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LAURENT BUSÉ, MARC CHARDIN, AND JEAN-PIERRE JOUANOLOU

ABSTRACT. Recently, a method to compute the implicit equation of a parametrized hypersurface has been developed by the authors. We address here some questions related to this method. First, we prove that the degree estimate for the stabilization of the MacRae's invariant of $\mathrm{Sym}_A(I)_\nu$ is optimal. Then, we show that the extraneous factor that may appear in the process splits into a product a linear forms in the algebraic closure of the base field, each linear form being associated to a non complete intersection base point. Finally, we make a link between this method and a resultant computation for the case of rational plane curves and space surfaces.

1. INTRODUCTION

Let k be a field, and $A := k[X_1, \ldots, X_n]$ be a polynomial ring in $n \geq 2$ variables, with its standard $\mathbb{N}$-grading $\deg(X_i) = 1$ for all $i = 1, \ldots, n$. Suppose given an integer $d \geq 1$ and $n+1$ homogeneous polynomials $f_1, \ldots, f_{n+1} \in A_d$, not all zero. We will denote by I the ideal of A generated by these polynomials and set $X := \mathrm{Proj}(A/I) \subseteq \mathbb{P}_k^{n-1} = \mathrm{Proj}(A)$. Let $T_1, \ldots, T_{n+1}$ be $n+1$ new indeterminates, $B := k[T_1, \ldots, T_{n+1}]$ with its standard grading, and consider the rational map

$$(1) \qquad \lambda : \mathbb{P}_k^{n-1} \to \mathbb{P}_k^n : x \mapsto (f_1(x) : f_2(x) : \cdots : f_{n+1}(x)).$$

We will always assume throughout this paper that λ is generically finite onto its image, or equivalently that the closed image of λ is a hypersurface $\mathcal{H}$, and that the base locus X of λ is supported on a finite set of points. We will focus on the computation of an equation H of the closed image $\mathcal{H}$ following a technique developed in [6, 4]. It is based on the computation of MacRae's invariants of certain graded parts of the symmetric algebra $\mathrm{Sym}_A(I)$, which are B-modules, by means of the determinant of the corresponding graded parts of a $\mathcal{Z}$-approximation complex (see [9] for the definition and a detailed study of these complexes).

The closure Γ of the graph of λ is a geometrically irreducible and reduced subscheme of $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^n$; its image under the projection $\pi : \mathbb{P}_k^{n-1} \times \mathbb{P}_k^n \to \mathbb{P}_k^n$ is $\mathcal{H}$.

The natural bigraded epimorphisms

$$A \otimes_k B \xrightarrow{\phi} \mathrm{Sym}_A\left(I(d)\right) \xrightarrow{\psi} \mathrm{Rees}_A\left(I(d)\right)$$

correspond to the inclusions of schemes $\Gamma \subseteq V \subseteq \mathbb{P}_k^{n-1} \times \mathbb{P}_k^n$ where V is the biprojective scheme defined by $\mathrm{Sym}_A(I)$.

By definition $\mathrm{Ker}(\phi)$ is generated by the elements of bi-degree $(*, 1)$ of the bigraded prime ideal $\mathfrak{P} := \mathrm{Ker}(\psi \circ \phi)$. In other words, $\mathrm{Ker}(\phi)$ is generated by the syzygies of the f_i's : the elements $\sum_{i=1}^{n+1} a_i T_i \in A \otimes_k B_1$ such that $\sum_{i=1}^{n+1} a_i f_i = 0$.

As the construction of symmetric algebras and Rees algebras commute with localization, and both algebras are the quotient of a polynomial extension of the base ring by the Koszul syzygies on a minimal set of generators in the case of a complete intersection ideal, it follows that Γ and V coincide on $(\mathbb{P}_k^{n-1} - S) \times \mathbb{P}_k^n$, where S is the (possibly empty) set of non locally complete intersection points of X.

Recall that for a point $x := \mathrm{Proj}(A/\mathfrak{p}) \in X$, the two notions of multiplicity :

- the "degree", that we will denote by $d_x = d_{\mathfrak{p}}$, which equals $\ell_{A_{\mathfrak{p}}}(A/I)_{\mathfrak{p}} = \ell_{\mathcal{O}_{\mathbb{P}_k^{n-1},x}}(\mathcal{O}_{X,x})$,
- and the "Hilbert-Samuel multiplicity" $e_x = e_{\mathfrak{p}}$ given by the asymptotic behavior for $t \gg 0$ of $\ell_{A_{\mathfrak{p}}}(I^{t-1}/I^t)_{\mathfrak{p}} = \ell_{\mathcal{O}_{\mathbb{P}_k^{n-1},x}}(\mathcal{I}_{X,x}^{t-1}/\mathcal{I}_{X,x}^t)$ (e.g. [6, §2.2]),

satisfies $e_{\mathfrak{p}} \geq d_{\mathfrak{p}}$ and this inequality is an equality if and only if $I_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$ can be generated by a regular sequence. More precisely, $e_{\mathfrak{p}}$ is the smallest degree of a $\mathfrak{p}$-primary complete intersection ideal contained in $I_{\mathfrak{p}}$, if the field k is infinite (see e.g. [3, §7]).

