



**HAL**  
open science

# Complement to the implicitization of rational hypersurfaces by means of approximation complexes

Laurent Busé, Marc Chardin, Jean-Pierre Jouanolou

► **To cite this version:**

Laurent Busé, Marc Chardin, Jean-Pierre Jouanolou. Complement to the implicitization of rational hypersurfaces by means of approximation complexes. 2006. inria-00104191v1

**HAL Id: inria-00104191**

**<https://inria.hal.science/inria-00104191v1>**

Preprint submitted on 6 Oct 2006 (v1), last revised 13 Sep 2007 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**COMPLEMENT TO THE IMPLICITIZATION OF RATIONAL  
HYPERSURFACES BY MEANS OF APPROXIMATION  
COMPLEXES**

LAURENT BUSÉ, MARC CHARDIN, AND JEAN-PIERRE JOUANOLOU

ABSTRACT. Recently, a method to compute the implicit equation of a parameterized hypersurface has been developed by the authors. We address here some questions related to this method. First, we prove that the refined degree estimate of [3] for the stabilization of the Mac Rae invariant of  $\text{Sym}_A(I)_\nu$  is optimal. When the base points of a parameterization are complete intersections the method gives exactly an implicit equation, but if some of the base points are only almost complete intersections, then it gives an implicit equation times an extraneous factor. We show that this extraneous factor splits into a product a linear forms in the algebraic closure of the base field, each linear form being associated to a non complete intersection base points. Finally, we make a link between this method and a resultant computation for the case of rational plane curves and for particular cases of space surfaces. As a consequence, we prove a conjecture stated in [4].

1. INTRODUCTION

Let  $k$  be a field, and  $A := k[X_1, \dots, X_n]$  be a polynomial ring in  $n \geq 2$  variables, with its standard  $\mathbb{N}$ -grading  $\deg(X_i) = 1$  for all  $i = 1, \dots, n$ . We suppose given an integer  $d \geq 1$  and  $n + 1$  homogeneous polynomials  $f_1, \dots, f_{n+1} \in A_d$ , non all zero. We will denote by  $I$  the ideal of  $A$  generated by these polynomials and set  $X := \text{Proj}(A/I) \subseteq \mathbb{P}_k^{n-1} = \text{Proj}(A)$ . Let  $T_1, \dots, T_{n+1}$  be  $n + 1$  new indeterminates,  $B := k[T_1, \dots, T_{n+1}]$  with its standard grading, and consider the rational map

$$(1) \quad \begin{aligned} \lambda : \mathbb{P}_k^{n-1} &\rightarrow \mathbb{P}_k^n \\ x &\mapsto (f_1(x) : f_2(x) : \dots : f_{n+1}(x)). \end{aligned}$$

We will always assume throughout this paper that  $\lambda$  is generically finite onto its image, or equivalently that the closed image of  $\lambda$  is a hypersurface  $\mathcal{H}$ , and that the base locus  $X$  of  $\lambda$  is supported on a finite set of points. We will focus on the computation of an equation  $H$  of the closed image  $\mathcal{H}$  following a technique developed in [5, 3]. It is based on the computation of a MacRae's invariant of certain graded parts of the symmetric algebra  $\text{Sym}_A(I)$  by means of the determinant of the corresponding graded parts of a so-called  $\mathcal{Z}$ -approximation complex.

The closure  $\Gamma$  of the graph of  $\lambda$  is a geometrically irreducible and reduced subscheme of  $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^n$  whose image under the second projection  $\pi : \mathbb{P}_k^{n-1} \times \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$  is  $\mathcal{H}$ .

The natural bigraded epimorphisms :

$$A \otimes_k B \xrightarrow{\phi} \text{Sym}_A(I) \xrightarrow{\psi} \text{Rees}_A(I)$$

---

*Date:* October 6, 2006.

correspond to the inclusions of schemes

$$\Gamma \subseteq V \subseteq \mathbb{P}_k^{n-1} \times \mathbb{P}_k^n$$

where  $V$  is the biprojective scheme defined by  $\text{Sym}_A(I)$ .

By definition  $\text{Ker}(\phi)$  is generated by the elements of bidegree  $(*, 1)$  of the bigraded prime ideal  $\mathfrak{P} := \text{Ker}(\psi \circ \phi)$ . In other words,  $\text{Ker}(\phi)$  is generated by the syzygies of the  $f_i$ 's : the elements  $\sum_{i=1}^{n+1} a_i T_i \in A \otimes_k B_1$  such that  $\sum_{i=1}^{n+1} a_i f_i = 0$ .

