

for  $n \in \mathbb{N}^*$  and  $t_k^n = Tk/2^n$ , and

$$\psi_{x,y}(t) = x \otimes (tx^{-1} \otimes y) \text{ for } x, y \in T_1(\mathbb{R}^2) \text{ and } t \in [0, 1].$$

The derivative of  $\mathbf{x}^n$  is

$$\frac{d\mathbf{x}_t^n}{dt} = x^{-1} \otimes y.$$

Hence,

$$\begin{aligned} \int_{t_k^n}^{t_{k+1}^n} \mathfrak{E}_{T_1(\mathbb{R}^2)}(f)(\mathbf{x}_s^n) \frac{d\mathbf{x}_s^n}{ds} ds &\approx \sum_{i,\ell=1,2} f_i^\ell(x_{t_k^n}) \pi_{e_i}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) \check{e}_\ell \\ &+ \sum_{i,k=1,2} \frac{\partial f_i^k}{\partial x_j}(x_{t_k^n}) \pi_{e_i \otimes e_j}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) \\ &+ \sum_{i,j,k=1,2} \frac{1}{2} \frac{\partial f_i^\ell}{\partial x_j}(x_{t_k^n}) \pi_{e_i \otimes e_j}(\mathfrak{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n)) \check{e}_\ell \\ &+ \sum_{i,j,\ell,m=1,2} f_i^\ell(x_{t_k^n}) f_j^m(x_{t_k^n}) \pi_{e_i \otimes e_j}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n) \check{e}_\ell \otimes \check{e}_\ell \\ &= \mathfrak{F}(f, \mathbf{x}; t_k^n, t_{k+1}^n) + \frac{1}{2} \nabla f(x_{t_k^n}) \mathfrak{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n). \end{aligned}$$

Here, the sign  $\approx$  means that the expressions differs by a term smaller than  $C2^{-n\theta}$  for some  $\theta > 1$ , and  $C$  a constant that does not depend on  $n$ . In addition, let us note that, if

$$\beta_k = \frac{1}{2} \nabla f(x_{t_k^n}) \mathfrak{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}^n),$$

then

$$\begin{aligned} \bigotimes_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} \mathfrak{E}_{T_1(\mathbb{R}^2)}(f)(\mathbf{x}_s^n) \frac{d\mathbf{x}_s^n}{ds} ds &= \bigotimes_{k=0}^{2^n-1} \mathfrak{F}(f, \mathbf{x}; t_k^n, t_{k+1}^n) \\ &+ \sum_{k=0, \dots, 2^n-1, \ell > k} \beta_k \otimes \beta_\ell + \sum_{k,\ell=0, \dots, 2^n-1, \ell > k} \mathfrak{F}(f, \mathbf{x}; t_k^n, t_{k+1}^n) \otimes \beta_\ell \\ &+ \sum_{k,\ell=0, \dots, 2^n-1, \ell > k} \beta_k \otimes \mathfrak{F}(f, \mathbf{x}; t_\ell^n, t_{\ell+1}^n) + \sum_{k=0}^{2^n-1} \beta_k. \end{aligned}$$

If  $x$  has a quadratic variation  $\mathfrak{Q}(x)$ , then the component of  $\mathfrak{Q}(x)$  are  $2\alpha$ -Hölder continuous. In addition, the components of  $\nabla f$  belongs in

$\text{Lip}(\gamma - 1; \mathbb{R}^2 \rightarrow \mathbb{R}^2)$ . Hence, since  $\mathfrak{Q}_n(x)$  converges to  $\mathfrak{Q}(x)$  and

$$\left| \nabla f(x_{t_k^n}) \mathfrak{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}) - \int_{t_k^n}^{t_{k+1}^n} \nabla f(x_s) d\mathfrak{Q}_n(x; s) \right| \leq \Delta_n t^{\alpha(2+\gamma)} \|f\|_{\text{Lip}} \|\mathbf{x}\|_{\alpha}^{\alpha(2+\gamma)},$$

we easily get the convergence of the last term to the Young integral  $\frac{1}{2} \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r)$ . In addition, using similar computations as in Section 7.5, we get that

$$\begin{aligned} & \bigotimes_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s^n) \frac{d\mathbf{x}_s^n}{ds} ds \\ & \xrightarrow{n \rightarrow \infty} \mathfrak{I}_b(\mathbf{x}; 0, T) \stackrel{\text{def}}{=} \mathfrak{I}(\mathbf{x}; 0, T) + \frac{1}{2} \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \\ & \quad + \frac{1}{2} \int_0^T f(x_t) \otimes \left( \int_0^t \nabla f(x_r) d\mathfrak{Q}(x; r) \right) d\mathbf{x}_t \\ & \quad + \frac{1}{2} \int_0^T \nabla f(x_t) \otimes \left( \int_0^t \nabla f(x_r) d\mathbf{x}_r \right) d\mathfrak{Q}(x; t) \\ & \quad + \frac{1}{8} \left( \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \right) \otimes \left( \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \right). \end{aligned}$$

Note that we get the same expression if one uses  $\varphi_{\mathbf{x}_{t_k^n}, \mathbf{x}_{t_{k+1}^n}}$  to link the point, but one considers the derivative of this function in  $\mathbb{T}_0(\mathbb{R}^2)$  (see (49)) and not its covariant derivative (see (50)).

*Remark 14.* We get a different integral than the one given by  $\mathfrak{I}(\mathbf{x} + \mathfrak{Q}(x))$  which is such that  $\pi_{\mathbb{R}^2}(\mathfrak{I}(\mathbf{x} + \mathfrak{Q}(x))) = \pi_{\mathbb{R}^2}(\mathfrak{I}_b(\mathbf{x}))$  but

$$\pi_{\mathbb{R}^2 \otimes \mathbb{R}^2}(\mathfrak{I}(\mathbf{x} + \mathfrak{Q}(x); 0, t)) = \pi_{\mathbb{R}^2 \otimes \mathbb{R}^2}(\mathfrak{I}_b(\mathbf{x}; 0, t)) + \int_0^t f(x_s) \otimes f(x_s) d\mathfrak{Q}(x; s).$$

For the Brownian motion or semi-martingales, it corresponds to the *backward Itô integral*, which is the integral defined as the limit in probability of

$$\sum_{k=0}^{2^n-1} f(B_{t_{k+1}^n})(B_{t_{k+1}^n} - B_{t_k^n}).$$

Because of the convergence result and the Wong-Zakai theorem, the integral  $\mathfrak{I}(\mathbf{x}; 0, T)$  may be interpreted as a Stratonovich integral.



To construct an integral similar to the one from Itô, and in view of (51), it is natural to use

$$\begin{aligned}
\mathfrak{J}_{\text{It}\hat{o}}(\mathbf{x}; 0, T) &= \lim_{n \rightarrow \infty} \bigotimes_{k=0}^{2^n-1} \mathfrak{E}_{\mathbb{A}(\mathbb{R}^2)}(f)(\mathbf{x}_{t_k^n}^n) \mathbf{x}_{t_k^n, t_{k+1}^n}^n \\
&= \lim_{n \rightarrow \infty} \bigotimes_{k=0}^{2^n-1} \left( \mathfrak{F}(f, \mathbf{x}; t_k^n, t_{k+1}^n) - \frac{1}{2} \nabla f(x_{t_k^n}) \mathfrak{s}(\mathbf{x}_{t_k^n, t_{k+1}^n}) \right) \\
&= \mathfrak{J}(\mathbf{x}; 0, T) - \frac{1}{2} \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \\
&\quad - \frac{1}{2} \int_0^T f(x_t) \otimes \left( \int_0^t \nabla f(x_r) d\mathfrak{Q}(x; r) \right) d\mathbf{x}_t \\
&\quad - \frac{1}{2} \int_0^T \nabla f(x_t) \otimes \left( \int_0^t \nabla f(x_r) d\mathbf{x}_r \right) d\mathfrak{Q}(x; t) \\
&\quad + \frac{1}{8} \left( \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \right) \otimes \left( \int_0^T \nabla f(x_r) d\mathfrak{Q}(x; r) \right).
\end{aligned}$$

Hence, we see that for paths with quadratic variation, several integrals naturally arise. But these “backward” and “Itô” integrals are not geometric rough paths.

**8.4. Link with stochastic integrals.** Itô and Stratonovich integrals are defined as limit in probability of Riemann sum. On the other hand, the rough path theory gives a pathwise definition of the integral, but the price to pay is to add a supplementary information. Is there some link between the two integrals?

Let  $B$  be a  $d$ -dimensional Brownian motion (a semi-martingale may be used as well). A natural way to construct a rough path  $\mathbf{B}$  lying above  $B$  is to set

$$\pi_{e_i \otimes e_j}(\mathbf{B}_t) = \int_0^t (B_r^i - B_0^i) \circ dB_r^j.$$

for  $i, j = 1, \dots, d$ . For the construction of  $\mathbf{B}$  as a rough path, see for example [Sip93, LQ02, Lej03a, CL05]. The process  $\log(\mathbf{B})$  is called the *Brownian motion on the Heisenberg group*, and add been widely studied (See references in Section A.3).