We recall that if M is a non zero $\mathbb{Z}$-graded A-module of finite type then its initial degree, denoted $\mathrm{indeg}(M)$, is defined as the smallest integer ν such that the degree ν part of M is non-zero.

For $\nu \in \mathbb{N}$, we define $\mathrm{Sym}_A(I)_\nu$ as the classes of elements of $B \otimes_k A_\nu$ in $\mathrm{Sym}_A(I)$. In other words, $\mathrm{Sym}_A(I)_\nu$ is the cokernel of the map :

$$B \otimes_k \mathrm{Syz}(f_1, \ldots, f_{n+1})_{\nu+d} \longrightarrow B \otimes_k A_\nu$$

sending $b \otimes (a_1, \ldots, a_{n+1})$ with $\deg(a_i) = \nu$ (we follow here the usual convention for the grading of syzygy modules) to $b \otimes \sum_{i=1}^{n+1} a_i T_i$.

The following theorem is proved in [6, 4], where the notation $\mathfrak{S}(-)$ stands for the MacRae's invariant, as defined and studied in [12], of a B-module. Recall that, as B is regular, any finitely generated torsion B-module has a MacRae's invariant. If M is a B-module that is not torsion, we set $\mathfrak{S}(M) := 0$ (observe that the multiplicativity property of MacRae's invariants is preserved by this extension).

Theorem 1 ([6, 4]). *Set $I_X := I :_A \mathfrak{m}^\infty$. If X is locally defined by at most n equations[1] then, for any integer $\nu \geq \eta := (n-1)(d-1) - \mathrm{indeg}(I_X)$, one has*

$$\left(H^{\deg(\lambda)}\right) \supseteq \mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Sym}_A(I)_\eta) \simeq B\left(-d^{n-1} + \sum_{x \in X} d_x\right)$$

where the last isomorphism is a graded isomorphism of B-modules. Moreover, the four following statements are equivalent

  (i) *$I \subseteq A$ is of linear type on the punctured spectrum,*
  (ii) *X is locally a complete intersection,*
  (iii) *$V = \Gamma$,*
  (iv) *$\left(H^{\deg(\lambda)}\right) = \mathfrak{S}(\mathrm{Sym}_A(I)_\nu)$ for all $\nu \geq \eta$.*

Before going further, let us make two comments about the hypothesis of this theorem. First, it is proved in [4] that if X is not locally defined by n equations at some base point x (it is then locally defined by exactly $n + 1$ equations) then the $\mathcal{Z}$-approximation complex used to get a resolution of the symmetric algebra is

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[1]*i.e. $\mathcal{I}_{X,x} \subseteq \mathcal{O}_{\mathbb{P}_k^{n-1},x}$ is generated by n elements for any $x \in X$*

no longer acyclic in high degrees (actually acyclicity and acyclicity in high degrees are equivalent in this case). Moreover, notice that in this case, $\pi(V) = \mathbb{P}_k^n$ as V contains $\{x\} \times \mathbb{P}_k^n$. Second, the degree bound η is sharp in the following sense

Proposition 2. *Under the hypotheses of Theorem 1,*

$$\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Sym}_A(I)_\eta) \text{ if and only if } \nu \geq \eta.$$

We will use the following lemma in the proof,

Lemma 3. *If $J \subset I$ is a complete intersection ideal generated by $n-1$ forms of degree d and $l \in \mathbb{Z}$, then*

$$l < \mathrm{indeg}(I_X) \;\Leftrightarrow\; (I_X)_l = 0 \;\Leftrightarrow\; (I_X/J)_l = 0.$$

Proof of Lemma 3. Notice that as J is unmixed and $\dim(A/J) > 0$, it suffices to show that $J_l \neq (I_X)_l$ for $l = \mathrm{indeg}(I_X)$. This is clearly the case if $\mathrm{indeg}(I_X) < d = \mathrm{indeg}(J)$. If $\mathrm{indeg}(I_X) = d$, $J_d = (I_X)_d$ implies that $J_d = I_d$, which in turn shows that λ factors through a rational map to $\mathbb{P}_k^{n-2}$, contradicting the assumption that the closed image of λ has dimension $n-1$. $\qquad\square$

Proof of Proposition 2. The case $n = 2$ follows immediately from [4, §2], so we can assume that $n \geq 3$. By faithfully flat base change $k \to \overline{k}$, we may assume that k is algebraically closed, and in particular infinite. As $\mathrm{Sym}_A(I)_\nu = 0$ for $\nu < 0$ it suffices to prove that if $\nu \geq 0$ then

$$\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Sym}_A(I)_\eta) \Leftrightarrow \nu \geq \eta := (n-1)(d-1) - \mathrm{indeg}(I_X).$$

Let $\nu \geq 0$, $\mu := (n-1)(d-1) - \nu$ and $D := \deg(\det(\mathcal{Z}_\eta))(= d^{n-1} - \sum_{x \in X} d_x)$. As $\mathrm{codim}(I) = n-1$, I is generated in degree d and k is infinite, there exists a complete intersection ideal $J \subset I$ generated by $n-1$ forms of degree d. We will now prove that, if $\mathcal{Z}_\nu$ is generically acyclic, one has $\deg(\det(\mathcal{Z}_\nu)) = D - \dim(I_X/J)_{\mu-1}$. This together with Lemma 3 establishes our claim.