As the construction of symmetric algebras and Rees algebras commute with localization, and both algebras are the quotient of a polynomial extension of the base ring by the Koszul syzygies on a minimal set of generators in the case of a complete intersection ideal, it follows that  $\Gamma$  and  $V$  coincide on  $(\mathbb{P}_k^{n-1} - S) \times \mathbb{P}_k^n$ , where  $S$  is the (possibly empty) set of non locally complete intersection points of  $X$ .

Recall that for a point  $x := \text{Proj}(A/\mathfrak{p}) \in X$ , the two notions of multiplicity :

- the “degree”, that we will denote by  $d_x = d_{\mathfrak{p}}$ , which equals  $\ell_{A_{\mathfrak{p}}}(A/I)_{\mathfrak{p}} = \ell_{\mathcal{O}_{\mathbb{P}_k^{n-1}, x}}(\mathcal{O}_{X,x})$ ,
- and the “Hilbert-Samuel multiplicity”  $e_x = e_{\mathfrak{p}}$  given by the asymptotic behavior for  $t \gg 0$  of  $\ell_{A_{\mathfrak{p}}}(I^{t-1}/I^t)_{\mathfrak{p}} = \ell_{\mathcal{O}_{\mathbb{P}_k^{n-1}, x}}(\mathcal{I}_{X,x}^{t-1}/\mathcal{I}_{X,x}^t)$  (see [5, §2.2] for more details in our setting),

satisfies  $e_{\mathfrak{p}} \geq d_{\mathfrak{p}}$  and this inequality is an equality if and only if  $I_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$  can be generated by a regular sequence. More precisely,  $e_{\mathfrak{p}}$  is the smallest degree of a  $\mathfrak{p}$ -primary complete intersection ideal contained in  $I_{\mathfrak{p}}$ , if the field  $k$  is infinite (see e.g. [2, §7]).

We recall that if  $M$  is a non zero  $\mathbb{Z}$ -graded  $A$ -module of finite type then its initial degree, denoted  $\text{indeg}(M)$ , is defined as the smallest integer  $\nu$  such that the degree  $\nu$  part of  $M$  is non-zero.

For  $\nu \in \mathbb{N}$ , we define  $\text{Sym}_A(I)_{\nu}$  as the classes of elements of  $B \otimes_k A_{\nu}$  in  $\text{Sym}_A(I)$ . In other words,  $\text{Sym}_A(I)_{\nu}$  is the cokernel of the map :

$$B \otimes_k \text{Syz}(f_1, \dots, f_{n+1})_{\nu+d} \longrightarrow B \otimes_k A_{\nu}$$

sending  $b \otimes (a_1, \dots, a_{n+1})$  with  $\deg(a_i) = \nu$  (we follow here the usual convention for the grading of syzygy modules) to  $b \otimes \sum_{i=1}^{n+1} a_i T_i$ .

The following theorem is proved in [5, 3], where the notation  $\mathfrak{S}(-)$  stands for the MacRae's invariant as defined and studied in [8].

**Theorem 1** ([5, 3]). *Set  $I_X := I :_A \mathfrak{m}^{\infty}$ . If  $X$  is locally defined by at most  $n$  equations<sup>1</sup> then, for any integer  $\nu$  such that*

$$\nu \geq \eta := (n-1)(d-1) - \text{indeg}(I_X),$$

*one has*

$$H^{\text{deg}(\lambda)} \supseteq \mathfrak{S}(\text{Sym}_A(I)_{\nu}) = \mathfrak{S}(\text{Sym}_A(I)_{\eta}) \simeq k[\mathbf{T}] \left( -d^{n-1} + \sum_{x \in X} d_x \right)$$

*where the last isomorphism is a graded isomorphism of  $k[\mathbf{T}]$ -modules. Moreover, the four following statements are equivalent*

- $I \subseteq A$  is of linear type on the punctured spectrum,
- $X$  is locally a complete intersection,

<sup>1</sup>i.e.  $\mathcal{I}_{X,x} \subseteq \mathcal{O}_{\mathbb{P}_k^{n-1}, x}$  is generated by  $n$  elements for any  $x \in X$

- (iii)  $V = \Gamma$ ,
- (iv)  $H^{\deg(\lambda)} = \mathfrak{S}(\mathrm{Sym}_A(I)_\nu)$  for all  $\nu \geq \eta$ .

