The Calculus of Algebraic Constructions

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Abstract : This paper is concerned with the foundations of the Calculus of Algebraic Constructions (CAC), an extension of the Calculus of Constructions by inductive data types. CAC generalizes inductive types equipped with higher-order primitive recursion, by providing definitions of functions by pattern-matching which capture recursor definitions for arbitrary non-dependent and non-polymorphic inductive types satisfying a strictly positivity condition. CAC also generalizes the first-order framework of abstract data types by providing dependent types and higher-order rewrite rules.

1 Introduction

Proof assistants allow one to build complex proofs by using macros, called tactics, which generate proof terms representing the sequence of deduction rules used in the proof. These proof terms are then "type-checked" in order to ensure the correct use of each deduction step. As a consequence, the correctness of the proof assistant, hence of the verification itself, relies solely on the correctness of the type-checker, but not on the tactics themselves. This approach has a major problem: proof objects may become very large. For example, proving that 0+100 equals its normal form 100 in some encoding of Peano arithmetic will generate a proof of a hundred steps, assuming + is defined by induction on its second argument. Such proofs occur in terms, as well as in subterms of a dependent type. Our long term goal is to cure this situation by restoring the balance between computations and deductions, as argued in [14]. The work presented in this paper intends to be a first important step towards this goal. To this end, we will avoid encodings by incorporating to the Calculus of Constructions (CC) [9] user-defined function symbols defined by sets of first and higher-order rewrite rules. These rules will be used in conjunction with the usual proof reduction rule that reduces subterms in dependent types:

*This work was partly supported by the Grants-in-aid for Scientific Research of Ministry of Education, Science and Culture of Japan, and the Oogata-kenkyuu-jyosei grant of Keio University.

$$\frac{\Gamma \ \vdash \ M : T \quad T \longleftrightarrow^*_{R \cup \beta} T'}{\Gamma \ \vdash \ M : T'}$$

Since the pioneer work by Breazu-Tannen in 1988 [5] on the confluence of the combination of the simply-typed λ-calculus with first-order algebraic rewriting, soon followed, as for the strong normalization, by Breazu-Tannen and Gallier [6] and, independently, by Okada [21], this question has been very active. We started our program at the beginning of the decade, by developing the notion of abstract data type system [18], in which the user defined computations could be described by using rewrite rules belonging to the so-called *General Schema*, a generalization of higher-order primitive recursion. This work was done in the context of a bounded polymorphic type discipline, and was later extended to CC [1].

In [4], we introduced, in the context of the simply-typed λ-calculus, a new and more flexible definition of the General Schema to capture the rewrite rules defining recursors for strictly positive inductive types [10], problem left open in [18]. In this paper, we similarly equip CC with non-dependent and non-polymorphic inductive types, and first and higher-order rewriting. Our main result is that this extension is compatible with CC.

In [10], strictly positive inductive types can be dependent and polymorphic. Hence, further work will be needed to reach the expressive power of the Calculus of Inductive Constructions [22], implemented in the Coq proof assistant [3], all the more so since it handles strong elimination, that is the possibility to define types by induction. But our new General Schema seems powerful and flexible enough to be further extended to such a calculus, hence resulting in to a simpler strong normalization proof.

As a consequence of our result, it will become possible to develop a new version of the Coq proof assistant, in which the user may define functions by pattern-matching and then develop libraries of decision procedures using this kind of functional style. Ensuring the consistency of the underlying proof theory requires a proof that the user-defined rules obey the General Schema, a task that can be easily automated. Note also that, since most of the time, when one develops proofs, the efficiency of rewriting does not really matter, the type-checker of the proof development system can be kept small and not too difficult to certify, hence conforming to the idea of relying on a small easy-to-check kernel.

2 Definition of the calculus

2.1 Syntax

Definition 1 (Algebraic types) *Given a set $\mathcal{S}$ of* sorts*, we define the sets $\mathcal{T}_\mathcal{S}$ of* algebraic types*:*

$$s := \mathtt{s} \mid (s \to s)$$

where $\mathtt{s}$ *ranges over $\mathcal{S}$ and $\to$ associates to the right such that $s_1 \to (s_2 \to s_3)$ can be written $s_1 \to s_2 \to s_3$. An algebraic type $s_1 \to \ldots \to s_n$ is* first-order *if each s_i is a sort, otherwise it is* higher-order*.*

Definition 2 (Constructors) *We assume that each sort* $\mathtt{s}$ *has an associated set* $\mathcal{C}(\mathtt{s})$ *of* constructors. *Each constructor* C *is equipped with an algebraic type* $\tau(C)$ *of the form* $s_1 \to \ldots \to s_n \to \mathtt{s}$*;* n *is called the* arity *of* C*, and* $\mathtt{s}$ *its* output type. *We denote by* $\mathcal{C}^n$ *the set of constructors of arity* n.

A constructor C *is* first-order *if its type is first-order, otherwise it is* higher-order. *Constructor declarations define a quasi-ordering on sorts:* $\mathtt{s} \geq_{\mathcal{S}} \mathtt{t}$ *if and only if* $\mathtt{t}$ *occurs in the type of a constructor belonging to* $\mathcal{C}(\mathtt{s})$. *In the following, we will assume that* $>_{\mathcal{S}}$ *is well-founded, ruling out mutually inductive sorts.*

Definition 3 (Algebraic signature) *Given a non empty sequence* $s_1, \ldots, s_n, s$ *of algebraic types, we denote by* $\mathcal{F}_{s_1,\ldots,s_n,s}$ *the set of* function symbols *of* arity n, *of* type $\tau(f) = s_1 \to \ldots \to s_n \to s$ *and of* output type s. *We will denote by* $\mathcal{F}^n$ *the set of function symbols of arity* n, *and by* $\mathcal{F}$ *the set of all function symbols. Function symbols with a first-order (resp. higher-order) type are called* first-order *(resp.* higher-order*).*

Here are familiar examples of sorts and functions:

(i) the sort $\mathtt{bool}$ of booleans whose constructors are $\mathtt{true} : \mathtt{bool}$ and $\mathtt{false} : \mathtt{bool}$; $\mathtt{if}_t$ of arity 3 is a defined function of type $\mathtt{bool} \to t \to t \to t$, for any algebraic type t;

(ii) the sort $\mathtt{nat}$ of natural numbers whose constructors are $\mathtt{0} : \mathtt{nat}$ and $\mathtt{s} : \mathtt{nat} \to \mathtt{nat}$; $+$ of arity 2 is a defined function of type $\mathtt{nat} \to \mathtt{nat} \to \mathtt{nat}$;

(iii) the sort $\mathtt{list}_t$ of lists of elements of an algebraic type t whose constructors are $\mathtt{nil}_t : \mathtt{list}_t$ and $\mathtt{cons}_t : t \to \mathtt{list}_t \to \mathtt{list}_t$; $\mathtt{append}_t$ of arity 2 is a defined function of type $\mathtt{list}_t \to \mathtt{list}_t \to \mathtt{list}_t$, while $\mathtt{map}_{t,t'}$ of arity 2 is a defined function of type $(t \to t') \to \mathtt{list}_t \to \mathtt{list}_{t'}$;

(iv) the sort $\mathtt{ord}$ of ordinals whose constructors are $\mathtt{0}_{\mathtt{ord}} : \mathtt{ord}$, $\mathtt{s}_{\mathtt{ord}} : \mathtt{ord} \to \mathtt{ord}$ and $\mathtt{lim}_{\mathtt{ord}} : (\mathtt{nat} \to \mathtt{ord}) \to \mathtt{ord}$.

