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# Binding Logic: proofs and models

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**Abstract.** We define an extension of predicate logic, called *Binding Logic*, where variables can be bound in terms and in propositions. We introduce a notion of model for this logic and prove a soundness and completeness theorem for it. This theorem is obtained by encoding this logic back into predicate logic and using the classical soundness and completeness theorem there.

In predicate logic, only a quantifier can bind a variable. Function and predicate symbols cannot. In contrast, in the informal language of mathematics and in languages used in computer science, binders are pervasive, as they are used to express functions, relations, definite descriptions, fix-points, sums, products, integrals, derivatives, etc.

There are many ways to express such constructions without explicit binding symbols using, for instance, combinators [7], de Bruijn indices [4], higher-order abstract syntax [18, 21], ... However, as binders are everywhere in mathematics and in computer science, we believe that they should be studied as a basic notion, and not just encoded. The first contribution of this paper is to define a logic with binders, called *Binding Logic*, that is a simple extension of predicate logic: proof rules are the same and no extra axiom is added.

A first question, about binding logic, is that of the provability of the extensionality scheme

$$(\forall x (t = u)) \Rightarrow \Lambda x t = \Lambda x u$$

where  $\Lambda$  is a unary function symbol binding one variable in its argument. This scheme cannot be derived in a simple way from the axiom of equality [15]:

$$\forall y \forall z (y = z \Rightarrow \Lambda x y = \Lambda x z)$$

because substituting  $t$  for  $y$  and  $u$  for  $z$  renames the variable  $x$  to avoid captures. We shall prove that this scheme is indeed independent. Thus, if we want binders to be extensional, we must add this scheme, but Binding logic can also be used, if we do not, for instance, when we define weak reduction in  $\lambda$ -calculus or when we want to distinguish various algorithms that implement the same function.

To prove such independence results, we want to construct counter-models. Thus the notion of model of predicate logics needs to be extended to Binding logic. The last and main contribution of this paper is to define such a notion of model, and prove that is sound and complete with respect to deduction.

In predicate logic, a term  $t$  that contains a free variable  $x$  does not have a unique denotation in a given model  $\mathcal{M}$ . Instead, for each element  $a$  of the domain  $M$  of this model, we can define a denotation  $\llbracket t \rrbracket_{a/x}$ . It is tempting to define the absolute denotation of the term  $t$  as the function from  $M$  to  $M$  mapping  $a$  to  $\llbracket t \rrbracket_{a/x}$ . Then, we would obtain the denotation of  $\lambda x t$  by applying the denotation of the symbol  $\lambda$  to the function denoted by the term  $t$ . The denotation of the symbol  $\lambda$  would then be a functional  $\hat{\lambda}$ . However, with such a notion of model, some non provable propositions are valid in all models, e.g. the extensionality scheme. Indeed, if  $t$  and  $u$  are two terms containing the free variable  $x$  such that  $\forall x (t = u)$  is valid, then  $\llbracket t \rrbracket_{a/x}$  and  $\llbracket u \rrbracket_{a/x}$  are always equal. Hence the terms  $t$  and  $u$  denote the same function, and thus the terms  $\lambda x t$  and  $\lambda x u$  have the same denotation. As we want to have the choice to take this extensionality scheme or not, it should be valid in some models, but not all. In other words, extensionality should not be wired in the notion of model. Thus we need to consider a more general notion of model, leaving room to interpret a term containing a free variable as an “intensional function”.

When, in the term  $\lambda x t$ , the body  $t$  contains other free variables than  $x$ , we interpret it as an intensional function of several arguments. Thus, to be able to define the denotation of the term  $\lambda x t$ , we must be able to apply the functional  $\hat{\lambda}$  to intensional functions of various arities. Hence, we shall define the denotation of  $\lambda$  as a family of functionals  $\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2, \dots$  each of these taking as arguments functions of some arity.

Once the notion of model is defined, we enter the proof of the completeness theorem. The proof proceeds by giving a provability preserving translation from binding logic to a formulation of predicate logic, called *Deduction modulo*. This translation is based on de Bruijn indices and explicit substitutions. Then, we relate the models of a theory in binding logic with those of its translation in Deduction modulo. It should be noticed that de Bruijn indices and explicit substitutions are used in the proof of the completeness theorem, but not in the definition of the notion of model itself. Hence we can imagine other completeness proofs using other methods such as a translation to higher-order logic, based on higher-order abstract syntax or a direct construction of a model *à la* Henkin. A potential difficulty in such a direct proof is to construct the elements of the model, that have to be intensional, and thus must differ from the functions used in Henkin’s construction [14, 2].