The continuity result of the rough path integral and the Wong-Zakai theorem allows us to identify the integral  $\mathfrak{J}(\mathbf{B}; 0, T)$  with the Stratonovich integral given by

$$\int f(B_s) \circ dB_s = \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \frac{1}{2} (f(B_{t_{k+1}^n}) + f(B_{t_k^n})) (B_{t_{k+1}^n} - B_{t_k^n})$$

where the limit is a limit in probability. We will see here that there is another relationship between the two integrals without invoking this continuity result, and that the construction of the Stratonovich and Itô

integrals (although under stronger condition on the function  $f$  than the one required by the “classical” theory) can be deduced from the rough paths theory.

The theory of rough paths also gives a better intuitive understanding of the counter-examples to the Wong-Zakai theorem (see [McS72, IW89] for SDEs and [LL06] in the context of rough paths).

The projection on  $\mathbb{R}^d$  of  $\mathfrak{I}(\mathbf{B}; 0, T)$  is given by

$$(59) \quad \begin{aligned} & \pi_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B}; 0, T)) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left( f(B_{t_k^n})(B_{t_{k+1}^n} - B_{t_k^n}) + \nabla f(B_{t_k^n}) \pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{B}_{t_k^n, t_{k+1}^n}) \right) \end{aligned}$$

which we rewrite using  $\mathfrak{a}$  and  $\mathfrak{s}$  as

$$\begin{aligned} \pi_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B}; 0, T)) &= \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left( f(B_{t_k^n})(B_{t_{k+1}^n} - B_{t_k^n}) + \nabla f(B_{t_k^n}) \mathfrak{a}(\mathbf{B}_{t_k^n, t_{k+1}^n}) \right. \\ & \quad \left. + \nabla f(B_{t_k^n}) \mathfrak{s}(\mathbf{B}_{t_k^n, t_{k+1}^n}) \right). \end{aligned}$$

But we have seen that

$$f(B_{t_k^n})(B_{t_{k+1}^n} - B_{t_k^n}) + \nabla f(B_{t_k^n}) \mathfrak{s}(\mathbf{B}_{t_k^n, t_{k+1}^n}) \approx \int_{t_k^n}^{t_{k+1}^n} f(B_s^{\Pi^n}) dB_s^{\Pi^n}$$

with  $B^{\Pi^n}$  the piecewise linear approximation of  $B$  along the dyadic partition  $\Pi^n$ , and  $\approx$  meaning that the difference between the two terms is less than  $C2^{-n\theta}$  with  $\theta > 1$ .

On the other hand, using the relation  $f(x) - f(y) = \int_0^1 \nabla f(x + \tau(y-x))(y-x) d\tau$  and the change of variable  $\tau' = 2^n \tau$ , we get that for  $k = 0, \dots, 2^n$ ,

$$\begin{aligned} & \sum_{i=1}^d (f_i(B_{t_{k+1}^n}) - f_i(B_{t_k^n}))(B_{t_{k+1}^n}^i - B_{t_k^n}^i) \\ &= \int_{t_k^n}^{t_{k+1}^n} \sum_{i,j=1}^d \frac{\partial f_i}{\partial x_j}(B_s^{\Pi^n})(B_{t_{k+1}^n}^j - B_{t_k^n}^j)(B_{t_{k+1}^n}^i - B_{t_k^n}^i) 2^n ds \\ & \quad \approx \nabla f(B_{t_k^n})(B_{t_{k+1}^n} - B_{t_k^n}) \otimes (B_{t_{k+1}^n} - B_{t_k^n}). \end{aligned}$$

With (44),  $\mathfrak{s}(\mathbf{B}_{s,t}) = \frac{1}{2}(B_t - B_s) \otimes (B_t - B_s)$ . This implies that

$$\sum_{i=1}^d \frac{1}{2} (f_i(B_{t_{k+1}^n}) - f_i(B_{t_k^n}))(B_{t_{k+1}^n}^i - B_{t_k^n}^i) \approx \nabla f(B_{t_k^n}) \mathfrak{s}(\mathbf{B}_{t_k^n, t_{k+1}^n}).$$

On the other hand, let us remark that if  $M_k = \mathfrak{a}(\mathbf{B}_{t_k^n, t_{k+1}^n})$ , then  $(\sum_{\ell=0}^k M_\ell)_{k=0, \dots, 2^n}$  forms a martingale with respect to  $(\mathcal{F}_k)_{k=0, \dots, 2^n}$ , where

$(\mathcal{F}_t)_{t \geq 0}$  is the filtration of the Brownian motion. In addition,

$$\mathbb{E} [(M_k)^2] \leq \frac{6T^2}{2^{2n}}.$$

Hence,

$$\mathbb{E} \left[ \left( \sum_{k=0}^{2^n-1} \nabla f(B_{t_k^n}) \mathbf{a}(\mathbf{B}_{t_k^n, t_{k+1}^n}) \right)^2 \right] \leq \frac{6T^2}{2^n} \|\nabla f\|_\infty$$

and the latter term converges to 0 in probability. The convergence of the Stratonovich integral in probability follows from the last convergence and the almost sure convergence of the rough path approximation given in (59).

Regarding the Itô integrals, we lift the Brownian motion  $B$  as a Brownian motion  $\mathbf{B}'$  with  $\pi_{\mathbb{R}^2}(\mathbf{B}') = B$  and

$$\pi_{e_i \otimes e_j}(\mathbf{B}'_t) = \int_0^t (B_r^i - B_0^i) dB_r^j = \int_0^t (B_r^i - B_0^i) \circ dB_r^j - \delta_{i,j}t.$$

However, let us note that the anti-symmetric part  $\mathbf{a}(\mathbf{B}')$  is equal to the anti-symmetric part of  $\mathbf{a}(\mathbf{B})$ . Indeed, due to the Wong-Zakai theorem [IW89],  $\mathbf{B}$  is a geometric rough path, while  $\mathbf{B}'$  is not a geometric rough path. From the previous computations, we get easily that

$$\pi_{\mathbb{R}^d}(\mathfrak{J}(\mathbf{B}'; 0, T)) = \mathfrak{J}_{\mathbb{R}^d}(\mathfrak{J}(\mathbf{B}; 0, T)) - \sum_{i=1}^d \int_0^T \frac{\partial f_i}{\partial x_i}(B_s) ds = \int_0^T f(B_s) dB_s$$

and thus  $\mathbf{B}'$  gives rise to the Itô integral. The effect of the bracket terms  $t \mapsto \langle B^i, B^j \rangle_t = \delta_{i,j}t$  on  $\mathfrak{J}(\mathbf{B}')$  with respect to  $\mathfrak{J}(\mathbf{B})$  is studied in Section 7.5.

## 9. SOLVING A DIFFERENTIAL EQUATIONS

The theory of rough paths may be applied to solve differential equations, since one can transform integrals into differential equations using a fixed point principle. Indeed, as noted in Section 8.2, most of the ideas from the rough paths theory comes from the developments around iterated integrals as a way to deal formally with ordinary differential equations. Thus, the algebraic structures we used were introduced in the context of differential equations, not integrals (see for example [Mag54, Che57, Str87]... and also [Yam79, FNC82, BA89, Cas93]... on Stratonovich stochastic differential equations).

We wish now to consider the following differential equation

$$(60) \quad y_t = y_0 + \int_0^t g(y_s) dx_s,$$

where  $x$  is an irregular path. We assume that  $x$  lives in  $\mathbb{R}^d$ , and  $y$  lives in  $\mathbb{R}^m$ . We denote by  $\{e_1, \dots, e_d\}$  (resp.  $\{\bar{e}_1, \dots, \bar{e}_m\}$ ) the canonical

basis of  $\mathbb{R}^d$  (resp.  $\mathbb{R}^m$ ). If one wishes to interpret this integral as a rough path, one has first to transform the vector field

$$g(z) = \sum_{\substack{i=1,\dots,d \\ k=1,\dots,m}} \bar{e}_k g_i^k(z) \frac{\partial}{\partial x_i}$$

into a differential form  $h$  which is integrated along a path  $(x, y)$  living in  $\mathbb{R}^d \oplus \mathbb{R}^m$ . For this, the natural extension is

$$h(z, z') = \sum_{\substack{i=1,\dots,d \\ k=1,\dots,m}} \bar{e}_k g_i^k(z') e^i + \sum_{i=1}^d e_i \cdot e^i, \quad z \in \mathbb{R}^d, \quad z' \in \mathbb{R}^m.$$

Hence, if  $x$  is smooth and (60) has a smooth solution  $y$ ,

$$(x_t, y_t) = (x_0, y_0) + \int_0^t h(x_s, y_s) d(x_s, y_s) = (x_0, y_0) + \int_{(x,y)|_{[0,t]}} h.$$