Recall that for a finitely generated graded module M one has $H_M(l) = P_M(l) + \sum_i (-1)^i h_{\mathfrak{m}}^i(M)_l$ ($l \in \mathbb{Z}$), where P_M, respectively H_M, denotes the Hilbert polynomial, respectively the Hilbert function, of M and $h_{\mathfrak{m}}^i(M)_l := \dim_k H_{\mathfrak{m}}^i(M)_l$. We will denote by Z_i and H_i, the i-th module of cycles and i-th module of homology of the Koszul complex $K(f_1, \ldots, f_{n+1}; A)$, respectively. The degree ν part of the $\mathcal{Z}$-complex given by the f_i's will be denoted by $\mathcal{Z}_\nu$. Recall that $(\mathcal{Z}_i)_\nu = B \otimes_k (Z_i)_{\nu+id}$.

As $\mu \leq (n-1)(d-1) \leq (n-1)d$, [4, 4.1] shows that $H_{Z_0}(\nu) = P_{Z_0}(\nu)$,

$$H_{Z_1}(\nu + d) = P_{Z_1}(\nu + d) + h_{\mathfrak{m}}^0(H_1)_{\mu+d-1} - H_{H_2}(\mu + d - 1),$$

$$H_{Z_2}(\nu + 2d) = P_{Z_2}(\nu + 2d) + h_{\mathfrak{m}}^0(H_0)_{\mu-1} - H_{H_1}(\mu - 1) + H_{H_2}(\mu - 1),$$

$$H_{Z_i}(\nu + id) = P_{Z_i}(\nu + id) + \sum_j (-1)^{i+j} H_{H_j}(\mu - (i-2)d - 1) \text{ for } 3 \leq i \leq n$$

(notice that H_p coincides with Z_p in degree at most $(p+1)d - 1$, and therefore $H_{\mathfrak{m}}^n(Z_i)_{\nu+id} \simeq (Z_{n-i})^*_{\mu-(i-2)d-1}$ coincides with $(H_{n-i})^*_{\mu-(i-2)d-1}$).

Set $\chi(l) := \sum_j (-1)^j H_{H_j}(l)$ ($l \in \mathbb{Z}$) and remark that

$$H_{Z_2}(\nu + 2d) = P_{Z_2}(\nu + 2d) + \chi(\mu - 1) - H_{R/I_X}(\mu - 1),$$

$$H_{Z_i}(\nu + id) = P_{Z_i}(\nu + id) + (-1)^i \chi(\mu - (i-2)d - 1) \text{ for } 3 \leq i \leq n.$$

Now $\sum_i (-1)^i P_{Z_i}(l+id) = 0$ for any l and $P(l) := \sum_i (-1)^{i+1} i P_{Z_i}(l+id)$ is constant and equal to D. If $\mathcal{Z}_\nu$ is generically acyclic, then $\sum_i (-1)^i H_{Z_i}(\nu + id) = 0$ and we can write, using our expressions for H_{Z_i},

$$
\begin{aligned}
\deg(\det(\mathcal{Z}_\nu)) &= \sum_i (-1)^{i-1} i H_{Z_i}(\nu + id) \\
&= P(\nu) + \sum_i (-1)^{i-1}(i-1)[H_{Z_i}(\nu+id) - P_{Z_i}(\nu+id)] \\
&= D + H_{R/I_X}(\mu - 1) - \sum_{i \geq 0}(i+1)\chi(\mu - id - 1).
\end{aligned}
$$

It follows that $\deg(\det(\mathcal{Z}_\nu)) = D + H_{R/I_X}(\mu - 1) - \sum_{i \geq 0}(i+1)\chi(\mu - id - 1)$.

The Hilbert series of χ only depends on d (the degree of the f_i's) and n and is given by

$$
\sum \chi(l) t^l = \frac{(1 - t^d)^{n+1}}{(1 - t)^n},
$$

which shows that the series corresponding to $F(l) := \sum_{i \geq 0}(i+1)\chi(l - id)$ is

$$
\frac{(1 - t^d)^{n+1}}{(1 - t)^n} \sum_{i \geq 0}(i+1)t^{id} = \frac{(1 - t^d)^{n-1}}{(1 - t)^n}.
$$

Hence $F = H_{R/J}$ and $\deg(\det(\mathcal{Z}_\nu)) = D + H_{R/I_X}(\mu - 1) - H_{R/J}(\mu - 1)$. $\qquad\square$

Remark 4. *Under the hypotheses of Theorem 1, set $\varepsilon := \operatorname{indeg}(I_X)$. One can expect that $\operatorname{Sym}_A(I)_{\eta-1}$ is not a torsion B-module unless I is n-generated. This is indeed the case if I is saturated, or if I has no syzygy of degree less then $d + \varepsilon$, or if $I_X = A$ (i.e. $X = \emptyset$).*