Before going further, let us make two comments about the hypothesis of this theorem. First, it is proved in [3] that if  $X$  is not locally defined by  $n$  equations at some base point  $x$  (it is then locally defined by exactly  $n + 1$  equations) then the  $\mathcal{Z}$ -approximation complex used to get a resolution of the symmetric algebra is no longer acyclic in high degrees (actually acyclicity and acyclicity in high degrees are equivalent in this case). Moreover, notice that in this case,  $\pi(V) = \mathbb{P}_k^n$  as  $V$  contains  $\{x\} \times \mathbb{P}_k^n$ . Second, the degree bound  $\eta$  is sharp in the following sense

**Proposition 2.** *Under the hypotheses of Theorem 1,*

$$\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Sym}_A(I)_\eta) \text{ if and only if } \nu \geq \eta.$$

We will use the following lemma in the proof,

**Lemma 3.** *If  $J \subset I$  is a complete intersection ideal generated by  $n - 1$  forms of degree  $d$  and  $l \in \mathbb{Z}$ , then*

$$l < \mathrm{indeg}(I_X) \Leftrightarrow (I_X)_l = 0 \Leftrightarrow (I_X/J)_l = 0.$$

*Proof of Lemma 3.* Notice that as  $J$  is unmixed and  $\dim(A/J) > 0$ , it suffices to show that  $J_l \neq (I_X)_l$  for  $l = \mathrm{indeg}(I_X)$ . This is clearly the case if  $\mathrm{indeg}(I_X) < d = \mathrm{indeg}(J)$ . If  $\mathrm{indeg}(I_X) = d$ ,  $J_d = (I_X)_d$  implies that  $J_d = I_d$ , which in turn shows that  $\lambda$  factors through a rational map to  $\mathbb{P}_k^{n-2}$ , contradicting the assumption that the closed image of  $\lambda$  has dimension  $n - 1$ .  $\square$

*Proof of Proposition 2.* The case  $n = 2$  follows immediately from [3, §2], so we can assume that  $n \geq 3$ . By faithfully flat base change  $k \rightarrow \bar{k}$ , we may assume that  $k$  is algebraically closed, and in particular infinite. As  $\mathrm{Sym}_A(I)_\nu = 0$  for  $\nu < 0$  it suffices to prove that if  $\nu \geq 0$  then

$$\mathfrak{S}(\mathrm{Sym}_A(I)_\nu) = \mathfrak{S}(\mathrm{Sym}_A(I)_\eta) \Leftrightarrow \nu \geq \eta := (n - 1)(d - 1) - \mathrm{indeg}(I_X).$$

Let  $\nu \geq 0$ ,  $\mu := (n - 1)(d - 1) - \nu$  and  $D := \deg(\det(\mathcal{Z}_\eta)) = d^{n-1} - \sum_{x \in X} d_x$ . As  $\mathrm{codim}(I) = n - 1$ ,  $I$  is generated in degree  $d$  and  $k$  is infinite, there exists a complete intersection ideal  $J \subset I$  generated by  $n - 1$  forms of degree  $d$ . We will now prove that, if  $\mathcal{Z}_\nu$  is generically acyclic, one has

$$\deg(\det(\mathcal{Z}_\nu)) = D - \dim(I_X/J)_{\mu-1}.$$

This together with Lemma 3 establishes our claim.

Recall that for a finitely generated graded module  $M$  one has  $H_M(l) = P_M(l) + \sum_i (-1)^i h_m^i(M)_l$  ( $l \in \mathbb{Z}$ ), where  $P_M$ , respectively  $H_M$ , denotes the Hilbert polynomial, respectively the Hilbert function, of  $M$  and  $h_m^i(M)_l := \dim_k H_m^i(M)_l$ . We will denote by  $Z_i$  and  $H_i$ , the  $i$ -th module of cycles and  $i$ -th module of homology of the Koszul complex  $K(f_1, \dots, f_{n+1}; A)$ , respectively. The degree  $\nu$  part of the  $\mathcal{Z}$ -complex given by the  $f_i$ 's will be denoted by  $\mathcal{Z}_\nu$ . Recall that  $(\mathcal{Z}_i)_\nu = B \otimes_k (Z_i)_{\nu+id}$ .

As  $\mu \leq (n - 1)(d - 1) \leq (n - 1)d$ , [3, 4.1] shows that

$$\begin{aligned} H_{Z_0}(\nu) &= P_{Z_0}(\nu), \\ H_{Z_1}(\nu + d) &= P_{Z_1}(\nu + d) + h_m^0(H_1)_{\mu+d-1} - H_{H_2}(\mu + d - 1), \\ H_{Z_2}(\nu + 2d) &= P_{Z_2}(\nu + 2d) + h_m^0(H_0)_{\mu-1} - H_{H_1}(\mu - 1) + H_{H_2}(\mu - 1), \\ H_{Z_i}(\nu + id) &= P_{Z_i}(\nu + id) + \sum_j (-1)^{i+j} H_{H_j}(\mu - (i - 2)d - 1) \text{ for } 3 \leq i \leq n \end{aligned}$$

(notice that  $H_p$  coincides with  $Z_p$  in degree at most  $(p+1)d-1$ , and therefore  $H_m^n(Z_i)_{\nu+id} \simeq (Z_{n-i})_{\mu-(i-2)d-1}^*$  coincides with  $(H_{n-i})_{\mu-(i-2)d-1}^*$ ).