In order to deal with an irregular path  $x$ , the last integrals will be defined as a rough path, which means that we shall consider a rough path  $\mathbf{z}$  living above  $(x, y)$ , in the tensor space  $\mathbb{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)$ . We have also to extend the differential form  $h$ . For  $(z, z') \in \mathbb{R}^d \oplus \mathbb{R}^m$ ,

$$\begin{aligned} \mathfrak{E}_{\mathbb{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)}(h)(z, z') &= h(z, z') + \sum_{\substack{i=1,\dots,d \\ k,\ell=1,\dots,m}} \bar{e}_k \frac{\partial g_i^k}{\partial x_\ell}(z') \bar{e}^\ell \otimes e^i \\ &+ \sum_{\substack{k,\ell=1,\dots,m \\ i,j=1,\dots,d}} \bar{e}_k \otimes \bar{e}_\ell g_i^k(z') g_j^\ell(z') e^i \otimes e^j + \sum_{i,j=1,\dots,d} e_i \otimes e_j \cdot e^i \otimes e^j \\ &+ \sum_{\substack{k=1,\dots,m \\ i,j=1,\dots,d}} \bar{e}_k \otimes e_j g_i^k(z) e^i \otimes e^j + \sum_{\substack{k=1,\dots,m \\ i,j=1,\dots,d}} e_i \otimes \bar{e}_k g_j^k(z) e^i \otimes e^j. \end{aligned}$$

The idea is now to apply a Picard iteration scheme. Define by  $\mathfrak{J}$  the integral with respect to the differential form  $h$ . If  $\mathbf{z}^0$  is a rough path in  $C^\alpha([0, T]; \mathbb{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m))$  lying above  $(x, y^0)$  for some path  $y^0 \in C^\alpha([0, T]; \mathbb{R}^m)$  and  $\pi_{\mathbb{T}_1(\mathbb{R}^d)}(\mathbf{z}) = \mathbf{x}$ , then set recursively  $\mathbf{z}^{k+1} = \mathfrak{J}(\mathbf{z}^k)$ . The problem is to study the convergence of  $(\mathbf{z}^k)_{k \in \mathbb{N}}$ .

**Definition 8.** A solution of (60) is a rough path  $\mathbf{z}$  living in  $\mathbb{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)$  with  $\mathbf{z}_0 = (x_0, y_0, 0)$  and such that  $\mathfrak{J}(\mathbf{z}; s, t) = \mathbf{z}_{s,t}$  for all  $0 \leq s \leq t \leq T$ .

Let us start our study by the following observation: from the choice of  $h$ ,  $\pi_{\mathbb{T}_1(\mathbb{R}^d)}(\mathbf{z}^k)$  is equal to  $\mathbf{x}$ , whatever  $k$ . In addition, to compute  $\mathbf{z}^{k+1}$ , we need  $\mathbf{x} = \pi_{\mathbb{T}_1(\mathbb{R}^d)}(\mathbf{z}^k)$ ,  $\pi_{\mathbb{R}^m}(\mathbf{z}^k)$  and  $\pi_{\mathbb{R}^m \otimes \mathbb{R}^d}(\mathbf{z}^k)$ . If  $\mathbf{z}^k$  lies above  $(x, y^k)$ , the last term corresponds to the iterated integrals of  $y^k$  against  $x$ .

The the proofs, the reader is referred to [Lyo98, Section 4.1, p. 296], [LQ02, Chapter 6, p. 148] and to [LV06].

**Theorem 4.** *Let  $\mathbf{x}$  be a rough path in  $C^\alpha([0, T]; \mathbf{T}_1(\mathbb{R}^d))$ . Let  $g_1, \dots, g_d$  be vector fields on  $\mathbb{R}^m$  with a bounded derivatives which are  $\gamma$ -Hölder continuous with  $\alpha(\gamma + 1) > 1$ . Then there exists at least one solution to (60) in  $C^\alpha([0, T]; \mathbf{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m))$ .*

*If  $g_1, \dots, g_d$  are vector fields on  $\mathbb{R}^m$  that are two times differentiable with, for  $i, j, k = 1, \dots, d$ ,  $\partial_{x_j} g_i$  which is bounded and  $\partial_{x_k, x_j}^2 g_i$  which is bounded and  $\gamma$ -Hölder continuous with  $\alpha(1 + \gamma) > 1$ , then the solution of (60) is unique and  $\mathbf{x} \mapsto \mathbf{z}$  is continuous from  $(C^\alpha([0, T]; \mathbf{T}_1(\mathbb{R}^d)), \|\cdot\|_\alpha)$  to  $(C^\alpha([0, T]; \mathbf{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)), \|\cdot\|_\alpha)$ .*

*Remark 15.* The map  $\mathbf{x} \mapsto \mathbf{z}$  is called the *Itô map*. Its differentiability is studied in [LQ02] and in [LL05].

Here again, because of the continuity of  $\mathbf{x} \mapsto \mathbf{z}$ , we get that if  $\mathbf{x}$  belongs to  $C^\alpha([0, T]; \mathbf{G}(\mathbb{R}^d))$ , then  $\mathbf{z} \in C^\alpha([0, T]; \mathbf{G}(\mathbb{R}^d \oplus \mathbb{R}^m))$  and if  $\mathbf{x}$  belongs to  $C^{0,\alpha}([0, T]; \mathbf{G}(\mathbb{R}^d))$ , then  $\mathbf{z} \in C^\alpha([0, T]; \mathbf{G}(\mathbb{R}^d \oplus \mathbb{R}^m))$ .

Finally, the solution of (60) may also be interpreted using an Euler scheme, as in [FV06a], following [Dav05].

## APPENDIX A. CARNOT GROUPS AND THE HEISENBERG GROUP

**A.1. Carnot groups and homogeneous gauges and norms.** Let  $(\mathbf{G}, \times)$  be a Lie group, and  $(\mathfrak{g}, [\cdot, \cdot])$  be its Lie algebra  $\mathbf{G}$  is a *Carnot group of step  $k$*  [Mon02, Bau04] if for some positive integer  $k$ ,  $\mathfrak{g} = \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \dots \oplus \mathbf{V}_k$  — this decomposition being called a *stratification* — with

$$[\mathbf{V}_1, \mathbf{V}_i] = \mathbf{V}_{i+1} \text{ for } i = 1, \dots, k-1 \text{ and } [\mathbf{V}_1, \mathbf{V}_k] = \{0\}, \text{ and}$$

where  $[\mathbf{V}_i, \mathbf{V}_j] = \{[x, y] \mid x \in \mathbf{V}_i, y \in \mathbf{V}_j\}$ . A Carnot group is naturally equipped with a *dilatation* operator  $\delta_\lambda(x) = (\lambda^{\alpha_1} x^1, \dots, \lambda^{\alpha_k} x^k)$  with  $x^i \in \exp(\mathbf{V}_i)$  and some positive real numbers  $\alpha_1, \dots, \alpha_k$ , where  $\exp$  is the map from  $\mathfrak{g}$  to  $\mathbf{G}$ . This dilatation operator shall verify  $\delta_\lambda(x \times y) = \delta_\lambda(x) \times \delta_\lambda(y)$ . If the dimension of  $\mathbf{V}_1$  is finite, the real number  $N = \alpha_1 \dim(\mathbf{V}_1) + \dots + \alpha_k \dim(\mathbf{V}_k)$  is called the *homogeneous dimension*.

On  $\mathbf{G}$  equipped with a dilatation operator  $\delta$ , an *homogeneous gauge* is a continuous function which maps  $x$  into a non-negative real number  $\|x\|$  such that  $\|x\| = 0$  if and only if  $x$  is the neutral elements of  $\mathbf{G}$ , and for all  $\lambda \in \mathbb{R}$ ,  $\|\delta_\lambda(x)\| = |\lambda| \cdot \|x\|$ .

An homogeneous gauge is an *homogeneous norm* if  $\|x^{-1}\| = \|x\|$  for all  $x \in \mathbf{G}$ . In addition, this homogeneous norm is said to be *sub-additive* if  $\|x \times y\| \leq \|x\| + \|y\|$  for all  $x, y \in \mathbf{G}$ .

If  $\mathbf{V}_1$  is of finite dimension, then a homogeneous norm always exists [FS82]. For this, equip the Lie algebra  $\mathfrak{g}$  with the Euclidean norm  $|\cdot|$  and denote by  $\exp$  the canonical diffeomorphism from  $\mathfrak{g}$  to  $\mathbf{G}$ . For  $x \in \mathfrak{g}$ , let  $r(x)$  be the smallest positive real such that  $|\delta_{r(x)} x| = 1$ , which exists, since  $|\delta_r x|$  is increasing from  $[0, +\infty)$  to  $[0, +\infty)$ . Then, for  $y \in \mathbf{G}$ ,  $\|y\| = 1/r(\exp^{-1} y)$  defines a symmetric homogeneous norm.

Two homogeneous gauges  $\|\cdot\|$  and  $\|\cdot\|'$  are said to be equivalent if for some constants  $C$  and  $C'$ ,  $C\|x\|' \leq \|x\| \leq C'\|x\|'$  for all  $x \in G$ .

**Proposition 10** ([Goo77]). *If the dimension of  $V_1$  is finite, then all the homogeneous gauges are equivalent. In addition, for a homogeneous gauge  $\|\cdot\|$ , there exists some constant  $C$  and  $C'$  such that  $\|x^{-1}\| \leq C\|x\|$  and*

$$|x \times y| \leq C'(|x| + |y|),$$

for all  $x, y \in G$ .