It is known that the degree of the hypersurface $\mathcal{H}$, closure of the image of λ, is given by the formula (see for instance [13, Section 6] for an algebraic proof, or [6, 2.5] for a more geometrical approach)

$$
\deg(\lambda)\deg(\mathcal{H}) = d^{n-1} - \sum_{x \in X} e_x.
$$

Therefore, under the assumption of Theorem 1, for all $\nu \geq \eta$ the MacRae's invariant $\mathfrak{S}(\operatorname{Sym}_A(I)_\nu)$ is a principal ideal of B whose generator (unique up to multiplication by a non-zero constant in k) equals $H^{\deg(\lambda)}$ times an *extraneous homogeneous factor* $G \in B$ whose total degree is equal to $\sum_{x \in X}(e_x - d_x)$ and whose support is the codimension 1 part of $\pi(V - \Gamma)$.

In the following section 2 we give an explicit description of this extraneous factor G. Then, in section 3 we relate the computation of the MacRae's invariant $\mathfrak{S}(\operatorname{Sym}_A(I)_\nu)$ to a resultant computation in the case of algebraic plane curves ($n = 2$) and space surfaces ($n = 3$).

2. Description of the extraneous factor

Recall that $S \subset X$ is the set of points where X is not locally a complete intersection. Assume that $x = \operatorname{Proj}(A/\mathfrak{p}) \in S$ is a base point locally defined by exactly n equations. Then one of the generators of I may be written in terms of the other ones in any sufficiently small open neighborhood of x (or equivalently in $A_\mathfrak{p}$), and this unique relation (up to multiplication by a function invertible in $A_\mathfrak{p}$) specializes

at x, thus defining a non zero linear form $L_x \in B_1 \otimes_k k(x)$. It turns out that the extraneous factor G can be described in terms of these linear forms.

Proposition 5. *If the field k is algebraically closed and X is locally generated by at most n elements then, up to multiplication by a non-zero constant in k, then*

$$G = \prod_{x \in S} L_x^{e_x - d_x} \in B.$$

Moreover, all syzygies $a_1 T_1 + \cdots + a_{n+1} T_{n+1} \in A \otimes_k B_1 \simeq A^{n+1}$ specializes at x to a multiple of L_x, and at least one does not specialize to 0.

Proof. Let K be the kernel of the canonical epimorphism $\mathrm{Sym}_A(I) \to \mathrm{Rees}_A(I)$. The multiplicativity of MacRae's invariants shows the equality of ideals of B
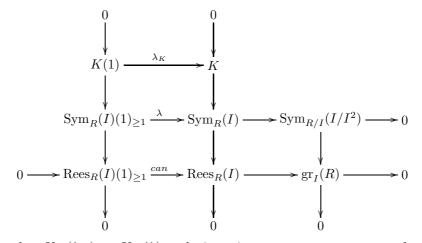
$$\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Rees}_A(I)_\nu)\mathfrak{S}(K_\nu)$$

for all integers $\nu \geq 0$. Since $\mathfrak{S}(\mathrm{Rees}_A(I)_\nu)) = (H^{\deg(\lambda)})$ for all $\nu \geq 0$ (see for instance the proof of [6, Theorem 2.5]) and $\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = (GH^{\deg(\lambda)})$ for all $\nu \geq \eta$ by Theorem 1, we deduce that $\mathfrak{S}(K_\nu) = (G)$ for all $\nu \geq \eta$.

Fix an integer $\nu \geq \eta$. The divisor $\mathfrak{S}(K_\nu) = (G)$ is supported on $\pi(V - \Gamma)$ and we know that V and Γ coincides outside $S \times \mathbb{P}_k^n$. Moreover, V is defined by linear forms in the T_i's, in particular $V - \Gamma$ is supported on a union of linear spaces over the points of S, therefore its closure is a scheme whose support is a union of linear spaces $\{x\} \times V_x$ for $x \in S$. Now $V_x \subseteq W_x := \{L_x = 0\}$. It follows that $G = \prod_{x \in S} L_x^{\mu_x}$ for some integers $\mu_x \geq 0$. The following lemma applied to A localized at the prime corresponding to $\{x\} \times W_x$ shows that $\mu_x = e_x - d_x$. $\square$

Lemma 6. *Let $(R, \mathfrak{n})$ be a Cohen-Macaulay local ring of dimension d, I be a $\mathfrak{n}$-primary ideal of R and K be the kernel of the canonical map $\mathrm{Sym}_R(I) \to \mathrm{Rees}_R(I)$. If I is generated by $d + 1$ elements and the $\mathcal{Z}$-complex of I is acyclic, then K has dimension $\leq d$ and its multiplicity in dimension d is equal to $e_I(R) - l_R(R/I)$.*

Proof. The commutative exact diagram

shows that $\mathrm{Ker}(\lambda_K) \simeq \mathrm{Ker}(\lambda)$ and gives rise to an exact sequence of graded $\mathrm{Sym}_R(I)$-modules

$$0 \longrightarrow \mathrm{Ker}(\lambda) \longrightarrow K(1) \xrightarrow{\lambda_K} K \longrightarrow \mathrm{Sym}_{R/I}(I/I^2) \longrightarrow \mathrm{gr}_I(R) \longrightarrow 0.$$