## 1 Binding logic

**Definition 1 (Language).** A language is a set of function symbols and predicate symbols, to each of which is associated a finite sequence of natural numbers called its binding arity and denoted by  $\langle k_1, \dots, k_n \rangle$  [19, 20, 13].

With respect of the standard vocabulary of predicate logic, a function symbol or a predicate symbol of arity  $n$  has binding arity  $\underbrace{\langle 0, \dots, 0 \rangle}_n$ .

**Definition 2 (Terms).** Let  $\mathcal{L}$  be a language and  $\mathcal{V}$  be an infinite set of variables. Terms are inductively defined as follows:

- variables are terms,
- if  $t_1, \dots, t_n$  are terms,  $f$  is a function symbol of binding arity  $\langle k_1, \dots, k_n \rangle$ ,  $x_1^1, \dots, x_{k_1}^1, \dots, x_1^n, \dots, x_{k_n}^n$  are distinct variables, then:

$$f(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n)$$

is a term.

In the term  $f(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n)$  the variables  $x_1^i \cdots x_{k_i}^i$  are simultaneously bound in the terms  $t_i$ .

A classical example is the binder of the lambda calculus; with our notations, the lambda term  $\lambda x t$  is denoted  $\lambda(x t)$  where the function symbol  $\lambda$  is of binding arity  $\langle 1 \rangle$ .

**Definition 3 (Propositions).** Propositions are inductively defined as follows:

- if  $t_1, \dots, t_n$  are terms,  $P$  is a predicate symbol of binding arity  $\langle k_1, \dots, k_n \rangle$ ,  $x_1^1, \dots, x_{k_1}^1, \dots, x_1^n, \dots, x_{k_n}^n$  are distinct variables then:

$$P(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n)$$

is a proposition,

- if  $A$  and  $B$  are propositions, then  $A \Rightarrow B$ ,  $A \wedge B$ ,  $A \vee B$ ,  $\perp$ ,  $\forall x A$  and  $\exists x A$  are propositions.

Free and bound variables are defined as usual.

**Definition 4 (Grafting).** Let  $t$  be a term (resp. a proposition) and  $\theta = t_1/x_1, \dots, t_n/x_n$  be a finite function mapping variables to terms. The term (resp. the proposition)  $\langle \theta \rangle t$  is defined as follows:

- $\langle \theta \rangle x_i = t_i$ ,
- $\langle \theta \rangle x = x$  if  $x$  is not in the domain of  $\theta$ ,
- $\langle \theta \rangle f(x_1^1 \cdots x_{k_1}^1 u_1, \dots, x_1^n \cdots x_{k_n}^n u_n)$   
 $= f(x_1^1 \cdots x_{k_1}^1 \langle \theta|_{V \setminus \{x_1, \dots, x_n\}} \rangle u_1, \dots, x_1^n \cdots x_{k_n}^n \langle \theta|_{V \setminus \{x_1, \dots, x_n\}} \rangle u_n)$ ,  
where  $f$  is a function symbol or a predicate symbol,
- $\langle \theta \rangle (A \Rightarrow B) = \langle \theta \rangle A \Rightarrow \langle \theta \rangle B$ ,  $\langle \theta \rangle (A \wedge B) = \langle \theta \rangle A \wedge \langle \theta \rangle B$ ,  
 $\langle \theta \rangle (A \vee B) = \langle \theta \rangle A \vee \langle \theta \rangle B$ ,  $\langle \theta \rangle \perp = \perp$ ,  
 $\langle \theta \rangle \forall x A = \forall x \langle \theta \rangle A$ ,  $\langle \theta \rangle \exists x A = \exists x \langle \theta \rangle A$ .

Grafting allows captures. We introduce next substitution, that avoids them.

**Definition 5 ( $\alpha$ -equivalence).** The  $\alpha$ -equivalence relation is inductively defined as follows:

- $x \sim x$ ,
- if  $f$  is a function symbol or a predicate symbol, then:

$$f(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n) \sim f(x_1'^1 \cdots x_{k_1}'^1 t'_1, \dots, x_1'^n \cdots x_{k_n}'^n t'_n)$$

if for all  $i$   $\langle y_1/x_1^i, \dots, y_{k_i}/x_{k_i}^i \rangle t_i \sim \langle y_1/x_1'^i, \dots, y_{k_i}/x_{k_i}'^i \rangle t'_i$  for a sequence of distinct variables  $y_1, \dots, y_{k_i}$  not appearing (free or bound) in  $t_i$  and  $t'_i$ ,

- $A \Rightarrow B \sim A' \Rightarrow B'$ ,  $A \wedge B \sim A' \wedge B'$  and  $A \vee B \sim A' \vee B'$ , if  $A \sim A'$  and  $B \sim B'$ ,
- $\perp \sim \perp$ ,
- $\forall x A \sim \forall x' A'$  and  $\exists x A \sim \exists x' A'$  if  $\langle y/x \rangle A \sim \langle y/x' \rangle A'$  for a variable  $y$  not appearing (free or bound) in  $A$  and  $A'$ .