*Proof.* If  $\exp^{-1}(x)$  is decomposed as  $y_1, \dots, y_k$  with  $y_i \in V_i$  for  $x \in G$ , then set  $|x|' = \sum_{i=1}^k |y_i|^{1/i}$ , where  $|\cdot|$  denotes the Euclidean norm on each of the finite-dimensional vector space  $V_i$ . It is easily verified that  $|x|$  is a homogeneous gauge. Let  $\|\cdot\|$  be another homogeneous gauge. Set  $\varphi(x) = \|x\|/|x|'$ . Then  $\varphi$  and  $1/\varphi$  are continuous on  $G \setminus \{1\}$ , where 1 is the neutral element of  $G$ . As  $\{x \in G \mid |x|' = 1\}$  is compact, we easily get that  $\varphi$  and  $1/\varphi$  are bounded, and then that for some constants  $C$  and  $C'$ ,  $C \leq \|x\| \leq C'$  when  $|x|' = 1$ . This implies that  $\|\cdot\|$  and  $|\cdot|'$  are equivalent by using the dilatation  $\delta_{1/|x|'}$  for a general  $x$ .

The other results are proved in a similar way by using  $\varphi(x) = \|x^{-1}\|/\|x\|$  and  $\varphi(x, y) = \|x \times y\|/(\|x\| + \|y\|)$ .  $\square$

It follows that any homogeneous gauge can be transformed in an equivalent homogeneous norm by setting  $\|x\|' = \|x\| + \|x^{-1}\|$ .

The notion a *Lipschitz* function is then extended to homogeneous gauges.

**Definition 9.** If  $(G, \times)$  and  $(G', \times)$  are two nilpotent Carnot groups with homogeneous gauges  $\|\cdot\|$  and  $\|\cdot\|'$ , then  $f : G \rightarrow G'$  is said to be *Lipschitz* if for some constant  $C$ ,

$$\|f(x)^{-1} \times f(y)\|' \leq C\|x^{-1} \times y\|$$

for all  $x, y \in G$ .

The group  $(A(\mathbb{R}^2), \boxplus)$  (and thus  $(G(\mathbb{R}^2), \otimes)$ ) is obviously a Carnot group of step 2 with  $V_1 = \mathbb{R}^2$  and  $V_2 = [\mathbb{R}^2, \mathbb{R}^2]$ , and  $\delta_\lambda(x) = (\lambda x^1, \lambda^2 x^2)$ . Its homogeneous dimension is 4. Homogeneous norms and gauges are easily constructed. It is sufficient to consider  $\|x\| = |x^1| + \sqrt{|x^2|}$ ,  $\|x\| = \max\{|x^1|, \sqrt{|x^2|}\}$  either on  $A(\mathbb{R}^2)$  or  $G(\mathbb{R}^2)$ . Of course, if  $\|\cdot\|$  is a homogeneous gauge on  $A(\mathbb{R}^2)$ , then  $\|\cdot\|'$  defined by  $\|x\|' = \|\log(x)\|$  is a homogeneous gauge on  $G(\mathbb{R}^2)$ .

**A.2. The Heisenberg as a matrix group.** It is possible to embed the elements of  $A(\mathbb{R}^2)$  and  $G(\mathbb{R}^2)$  into matrices. For references on the Heisenberg group, see for example [Fol89, Mon02, Bau04], ... The Lie algebra of the Heisenberg group (and then the group itself) is connected to quantum mechanics, since it can be identified with the Lie algebra generated by the three following operators on functions:  $\mathbf{1}f(x) = f(x)$ ,

$qf(x) = xf(x)$  and  $pf(x) = \frac{df(x)}{dx}$ , for which  $[p, q] = \mathbf{1}$  and  $[p, \mathbf{1}] = [q, \mathbf{1}] = 0$ .

The Heisenberg group  $H_3(\mathbb{R})$  is the matrix group formed with the matrices of type

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}^3.$$

Let us remark that

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & aa'+bb'+c+c' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{bmatrix},$$

and that  $M^{-1} = \begin{bmatrix} 1 & -a & -c+a^2+b^2 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$ . Thus,  $(H_3(\mathbb{R}), \times)$  is a (non-commutative) group of matrices, called the *Heisenberg group*, which is also a Lie group.

Let also  $\mathfrak{h}_3(\mathbb{R})$  be the set of matrices of type

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, \quad a, b, c \in \mathbb{R}^3.$$

For  $M \in \mathfrak{h}_3(\mathbb{R})$ ,  $M^p = 0$  as soon as  $p \geq 3$ , and  $\exp(M)$  belongs to  $H_3(\mathbb{R})$ . In addition,

$$\exp(tM) = \begin{bmatrix} 1 & ta & \frac{t^2}{2}ab + tc \\ 0 & 1 & tb \\ 0 & 0 & 1 \end{bmatrix}.$$

With the bracket  $[M, N] = MN - NM$ ,  $\mathfrak{h}_3(\mathbb{R})$  is the Lie algebra of  $H_3(\mathbb{R})$ .

In fact,  $G(\mathbb{R}^2)$  (resp.  $\mathfrak{g}(\mathbb{R}^2)$ ) and  $H_3(\mathbb{R})$  (resp.  $\mathfrak{h}_3(\mathbb{R})$ ) are in one-to-one correspondence.

If  $x \in \mathfrak{g}(\mathbb{R}^2)$  with  $x = ae_1 + be_2 + c[e_1, e_2]$ , then

$$\begin{aligned} \exp(tx) &= 1 + tae_1 + tbe_2 + \left( \frac{t^2}{2} + tce_1 \otimes e_2 \right) ae_1 \otimes e_2 \\ &\quad + \left( \frac{t^2}{2}ab - tc \right) e_2 \otimes e_1 + \frac{t^2}{2}a^2e_1 \otimes e_1 + \frac{t^2}{2}b^2e_2 \otimes e_2. \end{aligned}$$

Hence, for  $x \in \mathfrak{g}(\mathbb{R}^2)$ , set

$$\mathfrak{M}(x) = \exp \begin{bmatrix} 0 & \pi_{e_1}(x) & \pi_{[e_1, e_2]}(x) \\ 0 & 0 & \pi_{e_2}(x) \\ 0 & 0 & 0 \end{bmatrix}$$

which is a one-to-one map from  $\mathfrak{g}(\mathbb{R}^2)$  to  $H_3(\mathbb{R})$ . We remark then that for all  $x, x' \in \mathfrak{g}(\mathbb{R}^2)$ ,  $\mathfrak{M}(x \boxplus x') = \mathfrak{M}(x) \times \mathfrak{M}(x')$ . This means that  $\mathfrak{M} : (\mathfrak{g}(\mathbb{R}^2), \boxplus) \rightarrow (H_3(\mathbb{R}), \times)$  is a group homomorphism.

### A.3. The Brownian motion on the Heisenberg group. .

We have seen in Section 8.4 that the Brownian motion is naturally lifted as a rough path and then that the integrals correspond to the usual Itô or Stratonovich integrals.

The tangent plane of  $A(\mathbb{R}^2)$  may be identified with  $A(\mathbb{R}^2)$ , and we denote by  $\partial_x, \partial_y$  and  $\partial_z$  the basis of  $T_x A(\mathbb{R}^2)$  at a point  $x$  which is deduced from the canonical coordinates  $e_1, e_2$  and  $[e_1, e_2]$ .

Let  $V^1, V^2$  and  $V^3$  be the left invariant vector fields that goes through 0 and that coincide respectively with  $\partial_x, \partial_y$  and  $\partial_z$  at this point.

For example, for  $a \in A(\mathbb{R}^2)$  and all  $x \in A(\mathbb{R}^2)$  and smooth function  $f$  on  $A(\mathbb{R}^2)$ ,  $V^i f(a \boxplus x) = V^i f \circ L_a(x)$  where  $L_a(x) = a \boxplus x$  for  $i = 1, 2, 3$ . Hence, we deduce easily that the  $V^i$ 's are decomposed in the basis  $\{\partial_x, \partial_y, \partial_z\}$  as

$$V^1 = \partial_x - \frac{1}{2}y\partial_z, \quad V^2 = \partial_y + \frac{1}{2}x\partial_y \text{ and } V^3 = \partial_z.$$

We remark that  $[V^1, V^2] = V^3$  and  $[V^i, V^j] = 0$  in all the other cases. The tangent plane at any point of  $A(\mathbb{R}^2)$  is then equipped with a scalar product  $\langle \cdot, \cdot \rangle$  such that  $\langle V^i, V^j \rangle = \delta_{i,j}$  for  $i, j = 1, 2, 3$ , *i.e.*, for which  $\{V^1, V^2, V^3\}$  forms an orthonormal basis. With this scalar product,  $A(\mathbb{R}^2)$  becomes a Riemannian manifold.

Let  $B = (B^1, B^2)$  be a two dimensional Brownian motion, and  $B^n = (B^{n,1}, B^{n,2})$  for  $n = 1, 2, \dots$  be a family of piecewise linear approximation of  $B$  along a family of deterministic partitions whose mesh decreases to 0.