Furthermore, if the $\mathcal{Z}$-complex of I is acyclic, then $H_1(\mathcal{M}) \simeq \mathrm{Ker}(\lambda)$ and $H_i(\mathcal{M}) = 0$ for $i \geq 2$. Hence the above exact sequence of graded $\mathrm{Sym}_R(I)$-modules of finite type supported on $V(I)$ implies an equality of Poincaré series :

$$(1 - t^{-1})S(K, t) = S(H_0(\mathcal{M}), t) - S(H_1(\mathcal{M}), t) - S(\mathrm{gr}_I(R), t)$$

in the $\mathbb{Z}[[t]][t^{-1}]$-module $K_0(R/I)[[t]][t^{-1}]$. If I is generated by $a := (a_1, \ldots, a_m)$, one has

$$\sum_i (-1)^i S(H_i(\mathcal{M}), t) = \frac{1}{(1-t)^m} \sum_i (-1)^i S(H_i(a; R), t) t^i.$$

This gives

$$S(K, t) = \frac{t}{(1-t)} S(\mathrm{gr}_I(R), t) - \frac{t}{(1-t)^{m+1}} \sum_i (-1)^i S(H_i(a; R), t) t^i.$$

Set $d := \dim R$. As I is $\mathfrak{n}$-primary, $K_0(R/I) = \mathbb{Z}$ and $S(M, t) := \sum_{n \in \mathbb{Z}} l_R(M_n) t^n$ for a finitely generated graded R/I-module M. Recall that $S(\mathrm{gr}_I(R), t) = \frac{Q(t)}{(1-t)^d}$ with $Q(1) = e_I(R)$ and $S(K, t) = \frac{P(t)}{(1-t)^{d+1}}$ for some $P \in \mathbb{Z}[t]$. It follows that

$$(*) \qquad (1-t)^{m-d}P(t) = t(1-t)^{m-d}Q(t) - t\left(\sum_{i=0}^m (-1)^i l_R(H_i(a; R)) t^i\right).$$

If further R is Cohen-Macaulay and $m = d + 1$, then $H_j(a; R) = 0$ for $j \notin \{0, 1\}$. Setting $t = 1$ in formula $(*)$ shows that $l_R(H_1(a; R)) = l_R(H_0(a; R)) = l_R(R/I)$, hence

$$P(t) = tQ(t) - \frac{t}{1-t}(l_R(H_0(a; R)) - l_R(H_1(a; R))t) = t(Q(t) - l_R(R/I))$$

which gives $P(1) = e_I(R) - l_R(R/I)$. $\qquad\qquad\square$

3. Link with some resultants

We begin this section by focusing on the particular case where the ideal $I \subset A = k[X_1, \ldots, X_n]$ generated by the polynomials $f_1, \ldots, f_{n+1}$ (that define the map λ in (1)) has projective dimension 1. From Hilbert-Burch Theorem (see for instance [8, 20.15]), I admits a finite free graded resolution of the form

$$(2) \qquad 0 \to \bigoplus_{i=1}^n A(-\mu_i - d) \xrightarrow{M} A(-d)^{n+1} \xrightarrow{[f_1\ f_2\ \cdots\ f_{n+1}]} A \to A/I \to 0$$

where $M = [p_{i,j}]_{i=1,\ldots,n+1,\ j=1,\ldots,n}$, with the properties that $\sum_{i=1}^n \mu_i = d$ and the ideal of n-minors of M is equal to I.

When the ideal I has projective dimension 1 our approach to the implicitization problem via Theorem 1 coincides with a resultant computation. As a consequence of the Auslander-Buchsbaum formula, the ideal I may have projective dimension 1 only if $n = 2$ or $n = 3$ (remember that we assumed that $\dim(A/I) \leq 1$).

More precisely, define the polynomials

$$L_i := \sum_{j=1}^{n+1} p_{j,i} T_j \in A[T_1, \ldots, T_{n+1}] \quad \text{for all}\ \ i = 1, \ldots, n$$

that are respectively bi-homogeneous of bi-degree $(\mu_i; 1)$ in $A[\mathbf{T}] = k[\mathbf{X}][\mathbf{T}]$. We denote by $\mathrm{Res}(L_1, \ldots, L_n)$ the resultant of $L_1, \ldots, L_n$ w.r.t. the homogeneous variables $X_1, \ldots, X_n$.

Proposition 7 $(n = 2, 3)$. *Assume that the ideal I is locally generated by at most n elements and that (2) is a free resolution of I. Then, for all $\nu \geq \eta$*

$$(\mathrm{Res}(L_1, \ldots, L_n)) = \mathfrak{S}(\mathrm{Sym}_A(I)_\nu) \subset B.$$

Proof. Let J be the ideal generated by polynomials $L_1, \ldots, L_n$ in the polynomial ring $A[T_1, \ldots, T_{n+1}]$ and denote by Q the quotient algebra $A[T_1, \ldots, T_{n+1}]/J$.