Set  $\chi(l) := \sum_j (-1)^j H_{H_j}(l)$  ( $l \in \mathbb{Z}$ ) and remark that

$$\begin{aligned} H_{Z_2}(\nu+2d) &= P_{Z_2}(\nu+2d) + \chi(\mu-1) - H_{R/I_X}(\mu-1), \\ H_{Z_i}(\nu+id) &= P_{Z_i}(\nu+id) + (-1)^i \chi(\mu - (i-2)d - 1) \text{ for } 3 \leq i \leq n. \end{aligned}$$

Now  $\sum_i (-1)^i P_{Z_i}(l+id) = 0$  for any  $l$  and  $P(l) := \sum_i (-1)^{i+1} P_{Z_i}(l+id)$  is constant and equal to  $D$ . If  $\mathcal{Z}_\nu$  is generically acyclic, then  $\sum_i (-1)^i H_{Z_i}(\nu+id) = 0$  and we can write using our expressions for  $H_{Z_i}$  :

$$\begin{aligned} \deg(\det(\mathcal{Z}_\nu)) &= \sum_i (-1)^{i-1} i H_{Z_i}(\nu+id) \\ &= P(\nu) + \sum_i (-1)^{i-1} (i-1) [H_{Z_i}(\nu+id) - P_{Z_i}(\nu+id)] \\ &= D + H_{R/I_X}(\mu-1) - \sum_{i \geq 0} (i+1) \chi(\mu - id - 1). \end{aligned}$$

It follows that

$$\deg(\det(\mathcal{Z}_\nu)) = D + H_{R/I_X}(\mu-1) - \sum_{i \geq 0} (i+1) \chi(\mu - id - 1).$$

The Hilbert series of  $\chi$  only depends on  $d$  (the degree of the  $f_i$ 's) and  $n$  and is given by

$$\sum \chi(l) t^l = \frac{(1-t^d)^{n+1}}{(1-t)^n},$$

which shows that the series corresponding to  $F(l) := \sum_{i \geq 0} (i+1) \chi(l-id)$  is

$$\frac{(1-t^d)^{n+1}}{(1-t)^n} \sum_{i \geq 0} (i+1) t^{id} = \frac{(1-t^d)^{n-1}}{(1-t)^n}.$$

Hence  $F = H_{R/J}$  and  $\deg(\det(\mathcal{Z}_\nu)) = D + H_{R/I_X}(\mu-1) - H_{R/J}(\mu-1)$  as claimed.  $\square$

**Remark 4.** *Under the hypotheses of Theorem 1, set  $\varepsilon := \text{indeg}(I_X)$ . One can expect that  $\mathfrak{S}(\text{Sym}_A(I)_{\eta-1})$  is not a torsion module unless  $I$  is  $n$  generated. This is indeed the case if  $I$  is saturated, or if  $I$  has no syzygy of degree less than  $d + \varepsilon$ , or if  $I_X = A$  (i.e.  $X = \emptyset$ ).*

It is known that the degree of the hypersurface  $\mathcal{H}$ , closure of the image of  $\lambda$ , is given by the formula (see for instance [5, 2.5])

$$\deg(\lambda) \deg(\mathcal{H}) = d^{n-1} - \sum_{x \in X} e_x.$$

In other words, we have a graded isomorphism of  $k[\mathbf{T}]$ -modules

$$(2) \quad H^{\deg(\lambda)} \simeq k[\mathbf{T}] \left( -d^{n-1} + \sum_{x \in X} e_x \right).$$

Therefore, under the assumption of Theorem 1, for all  $\nu \geq \eta$  the MacRae's invariant  $\mathfrak{S}(\text{Sym}_A(I)_\nu)$  is a principal ideal of  $k[\mathbf{T}]$  whose generator (unique up to multiplication by a non-zero constant in  $k$ ) equals  $H^{\deg(\lambda)}$  times an *extraneous homogeneous*

factor, say  $G \in B$  whose total degree is

$$\deg(G) = \sum_{x \in X} (e_x - d_x)$$

and whose support is the codimension 1 part of  $\pi(V - \Gamma)$ .