Definition 4 (Terms) *The set* $Term$ *of CAC* terms *is inductively defined as:*

$$a := x \mid \mathtt{s} \mid \star \mid \Box \mid \lambda x{:}a.a \mid \Pi x{:}a.a \mid (a\ a) \mid C(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n)$$

where $\mathtt{s}$ *ranges over* $\mathcal{S}$*,* C *over* $\mathcal{C}^n$*,* f *over* $\mathcal{F}^n$ *and* x *over* Var*, a set of variables made of two disjoint infinite sets* $Var^{\Box}$ *and* $Var^{\star}$*. The application* $(a\ b)$ *associates to the left such that* $(a_1\ a_2)\ a_3$ *can be written* $a_1\ a_2\ a_3$*. The sequence of terms* $a_1 \ldots a_n$ *is denoted by the vector* $\vec{a}$ *of length* $|\vec{a}| = n$*. A term* $C(\vec{a})$ *(resp.* $f(\vec{a})$*) is said to be* constructor headed *(resp.* function headed*).*

After Dewey, the set $Pos(a)$ of *positions* in a term a is a language over the alphabet $\mathbb{N}^+$ of strictly positive natural numbers. Note that abstraction and product have two arguments, the type and the body. The *subterm* of a term a at position $p \in Pos(a)$ is denoted by $a|_p$ and the term obtained by replacing $a|_p$ by a term b is written $a[b]_p$. We write $a \trianglerighteq b$ if b is a subterm of a.

We note by $FV(a)$ and $BV(a)$ the sets of respectively free and bound variables occurring in a term a, and by $Var(a)$ their union. By convention, *bound and free variables will always be assumed different.* As in the untyped λ-calculus,

Figure 1: Typing rules of CAC

$$\text{(ax)} \qquad \vdash \star : \Box$$

$$\text{(sort)} \qquad \vdash \mathtt{s} : \star \qquad (\mathtt{s} \in \mathcal{S})$$

$$\text{(var)} \qquad \frac{\Gamma \vdash c : p}{\Gamma, x : c \vdash x : c} \qquad (x \in Var^p \setminus dom(\Gamma),\ p \in \{\star, \Box\})$$

$$\text{(weak)} \qquad \frac{\Gamma \vdash a : b \quad \Gamma \vdash c : p}{\Gamma, x : c \vdash a : b} \qquad (x \in Var^p \setminus dom(\Gamma),\ p \in \{\star, \Box\})$$

$$\text{(cons)} \qquad \frac{\Gamma \vdash a_1 : s_1 \quad \ldots \quad \Gamma \vdash a_n : s_n}{\Gamma \vdash C(a_1, \ldots, a_n) : \mathtt{s}} \qquad (C \in \mathcal{C}^n,\ \tau(C) = s_1 \to \ldots \to s_n \to \mathtt{s})$$

$$\text{(fun)} \qquad \frac{\Gamma \vdash a_1 : s_1 \quad \ldots \quad \Gamma \vdash a_n : s_n}{\Gamma \vdash f(a_1, \ldots, a_n) : s} \qquad (f \in \mathcal{F}_{s_1,\ldots,s_n,s},\ n \geq 0)$$

$$\text{(abs)} \qquad \frac{\Gamma, x : a \vdash b : c \quad \Gamma \vdash (\Pi x{:}a.c) : q}{\Gamma \vdash (\lambda x{:}a.b) : (\Pi x{:}a.c)} \qquad (x \notin dom(\Gamma),\ q \in \{\star, \Box\})$$

$$\text{(app)} \qquad \frac{\Gamma \vdash a : (\Pi x{:}b.c) \quad \Gamma \vdash d : b}{\Gamma \vdash (a\ d) : c\{x \mapsto d\}}$$

$$\text{(conv)} \qquad \frac{\Gamma \vdash a : b \quad \Gamma \vdash b' : p}{\Gamma \vdash a : b'} \qquad (p \in \{\star, \Box\},\ b \longrightarrow_\beta^* b' \text{ or } b' \longrightarrow_\beta^* b \text{ or } b \longrightarrow_R^* b' \text{ or } b' \longrightarrow_R^* b)$$

$$\text{(prod)} \qquad \frac{\Gamma \vdash a : p \quad \Gamma, x : a \vdash b : q}{\Gamma \vdash (\Pi x{:}a.b) : q} \qquad (x \notin dom(\Gamma),\ p, q \in \{\star, \Box\})$$

terms that only differ from each other in their bound variables will be identified, an operation called α-*conversion*. A *substitution* θ of domain $dom(\theta) = \{\vec{x}\}$ is written $\{\vec{x} \mapsto \vec{b}\}$. Substitutions are written in postfix notation, as in $a\theta$.

Finally, we traditionally consider that $(b\ \vec{a})$, $\lambda\vec{x}{:}\vec{a}.b$ and $\Pi\vec{x}{:}\vec{a}.b$, denote all three the term b if $\vec{a}$ is the empty sequence, and the respective terms $(\ldots((b\ a_1)\ a_2)\ldots a_n)$, $\lambda x_1 : a_1.(\lambda x_2 : a_2.(\ldots(\lambda x_n : a_n.b)\ldots))$ and $\Pi x_1 : a_1.(\Pi x_2 : a_2.(\ldots(\Pi x_n : a_n.b)\ldots))$ otherwise. We also write $a \to b$ for the term $\Pi x{:}a.b$ when $x \notin FV(b)$. This abbreviation allows us to see algebraic types as terms of our calculus.

2.2 Typing rules

Definition 5 (Typing rules) *A* declaration *is a pair $x{:}a$ made of a variable x and a term a. An* environment Γ *is a (possibly empty) ordered sequence of declarations of the form $x_1{:}a_1, \ldots, x_n{:}a_n$, where all x_i are distinct; $dom(\Gamma) = \{x_1, \ldots, x_n\}$ is its domain, $FV(\Gamma) = \bigcup_{x:a \in \Gamma} FV(a)$ is its set of free variables, and $\Gamma(x_i) = a_i$. A* typing judgement *is a triple $\Gamma \vdash a : b$ made of an environment Γ and two terms a, b. A term a has* type b in an environment Γ *if the judgement $\Gamma \vdash a : b$ can be deduced by the rules of Figure 1. An environment is* valid *if* $\star$

can be typed in it. An environment is algebraic *if every declaration has the form x:c, where c is an algebraic type.*

The rules (sort), (cons) and (fun) are added to the rules of CC [9]. The (conv) rule expresses that types depend on reductions via terms. In CC, the relation used in the side condition is the monotonic, symmetric, reflexive, transitive closure of the β-rewrite relation $(\lambda x{:}a.b)\ c \longrightarrow_\beta b\{x \mapsto c\}$.

In our calculus, there are two kinds of computation rules: β- (or proof-) reduction and the user-defined rewrite rules, denoted by $\longrightarrow_R$. This contrasts with the other calculi of constructions, in which the meaning of (conv) is fixed by the designer of the language, while it depends on the user in our system. The unusual form of the side condition of our conversion rule is due to the fact that no proof of subject reduction is known for a conversion rule with the more natural side condition $b \longleftrightarrow^*_{\beta R} b'$. See [1] for details.