On the one hand, by definition of the symmetric algebra $Q \simeq \mathrm{Sym}_A(I)$ as bi-graded modules. On the other hand, $\mathrm{Res}(L_1, \ldots, L_n) = \mathfrak{S}(Q_\nu)$, for all $\nu \gg 0$, if the sequence $(L_1, \ldots, L_n)$ is regular in $A[T_1, \ldots, T_{n+1}]$ (see e.g. [11, §3.5]). We deduce that the claimed result is proved if we show that $\mathrm{Sym}_A(I) \simeq Q$ is a complete intersection. Recall Avramov's criterion [2]: $\mathrm{Sym}_A(I)$ *is a complete intersection if and only if, for all integer $r \in \{1, \ldots, n\}$, $\mathrm{depth}_{I_r(M)}(A) \geq n - r + 1$, where $I_r(M)$ denotes the ideal of A generated by all the $r \times r$-minors of the matrix M* (notice that the fact that B is a complete intersection also follows from [1, 2.1]).

In the case $n = 2$, M is a 3×2 matrix and, by Hilbert-Burch Theorem, $\mathrm{depth}_{I_2(M)}(A) \geq 2$. Since $I_2(M) \subset I_1(M)$, $\mathrm{depth}_{I_1(M)}(A) \geq \mathrm{depth}_{I_2(M)}(A)$ and we obtain that $\mathrm{Sym}_A(I)$ is a complete intersection in this case.

For $n = 3$, the Hilbert-Burch Theorem shows that $\mathrm{depth}_{I_3(M)}(A) = \mathrm{depth}_I(A) \geq 2$ since we assumed that I defines isolated points. It follows that

$$\mathrm{depth}_{I_1(M)} \geq \mathrm{depth}_{I_2(M)} \geq \mathrm{depth}_{I_3(M)} \geq 2.$$

Moreover, as I is supposed to be locally generated by at most 3 elements, basic properties of Fitting ideals [8, §20.2] imply that $V(I_1(M)) = \emptyset$, i.e. $\mathrm{depth}_{I_1(M)}(A) \geq 3$. Therefore, $\mathrm{Sym}_A(I)$ is also a complete intersection in this case[2]. $\square$

In the case $n = 2$ (λ parametrizes a plane curve), the ideal I always admits a free resolution of the form (2) and is always locally generated by at most 2 elements; therefore, Proposition 7 shows that as soon as λ is generically finite, the resultant of L_1 and L_2 always compute an implicit equation of the image of λ to the power $\deg(\lambda)$, without any extraneous factor.

In the case $n = 3$ (λ parametrizes a space surface), the situation is a little more intricate. The ideal I admits a resolution of the form (2) if it defines some isolated base points (at least one) and if it is saturated with respect to the maximal ideal (X_1, X_2, X_3) of A (see for instance [8, §20.4]). Proposition 5 and Proposition 7 prove the following result that has been conjectured in [5, Conjecture 5.1]:

Corollary 8. *If the field k is algebraically closed, the ideal I has codimension 2, is saturated and is locally generated by at most 3 elements, then I admits a resolution as in (2) and, with the above notation, one has*[3]

$$\mathrm{Res}(L_1, L_2, L_3) = H(T_1, \ldots, T_4)^{\deg(\lambda)} \prod_{x \in V(I)} L_x(T_1, \ldots, T_4)^{e_x - d_x},$$

where $H \in B$ denotes an implicit equation of the image of λ.

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[2]Notice that if I is locally minimally generated by four elements at some point, then $V(I_1(M))$ is non-empty, $\mathrm{depth}_{I_1(M)}(A) = 2$ and hence $\mathrm{Res}(L_1, \ldots, L_n) = 0$ in B.

[3]Observe that this equality is rigorously an equality of divisors since H and the $L_{\mathfrak{p}}$'s are defined up to multiplication by a nonzero element in k and have not been explicitly chosen.

Example 9. *Consider the following 4×3-matrix*

$$M := \begin{bmatrix} X_1^2 + X_3^2 & X_2 & X_2X_3 \\ X_1^2 - X_1X_2 & X_1 + 2X_2 & X_1^2 + X_3^2 \\ X_1X_2 + X_1X_3 & 2X_2 + X_3 & X_2^2 \\ X_1X_3 + X_2^2 & X_3 & X_1X_2 \end{bmatrix}$$

whose ideal $I_3(M) \subset A$ of 3×3 minors is of codimension 2. This matrix fits into (2) and hence defines the parametrization of a surface $\mathcal{H}$ in $\mathbb{P}^3$. Since $V(I_1(M)) = \emptyset$, Corollary 8 applies and the resultant of L_1, L_2, L_3 is the product of two irreducible polynomials: an implicit equation of $\mathcal{H}$ which has degree 6, and an extraneous factor $G := T_2^2 + 2T_3T_4 + T_3^2 + T_4^2$. As predicted by Proposition 5, G splits into a product of two linear forms over the complex numbers field $\mathbb{C}$:

$$G = (T_2 - iT_3 - iT_4)(T_2 + iT_3 + iT_4) \in \mathbb{C}[T_1, T_2, T_3, T_4]$$

where $i \in \mathbb{C}$, $i^2 = -1$. Also, evaluating the matrix M at the two projective points $(X_1 : X_2 : X_3) = (1 : 0 : \epsilon i)$, $\epsilon = \pm 1$, we find

$$M(1 : 0 : \epsilon i) := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \epsilon i & \epsilon i & 0 \\ \epsilon i & \epsilon i & 0 \end{bmatrix}.$$

Hence these two points are base points of the parametrization (the last column is the null vector) which are almost complete intersection since the rank of $M(1 : 0 : \epsilon i)$ is 1. The linear factors of G can be read from a non-zero column of $M(1 : 0 : \epsilon i)$.