It is easy to check that  $\alpha$ -equivalence is an equivalence relation. From now on, terms will be considered up to  $\alpha$ -equivalence.

**Definition 6 (Substitution).** Let  $t$  be a term (resp. a proposition) and  $\theta = t_1/x_1, \dots, t_n/x_n$  a finite function mapping variables to terms. The term (resp. the proposition)  $\theta t$  is defined as follows:

- $\theta x_i = t_i$ ,
- $\theta x = x$  if  $x$  is not in the domain of  $\theta$ ,
- if  $f$  is a function symbol or a predicate symbol,  
 $\theta f(x_1^1 \cdots x_{k_1}^1 u_1, \dots, x_1^n \cdots x_{k_n}^n u_n) =$   
 $f(y_1^1 \cdots y_{k_1}^1 \theta \langle y_1^1/x_1^1, \dots, y_{k_1}^1/x_{k_1}^1 \rangle u_1, \dots, y_1^n \cdots y_{k_n}^n \theta \langle y_1^n/x_1^n, \dots, y_{k_n}^n/x_{k_n}^n \rangle u_n)$   
 where  $y_1^1, \dots, y_{k_1}^1, \dots, y_1^n, \dots, y_{k_n}^n$  are distinct variables appearing neither free nor bound neither in  $f(x_1^1 \cdots x_{k_1}^1 u_1, \dots, x_1^n \cdots x_{k_n}^n u_n)$  nor in  $\theta$ .
- $\theta(A \Rightarrow B) = \theta A \Rightarrow \theta B$ ,  $\theta(A \wedge B) = \theta A \wedge \theta B$ ,  $\theta(A \vee B) = \theta A \vee \theta B$ ,  $\theta \perp = \perp$ ,  
 $\theta \forall x A = \forall y \theta \langle y/x \rangle A$  where  $y$  is a variable appearing neither free nor bound neither in the proposition nor in  $\theta$ ,  
 $\theta \exists x A = \exists y \theta \langle y/x \rangle A$  where  $y$  is a variable appearing neither free nor bound neither in the proposition nor in  $\theta$ .

Notice that if  $\theta = t_1/x_1, \dots, t_n/x_n$  is a finite function mapping variables to terms and  $t$  a term or a proposition, we write  $\langle \theta \rangle t$  or  $\langle t_1/x_1, \dots, t_n/x_n \rangle t$  for the grafting of  $\theta$  in  $t$  and  $\theta t$  or  $(t_1/x_1, \dots, t_n/x_n)t$  for the substitution of  $\theta$  in  $t$ .

**Definition 7 (Sequent calculus).** A sequent is a pair of multisets of propositions. The sequent calculus rules of binding logic are given figure 1. There rules are the same than those of predicate logic, except that, in the quantifier rules, the substitution avoids captures not only with respect to quantified variables, but also with respect to variables bound by the function and predicate symbols.

$$\begin{array}{c}
\frac{}{A \vdash A} \text{ axiom} \\
\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left} \\
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left} \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow\text{-left} \\
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\text{-left} \\
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\text{-left} \\
\frac{}{\Gamma, \perp \vdash \Delta} \perp\text{-left} \\
\frac{\Gamma, (t/x)A \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall\text{-left} \\
\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists\text{-left if } x \notin FV(\Gamma\Delta)
\end{array}
\qquad
\begin{array}{c}
\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta} \text{ cut} \\
\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right} \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right} \\
\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow\text{-right} \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge\text{-right} \\
\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee\text{-right} \\
\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta} \forall\text{-right if } x \notin FV(\Gamma\Delta) \\
\frac{\Gamma \vdash (t/x)A, \Delta}{\Gamma \vdash \exists x A, \Delta} \exists\text{-right}
\end{array}$$

**Fig. 1.** Sequent calculus of Binding Logic

*Example 1.* The equational theory of  $\lambda$ -calculus can be defined in binding logic as follows. The language contains a predicate  $=$  of arity  $\langle 0, 0 \rangle$ , a function symbol  $\alpha$  of arity  $\langle 0, 0 \rangle$  and a function symbol  $\lambda$  of arity  $\langle 1 \rangle$ . We take the axioms of equality: reflexivity, symmetry, transitivity, compatibility with  $\alpha$  and  $\lambda$  [15]:

$$\forall y \forall z (y = z \Rightarrow \lambda x y = \lambda x z)$$

and the axiom scheme  $\beta$

$$\forall x (\alpha(\lambda x t, x) = t)$$

Notice here that we take a different axiom for each term  $t$ , but that the argument of the function is simply expressed by a variable.