In the following section 2 we give an explicit description of this extraneous factor  $G$ . Then, in section 3 we relate the computation of the MacRae's invariant  $\mathfrak{S}(\text{Sym}_A(I)_\nu)$  to a resultant computation in the case of algebraic plane curves ( $n = 2$ ) and particular cases of space surfaces ( $n = 3$ ). As a byproduct, we will prove a conjecture stated in [4].

## 2. DESCRIPTION OF THE EXTRANEOUS FACTOR

Recall that  $S \subset X$  is the set of points where  $X$  is not locally a complete intersection. Assume that  $x = \text{Proj}(A/\mathfrak{p}) \in S$  is a base point locally defined by exactly  $n$  equations. Then one of the generators of  $I$  may be written in terms of the other ones in any sufficiently small open neighborhood of  $x$  (or equivalently in  $A_{\mathfrak{p}}$ ), and this unique relation (up to multiplication by a function invertible in  $A_{\mathfrak{p}}$ ) specializes at  $x$ , to give a non zero linear form  $L_x \in B_1 \otimes_k k(x)$ .

**Proposition 5.** *If the field  $k$  is algebraically closed and  $X$  is locally generated by at most  $n$  elements then, up to multiplication by a non-zero constant in  $k$ , we have*

$$G = \prod_{x \in S} L_x^{e_x - d_x} \in B.$$

Moreover, all syzygies  $a_1 T_1 + \dots + a_{n+1} T_{n+1} \in A \otimes_k B_1 \simeq A^{n+1}$  specializes at  $x$  to a multiple of  $L_x$ , and at least one does not specialize to 0.

*Proof.* We know that  $G$  defines a divisor supported on  $\pi(V - \Gamma)$ , and that  $V$  and  $\Gamma$  coincides outside  $S \times \mathbb{P}_k^n$ . Moreover,  $V$  is defined by linear forms in the  $T_i$ 's, in particular  $V - \Gamma$  is supported on a union of linear spaces over the points of  $S$ , therefore its closure is a scheme whose support is a union of linear spaces  $\{x\} \times V_x$  for  $x \in S$ . Now  $V_x \subseteq \{L_x = 0\}$ . It follows that  $G = \prod_{x \in S} L_x^{\mu_x}$  for some integers  $\mu_x \geq 0$  (which are either 0 if  $V_x \neq L_x$  or equal to  $\ell_{\mathcal{O}_{\mathbb{P}_k^{n-1} \times \mathbb{P}_k^n, \{x\} \times V_x}}(\mathcal{O}_{V, \{x\} \times V_x})$  else).

To determine  $\mu_x$ , we remark that this integer only depends on  $I_{\mathfrak{p}}$  and not on the whole ideal  $I$ . Let  $I_x := I_{\mathfrak{p}} \cap A$  be the defining ideal of the fat point defined by  $I$  locally at  $x$ . For  $d' \geq d$  at least equal to the maximal degree of a minimal generator of  $I_x$ , we may choose  $n + 1$  homogeneous forms  $g_1, \dots, g_{n+1}$  of degree  $d'$  in  $I_x$  such that  $I_x = (g_1, \dots, g_{n+1}) :_A \mathfrak{m}^\infty$ .

The elements  $g_1, \dots, g_{n+1}$  define a parameterization  $\lambda'$  of a hypersurface  $H'$  (for instance by [5, 2.5]), and, by construction,  $x$  is the unique base point of this parameterization. In this case we have seen that the extraneous factor, whose associated divisor has a contributions at  $x$  equal in both cases, has degree  $e'_x - d'_x = e_x - d_x > 0$  (the two invariants are local invariants at  $x$ , and only  $x$  contributes to  $G'$ ). This implies that  $V_x = L_x$  (so that all syzygies specializes to a multiple of  $L_x$  at  $x$ , and at least one not to 0) and  $\mu_x = e_x - d_x$ .  $\square$

## 3. LINK WITH RESULTANTS

In this section, we focus on the particular case where the ideal  $I \subset A = k[X_1, \dots, X_n]$  generated by the polynomials  $f_1, \dots, f_{n+1}$  defining the rational map (1) has projective dimension one. From the well-known Hilbert-Burch Theorem,  $I$  admits a finite free graded resolution of the form

$$(3) \quad 0 \rightarrow \bigoplus_{i=1}^n A(-\mu_i - d) \xrightarrow{M} A(-d)^{n+1} \xrightarrow{(f_1 \ f_2 \ \cdots \ f_{n+1})} A \rightarrow A/I \rightarrow 0$$

where  $\sum_{i=1}^n \mu_i = d$  and  $M$  is a matrix

$$M := \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n+1,1} & p_{n+1,2} & \cdots & p_{n+1,n} \end{pmatrix}$$

with entries  $p_{i,j} \in A$  of degree  $\mu_i$  respectively.