From now on we will bring the case $n = 3$ into focus. It is then possible to weaken the hypotheses of Proposition 7 using the fact that the first syzygy module of $I = (f_1, f_2, f_3, f_4)$ becomes a locally free module of rank 3 after localization by any non-zero linear form $l \in A_1$ (by Hilbert's syzygy theorem applied to localizations of $A_{(l)} \simeq k[X_1, X_2]$), hence a free $A_{(l)}$-module of rank 3, by Quillen-Suslin Theorem. For simplicity, assume that $l = X_3$. Then $I_{(X_3)} \subset A_{(X_3)} \simeq k[X_1, X_2]$ admits a resolution of the form

$$(3) \qquad 0 \to A_{(X_3)}{}^3 \xrightarrow{M} A_{(X_3)}{}^4 \xrightarrow{[\tilde{f}_1 \ \tilde{f}_2 \ \tilde{f}_3 \ \tilde{f}_4]} I_{(X_3)} \to 0$$

where $\tilde{f}_i(X_1, X_2) = f_i(X_1, X_2, 1)$ (observe that $\max_i(\deg(\tilde{f}_i)) = d$, since we assumed that I has codimension at least 2) and $M = [p_{i,j}]_{i=1,\dots,4,\, j=1,\dots,3}$. For all $j = 1, 2, 3$, we define $L_j(X_1, X_2; \mathbf{T}) = \sum_{i=1}^4 p_{i,j}T_i \in A_{(X_3)}[X_1, X_2]$ and its homogenization in the variable X_3 w.r.t. the variables X_1, X_2

$$L_j^h(\mathbf{X}; \mathbf{T}) = \sum_{i=1}^4 X_3^{\deg_{X_1,X_2}(L_j) - \deg_{X_1,X_2}(p_{i,j})} p_{i,j}T_i = \sum_{i=1}^4 p_{i,j}^h(\mathbf{X})T_i \in A[\mathbf{T}].$$

Set $d_j = \deg_{\mathbf{X}}(L_j^h) = \deg_{X_1,X_2}(L_j)$ and $M^h = [p_{i,j}^h]_{i=1,\dots,4,\, j=1,\dots,3}$.

Proposition 10. *Assume I is locally generated by at most 3 elements off $V(X_3)$. With the above notation, consider the two conditions:*
  (a) *$V(I_1(M^h), X_3) = \emptyset$ and $V(X_3) \nsubseteq V(I_2(M^h))$,*
  (b) *there exists homogeneous polynomials $P_1, P_2, P_3 \in k[X_1, X_2]$ without common factor and a linear form $Q \in k[\mathbf{T}]$ such that $L_i^h(X_1, X_2, 0, \mathbf{T}) = P_iQ$ for all $i = 1, 2, 3$.*

Then

$$\mathrm{Res}(L_1^h, L_2^h, L_3^h) \neq 0 \in B \iff \text{(a) holds or (b) holds.}$$

Furthermore, if k is algebraically closed and $\mathrm{Res}(L_1^h, L_2^h, L_3^h)$ is nonzero then it is equal to

$$H(\mathbf{T})^{\deg(\lambda)} \prod_{x \in V(I_2(M^h)) \setminus V(X_3)} L_x(\mathbf{T})^{e_x - d_x} \prod_{x \in V(I_2(M^h)) \cap V(X_3)} l_x(\mathbf{T})^{\mu_x}$$

where l_x is a linear form such that $\langle l_x \rangle_k = \langle L_1^h(x), L_2^h(x), L_3^h(x) \rangle_k$, and

$$\sum_{x \in V(I_2(M^h)) \cap V(X_3)} \mu_x = d_1 d_2 + d_2 d_3 + d_1 d_3 - d^2 + \sum_{x \in V(I) \setminus V(X_3)} d_x.$$

Proof. We begin by proving the characterization for $\mathrm{Res}(L_1^h, L_2^h, L_3^h) \neq 0$. Observe that if (b) holds then $V(X_3) \subseteq V(I_2(M^h))$; in particular (a) and (b) do not overlap. Also Avramov's criterion implies that (L_1^h, L_2^h, L_3^h) is a $A[\mathbf{T}]$-regular sequence if and only if (a) holds, in which case $\mathrm{Res}(L_1^h, L_2^h, L_3^h) \neq 0$.