The structural properties of CC are also true in CAC. See [1] and [2] for details. We just recall the different term classes that compose the calculus.

Definition 6 *Let Kind be the set* $\{K \in Term \mid \exists\Gamma, \Gamma \vdash K{:}\Box\}$ *of* kinds*, Constr be the set* $\{T \in Term \mid \exists\Gamma, \exists K \in Kind, \Gamma \vdash T{:}K\}$ *of* type constructors*, Type be the set* $\{\tau \in Term \mid \exists\Gamma, \Gamma \vdash \tau{:}\star\}$ *of* types*, Obj be the set* $\{u \in Term \mid \exists\Gamma, \exists\tau \in Type, \Gamma \vdash u{:}\tau\}$ *of* objects*, and Thm be the set Constr* $\cup$ *Kind of* theorems*.*

Lemma 7 *Kinds, type constructors and objects can be characterized as follows:*

- $K := \star \mid \Pi x{:}\tau.K \mid \Pi\alpha{:}K.K$
- $T := \mathtt{s} \mid \alpha \mid \Pi x{:}\tau.\tau \mid \Pi\alpha{:}K.\tau \mid \lambda x{:}\tau.T \mid \lambda\alpha{:}K.T \mid (T\ u) \mid (T\ T)$
- $u := x \mid C(u_1, \ldots, u_n) \mid f(u_1, \ldots, u_n) \mid \lambda x{:}\tau.u \mid \lambda\alpha{:}K.u \mid (u\ u) \mid (u\ T)$

where $\alpha \in Var^\Box$ *and* $x \in Var^\star$*.*

2.3 Inductive types

Inductive types have been introduced in CC for at least two reasons: firstly, to ease the user's description of his/her specification by avoiding the complicated impredicative encodings which were necessary before; secondly, to transform inductive proofs into inductive procedures via the Curry-Howard isomorphism. The logical consistency of the calculus follows from the existence of a least fixpoint, a property which is ensured syntactically in the Calculus of Inductive Constructions by restricting oneself to strictly positive types [10].

Definition 8 (Positive and negative type positions) *Given an algebraic type s, its sets of positive and negative positions are inductively defined as follows:*

$$Pos^+(s \in \mathcal{S}) = \epsilon \qquad Pos^-(s \in \mathcal{S}) = \emptyset$$

$$Pos^+(s \to t) = 1{\cdot}Pos^-(s) \cup 2{\cdot}Pos^+(t) \quad Pos^-(s \to t) = 1{\cdot}Pos^+(s) \cup 2{\cdot}Pos^-(t)$$

Given an algebraic type t, we say that s does occur positively *in t if s occurs in t, and each occurrence of s in t is at a positive position.*

Definition 9 (Inductive sorts) *Let* $\mathtt{s}$ *be a sort whose constructors are* $C_1, \ldots, C_n$ *and suppose that* C_i *has type* $s^i_1 \to \ldots \to s^i_{n_i} \to \mathtt{s}$*. Then we say that:*

(i) $\mathtt{s}$ *is a* basic inductive sort *if each* s^i_j *is* $\mathtt{s}$ *or a basic inductive sort smaller than* $\mathtt{s}$ *in* $<_{\mathcal{S}}$*,*

(ii) $\mathtt{s}$ *is a* strictly positive inductive sort *if each* s^i_j *is either a strictly positive inductive sort smaller than* $\mathtt{s}$ *in* $<_{\mathcal{S}}$*, or of the form* $s'_1 \to \ldots \to s'_p \to \mathtt{s}$ *where each* s'_k *is built from strictly positive inductive sorts smaller than* $\mathtt{s}$ *in* $<_{\mathcal{S}}$*.*

In the following, we will assume that every inductive sort of a user specification is strictly positive.

The sort $\mathtt{nat}$ whose constructors are $\mathtt{0} : \mathtt{nat}$ and $\mathtt{s} : \mathtt{nat} \to \mathtt{nat}$ is a basic sort. The sort $\mathtt{ord}$ whose constructors are $\mathtt{0}_{\mathtt{ord}} : \mathtt{ord}$, $\mathtt{s}_{\mathtt{ord}} : \mathtt{ord} \to \mathtt{ord}$ and $\mathtt{lim}_{\mathtt{ord}} : (\mathtt{nat} \to \mathtt{ord}) \to \mathtt{ord}$ is a strictly positive sort, since $\mathtt{ord} >_{\mathcal{S}} \mathtt{nat}$.

Definition 10 (Strictly positive recursors) *Let* $\mathtt{s}$ *be a strictly positive inductive sort generated by the constructors* $C_1, \ldots, C_n$ *of respective types* $s^i_1 \to \ldots \to s^i_{n_i} \to \mathtt{s}$*. The associated recursor* $rec^{\mathtt{s}}_t$ *of algebraic output type* t *is a function symbol of arity* $n+1$*, and type* $\mathtt{s} \to t_1 \to \ldots \to t_n \to t$ *where* $t_i = s^i_1 \to \ldots \to s^i_{n_i} \to s^i_1\{\mathtt{s} \mapsto t\} \to \ldots \to s^i_{n_i}\{\mathtt{s} \mapsto t\} \to t$*. It is defined by the rewrite rules:*

$$rec^{\mathtt{s}}_t(C_i(\vec{a}), \vec{b}) \longrightarrow b_i\, \vec{a}\, \vec{d} \quad \textit{where}$$

$d_j = a_j$ *if* $\mathtt{s}$ *is not in* s^j_i*, otherwise* $s^i_j = s'_1 \to \ldots \to s'_p \to \mathtt{s}$ *and* $d_j = \lambda \vec{x}{:}\vec{s'}\{\mathtt{s} \mapsto t\}.rec^{\mathtt{s}}_t(a_j\ \vec{x}, \vec{b})$.

Via the Curry-Howard isomorphism, a recursor of a sort $\mathtt{s}$ corresponds to the structural induction principle associated to the set of elements built from the constructors of $\mathtt{s}$. Strictly positive types are found in many proof assistants based on the Curry-Howard isomorphism, e.g. in Coq [3]. Here are a few recursors:

$$\begin{array}{rcl@{\qquad}rcl}
\mathtt{rec}^t_{\mathtt{bool}}(\mathtt{true}, u, v) & \longrightarrow & u & \mathtt{rec}^t_{\mathtt{nat}}(\mathtt{0}, u, v) & \longrightarrow & u \\
\mathtt{rec}^t_{\mathtt{bool}}(\mathtt{false}, u, v) & \longrightarrow & v & \mathtt{rec}^t_{\mathtt{nat}}(\mathtt{s}(n), u, v) & \longrightarrow & v\ n\ \mathtt{rec}^t_{\mathtt{nat}}(n, u, v)
\end{array}$$

$$\begin{array}{rcl}
\mathtt{rec}^t_{\mathtt{ord}}(\mathtt{0}_{\mathtt{ord}}, u, v, w) & \longrightarrow & u \\
\mathtt{rec}^t_{\mathtt{ord}}(\mathtt{s}_{\mathtt{ord}}(n), u, v, w) & \longrightarrow & v\ n\ \mathtt{rec}^t_{\mathtt{ord}}(n, u, v, w) \\
\mathtt{rec}^t_{\mathtt{ord}}(\mathtt{lim}_{\mathtt{ord}}(f), u, v, w) & \longrightarrow & w\ f\ \lambda n{:}\,\mathtt{nat}.\mathtt{rec}^t_{\mathtt{ord}}(f\, n, u, v, w)
\end{array}$$

$\mathtt{rec}^t_{\mathtt{bool}}$ is $\mathtt{if}_t$, and $\mathtt{rec}^t_{\mathtt{nat}}$ is Gödel's higher-order primitive recursion operator.