As a consequence of the Auslander-Buchsbaum formula, the ideal  $I$  may have projective dimension one only if  $n = 2$  or  $n = 3$  (remember that we assumed that  $\dim(A/I) \leq 1$ ). It turns out that in these two cases, corresponding respectively to plane curves and space surfaces which are of great interest for applications in Computer Aided Geometric Design, our approach to the implicitization problem via Theorem 1 coincides with a resultant computation. More precisely, defining the polynomials

$$L_i := \sum_{j=1}^{n+1} p_{j,i} T_j \in A[\mathbf{T}] \quad \text{for all } i = 1, \dots, n$$

where  $L_i$  is a bi-homogeneous polynomial of bi-degree  $(\mu_i; 1)$  in  $A[\mathbf{T}] = k[\mathbf{X}][\mathbf{T}]$ , we have the

**Proposition 6** ( $n = 2$  or  $3$ ). *Assume that the ideal  $I$  is locally generated by at most  $n$  elements and has projective dimension one with the free resolution (3). Then, with the above notation we have*

$$(\text{Res}(L_1, \dots, L_n)) = \mathfrak{S}(\text{Sym}_A(I)_\nu) \subset k[\mathbf{T}]$$

for all  $\nu \geq \eta$ , where this resultant eliminates the homogeneous variables  $X_1, \dots, X_n$ .

*Proof.* Define by  $J$  the ideal generated by polynomials  $L_1, \dots, L_n$  in the polynomial ring  $A[T_1, \dots, T_n]$  and denote by  $B$  the quotient algebra  $A[T_1, \dots, T_n]/J$ .

On the one hand, we have, by definition of the symmetric algebra,  $B \simeq \text{Sym}_A(I)$  as bi-graded  $A[\mathbf{T}]$ -modules and on the other hand we know that  $\text{Res}(L_1, \dots, L_n) = \mathfrak{S}(B_\nu)$ , for all  $\nu \gg 0$ , as soon as the sequence  $(L_1, \dots, L_n)$  is regular in  $A[\mathbf{T}]$  (see e.g. [7, §3.5]). We deduce that the claimed result will be proved if we show that  $\text{Sym}_A(I) \simeq B$  is a complete intersection. To do this, we use the following criterion due to Avramov [1]: *Sym<sub>A</sub>(I) is a complete intersection if and only if for all integer  $r \in \{1, \dots, n\}$  we have  $\text{depth}_{I_r(M)}(A) \geq n - r + 1$  where  $I_r(M)$  denotes the ideal of  $A$  generated by all the  $r \times r$ -minors of the matrix  $M$ .*

In the case  $n = 2$ ,  $M$  is a  $2 \times 3$  matrix and the Hilbert-Burch Theorem asserts that  $\text{depth}_{I_2(M)}(A) \geq 2$ . Since  $I_2(M) \subset I_1(M)$ , we have  $\text{depth}_{I_1(M)}(A) \geq \text{depth}_{I_2(M)}(A)$  and we obtain that  $\text{Sym}_A(I)$  is a complete intersection in this case.

For  $n = 3$ , the Hilbert-Burch Theorem shows that  $\text{depth}_{I_3(M)} = \text{depth}_I(A) \geq 2$  since we assumed that  $I$  defines isolated points. It follows that

$$\text{depth}_{I_1(M)} \geq \text{depth}_{I_2(M)} \geq \text{depth}_{I_3(M)} \geq 2.$$

Moreover, as  $I$  is supposed to be locally generated by at most 3 elements we deduce that  $V(I_1(M)) = \emptyset$  (a fact that we have already noted in section 2), i.e.  $\text{depth}_{I_1(M)}(A) \geq 3$ . Therefore,  $\text{Sym}_A(I)$  is also a complete intersection in this case<sup>2</sup>.  $\square$

As a consequence, note that  $\text{Res}(L_1, \dots, L_n)$  only depends, up to a non-zero multiplicative constant in  $k$ , on the first module of syzygies of  $I$  and not on the specific choice of the matrix  $M$ . Before illustrating this proposition with an example, we briefly explicit its consequences for the implicitization problem for rational algebraic plane curves and space surfaces.

*The case of plane curves ( $n = 2$ ).* When  $\lambda$  parameterizes a plane curve, the ideal  $I$  always admits a free resolution of the form (3) by the Hilbert's syzygies Theorem. Moreover,  $I$  is also always locally generated by at most 2 elements. Therefore, this proves that as soon as  $\lambda$  is generically finite, the resultant of  $L_1$  and  $L_2$  (which always exists) will always compute an implicit equation of the image of  $\lambda$  to the power  $\deg(\lambda)$  without any extraneous factor. We thus recover the "moving lines" method of Sederberg.