($\Leftarrow$): We may assume that (b) holds. For $i = 1, 2, 3$,

$$L_i^h(\mathbf{X}, \mathbf{T}) = P_i(X_1, X_2)Q(\mathbf{T}) + X_3 R_i(\mathbf{X}, \mathbf{T}).$$

As (L_1, L_2, L_3) form a $A_{(X_3)}[\mathbf{T}]$-regular sequence, $Q \neq 0$ and $V(P_1, P_2, P_3) = \emptyset \subset \mathbb{P}^1$ we deduce that $\mathrm{Res}(L_1^h, L_2^h, L_3^h) \neq 0$.

($\Rightarrow$): We proceed by contradiction and assume that both (a) and (b) do not hold. If $x \in V(I_1(M^h), X_3) \neq \emptyset$ then $L_i^h(x; \mathbf{T}) = 0$, $i = 1, 2, 3$, hence $\mathrm{Res}(L_1^h, L_2^h, L_3^h) = 0$. Else, since the specialization $X_3 = 0$ makes M^h a matrix of rank ≤ 1 and the ring $k[X_1, X_2][\mathbf{T}]$ is factorial, there exists polynomials $(P_i(X_1, X_2))_{i=1,2,3}$ and $Q(X_1, X_2, \mathbf{T})$ such that

$$L_i^h(X_1, X_2, 0, \mathbf{T}) = P_i(X_1, X_2)Q(X_1, X_2, \mathbf{T}).$$

Since (b) does not hold, either the P_i's have a common root, or Q has a positive degree w.r.t. the variables X_1, X_2. It follows that $\mathrm{Res}(L_1^h, L_2^h, L_3^h) = 0$ since $L_i^h \in (P_1, P_2, P_3, X_3) \cap (Q, X_3)$, $i = 1, 2, 3$.

We now turn to the proof of the second statement. Let $\nu \geq \eta$ be an integer. From resultant theory it follows that $\mathrm{Res}(L_1^h, L_2^h, L_3^h)$ is a generator of the principal B-ideal $\mathfrak{S}(A[\mathbf{T}]_\nu / (L_1^h, L_2^h, L_3^h)_\nu)$. Consider the canonical exact sequence of B-modules

$$(4) \quad 0 \to H^0_{(X_3)}(A[\mathbf{T}]_\nu / (L_1^h, L_2^h, L_3^h))_\nu \to A[\mathbf{T}]_\nu / (L_1^h, L_2^h, L_3^h)_\nu \xrightarrow{\phi}$$
$$\mathrm{Sym}_A(I)_\nu / H^0_{(X_3)}(\mathrm{Sym}_A(I))_\nu \to 0.$$

From Proposition 5 and its proof, we deduce that the MacRae's invariant of the last term of (4) is generated by

$$H(\mathbf{T})^{\deg(\lambda)} \prod_{x \in V(I_2(M^h)) \setminus V(X_3)} L_x(\mathbf{T})^{e_x - d_x}.$$

Furthermore, the MacRae's invariant of the first term of (4) is supported above each point $x \in V(X_3) \subset \mathbb{P}^2$ such that the linear space $\{L_1^h(x) = L_2^h(x) = L_3^h(x) = 0\}$ has codimension one (see for instance [10]); we deduce that it is generated by $\prod_{x \in V(I_2(M^h)) \cap V(X_3)} l_x(\mathbf{T})^{\mu_x}$. This proves the claimed formula for $\mathrm{Res}(L_1^h, L_2^h, L_3^h)$ by multiplicativity of MacRae's invariants applied on (4). The last claim is a straightforward computation. $\qquad \square$

The above proposition answers questions raised in [7, §5]. Let us emphasize that $\mathrm{Res}(L_1^h, L_2^h, L_3^h)$ and a generator of $\mathfrak{S}(\mathrm{Sym}_A(I)_\nu)$, with $\nu \geq \eta$, may only differ by linear forms which are associated to base points supported on the line $V(X_3)$.

Example 11. *Consider the example [7, Example 2.2] of a canonical Steiner surface. The first syzygy module of the $\tilde{f}_i$'s is given by the matrix*

$$M = \begin{bmatrix} 0 & 0 & 1 \\ X_1 X_2 & 1 + X_2^2 & -X_1 \\ 1 + X_1^2 & X_1 X_2 & 0 \\ -2X_1 & -2X_2 & 0 \end{bmatrix}.$$

One can verify that L_1^h, L_2^h, L_3^h is a regular sequence by Avramov's criterion, and compute

$$\mathrm{Res}(L_1^h, L_2^h, L_3^h) = T_2^4 \, H(T_1, T_2, T_3, T_4),$$

where $H = 0$ is an implicit equation of the Steiner surface. Observe that the extraneous factor T_2^4 is associated to the base point $X_1 = X_3 = 0$. Also, notice that $\mathfrak{S}(\mathrm{Sym}_A(I)_\nu)$ is generated by H for all $\nu \geq \eta = 2$.

The choice of the matrix M is crucial: from a given M it is easy to get another choice such that $\mathrm{Res}(L_1^h, L_2^h, L_3^h) = 0$. This leads to the following question: is it always possible to find a matrix M such that $\mathrm{Res}(L_1^h, L_2^h, L_3^h) \neq 0$? The answer likely uses the condition that $\sum_{i=1}^3 \deg_{\mathbf{X}}(L_i)$ has to be minimal.

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