2.4 User-defined rules

First, we define the syntax of terms that may be used for rewrite rules:

Definition 11 (Rule terms) *Terms built up solely from constructors, function symbols and variables of* $Var^\star$*, are called* algebraic*. Their set is defined by the following grammar:*

$$a := x^\star \mid C(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n)$$

where $x^\star$ *ranges over* $Var^\star$*,* C *over* $\mathcal{C}^n$ *and* f *over* $\mathcal{F}^n$*. An algebraic term is* first-order *if its function symbols and constructors are first-order, and* higher-order *otherwise. The set of rule terms is defined by the following grammar:*

$$a := x^\star \mid \lambda x{:}\tau.a \mid (a\ a) \mid C(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n)$$

where $x^\star$ ranges over $Var^\star$, s over $\mathcal{T}_\mathcal{S}$, C over $\mathcal{C}^n$ and f over $\mathcal{F}^n$. A rule term is first-order *if it is a* first-order algebraic *term, otherwise it is* higher-order.

Definition 12 (Rewrite rules) *A* rewrite rule *is a pair $l \longrightarrow r$ of rule terms such that l is headed by a function symbol f which is said to be* defined*, and $FV(r) \subseteq FV(l)$. Given a set R of rewrite rules, a term a rewrites to a term b at position $m \in Pos(a)$ with the rule $l \longrightarrow r \in R$, written $a \longrightarrow_R^m b$ if $a|_m = l\theta$ and $b = a[r\theta]_m$ for some substitution θ.*

A rewrite rule is first-order *if l and r are both first-order, otherwise it is* higher-order*. A first-order rewrite rule $l \longrightarrow r$ is* conservative *if no (free) variable has more occurrences in r than in l. The rules induce the following quasi-ordering on function symbols: $f \geq_\mathcal{F} g$ iff g occurs in a defining rule of f.*

We assume that first-order function symbols are defined only by first-order rewrite rules. Of course, it is always possible to treat a first-order function symbol as an higher-order one. Here are examples of rules:

$$\begin{array}{rl@{\qquad}rl}
\mathtt{if}_t(\mathtt{true}, u, v) & \longrightarrow u & \mathtt{map}_{t,t'}(f, \mathtt{nil}_t) & \longrightarrow \mathtt{nil}_{t'} \\
\mathtt{if}_t(\mathtt{false}, u, v) & \longrightarrow v & \mathtt{map}_{t,t'}(f, \mathtt{cons}_t(x, l)) & \longrightarrow \mathtt{cons}_{t'}(f\ x, \mathtt{map}_{t,t'}(f, l)) \\
+(x, \mathtt{0}) & \longrightarrow x & \mathtt{ack}(\mathtt{0}, y) & \longrightarrow \mathtt{s}(y) \\
+(x, \mathtt{s}(y)) & \longrightarrow \mathtt{s}(+(x, y)) & \mathtt{ack}(\mathtt{s}(x), \mathtt{0}) & \longrightarrow \mathtt{ack}(x, \mathtt{s}(\mathtt{0})) \\
+(+(x, y), z) & \longrightarrow +(x, +(y, z)) & \mathtt{ack}(\mathtt{s}(x), \mathtt{s}(y)) & \longrightarrow \mathtt{ack}(x, \mathtt{ack}(\mathtt{s}(x), y))
\end{array}$$

Having rewrite rules in our calculus brings many benefits, in addition to obtaining proofs in which computational steps are transparent. In particular, it enhances the declarativeness of the language, as examplified by the Ackermann's function, for which the definition in Coq [3] must use two mutually recursive functions. For subject reduction, the following properties will be needed:

Definition 13 (Admissible rewrite rules) *A rewrite rule $l \longrightarrow r$, where l is headed by a function symbol whose output type is s, is* admissible *if and only if it satisfies the following conditions:*

- *there exists an algebraic environment Γ_l in which l is well-typed,*
- *for any environment Γ, $\Gamma \vdash l : s \Rightarrow \Gamma \vdash r : s$.*

We assume that rules use distinct variables and note by Γ_R the union of the Γ_l's.

2.5 Definition of the General Schema

Let us consider the example of a strictly positive recursor rule, for the sort `ord`:

$$\mathtt{rec}^t_{\mathtt{ord}}(\mathtt{lim}_{\mathtt{ord}}(f), u, v, w) \longrightarrow w\ f\ \lambda n{:}\mathtt{nat}.\mathtt{rec}^t_{\mathtt{ord}}(f\ n, u, v, w)$$

To prove the decreasingness of the recursive call arguments, one would like to compare $\mathtt{lim}_{\mathtt{ord}}(f)$ with f, and not $\mathtt{lim}_{\mathtt{ord}}(f)$ with $(f\ n)$. To this end, we introduce the notion of the *critical subterm* of an application, and then interpret a function call by the critical subterms of its arguments. Here, f will be the critical subterm of $(f\ n)$, hence resulting in the desired comparison.

Definition 14 (Γ,s-critical subterm) *Given an algebraic type s and an environment Γ, a term a is a* Γ,s-term *if it is typable in Γ by an algebraic type in which s occurs positively. A term b is a* Γ,s-subterm *of a term a, a $\trianglerighteq_{\Gamma,s}$ b, if b is a subterm of a, of which each superterm is a Γ,s-term. Writing a Γ,s-term a in its* application form $a_1 \dots a_n$, *where a_1 is not an application, its* Γ,s-critical subterm $\chi^s_\Gamma(a)$ *is the smallest Γ,s-subterm $a_1 \dots a_k$ (see Figure 2).*

For a higher-order function symbol, the arguments that have to be compared via their critical subterm, are said to be at *inductive positions*. They correspond to the arguments on which the function is inductively defined. Next, we define a notion of status that allows users to precise how to compare the arguments of recursive calls. Roughly speaking, it is a simple combination of multiset and lexicographic comparisons.

Definition 15 (Status orderings) *A* status *of arity n is a term of the form $lex(t_1, \dots, t_p)$ where t_i is either x_j for some $j \in [1..n]$, or a term of the form $mul(x_{k_1}, \dots, x_{k_q})$ such that each variable x_i, $1 \leq i \leq n$, occurs at most once. A position i is* lexicographic *if there exists j such that $t_j = x_i$. A* status term *is a status whose variables are substituted by arbitrary terms of CAC.*

Let $stat$ be a status of arity n, I be a subset of the lexicographic positions of $stat$, called inductive positions, *$S = \{>^i\}_{i\in I}$ a set of orders on terms indexed by I, and $>$ an order on terms. We define the corresponding* status ordering, *$>^S_{stat}$ on sequences of terms as follows:*

- $(a_1, \dots, a_n) >^S_{stat} (b_1, \dots, b_n)$ *iff* $stat\{\vec{x}\mapsto\vec{a}\} >^S_{stat} stat\{\vec{x}\mapsto\vec{b}\}$,
- $lex(c_1,\dots,c_p) >^S_{lex(t_1,\dots,t_p)} lex(d_1,\dots,d_p)$ *iff* $(c_1,\dots,c_p)\,(>^S_{t_1},\dots,>^S_{t_p})_{lex}\,(d_1,\dots,d_p)$,
- $>^S_{x_i}$ *is* $>^i$ *if $i \in I$, otherwise it is $>$,*
- $mul(c_1, \dots, c_q) >_{mul(x_{k_1},\dots,x_{k_q})} mul(d_1, \dots, d_q)$ *iff* $\{c_1, \dots, c_q\} >_{mul} \{d_1, \dots, d_q\}$.