*The case of space surfaces ( $n = 3$ ).* When  $\lambda$  parameterizes a space surface the situation is a little more intricate. It is known that the ideal  $I$  admits a resolution of the form (3) if it defines some isolated base points (at least one) and if it is saturated with respect to the maximal ideal  $(X_1, X_2, X_3)$  of  $A$ . So we just proved the

**Corollary 7.** *If the field  $k$  is algebraically closed, the ideal  $I$  has codimension 2, is saturated and is locally generated by at most 3 elements, then  $I$  admits a resolution like (3) and we have, with the above notation,*

$$\text{Res}(L_1, L_2, L_3) = S(T_1, \dots, T_4)^{\deg(\lambda)} \prod_{\substack{\mathfrak{p} \in \text{Proj}(A/I) \\ \text{such that } e_{\mathfrak{p}} > d_{\mathfrak{p}}}} L_{\mathfrak{p}}(T_1, \dots, T_4)^{e_{\mathfrak{p}} - d_{\mathfrak{p}}},$$

where  $S \in k[\mathbf{T}]$  denotes an implicit equation of the image of the rational map  $\lambda$ .

This corollary proves [4, conjecture 5.1] that we illustrate with the first following example which is taken from [4]. The second following example emphasizes that the extraneous factor  $G$  splits into a product of linear forms over the algebraic closure  $\bar{k}$  of the base field  $k$ , that is to say that it splits in  $B \otimes_k \bar{k}$  and not necessarily in  $B$ .

**Example 8.** *Consider the parameterization given by*

$$\begin{aligned} f_1 &= X_1 X_2^2 - X_2^3 - X_2 X_3^2, \\ f_2 &= X_2^3 - X_1 X_2 X_3 - X_2^2 X_3 + X_2 X_3^2 + X_3^3, \\ f_3 &= X_1 X_2 X_3 - 2X_2 X_3^2, \\ f_4 &= X_2^2 X_3 - 2X_2 X_3^2 + X_3^3. \end{aligned}$$

---

<sup>2</sup>Point out here that if  $I$  was locally minimally generated by four elements at some point, then  $V(I_1(M))$  would be non-empty,  $\text{depth}_{I_1(M)}(A) = 2$  and hence  $\text{Res}(L_1, \dots, L_n) \equiv 0$  in  $k[\mathbf{T}]$ .



Using `Macaulay2` [6], it is easy to compute that  $I$  is a saturated ideal with free resolution (3) where the matrix  $M$  is given by:

$$M = \begin{pmatrix} -X_3 & X_2 & 0 \\ -X_3 & -X_1 + X_2 + X_3 & -X_3 \\ X_2 - X_3 & -X_1 - X_2 + 2X_3 & -X_3 \\ X_3 & X_1 - X_3 & X_2 + X_3 \end{pmatrix}.$$

One can also show that  $V(I)$  consists of the two projective points  $p = (1 : 0 : 0)$  and  $q = (2 : 1 : 1)$ . The resultant of  $L_1, L_2, L_3$  is the determinant

$$\det \begin{pmatrix} 0 & -T_2 - T_3 + T_4 & 0 \\ T_3 & T_1 + T_2 - T_3 & T_4 \\ -T_1 - T_2 - T_3 + T_4 & T_2 + 2T_3 - T_4 & -T_2 - T_3 + T_4 \end{pmatrix}$$

since  $L_1, L_2$  and  $L_3$  are linear forms in the  $X_i$ 's. It can be factorized into the product

$$(-T_2T_3 - T_3^2 + T_1T_4 + T_2T_4 + 2T_3T_4 - T_4^2)(T_2 + T_3 - T_4)$$

where we easily identify  $G(T_1, \dots, T_4) = T_2 + T_3 - T_4$ . Looking at the matrix  $M$  evaluated at the point  $p$  we find

$$M(p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore the syzygy module at this point is generated by  $L_p := -T_2 - T_3 + T_4$  which equals  $-G$ , as predicted. It turns out that the rank of the matrix  $M$  evaluated at  $q$  is 2; this point is hence a complete intersection and we have  $e_q = d_q$ .