We may then add various extensionality axioms: the extensionality scheme

$$(\forall x (t = u)) \Rightarrow \lambda x t = \lambda x u$$

or the axiom  $\eta$

$$\forall x \forall y (\alpha(\lambda x y, x) = y)$$

or the strong extensionality axiom

$$(\forall x (\alpha(f, x) = \alpha(g, x))) \Rightarrow f = g$$

yielding different theories.

*Example 2.* The equational theory of the disjoint sum can be defined as follows. The language contains a predicate symbol  $=$  of arity  $\langle 0, 0 \rangle$ , function symbols  $i$  and  $j$  of arity  $\langle 0 \rangle$  and  $\delta$  of arity  $\langle 0, 1, 1 \rangle$ . The theory is formed with the axioms of equality and the schemes

$$\delta(i(x), x u, y v) = u$$

$$\delta(j(y), x u, y v) = v$$

*Example 3.* An elementary theory of derivatives can be defined as follows. Our goal here is to stick to Leibniz notations, i.e. to differentiate algebraic expressions with respect to some variable, not functions. The language contains a predicate symbol  $=$  of arity  $\langle 0, 0 \rangle$ , constants 0 and 1 of arity  $\langle \rangle$ , function symbols  $+$  and  $\times$  of arity  $\langle 0, 0 \rangle$  and a function symbol  $D$  of arity  $\langle 1, 0 \rangle$ .

The informal meaning is that  $D(x \ t, u)$  is the derivative of the expression  $t$  with respect to  $x$  taken at  $u$ . The term  $D(x \ t, u)$  is also written  $\frac{\partial t}{\partial x}(u)$ .

The axioms are

$$\forall z \left( \frac{\partial z}{\partial x}(x) = 1 \right)$$

$$\forall x \left( \frac{\partial y}{\partial x}(x) = 0 \right)$$

and the schemes

$$\forall x \left( \frac{\partial(t + u)}{\partial x}(x) = \frac{\partial t}{\partial x}(x) + \frac{\partial u}{\partial x}(x) \right)$$

$$\forall x \left( \frac{\partial(t \times u)}{\partial x}(x) = t \times \frac{\partial u}{\partial x}(x) + \frac{\partial t}{\partial x}(x) \times u \right)$$

## 2 Binding models

As already discussed in the introduction, the denotation of a term containing a free variable  $x$  in a model  $\mathcal{M}$  of domain  $M$  must be an “intensional function” from  $M$  to  $M$ . An intensional function can be any mathematical object, provided it can be applied to an element of  $M$  and composed with other intensional functions. Thus, besides  $M$ , we shall introduce in the definition of the model, another set  $M_1$  whose elements are called “intensional functions” and functions from  $M_1 \times M$  to  $M$ , and from  $M_1 \times M_1$  to  $M_1$  for application and composition.

When a term  $t$  contains more than one free variable, it must be interpreted as an intensional function of several arguments. Hence, we introduce also in the definition of the model sets  $M_2, M_3, \dots$ . The elements of  $M_n$  for  $n \geq 1$  are called “intensional functions of  $n$  arguments”. The set  $M$  is then written  $M_0$ . Among the elements of  $M_n$ , if  $n \geq 1$ , we need the  $n$  projections  $\mathbf{1}_n, \dots, \mathbf{n}_n$ . We also need functions to compose the intensional functions and to apply them to elements of  $M$ . The function  $\square_{0,n}$  applies an element of  $M_n$  to a  $n$ -uple of elements of  $M_0$ . The function  $\square_{p,n}$  composes an element of  $M_n$  with  $n$  elements of  $M_p$  and the result is an element of  $M_p$ .

This leads to the following definition.