Note that it boils down to the usual lexicographic ordering if $stat = lex(x_1, \dots, x_n)$ or to the multiset ordering if $stat = lex(mul(x_1, \dots, x_n))$. $>^S_{stat}$ is well-founded if so is $>$ and each $>^i$.

For example, let $>$ and $\succ$ be some orders, $stat = lex(x_2, mul(x_1, x_3))$, $I = \{1\}$, and $S = (\succ)$. Then, $(a_1, a_2, a_3) >^S_{stat} (b_1, b_2, b_3)$ iff $a_2 \succ b_2$, or else $a_2 = b_2$ and $\{a_1, a_3\} >_{mul} \{b_1, b_3\}$.

Definition 16 (Critical interpretation) *Given an environment Γ, the* critical interpretation function *$\phi_{f,\Gamma}$ of a function symbol $f \in \mathcal{F}_{s_1,\dots,s_n,s}$ is:*

- $\phi_{f,\Gamma}(a_1, \dots, a_n) = (\phi^1_{f,\Gamma}(a_1), \dots, \phi^n_{f,\Gamma}(a_n))$,
- $\phi^i_{f,\Gamma}(a_i) = a_i$ *if $i \notin Ind(f)$,*
- $\phi^i_{f,\Gamma}(a_i) = \chi^{s_i}_\Gamma(a_i)$ *if $i \in Ind(f)$.*

The critical ordering *associated to f is $>_{f,\Gamma} = \rhd^S_{stat_f}$, where $S = (\,\rhd_{\Gamma,s_i}\,)_{i\in Ind(f)}$.*

According to Definition 15, the critical ordering is nothing but the usual subterm ordering at non-inductive positions, and the critical subterm ordering of Definition 14 at inductive positions.

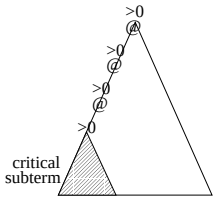

Figure 2: Critical subterm

We are now ready for describing the schema for higher-order rewrite rules. Given some lefthand side rule, we define a set of acceptable righthand sides, called computable closure. In the next section, we prove that it preserves strong normalization.

Definition 17 (Accessible subterms) *A term b is said to be* accessible *in a well-typed term c if it is a subterm of c which is typable by a basic inductive sort, or if there exists $p \in \mathcal{P}os(c)$ such that $c|_p = b$, and $\forall q < p$, $c|_q$ is headed by a constructor. b is said to be* accessible *in $\vec{c}$ if it is so in some $c \in \vec{c}$.*

Definition 18 (Computable closure) *Given an algebraic environment Γ containing Γ_R and a term $f(\vec{c})$ typable in Γ, the* computable closure *$\mathcal{CC}_{f,\Gamma}(\vec{c})$ of $f(\vec{c})$ in Γ is defined as the least set of Γ-terms containing all terms accessible in $\vec{c}$, all variables in $dom(\Gamma) \setminus FV(\vec{c})$, and closed under the following operations:*

(i) constructor application: let C be a constructor of type $s_1 \to \ldots \to s_n \to \mathtt{s}$; then $C(\vec{u}) \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ iff $u_i : s_i \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ for all $i \in [1..n]$,

(ii) defined application: let $g \in \mathcal{F}_{s_1,\ldots,s_n,t}$ such that $g <_{\mathcal{F}} f$; then $g(\vec{u}) \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ iff $u_i : s_i \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ for all $i \in [1..n]$,

(iii) application: let $u : s \to t \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ and $v : s \in \mathcal{CC}_{f,\Gamma}(\vec{c})$; then $(uv) \in \mathcal{CC}_{f,\Gamma}(\vec{c})$,

(iv) abstraction: let $u \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ and $x : s \in \Gamma$; then $\lambda x : s.u \in \mathcal{CC}_{f,\Gamma}(\vec{c})$,

(v) reduction: let $u \in \mathcal{CC}_{f,\Gamma}(\vec{c})$, and v be a reduct of u using a β-rewrite step or a higher-order rewrite rule for a function symbol $g <_{\mathcal{F}} f$; then $v \in \mathcal{CC}_{f,\Gamma}(\vec{c})$,

(vi) recursive call: let $\vec{c'}$ be a vector of n terms in $\mathcal{CC}_{f,\Gamma}(\vec{c})$ of respective types $s_1, \ldots, s_n$, such that $\phi_{f,\Gamma}(\vec{c}) = \vec{c} \;\; >_{f,\Gamma} \;\; \phi_{f,\Gamma}(\vec{c'})$; then $f(\vec{c'}) \in \mathcal{CC}_{f,\Gamma}(\vec{c})$.

A useful finite approximation of this infinite set is defined by the Coquand's notion of *structurally smaller* [7], where only cases (i), (iii), (v) (one β-step only) and (vi) are used, with a multiset status which forbids the use of nested recursions. Our definition is therefore richer for two independent reasons. Note further that Coquand restricts himself to the cases for which his ordering is well-founded, a property that we think related to the positivity condition.

This can also be compared with the current criterion used in Coq for accepting function definitions by fixpoint and constructor matching [11]. Functions are defined by induction on one argument at a time, this argument must be constructor headed, and recursive calls can be made only with its immediate subterms. We are now ready for defining the schema:

Definition 19 (General Schema) *A set R of rewrite rules satisfies the General Schema if*

(i) its first-order part is conservative and strongly normalizing,

(ii) each higher-order function $f \in \mathcal{F}_{s_1,\ldots,s_n,t}$ is defined by a set of admissible rewrite rules of the form $f(\vec{c}) \longrightarrow e$ such that $e \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ for some algebraic Γ containing Γ_R (the environment in which the rules of R are defined).

All pattern-matching definitions given so far satisfy the General Schema, including the first-order ones. We could have imposed that the first-order rules also satisfy the General Schema: this would have simplified our definition, but at the price of restricting the expressivity for the first-order rules. In our formulation, the strong normalization property of the first-order rules has to be proved beforehand. Tools exist that do the job automatically for many practical examples. Note that recursor rules of any strictly positive inductive type satisfy the General Schema:

Lemma 20 *The recursor rules for strictly positive inductive sorts satisfy the General Schema.*

2.6 CAC computations

Definition 21 (Reduction relation) *Given a set R of rewrite rules satisfying the General Schema, including the set Rec of recursor rules of a given user specification, the CAC rewrite relation is $\longrightarrow = \longrightarrow_\beta \cup \longrightarrow_R$. The CAC reduction relation is its reflexive and transitive closure denoted by $\longrightarrow^*$. Its transitive closure is denoted by $\longrightarrow^+$. Its reflexive, symmetric and transitive closure is denoted by $\longleftrightarrow^*$. A term is in* normal form *if it cannot be β-reduced, Rec-reduced or R-reduced. An* expansion *is the inverse of a reduction: a expanses to b if b reduces to a.*

Our calculus enjoys the subject reduction property, that is, preservation of types under reductions. The proof uses a weak version of confluence, see [1].