We then consider  $\mathbf{X}$  the solution of the Stratonovich SDE

$$\mathbf{X}_t = \int_0^t V^1(\mathbf{X}_s) \circ dB_s^1 + \int_0^t V^2(\mathbf{X}_s) \circ dB_s^2$$

as well as the solutions  $\mathbf{X}^n$  of the ordinary differential equations

$$\mathbf{X}_t^n = \int_0^t V^1(\mathbf{X}_s^n) \circ dB_s^{1,n} + \int_0^t V^2(\mathbf{X}_s^n) \circ dB_s^{2,n}.$$

Using the decomposition of the  $V^i$  on the coordinates  $\{\partial_x, \partial_y, \partial_z\}$ , we get that

$$\mathbf{X}_t = B_t^1 e_1 + B_t^2 e_2 + \mathfrak{A}(B^1, B^2; 0, t)[e_1, e_2]$$

where

$$\mathfrak{A}(B^1, B^2; 0, t) = \frac{1}{2} \int_0^t B_s^1 \circ dB_s^2 - \frac{1}{2} \int_0^t B_s^2 \circ dB_s^1$$

is the Lévy area of  $(B^1, B^2)$ . The process  $\mathbf{X}$ , we have already mentioned in Section 8.4 is the *Brownian motion on the Heisenberg group*. Similarly, we get that

$$\mathbf{X}_t^n = B_t^{1,n} e_1 + B_t^{2,n} e_2 + \mathfrak{A}(B^{1,n}, B^{2,n}; 0, t)[e_1, e_2]$$



and one knows from the Wong-Zakai theorem [IW89] that  $\mathbf{X}^n$  converges in probability to  $\mathbf{X}$  (with a dyadic partition, we get an almost sure convergence in the  $\alpha$ -Hölder norm for any  $\alpha < 1/2$  [Sip93, CL05]). Let us note that the piecewise smooth curves  $\mathbf{X}^n$  are horizontal curves, so that in this case, the natural approximation of  $\mathbf{X} \in C^\alpha([0, T]; A(\mathbb{R}^2))$  is provided by the piecewise linear approximations of  $(B^1, B^2)$  naturally lifted as paths in  $A(\mathbb{R}^2)$ . Many processes shares this property : see for example [CQ02, CL05, Lej06].

This is a special case of a Brownian motion in a Lie group. Its short time behavior and its density have been already widely studied: see for example [Gav77, A<sup>+</sup>81, Bis84, BA89, Bau04], ... From the Hörmander theorem, as  $\{V^1, V^2, [V^1, V^2]\}$  spans the tangent space at any point, one knows that for any  $t > 0$ ,  $\mathbf{X}_t$  has a density on the three dimensional space  $A(\mathbb{R}^2)$ , although it is constructed from a two dimensional Brownian motion. The infinitesimal generator of  $\mathbf{X}$  is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(V^1)^2 + \frac{1}{2}(V^2)^2 \\ &= \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2 + \frac{1}{2}x\partial_{zy}^2 - \frac{1}{2}y\partial_{zx}^2 + \frac{1}{8}(x^2 + y^2)\partial_z^2. \end{aligned}$$

This is an hypoelliptic generator.

## APPENDIX B. FROM ALMOST ROUGH PATHS TO ROUGH PATHS

**B.1. Theorems and proofs.** In this Section we prove Theorem 2 on almost rough paths, which we rewrite in a more general setting than with Hölder continuous norms.

We set  $\Delta_+ = \{(s, t) \in [0, T]^2 \mid 0 \leq s \leq t \leq T\}$ . A *control* is a function  $\omega : \Delta_+ \rightarrow \mathbb{R}_+$  such that  $\omega$  is continuous,  $\omega$  is super-additive, *i.e.*,

$$\forall 0 \leq s < t < u \leq T, \omega(s, t) + \omega(t, u) \leq \omega(s, u)$$

and  $\omega(t, t) = 0$  for all  $t \in [0, T]$ . If  $\omega$  is super-additive and  $\theta \geq 1$ , then  $\omega^\theta$  is also super-additive.

We recall that for  $x = (\xi, x^1, x^2)$  in  $T_\xi(\mathbb{R}^d)$  with  $\xi = 0$  or  $\xi = 1$ , we have defined  $\|x\| = \max\{|x^1|, \sqrt{\frac{1}{2}|x^2|}\}$ . We also set  $\|x\|_\star = \max\{|x^1|, |x^2|\}$ . These two norms are not equivalent, but they define the same topology.

For a continuous path  $\mathbf{x}$  with values in  $T_1(\mathbb{R}^2)$ , we introduce the norms

$$\|\mathbf{x}\|_{p, \omega} = \sup_{0 \leq s < t \leq T} \frac{\|\mathbf{x}_{s,t}\|}{\omega(s, t)^{1/p}}$$

and

$$\|\mathbf{x}\|_{\star, p, \omega} = \sup_{0 \leq s < t \leq T} \max \left\{ \frac{|\mathbf{x}_{s,t}^1|}{\omega(s, t)^{1/p}}, \frac{|\mathbf{x}_{s,t}^2|}{\omega(s, t)^{2/p}} \right\}$$

with  $\mathbf{x}^1 = \pi_{\mathbb{R}^d}(\mathbf{x})$  and  $\mathbf{x}^2 = \pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x})$ . Let us note that  $\|\mathbf{x}\|_{\star, p, \omega}$  is finite if and only if  $\|\mathbf{x}\|_{p, \omega}$  is finite. Hence, we denote by  $C^{p, \omega}([0, T]; \mathbb{T}_1(\mathbb{R}^d))$  the space of continuous paths with values in  $\mathbb{T}_1(\mathbb{R}^d)$  for which  $\|\mathbf{x}\|_{p, \omega}$  (or equivalently  $\|\mathbf{x}\|_{\star, p, \omega}$  is finite. We rewrite the first part of Theorem 2 with a control  $\omega$ .

*Remark 16.* The case of  $\alpha$ -Hölder continuous paths corresponds to  $\omega(s, t) = t - s$  and  $p = 1/\alpha$ . All the results we gave about the existence of the integral, solving a differential equation, ... may be written using a control  $\omega(s, t)$  instead of  $\omega(s, t) = t - s$  and the appropriate norms  $\|\cdot\|_{p, \omega}$  and  $\|\cdot\|_{\star, p, \omega}$ . Similarly, we are not bound to use dyadic partitions, although some results may be related to dyadic partitions (see for example [CL05] for an application to semi-martingales), and it is in generally computationally more simple.

**Theorem 5.** *Let  $(\mathbf{x}_{s,t})_{(s,t) \in \Delta_+}$  be a family of elements of  $\mathbb{T}_1(\mathbb{R}^d)$  such that for some  $\theta > 1$ ,*

$$(61) \quad \|\mathbf{x}\|_{p, \omega} < +\infty \text{ and } \|\mathbf{x}_{s,t} - \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}\|_{\star} \leq K\omega(s, t)^\theta$$

for all  $(s, t) \in \Delta_+$ . We call such a family an almost rough path controlled by  $\omega$ .

Then there exists a rough path  $\mathbf{y}$  in  $C^{p, \omega}([0, T]; \mathbb{T}_1(\mathbb{R}^d))$  such that

$$(62) \quad \|\mathbf{y}_{s,t} - \mathbf{x}_{s,t}\|_{\star} \leq C\omega(s, t)^\theta$$

for some constant  $C$  that depends only on  $K, \theta, p, \omega(0, T)$  and  $\|\mathbf{x}\|_{\star, p, \omega}$ . In addition,  $\mathbf{y}$  is unique up to the value of  $\mathbf{y}_0$ .

In addition, if  $\mathbf{x}_{s,t}$  belongs to  $\mathbb{G}(\mathbb{R}^d)$ , then  $\mathbf{y}$  is a weak geometric rough path with  $p$ -variation controlled by  $\omega$ .

*Proof.* Let us remark first that if  $\alpha^{(n)} = \bigotimes_{i=1}^n (1 + \alpha_i)$  with  $\alpha_i \in \mathbb{T}_0(\mathbb{R}^d)$ , then

$$\alpha^{(n)} = 1 + \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \sum_{j=i+1}^n \alpha_i \otimes \alpha_j.$$

Hence, if  $\bar{\alpha}^{(n)} = \bigotimes_{i=1}^n (1 + \bar{\alpha}_i)$  with  $\bar{\alpha}_k = \alpha_k + \zeta$  for some  $k \in \{1, \dots, n\}$  and  $\bar{\alpha}_i = \alpha_i, i \neq k$ , then

$$(63) \quad \bar{\alpha}^{(n)} = \alpha^{(n)} + \zeta + \zeta \otimes \sum_{j=k+1}^n \alpha_j + \sum_{i=1}^{k-1} \alpha_i \otimes \zeta.$$

Let us set for a partition  $\pi = \{t_k\}_{k=1}^{n+1}$  of  $[s, t]$  with  $t_1 = s$  and  $t_{n+1} = t$ ,

$$(64) \quad \mathbf{x}_{s,t}^{(\pi)} = \bigotimes_{k=1}^n \mathbf{x}_{t_k, t_{k+1}}.$$

We set  $\mathbf{x}_{s,t}^{1,(\pi)} = \pi_{\mathbb{R}^d}(\mathbf{x}_{s,t}^{(\pi)})$  and  $\mathbf{x}_{s,t}^1 = \pi_{\mathbb{R}^d}(\mathbf{x}_{s,t})$ . Let  $\{t_i\}$  be some point of  $\pi$  (except  $s, t$ ), and set  $\hat{\pi} = \pi \setminus \{t_i\}$ . Then

$$\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^{1,(\hat{\pi})} = \mathbf{x}_{t_{i-1}, t_i}^1 + \mathbf{x}_{t_i, t_{i+1}}^1 - \mathbf{x}_{t_{i-1}, t_{i+1}}^1.$$

For a partition  $\pi = \{t_i\}_{i=1,\dots,n+1}$  with  $n+1$  points in  $[s, t]$  and  $t_1 = s$ ,  $t_{n+1} = t$ , we pick a point  $t_i$  such that  $\omega(t_{i-1}, t_{i+1}) \leq 2\omega(s, t)/n$ . This is possible if  $n > 3$  thanks to Lemma 2.2.1 [Lyo98, p. 244]. Then

$$|\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^{1,(\widehat{\pi})}| \leq K \frac{2^\theta}{n^\theta} \omega(s, t)^\theta.$$

If  $n$  has 3 elements  $\{t_1, t_2, t_3\}$  with  $t_1 = s$  and  $t_3 = t$ , then  $|\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^1| \leq K\omega(s, t)^\theta$ . Thus, by summing from  $k = 1, \dots, n$  by choosing carefully an element of the partition to suppress, we get that

$$(65) \quad |\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^1| \leq 2^\theta \zeta(\theta) K \omega(s, t)^\theta$$

with  $\zeta(\theta) = \sum_{n \geq 1} 1/n^\theta$ . This is true for any partition  $\pi$ , whatever its size.