**Example 9.** Consider the following  $4 \times 3$ -matrix

$$M := \begin{pmatrix} X_1^2 + X_3^2 & X_2 & X_2X_3 \\ X_1^2 - X_1X_2 & X_1 + 2X_2 & X_1^2 + X_3^2 \\ X_1X_2 + X_1X_3 & 2X_2 + X_3 & X_2^2 \\ X_1X_3 + X_2^2 & X_3 & X_1X_2 \end{pmatrix}$$

whose ideal  $I_3(M) \subset A$  of  $3 \times 3$  minors is easily seen to have codimension 2. This matrix fits into the Hilbert-Burch exact complex (3) and thus defines homogeneous polynomials  $f_1, f_2, f_3$  and  $f_4$  of degree 5 in  $A$  which themselves parameterize a space rational surface.

Since we easily check that  $V(I_1(M)) = \emptyset$ , we can apply corollary 7. The computation of the resultant of  $L_1, L_2$  and  $L_3$  can be achieved via the computation of the determinant of the  $\mathcal{Z}$ -approximation complex (which is here nothing but the Koszul complex associated to the sequence  $L_1, L_2, L_3$  as explained in the proof of Proposition 6) taken in degree  $5 - 2 = 3$  (one can also recover this degree using resultant theory which gives  $2 + 2 + 1 - 2 = 3$ ). We find that this resultant is the ratio of two determinants which can be factored in  $\mathbb{Q}[T_1, T_2, T_3, T_4]$  as the product of two irreducible polynomials: an implicit equation of the parameterized surface, which is thus found to be properly parameterized and has degree 6, and an extraneous factor  $G := T_2^2 + 2T_3T_4 + T_3^2 + T_4^2$ . It turns out that, as predicted by Proposition 5,  $G$  splits into the product of two linear forms over the complex numbers field  $\mathbb{C}$ :

$$G = (T_2 - iT_3 - iT_4)(T_2 + iT_3 + iT_4) \in \mathbb{C}[T_1, T_2, T_3, T_4]$$

where  $i$  denotes the complex number such that  $i^2 = -1$  and  $i \neq 1$ . Also, evaluating the matrix  $M$  at the two projective points  $(X_1 : X_2 : X_3) = (1 : 0 : \epsilon i)$ , where  $\epsilon = \pm 1$ , we find

$$M(1 : 0 : \epsilon i) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \epsilon i & \epsilon i & 0 \\ \epsilon i & \epsilon i & 0 \end{pmatrix}$$

which shows that these two points are base points of the parameterization (the last column is the null vector) which are almost complete intersection since the rank of  $M(1 : 0 : \epsilon i)$  equals 1; moreover, the two linear forms in the factorization of  $G$  over  $\mathbb{C}$  can be read from a non-zero column of  $M(1 : 0 : \epsilon i)$ . As a consequence of our results, we also deduce that the other base points of this parameterization are complete intersection base points.

## REFERENCES

- [1] Luchezar L. Avramov. Complete intersections and symmetric algebras. *J. Algebra*, 73(1):248–263, 1981.
- [2] Nicolas Bourbaki. *Éléments de mathématique*. Masson, Paris, 1983. Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets. [Commutative algebra. Chapter 8. Dimension. Chapter 9. Complete Noetherian local rings].
- [3] Laurent Busé and Marc Chardin. Implicitizing rational hypersurfaces using approximation complexes. *J. Symbolic Computation* 40, 1150–1168 (2005).
- [4] Laurent Busé, David Cox, and Carlos D’Andrea. Implicitization of surfaces in  $\mathbb{P}^3$  in the presence of base points. *J. Algebra Appl.*, 2(2):189–214, 2003.
- [5] Laurent Busé and Jean-Pierre Jouanolou. On the closed image of a rational map and the implicitization problem. *J. Algebra*, 265(1):312–357, 2003.
- [6] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [7] Jean-Pierre Jouanolou. Aspects invariants de l’élimination. *Adv. Math.*, 114(1):1–174, 1995.
- [8] D. G. Northcott. *Finite free resolutions*. Cambridge University Press, Cambridge, 1976. Cambridge Tracts in Mathematics, No. 71.

INRIA SOPHIA-ANTIPOLIS, 2004 ROUTE DES LUCIOLES, B.P. 93, 06902 SOPHIA-ANTIPOLIS CEDEX, FRANCE

*E-mail address:* [lbuse@sophia.inria.fr](mailto:lbuse@sophia.inria.fr)

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, CNRS ET UNIVERSITÉ PIERRE ET MARIE CURIE, 4 PLACE JUSSIEU, F-75252 PARIS CEDEX 05, FRANCE

*E-mail address:* [chardin@math.jussieu.fr](mailto:chardin@math.jussieu.fr)

UNIVERSITÉ LOUIS PASTEUR, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE

*E-mail address:* [jouanol@math.u-strasbg.fr](mailto:jouanol@math.u-strasbg.fr)