Full confluence is proved after strong-normalization, by using Newman's Lemma, and by assuming there are no critical pairs between any two higher-order rules, and between the higher-order rules, the first-order rules and the β-reduction rule (by considering that the abstraction is an unary function symbol, and the application a binary one).

3 Strong-normalization

A term is *strongly normalizable* if any reduction issuing from it terminates. Strong-normalization and confluence together imply the logical soundness of the system as well as the decidability of type-checking. In this section, we investigate only the former. Let SN be the set of strongly normalizable terms.

To prove the strong normalization property for well-typed terms, we use the well known proof technique of Girard dubbed "reducibility candidates" [17], further extended by Coquand and Gallier to the Calculus of Constructions [8]. Note that these proofs use well-typed candidates, that is, sets of well-typed terms.

There exists proofs with lighter notations based on untyped candidates [16], but which do not allow one to reason about the type of the elements of a reducibility candidate, as it will be necessary to do with our extension of the General Schema. For a comprehensive survey of the method, see [15].

The strong normalization proof of Coquand and Gallier can easily be tailored to our need. It suffices to define an adequate interpretation for the inductive types, and to prove that, if the arguments of a function call belong to the interpretation of their type, then the function call itself belongs to the interpretation of its output type. We recall the definitions that are necessary for the understanding of our extension, and refer the reader to [8] for a complete exposition.

3.1 Interpretation of theorems

Definition 22 (Reducibility candidates) *We define the set Neutr of* neutral terms *as being the set of terms that are not an abstraction or constructor headed. Let* $\mathcal{T}_{\Delta,A} = \{\Delta' \vdash a \mid \Delta' \vdash a : A,\ \Delta' \supseteq \Delta\}$, $SN_{\Delta,A} = \{\Delta' \vdash a \in \mathcal{T}_{\Delta,A} \mid a \in SN\}$.

Given a valid environment Δ*, the family* $\mathcal{C}$ *of saturated sets* $\mathcal{C}_{\Delta,A}$ *where* A *is a* Δ*-theorem, is defined by the properties listed below.*

1. *If* $A = \Box$*, then* $\mathcal{C}_{\Delta,A}$ *is the set* $\{SN_{\Delta,\Box}\}$.
2. *If* A *is a* Δ*-type or a* Δ*-kind, then* $\mathcal{C}_{\Delta,A}$ *is the set of non empty sets* $S \subseteq SN_{\Delta,A}$ *such that the following properties hold:*
   - *(S1)* $S \supseteq \{\Delta' \vdash x\vec{a} \in \mathcal{T}_{\Delta,A} \mid x \in Var \text{ and } \vec{a} \in SN\}$.
   - *(S2) For every neutral term* t *such that* $\Delta' \vdash t \in \mathcal{T}_{\Delta,A}$*, if, for every immediate reduct* t' *of* t*,* $\Delta' \vdash t' \in S$*, then* $\Delta' \vdash t \in S$.
   - *(S3) Whenever* $\Delta' \vdash t \in S$ *and* $\Delta' \subseteq \Delta''$*, then* $\Delta'' \vdash t \in S$.
   - *(S4) Whenever* $\Delta' \vdash t \in S$ *and* t' *is a reduct of* t*, then* $\Delta' \vdash t' \in S$.
3. *If* A *is a type constructor of type* $\Pi x : B.C$ *in* Δ*, then* $\mathcal{C}_{\Delta,A}$ *is the set of functions with the following properties:*
   - *(a) If* B *is a kind, then*
     - $f \in \mathcal{C}_{\Delta,A}$ *is a function with domain* $\{(\Delta' \vdash T, S) \mid \Delta' \vdash T \in \mathcal{T}_{\Delta,B}$ *and* $S \in \mathcal{C}_{\Delta',T}\}$ *such that* $f(\Delta' \vdash T, S) \in \mathcal{C}_{\Delta',AT}$,
     - $f(\Delta' \vdash T_1, S_1) = f(\Delta' \vdash T_2, S_2)$ *whenever* $T_1 \longleftrightarrow^* T_2$.
   - *(b) If* B *is a type, then*
     - $f \in \mathcal{C}_{\Delta,A}$ *is a function with domain* $\mathcal{T}_{\Delta,B}$ *such that* $f(\Delta' \vdash t) \in \mathcal{C}_{\Delta',At}$,
     - $f(\Delta' \vdash t_1) = f(\Delta' \vdash t_2)$ *whenever* $t_1 \longleftrightarrow^* t_2$.

Compared to [8], we extended (S2) to neutral terms to take care of functions, and added (S4) to insure that reducibility candidates are stable by reduction.

Definition 23 (Interpretation of algebraic types) *Given a valid environment* Δ*, we define the interpretation of algebraic types as follows:*

- $can_{\Delta,\mathtt{s}} = \{\Delta' \vdash a \in SN_{\Delta,\mathtt{s}} \mid$ *if* $a \longrightarrow^* C(\vec{b})$ *and* $\tau(C) = s_1 \to \ldots \to s_n \to \mathtt{s}$*, then* $\Delta' \vdash b_i \in can_{\Delta,s_i}$ *for every* $i \in [1..n]\}$,
- $can_{\Delta,s\to t} = \{\Delta' \vdash a \in \mathcal{T}_{\Delta,s\to t} \mid \forall\ \Delta'' \subseteq \Delta',\ \forall\ \Delta'' \vdash b \in can_{\Delta,s}$, $\Delta'' \vdash ab \in can_{\Delta,t}\}$.

Let us justify the definition. Since $>_{\mathcal{S}}$ is assumed to be well-founded, our hypothesis is that the definition makes sense for every algebraic type built from sorts strictly smaller than a given sort $\mathtt{s}$. Let P be the set of subsets of $SN_{\Delta,\mathtt{s}}$ that contains all strongly normalizable terms that do not reduce to a term headed by a constructor of $\mathtt{s}$. P is a complete lattice for set inclusion. Given an element $X \in P$, we define the following function on algebraic types built from sorts smaller than $\mathtt{s}$: $R_X(\mathtt{s}) = X$, $R_X(\mathtt{t}) = can_{\Delta,\mathtt{t}}$ and $R_X(s \to t) = can_{\Delta,s\to t}$. Now, let $F : P \to P$, $X \mapsto X \cup Y$ where $Y = \{a \in SN_{\Delta,\mathtt{s}} \mid \text{ if } a \longrightarrow\!\!\!\!\!^{*} C(\vec{b})$ and $\tau(C) = s_1 \to \ldots \to s_n \to \mathtt{s}$ then $b_i \in R_X(s_i)$ for every $i \in [1..n]\}$. Since inductive sorts are assumed to be positive, one can show that F is monotone. Hence, from Tarski's Theorem, it has a least fixed point $can_{\Delta,\mathtt{s}} \in \mathcal{C}_{\Delta,\mathtt{s}}$.

Definition 24 (Well-typed substitutions) *Given two valid environments Δ and Γ, a substitution θ is a* well-typed substitution from Γ to Δ *if $dom(\theta) \subseteq dom(\Gamma)$ and, for every variable $x \in dom(\Gamma)$, $\Delta \vdash x\theta : \Gamma(x)\theta$.*

Definition 25 (Candidate assignments) *Given two valid environments Δ and Γ, and a well-typed substitution θ from Γ to Δ, a* candidate assignment compatible with θ *is a function ξ from $Var^{\Box}$ to the set of saturated sets such that, for every variable $\alpha \in dom(\Gamma) \cap Var^{\Box}$, $\xi(\alpha) \in \mathcal{C}_{\Delta,\alpha\theta}$.*

Compared to [8] where well-typed substitutions and candidate assignments are packaged together, we prefer to separate them since the former is introduced to deal with abstractions, while the latter is introduced to deal with polymorphism. We are now ready to give the definition of the interpretation of theorems.