Let us consider now a sequence of partitions  $\pi^n$  of  $[0, T]$  whose meshes decrease to 0. We set  $\pi^n[s, t] = (\pi^n \cap [s, t]) \cup \{s, t\}$ . Then for any  $(s, t) \in \Delta_+$ ,  $(\mathbf{x}_{s,t}^{1,(\pi^n[s,t])})_{n \in \mathbb{N}}$  has a convergent subsequence.

One can extract a subsequence  $(n_k)_{k \in \mathbb{N}}$  such that  $(\mathbf{x}_{s,t}^{1,(\pi^{n_k}[s,t])})_{k \in \mathbb{N}}$  converges for any  $(s, t) \in \Delta_+$ ,  $s, t \in \mathbb{Q}$ . We denote by  $y_{s,t}$  one of the possible limits for  $(s, t) \in \Delta_+$ ,  $s, t \in \mathbb{Q}$ . With set  $K_1 = K2^\theta \zeta(\theta)$  and with (65), we get that

$$(66) \quad |y_{s,t} - \mathbf{x}_{s,t}^1| \leq K_1 \omega(s, t)^\theta.$$

As  $\omega$  is continuous and  $\mathbf{x}_{s,t}^1$  converges to 0 as  $|t - s| \rightarrow 0$ , we may extend  $y$  by continuity on  $\Delta_+$ .

In addition, for  $0 \leq s < r < t \leq T$  and  $r \in \pi^n$ , then

$$\mathbf{x}_{s,t}^{1,(\pi^n[s,t])} = \mathbf{x}_{s,r}^{1,(\pi^n[s,r])} + \mathbf{x}_{r,t}^{1,(\pi^n[r,t])}.$$

Choosing the partitions  $\pi^n$  such that  $\pi^n \subset \pi^{n+1}$  and  $\pi^n \subset \mathbb{Q}$  for each  $\mathbb{Q}$ , we get that, by passing to the limit for  $r \in \pi^{n_{k_0}}$  for some  $k_0$  and  $s, t \in \mathbb{Q}$ , we get that  $y_{s,t} = y_{s,r} + y_{r,t}$ . Using the continuity of  $y$ , this is true for any  $0 \leq s < r < t \leq T$ . We define  $y_t = y_{0,t}$  and remark that  $y_{s,t} = y_t - y_s$ .

Now, let us consider another function  $z$  on  $[0, T]$  with values in  $\mathbb{R}^d$  and satisfying  $|z_t - z_s - \mathbf{x}_{s,t}^1| \leq 2^\theta \zeta(\theta) K \omega(s, t)^\theta$  for all  $(s, t) \in \Delta_+$ .

Since

$$|(y_t - y_s) - (z_t - z_s)| \leq |(y_r - y_s) - \mathbf{x}_{r,s}^1| + |(z_r - z_s) - \mathbf{x}_{r,s}^1|$$

for  $r \in [s, t]$ ,  $(s, t) \in \Delta_+$ ,

$$|(y_t - y_s) - (z_t - z_s)| \leq 2K_1 \omega(s, t)^\theta.$$

Thus,  $\widehat{y}_t = y_t - z_t$  is controlled by  $\omega^\theta$  with  $\theta > 1$  and is necessarily constant. Otherwise,

$$|\widehat{y}_t - \widehat{y}_0| \leq \sum_{k=0, \dots, 2^n-1, t_k^n \leq t} |\widehat{y}_{t_{k+1}^n} - \widehat{y}_{t_k^n}| \leq 2K_1 \omega(s, t) \sup_{k=0, \dots, 2^n-1} \omega(t_k^n, t_{k+1}^n)^{\theta-1}$$

and this converges to 0.

We have now to construct the second level of the rough path. For this purpose, we set  $\mathbf{z}_{s,t} = 1 + y_t - y_s + \pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x}_{s,t})$ , and, for a partition  $\pi$  with  $s$  and  $t$  as endpoints, we define  $\mathbf{z}_{s,t}^{(\pi)}$  as  $\mathbf{x}_{s,t}^{(\pi)}$  in (64) with  $\mathbf{z}$  instead of  $\mathbf{x}$ . Let us note that  $\mathbf{z}$  is also an almost rough path, since

$$\begin{aligned} \mathbf{z}_{s,t} - \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t} &= \mathbf{x}_{s,t}^2 - \mathbf{x}_{s,r}^2 - \mathbf{x}_{r,t}^2 - \mathbf{x}_{s,r}^1 \otimes \mathbf{x}_{r,t}^1 \\ &\quad - (\mathbf{z}_{s,r}^1 - \mathbf{x}_{s,r}^1) \otimes \mathbf{z}_{r,t}^1 + \mathbf{z}_{s,r}^1 \otimes (\mathbf{z}_{r,t}^1 - \mathbf{x}_{r,t}^1) \end{aligned}$$

and therefore with (66),

$$\|\mathbf{z}_{s,t} - \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t}\|_{\star} \leq K_2 \omega(s, t)^{\theta}$$

where  $K_2 = \{K + 2K_1(K_1 + \|\mathbf{x}\|_{\star, p, \omega})\omega(0, T)^{1/p}\} \omega(s, t)^{\theta}$ .

For  $0 \leq s < t \leq T$  and  $\pi = \{t_i\}_{i=1, \dots, n+1}$  a partition of  $[s, t]$  with  $n+1$  points and  $t_1 = s, t_{n+1} = t$ , then for some  $i \in \{2, \dots, n\}$  and  $\hat{\pi} = \pi \setminus \{t_i\}$ ,

$$\|\mathbf{z}_{s,t}^{(\pi)} - \mathbf{z}_{s,t}^{(\hat{\pi})}\|_{\star} \leq K_2 \omega(t_{i-1}, t_{i+1})^{\theta}.$$

One may choose  $t_i$  such that  $\omega(t_{i-1}, t_{i+1}) \leq 2\omega(s, t)/n$ . Hence, as previously,

$$(67) \quad \|\mathbf{z}_{s,t}^{(\pi)} - \mathbf{z}_{s,t}\|_{\star} \leq 2^{\theta} \zeta(\theta) K_2 \omega(s, t)^{\theta}.$$

Then, the same arguments apply and one can show that for all  $(s, t) \in \Delta_+$ , there exists  $\mathbf{y}_{s,t} \in \mathbf{T}_1(\mathbb{R}^d)$  such that  $\pi_{\mathbb{R}^d}(\mathbf{y}_{s,t}) = y_t - y_s$ , where  $y$  was the function previously defined at the first level, for all  $0 \leq s \leq r \leq t \leq T$ ,  $\mathbf{y}_{s,t} = \mathbf{y}_{s,r} \otimes \mathbf{y}_{r,t}$  and  $\|\mathbf{y}_{s,t} - \mathbf{x}_{s,t}\|_{\star} \leq K_3 \omega(s, t)^{\theta}$  with  $K_3 = K_2 2^{\theta} \zeta(\theta)$ . In particular,  $\mathbf{y}$  is continuous on  $\Delta_+$  and  $t \mapsto \mathbf{y}_{0,t}$  is a rough path in  $C^{p, \omega}([0, T]; \mathbf{T}_1(\mathbb{R}^d))$  lying above  $y$ .

Let  $\hat{\mathbf{y}}$  be another rough path in  $C^{p, \omega}([0, T]; \mathbf{T}_1(\mathbb{R}^d))$  lying above  $y$  and such that  $\|\hat{\mathbf{y}}_{s,t} - \mathbf{x}_{s,t}\|_{\star} \leq K_3 \omega(s, t)^{\theta}$ . Hence,

$$\|\mathbf{y}_{s,t} - \hat{\mathbf{y}}_{s,t}\|_{\star} \leq |\mathbf{y}_{s,r}^2 - \mathbf{z}_{s,r}^2| + |\mathbf{y}_{r,t}^2 - \mathbf{z}_{r,t}^2| + |\hat{\mathbf{y}}_{s,r}^2 - \mathbf{z}_{s,r}^2| + |\hat{\mathbf{y}}_{r,t}^2 - \mathbf{z}_{r,t}^2|$$

for all  $0 \leq s \leq r \leq t \leq T$ . As previously, it follows that  $\mathbf{y}_{s,t} = \hat{\mathbf{y}}_{s,t}$  for all  $(s, t) \in \Delta_+$ .