Definition 26 (Interpretation of theorems) *Given two valid environments Δ and Γ, a well-typed substitution θ from Γ to Δ, and a candidate assignment ξ compatible with θ, we define the interpretation of Γ-theorems as follows:*

- $[\![\Gamma \vdash \Box]\!]_{\Delta,\theta,\xi} = SN_{\Delta,\Box}$,
- $[\![\Gamma \vdash \star]\!]_{\Delta,\theta,\xi} = SN_{\Delta,\star}$,
- $[\![\Gamma \vdash \mathtt{s}]\!]_{\Delta,\theta,\xi} = can_{\Delta,\mathtt{s}}$,
- $[\![\Gamma \vdash \alpha]\!]_{\Delta,\theta,\xi} = \xi(\alpha)$,
- $[\![\Gamma \vdash \lambda x{:}\tau.T]\!]_{\Delta,\theta,\xi} =$ *the function which associates* $[\![\Gamma, x{:}\tau \vdash T]\!]_{\Delta',\theta\{x\mapsto t\},\xi}$ *to every* $\Delta' \vdash t \in \mathcal{T}_{\Delta,\tau\theta}$,
- $[\![\Gamma \vdash \lambda\alpha{:}K'.T]\!]_{\Delta,\theta,\xi} =$ *the function which associates* $[\![\Gamma, \alpha{:}K' \vdash T]\!]_{\Delta',\theta\{\alpha\mapsto T'\},\xi\{\alpha\mapsto S\}}$ *to every* $(\Delta' \vdash T', S) \in \{(\Delta' \vdash T', S) \mid \Delta' \vdash T' : K'\theta,\ \Delta' \supseteq \Delta,\ S \in \mathcal{C}_{\Delta',T'}\}$,
- $[\![\Gamma \vdash T\ t]\!]_{\Delta,\theta,\xi} = [\![\Gamma \vdash T]\!]_{\Delta,\theta,\xi}(\Delta \vdash t\theta)$
- $[\![\Gamma \vdash T\ T']\!]_{\Delta,\theta,\xi} = [\![\Gamma \vdash T]\!]_{\Delta,\theta,\xi}(\Delta \vdash T'\theta, [\![\Gamma \vdash T']\!]_{\Delta,\theta,\xi})$
- $[\![\Gamma \vdash \Pi x{:}\tau.A]\!]_{\Delta,\theta,\xi} = \{\Delta' \vdash a \in \mathcal{T}_{\Delta,\Pi x:\tau\theta.A\theta} \mid \forall \Delta'' \supseteq \Delta',\ \forall\ \Delta'' \vdash t \in [\![\Gamma \vdash \tau]\!]_{\Delta'',\theta,\xi},\ \Delta'' \vdash at \in [\![\Gamma, x{:}\tau \vdash A]\!]_{\Delta'',\theta\{x\mapsto t\},\xi}\}$,
- $[\![\Gamma \vdash \Pi\alpha{:}K.A]\!]_{\Delta,\theta,\xi} = \{\Delta' \vdash a \in \mathcal{T}_{\Delta,\Pi\alpha:K\theta.A\theta} \mid \forall \Delta'' \supseteq \Delta',\ \forall\ \Delta'' \vdash T \in [\![\Gamma \vdash K]\!]_{\Delta'',\theta,\xi},\ \forall S \in \mathcal{C}_{\Delta'',T},\ \Delta'' \vdash aT \in [\![\Gamma, \alpha{:}K \vdash A]\!]_{\Delta'',\theta\{\alpha\mapsto T\},\xi\{\alpha\mapsto S\}}\}$.

The last two cases correspond to the "stability by application". The well-definedness of this definition is insured by the following lemma.

Lemma 27 (Interpretation correctness) *Assume that* Δ *and* Γ *are two valid environments,* θ *is a well-typed substitution from* Γ *to* Δ*, and* ξ *is a candidate assignment compatible with* θ*. Then, for every* Γ*-theorem* A*,* $[\![\Gamma \vdash A]\!]_{\Delta,\theta,\xi} \in \mathcal{C}_{\Delta,A\theta}$.

We are now able to state the main lemma for the strong normalization theorem.

Definition 28 (Reducible substitutions) *Given two valid environments* Δ *and* Γ*, a well-typed substitution* θ *from* Γ *to* Δ*, and a candidate assignment* ξ *compatible with* θ*,* θ *is said to be* valid with respect to ξ *if, for every variable* $x \in dom(\Gamma)$*,* $\Delta \vdash x\theta \in [\![\Gamma \vdash \Gamma(x)]\!]_{\Delta,\theta,\xi}$.

Lemma 29 (Main lemma) *Assume that* $\Gamma \vdash a : b$*,* Δ *is a valid environment,* θ *is a well-typed substitution from* Γ *to* Δ*, and* ξ *is a candidate assignment compatible with* θ*. If* θ *is valid with respect to* ξ*, then* $\Delta \vdash a\theta \in [\![\Gamma \vdash b]\!]_{\Delta,\theta,\xi}$.

Proof: As in [8], by induction on the structure of the derivation. We give only the additional cases. The case (cons) is straightforward. The case (fun) is proved by Theorem 33 to come for the case of higher-order function symbols, and by [18] for the case of first-order function symbols. □

Theorem 30 (Strong normalization) *Assume that the higher-order rules satisfy the General Schema. Then, any well-typed term is strongly normalizable.*

Proof: Application of the Main Lemma, see [8] for details.

3.2 Reducibility of higher-order function symbols

One can see that the critical interpretation is not compatible with the reduction relation, and not stable by substitution either. We solve this problem by using yet another interpretation function for terms enjoying both properties and relating to the previous one as follows:

Definition 31 (Admissible recursive call interpretation) *A* recursive call interpretation *for a function symbol* f *is given by:*
(i) a function $\Phi_{f,\Gamma}$ *operating on arguments of* f*, for each environment* Γ*,*
(ii) a status ordering $\geq^S_{stat_f}$ *where* S *is a set of orders indexed by* $Ind(f)$*.*
A recursive call interpretation is admissible *if it satisfies the following properties:*
(Stability) *Assume that* $f(\vec{c'}) \in \mathcal{CC}_{f,\Gamma}(\vec{c})$*, hence* $\phi_{f,\Gamma}(\vec{c}) = \vec{c} >_{f,\Gamma} \phi_{f,\Gamma}(\vec{c'})$*,* Δ *is a valid environment, and* θ *is a well-typed substitution from* Γ *to* Δ *such that* $\vec{c}\theta$ *are strongly normalizable terms. Then,* $\Phi_{f,\Delta}(\vec{c}\theta) >^S_{stat_f} \Phi_{f,\Delta}(\vec{c'}\theta)$.
(Compatibility) *Assume that* s *is the output type of* f*,* $\vec{a}$ *and* $\vec{a'}$ *are two sequences of strongly normalizable terms such that* $\Delta \vdash f(\vec{a}) : s$ *and* $\vec{a} \longrightarrow\!\!\!\!\!^{*}\ \vec{a'}$*. Then,* $\Phi_{f,\Delta}(\vec{a}) \geq^S_{stat_f} \Phi_{f,\Delta}(\vec{a'})$.

The definition of the actual interpretation function, which is intricate, can be found in the full version of the paper. Before to prove the reducibility of higher-order function symbols, we need the following result.

Lemma 32 (Compatibility of accessibility with reducibility)
If $\Delta \vdash a \in can_{\Delta,A}$ *and* $b \in \mathcal{T}_{\Delta,B}$ *is accessible in* a*, then* $\Delta \vdash b \in can_{\Delta,B}$.

Theorem 33 (Reducibility of higher-order function symbols)
Assume that the higher-order rules satisfy the General Schema. Then, for every higher-order function symbol $f \in \mathcal{F}_{s_1,\ldots,s_n,s}$, $\Delta \vdash f(\vec{a}) \in can_{\Delta,s}$ *provided that* $\Delta \vdash f(\vec{a}) : s$ *and* $\Delta \vdash a_i \in can_{\Delta,s_i}$ *for every* $i \in [1..n]$.

Proof: The proof uses three levels of induction: on the function symbols ordered by $>_{\mathcal{F}}$, on the sequence of terms to which f is applied, and on the righthand side structure of the rules defining f. By induction hypothesis (1), any g occurring in the rules defining f satisfies the lemma.

We proceed to prove that $\Delta \vdash f(\vec{a}) \in can_{\Delta,s}$ by induction (2) on $(\Phi_{f,\Delta}(\vec{a}), \vec{a})$ with $(\geq^S_{stat_f}, (\longrightarrow^*)_{lex})_{lex}$ as well-founded order. Since $b = f(\vec{a})$ is a neutral term, by definition of reducibility candidates, it suffices to prove that every reduct b' of b belongs to $can_{\Delta,s}$.

If b is not reduced at its root then one a_i is reduced. Thus, $b' = f(\vec{a'})$ such that $\vec{a} \longrightarrow \vec{a'}$. As reducibility candidates are stable by reduction, $\Delta \vdash a'_i \in can_{\Delta,s_i}$, hence the induction hypothesis (2) applies since the interpretation is compatible with reductions.

If b is reduced at its root then $\vec{a} = \vec{c}\theta$ and $b' = e\theta$ for some terms $\vec{c}, e$ and substitution θ such that $f(\vec{c}) \longrightarrow e$ is the applied rule. θ is a well-typed substitution from Γ_R to Δ, and ξ is compatible with θ since $dom(\Gamma_R) \cap Var^{\square} = \emptyset$. We now show that θ is compatible with ξ. Let x be a free variable of e of type t. By definition of the General Schema, x is an accessible subterm of $\vec{c}$. Hence, by Lemma 32, $\Delta \vdash x\theta \in can_{\Delta,t}$ since, for every $i \in [1..n]$, $\Delta \vdash c_i\theta \in can_{\Delta,s_i}$.

Given an algebraic environment Γ containing Γ_R, let us show by induction (3) on the structure of $e \in \mathcal{CC}_{f,\Gamma}(\vec{c})$ that, for any well-typed substitution θ from Γ to Δ compatible with ξ, $e\theta \in can_{\Delta,t}$, provided that $c_i\theta \in can_{\Delta,s_i}$ for every $i \in [1..n]$.

Base case: either e is accessible in c_i, or e is a variable of $dom(\Gamma) \setminus FV(\vec{c})$. In the first case, this results from Lemma 32, and in the second case, this results from the fact that θ is compatible with ξ. Now, let us go through the different closure operations of the definition of $\mathcal{CC}_{f,\Gamma}()$.

(i) construction: $e = C(e_1, \ldots, e_p)$ and $\tau(C) = t_1 \to \ldots \to t_p \to \mathtt{t}$. $e\theta \in can_{\Delta,\mathtt{t}}$ since, by induction hypothesis (3), $e_i\theta \in can_{\Delta,t_i}$.

(ii) defined application: $e = g(e_1, \ldots, e_p)$ with $\tau(g) = t_1 \to \ldots \to t_p \to t$ and $g <_{\mathcal{F}} f$. By induction hypothesis (3), $e_i\theta \in can_{\Delta,t_i}$. Hence, $e\theta \in can_{\Delta,t}$, by [18] for first-order function symbols, or by induction hypothesis (1) for higher-order ones, since $g <_{\mathcal{F}} f$.

(iii) application: $e = u\ v$. $e\theta \in can_{\Delta,t}$ since, by induction hypothesis (3), $u\theta \in can_{\Delta,t' \to t}$ and $v\theta \in can_{\Delta,t'}$.

(iv) abstraction: $e = \lambda x{:}t_1.u$ and $t = t_1 \to t_2$ such that $\Gamma, x{:}t_1 \vdash u : t_2$. Let $v \in can_{\Delta,t_1}$. By induction hypothesis (3), $u\theta\{x \mapsto v\} \in can_{\Delta,x:t_1,t_2}$. Hence, $(\lambda x{:}t_1.u\theta)v \in can_{\Delta,x:t_1,t_2}$ and $e\theta \in can_{\Delta,t}$.

(v) reduction: e is a reduct of a term $u \in \mathcal{CC}_{f,\Gamma}(\vec{c})$. Since $\Gamma \vdash u{:}t$, by induction hypothesis (3), $u\theta \in can_{\Delta,t}$. Since reducibility candidates are stable by reduction, $e\theta \in can_{\Delta,t}$.

(vi) admissible recursive call: $e = f(\vec{c'})$ and $\phi_{f,\Gamma}(\vec{c}) = \vec{c} >_{f,\Gamma} \phi_{f,\Gamma}(\vec{c'})$. The induction hypothesis (1) applies since the interpretation is stable. □

This achieves the proof of the strong normalization property.

4 Conclusion and future work

We have defined an extension of the Calculus of Constructions by higher-order rewrite rules defining uncurried function symbols via the so called *General Schema* [4], which will allow a smooth integration in proof assistants like Coq, of function definitions by pattern-matching on the one hand, and decision procedures on the other hand. This result extends previous work by Barbanera et al. [1], by allowing for non-dependent and non-polymorphic inductive types. In our strong normalization proof based on Girard's reducibility candidates, we have indeed used a powerful generalization of the General Schema, of which the recursors for strictly positive inductive types are an instance, which is an important step of its own.

Several problems need to be solved to achieve our program, that is to extend the Coq proof assistant [3] with rewriting facilities. Firstly, to generalize our results to arbitrary positive inductive types, for which the type being defined may occur at any positive position of the argument types of its constructors. Secondly, to extend the results to dependent and polymorphic inductive types as defined by Coquand and Paulin in [10]. This is indeed the same problem, of defining and proving a generalization of the schema. Thirdly, to allow rewriting at the type level, enabling one to define types by induction. The corresponding recursor rules are called strong elimination [22]. We have already preliminary results in the latter two directions. Lastly, to accommodate the η-rule. By following [12], we plan to try the use of the η-rule as an expansion, instead of as a reduction. In this context, it would also be interesting to see to which extent the works by Nipkow [20] and Klop [19] on higher-order rewriting systems could be integrated in our framework. Fourthly, following [13], we also want to introduce modules in our calculus to be able to develop libraries of reusable parameterized proofs.

Acknowledgements: We want to thank Maribel Fernández for her careful reading, and the useful remarks by the anonymous referees.

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