This proves that  $\mathbf{y}$  is unique up to to an additive constant.

The question is now to know whether or not  $\mathbf{y}$  is also the limit of  $(\mathbf{x}^{(\pi^n)})_{n \in \mathbb{N}}$  for a family of partitions  $(\pi^n)_{n \in \mathbb{N}}$  whose meshes decrease to 0.

With the notation from the beginning of the proof, if  $\{\alpha_i\}_{i=1, \dots, n}$  is a family of elements in  $\mathbf{T}_0(\mathbb{R}^d)$  and  $\{\eta_i\}_{i=1, \dots, n}$  belongs to  $\mathbb{R}^d$ , then

$$\begin{aligned} \bigotimes_{i=1}^n (1 + \alpha_i + \eta_i) &= \bigotimes_{i=1}^n (1 + \alpha_i) + \sum_{i=1}^n \eta_i + \sum_{i=1}^{n-1} \eta_i \otimes \sum_{j=i+1}^n \alpha_j \\ &\quad + \sum_{i=1}^{n-1} \alpha_i \otimes \sum_{j=i+1}^n \eta_j + \sum_{i=1}^{n-1} \eta_i \otimes \sum_{j=i+1}^n \eta_j. \end{aligned}$$

Now, let us set  $\alpha_i = \mathbf{x}_{t_i, t_{i+1}}$  and  $\eta_i = \mathbf{y}_{t_i, t_{i+1}}^1$  for some partition  $\pi = \{t_i\}_{i=1, \dots, n+1}$  of  $[s, t]$ . Then for some constant  $C_1$ ,

$$\left| \sum_{i=1}^n \eta_i \right| \leq \sum_{i=1}^n C_1 \omega(t_i, t_{i+1})^\theta \leq C \omega(0, T) \sup_{i=1, \dots, n} \omega(t_i, t_{i+1})^{\theta-1}.$$

This last term converges to 0. Finally, let us remark that

$$\sum_{i=1}^{n-1} \alpha_i \otimes \sum_{j=i+1}^n \eta_j = \sum_{j=2}^n \left( \sum_{i=1}^{j-1} \alpha_i \right) \otimes \eta_j.$$

But from (65), for  $k \in \{2, \dots, n\}$ ,

$$\left| \sum_{i=1}^k \alpha_i \right| = |\mathbf{x}_{s, t_k}^{1, (\pi \cap [s, t_k])}| \leq K_1 \omega(s, t)^\theta + \|\mathbf{x}\|_{\star, p, \omega} \omega(s, t)^{1/p}.$$

It follows that for some constant  $C_2$  depending only on  $\|\mathbf{x}\|_{\star, p, \omega}$ ,  $K_1$ ,  $\omega(0, T)$ ,  $\theta$  and  $p$  that

$$\left| \sum_{j=2}^n \left( \sum_{i=1}^{j-1} \alpha_i \right) \otimes \eta_j \right| \leq C_2 \omega(s, t) \sup_{i=2, \dots, n} \omega(t_i, t_{i+1})^{\theta-1}.$$

Similarly,

$$\left| \sum_{i=1}^{n-1} \eta_i \otimes \sum_{j=i+1}^n \alpha_j \right| \leq C_2 \omega(s, t) \sup_{i=1, \dots, n-1} \omega(t_i, t_{i+1})^{\theta-1}$$

and

$$\left| \sum_{i=1}^{n-1} \eta_i \otimes \sum_{j=i+1}^n \eta_j \right| \leq K_1 \omega(s, t)^2 \sup_{i=1, \dots, n} \omega(t_i, t_{i+1})^{2\theta-2}.$$

It follows then that for some constant  $C_3$  depending on  $C_2$ ,  $K_1$ ,  $\theta$  and  $\omega(0, T)$ ,

$$\|\mathbf{x}_{s, t}^{(\pi)} - \mathbf{z}_{s, t}^{(\pi)}\|_{\star} \leq C_3 \omega(s, t) \sup_{i=1, \dots, n} \omega(t_i, t_{i+1})^{\theta-1}.$$

This proves that if  $(\pi^n)_{n \in \mathbb{N}}$  is a family of partitions whose meshes converge to 0 as  $n \rightarrow \infty$ , then  $\mathbf{x}_{s, t}^{(\pi^n)}$  converges to  $\mathbf{y}_{s, t}$ . In addition, combined with (66) and (67), this gives (62).

The last assertion of this theorem follows from the fact that  $\mathbf{x}^{(\pi)}$  belongs to  $G(\mathbb{R}^d)$  if  $\mathbf{x}_{s, t}$  belongs to  $G(\mathbb{R}^d)$ , which is a closed subgroup of  $T_1(\mathbb{R}^d)$ .  $\square$

**Theorem 6.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be two almost rough paths satisfying both (61) with the same constants  $K$  and  $\theta$ . Assume moreover that there exists an  $\varepsilon > 0$  such that*

$$\|\mathbf{x} - \mathbf{y}\|_{\star, p, \omega} \leq \varepsilon$$

and for all  $(s, t) \in \Delta_+$ ,

$$\|\mathbf{x}_{s, t} - \mathbf{y}_{s, t}\|_{\star} \leq \varepsilon \omega(s, t)^\theta.$$

Then the two rough paths  $\tilde{\mathbf{x}}$  and  $\tilde{\mathbf{y}}$  associated to  $\mathbf{x}$  and  $\mathbf{y}$  by Theorem 5 satisfy

$$\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_{*,p,\omega} \leq \varepsilon K',$$

where  $K'$  depends only on  $K$ ,  $\theta$ ,  $p$ ,  $\omega(0, T)$ ,  $\|\mathbf{x}\|_{*,p,\omega}$  and  $\|\mathbf{y}\|_{*,p,\omega}$ .

*Proof.* We use the same notations as previously. For a partition  $\pi = \{t_i\}_{i=1,\dots,n}$  of  $[s, t]$  with  $t_1 = s$ ,  $t_{n+1} = t$ , we consider  $\mathbf{x}_{s,t}^{(\pi)}$  and  $\mathbf{y}_{s,t}^{(\pi)}$  as above.

We pick a point  $t_i$  in  $\pi$  such that  $\omega(t_{i-1}, t_{i+1}) \leq 2\omega(s, t)/n$ . Hence, for

$$\xi = \mathbf{x}_{t_{i-1}, t_i} \otimes \mathbf{x}_{t_i, t_{i+1}} - \mathbf{x}_{t_{i-1}, t_{i+1}} - \mathbf{y}_{t_{i-1}, t_i} \otimes \mathbf{y}_{t_i, t_{i+1}} - \mathbf{y}_{t_{i-1}, t_{i+1}},$$

we get that, with (63),

$$\begin{aligned} \|\mathbf{x}_{s,t}^{(\pi)} - \mathbf{x}_{s,t}^{(\hat{\pi})} - (\mathbf{y}_{s,t}^{(\pi)} - \mathbf{y}_{s,t}^{(\hat{\pi})})\|_* \\ \leq \|\xi\|_* \left( 1 + \sum_{j=1,\dots,n} \sum_{j \neq i} |\mathbf{x}_{t_j, t_{j+1}}^1 - \mathbf{y}_{t_j, t_{j+1}}^1| \right) \end{aligned}$$

where  $\mathbf{x}_{s,t}^1$  (resp.  $\mathbf{y}_{s,t}^1$ ) is the projection of  $\mathbf{x}_{s,t}$  (resp.  $\mathbf{y}_{s,t}$ ) on  $\mathbb{R}^d$ .

With (65), we get that for some constant  $C$  that depends only on  $K$ ,  $\theta$ ,  $p$ ,  $\omega(0, T)$ ,  $\|\mathbf{x}\|_{p,\omega}$  and  $\|\mathbf{y}\|_{p,\omega}$ ,

$$\begin{aligned} \sum_{j=1,\dots,n} \sum_{j \neq i} |\mathbf{x}_{t_j, t_{j+1}}^1 - \mathbf{y}_{t_j, t_{j+1}}^1| &\leq |\mathbf{x}_{s,t}^{1,(\pi)}| + |\mathbf{y}_{s,t}^{1,(\pi)}| \\ &\leq (C\omega(s, t))^{\theta-1/p} + \|\mathbf{y}\|_{*,p,\omega} + \|\mathbf{x}\|_{*,p,\omega} \omega(s, t)^{1/p}. \end{aligned}$$

Thus, one get that for some constant  $K$ ,

$$\|\mathbf{x}_{s,t}^{(\pi)} - \mathbf{x}_{s,t}^{(\hat{\pi})} - (\mathbf{y}_{s,t}^{(\pi)} - \mathbf{y}_{s,t}^{(\hat{\pi})})\|_* \leq \varepsilon \frac{K}{n^\theta} \omega(s, t)^\theta.$$

It follows that by removing all successively all the points of  $\pi$  carefully,

$$\|\mathbf{x}_{s,t}^{(\pi)} - \mathbf{x}_{s,t} - (\mathbf{y}_{s,t}^{(\pi)} - \mathbf{y}_{s,t})\|_* \leq \varepsilon \zeta(\theta) K \omega(s, t)^\theta.$$

As we have seen that  $\mathbf{x}^{(\pi)}$  and  $\mathbf{y}^{(\pi)}$  converges to  $\tilde{\mathbf{x}}_{s,t}$  and  $\tilde{\mathbf{y}}_{s,t}$ , we deduce that

$$\|\tilde{\mathbf{x}}_{s,t} - \mathbf{x}_{s,t} - (\tilde{\mathbf{y}}_{s,t} - \mathbf{y}_{s,t})\|_* \leq \varepsilon \zeta(\theta) K \omega(s, t)^\theta.$$

The result is then easily deduced.  $\square$

**B.2. An algebraic interpretation.** We give now an algebraic interpretation of this construction, which is strongly inspired by the one given by M. Gubinelli in [Gub04].

Let us consider the sets

$$\begin{aligned} \Delta_1 &= [0, T], \quad \Delta_2 = \{(s, t) \mid 0 \leq s \leq t \leq T\} \\ \text{and } \Delta_3 &= \{(s, r, t) \mid 0 \leq s \leq r \leq t \leq T\}, \end{aligned}$$

as well as  $\mathcal{C}_i$  be the set of functions from  $\Delta_i$  to  $T_1(\mathbb{R}^d)$  for  $i = 1, 2, 3$ .

Let us introduce the operator from  $\mathcal{C}_1 \cup \mathcal{C}_2$  to  $\mathcal{C}_2 \cup \mathcal{C}_3$  defined by

$$\begin{aligned}\delta(\mathbf{x})_{s,t} &= \mathbf{x}_s^{-1} \otimes \mathbf{x}_t, \quad (s, t) \in \Delta_2, \quad \mathbf{x} \in \mathcal{C}_1, \\ \delta(\mathbf{x})_{s,r,t} &= \mathbf{x}_{s,t} - \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}, \quad (s, r, t) \in \Delta_3, \quad \mathbf{x} \in \mathcal{C}_2,\end{aligned}$$

so that  $\delta$  maps  $\mathcal{C}_i$  to  $\mathcal{C}_{i+1}$ ,  $i = 1, 2$ . Let us note that if  $\mathbf{x} \in \mathcal{C}_1$ , then  $\delta(\delta(\mathbf{x})) = 0$ , so that the range  $\text{Range}(\delta|_{\mathcal{C}_1})$  of  $\delta|_{\mathcal{C}_1}$  is contained in the Kernel  $\text{Ker}(\delta|_{\mathcal{C}_2})$  of  $\delta|_{\mathcal{C}_2}$ . Indeed, we get a better result.

**Lemma 21.** *The range of  $\delta|_{\mathcal{C}_1}$  is equal to the kernel of  $\delta|_{\mathcal{C}_2}$ , and  $\delta$  is injective from  $\mathcal{C}_1(x)$  into  $\mathcal{C}_2$  where  $\mathcal{C}_1(x)$  is the set of paths  $\mathbf{x}$  in  $\mathcal{C}_1$  with  $\mathbf{x}_0 = x$  for  $x \in \mathbb{T}_1(\mathbb{R}^2)$ . In particular, when restricted to  $\text{Range}(\delta|_{\mathcal{C}_1(x)})$ ,  $\delta$  is invertible.*

*Proof.* We have already seen the inclusion of  $\text{Range}(\delta|_{\mathcal{C}_1})$  in  $\text{Ker}(\delta|_{\mathcal{C}_2})$ .

Now, let  $\mathbf{x} \in \mathcal{C}_2$  be in  $\text{Ker}(\delta|_{\mathcal{C}_2})$ . Then set  $\mathbf{y}_t = \mathbf{x}_{0,t}$ . As  $\delta(\mathbf{x})_{0,s,t} = \mathbf{y}_t - \mathbf{y}_s \otimes \mathbf{x}_{s,t} = 0$ , we get that  $\mathbf{x}_{s,t} = \mathbf{y}_s^{-1} \otimes \mathbf{y}_t$  and thus  $\mathbf{x} = \delta(\mathbf{y})$ . This proves the result.

If two paths  $\mathbf{x}$  and  $\mathbf{y}$  are distinct in  $\mathcal{C}_1(x)$ , then  $\delta(\mathbf{x})_{0,t} = x^{-1} \otimes \mathbf{x}_t$  is different from  $\delta(\mathbf{y})_{0,t} = x^{-1} \otimes \mathbf{y}_t$  and  $\delta$  is injective from  $\mathcal{C}_1(x)$  into  $\mathcal{C}_2$ .  $\square$

Given a rough path  $\mathbf{x}$ , which then belongs to  $\mathcal{C}_1$  and a differential form  $f$ , the integral  $\mathfrak{I}(\mathbf{x}) = \int f(x) d\mathbf{x}$  is also a path in  $\mathcal{C}_1(0)$ . The idea is then to consider an approximation of  $\mathfrak{I}(\mathbf{x}; s, t)$  for  $t - s$  small, and to project it on the range of  $\delta|_{\mathcal{C}_1(x)}$ . Of course, the approximation of  $\mathfrak{I}(\mathbf{x}; s, t)$  shall be close enough to the range of  $\delta|_{\mathcal{C}_1(x)}$ .

For  $p \geq 1$  and  $\theta > 1$ , we define the distance  $d_{\theta, \omega}$  on  $\mathcal{C}_2$  by

$$D_{\star, \theta, \omega}(\mathbf{x}, \mathbf{y}) = \sup_{(s,t) \in \Delta^+} \frac{\|\mathbf{x}_{s,t} - \mathbf{y}_{s,t}\|_{\star}}{\omega(s, t)^{\theta}}.$$

To simplify the notation, we extend  $\delta|_{\mathcal{C}_2}$  as a function defined on  $\Delta^2 \times [0, T]$  by setting  $\delta(\mathbf{x})_{s,r,t} = 1$  if  $r \notin [s, t]$ . For a fixed  $r \in [0, T]$ ,  $\delta_{\cdot, r, \cdot}(\mathbf{x})$  is then a function in  $\mathcal{C}_2$ .

Theorem 5 is the rewritten the following way.

**Theorem 7.** *For  $K, K' > 0$  and  $\theta > 1$ , we denote by  $B(K, K', \theta, p, \omega)$  the subsets of functions  $\mathbf{x} \in \mathcal{C}_2$  for which*

$$\|\mathbf{x}\|_{\star, p, \omega} \leq K \text{ and } \sup_{r \in [0, T]} D_{\star, p, \omega}(\delta(\mathbf{x})_{\cdot, r, \cdot}, 0) \leq K'.$$

*Then to any  $\mathbf{x}$  in  $B(K, K', \theta, p, \omega)$  is associated a unique element  $\widehat{\mathbf{x}} \in \text{Ker}(\delta|_{\mathcal{C}_2})$ . In addition, for some constants  $C_1$  and  $C_2$  that depends only on  $K, K', \theta, p$  and  $\omega(0, T)$ ,*

$$\|\widehat{\mathbf{x}}\|_{\star, p, \omega} \leq C_1 \text{ and } D_{\star, p, \omega}(\mathbf{x}, \widehat{\mathbf{x}}) \leq C_2.$$

*In addition, if the distance  $\Theta_{\star, p, \theta, \omega}$  is defined on  $\cup_{K, K' > 0} B(K, K', \theta, p, \omega)$  by*

$$\Theta_{\star, p, \theta, \omega}(\mathbf{x}, \mathbf{y}) = \max\{\|\mathbf{x} - \mathbf{y}\|_{\star, p, \omega}, d_{\star, \theta, \omega}(\mathbf{x}, \mathbf{y})\},$$

*then this map  $\Pi : \mathbf{x} \mapsto \widehat{\mathbf{x}}$  is locally Lipschitz with respect to  $\Theta_{\star, p, \theta, \omega}$ .*

From the definition, an almost rough path  $\mathbf{x}$  of  $p$ -variation controlled by  $\omega$  belongs to  $\cup_{K, K' > 0} B(K, K', \theta, p, \omega)$ . It is then “projected” on an element  $\Pi(\mathbf{x})$  in  $\mathcal{C}_2$  in the kernel of  $\delta_{\mathcal{C}_2}$ , which is also equal to the image of  $\delta_{\mathcal{C}_1(1)}$ . The reciprocal image of  $\Pi(\mathbf{x})$  gives then a rough path in  $C^{p, \omega}([0, T]; T_1(\mathbb{R}^d))$ .

Given an element  $f$  in  $\text{Lip}(\gamma; \mathbb{R}^d \rightarrow \mathbb{R}^m)$  with  $\gamma > p - 1$ , the map  $\mathfrak{F}(f, \mathbf{x})$  defined by (54) define an element of  $\mathcal{C}_2$ . The integral  $\mathfrak{I}$  may then be defined as the composition of the maps

$$\mathfrak{I} = \delta_{\mathcal{C}_1(1)}^{-1} \circ \Pi \circ \mathfrak{F}(f, \cdot),$$

which corresponds to the construction given in Section 7.4.

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