**Definition 8 (Intensional functional structure).** A intensional functional structure is a family of sets  $M_0, M_1, M_2, \dots$  together with

- for each natural number  $n$ , elements  $\mathbf{1}_n, \dots, \mathbf{n}_n$  of  $M_n$ ,
- for each  $n$  and  $p$ , a function  $\square_{p,n}$  from  $M_n \times M_p^n$  to  $M_p$

verifying the properties

$$\begin{aligned} \mathbf{i}_n \square_{p,n} \langle a_1, \dots, a_n \rangle &= a_i \\ a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle &= a \\ (a \square_{p,n} \langle b_1, \dots, b_n \rangle) \square_{q,p} \langle c_1, \dots, c_p \rangle &= \\ &= a \square_{q,n} \langle b_1 \square_{q,p} \langle c_1, \dots, c_p \rangle, \dots, b_n \square_{q,p} \langle c_1, \dots, c_p \rangle \rangle \end{aligned}$$

*Remark 1.* Intensional function structures are multicategories with projections (see [16, 17] and also [3]) with a single object  $A$ , the set  $M_n$  being the set of multiarrows from  $A^n$  to  $A$ . If we wanted to define a model for many-sorted binding logic, then we would need many-sorted intensional functional structures that would exactly be multicategories with projections.

*Example 4.* Consider any set  $A$ . Let  $M_n$  be the functions from  $A^n$  to  $A$ ,

- let  $\mathbf{i}_n$  be the function mapping  $\langle a_1, \dots, a_n \rangle$  to  $a_i$ ,
- and  $\square_{p,n}$  be the function mapping  $f$  (from  $A^n$  to  $A$ ) and  $g_1, \dots, g_n$  (from  $A^p$  to  $A$ ) to the function mapping  $\langle a_1, \dots, a_p \rangle$  to  $f(g_1(a_1, \dots, a_p), \dots, g_n(a_1, \dots, a_p))$ .

This structure is an intensional functional structure, as the properties are obviously verified.

**Definition 9 (Binding model).**

A model of a language  $\mathcal{L}$  in binding logic is given by

1. an intensional functional structure,
2. for each function symbol of binding arity  $\langle k_1, \dots, k_n \rangle$  a sequence of functions  $\hat{f}_0, \dots, \hat{f}_p, \dots$  with  $\hat{f}_p$  in  $(\prod_{i=1}^n M_{p+k_i}) \rightarrow M_p$  such that

$$\hat{f}_p(a_1, \dots, a_n) \square_{q,p} \langle b_1, \dots, b_p \rangle = \hat{f}_q(a_1 \square_{q+k_1,p+k_1} \uparrow_{q,k_1,p} (\langle b_1, \dots, b_p \rangle), \dots)$$

where

$$\begin{aligned} \uparrow_{q,k,p} (\langle b_1, \dots, b_p \rangle) &= \langle \mathbf{1}_{q+k}, \dots, \mathbf{k}_{q+k}, b_1 \square_{q+k,q} S, \dots, b_p \square_{q+k,q} S \rangle \\ S &= \langle (\mathbf{1} + \mathbf{k})_{q+k}, \dots, (\mathbf{q} + \mathbf{k})_{q+k} \rangle \end{aligned}$$

3. for each predicate symbol a function  $\hat{P}$  in  $(\prod_{i=1}^n M_{k_i}) \rightarrow \{0, 1\}$ .

Notice that no variable can be bound outside a predicate symbol by a function symbol or a another predicate symbol. Hence the denotation of a predicate symbol is a single function and not a family of functions.

*Example 5.* If we have a single symbol  $A$  of binding arity  $\langle 1 \rangle$ , then the denotation of this symbol is a family of functions  $\hat{A}_i$  from  $M_{i+1}$  to  $M_i$  verifying, among others, the equations

$$\begin{aligned} \hat{A}_p(a) &= (\hat{A}_p(a) \square_{p+1,p} \langle \mathbf{2}, \dots, \mathbf{p} + \mathbf{1} \rangle) \square_{p,p+1} \langle \mathbf{1}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p} \rangle \\ &= \hat{A}_{p+1}(a \square_{p+2,p+1} \langle \mathbf{1}, \mathbf{3}, \mathbf{4}, \dots, \mathbf{p} + \mathbf{2} \rangle) \square_{p,p+1} \langle \mathbf{1}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p} \rangle \end{aligned}$$



Calling  $I_p$  the function from  $M_p$  to  $M_{p+1}$  mapping  $a$  to  $a \sqcup \langle \mathbf{1}, \mathbf{3}, \dots, \mathbf{p} + \mathbf{1} \rangle$  and  $D_p$  the function from  $M_{p+1}$  to  $M_p$  mapping  $a$  to  $a \sqcup \langle \mathbf{1}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p} \rangle$   
 We have

$$\hat{A}_p(a) = D_p(\hat{A}_{p+1}(I_{p+1}(a)))$$

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{I_1} & M_2 & \xrightarrow{I_2} & M_3 & \xrightarrow{I_3} & M_4 & \xrightarrow{I_4} & M_5 & \xrightarrow{I_5} & \dots \\ \hat{A}_0 \downarrow & & \hat{A}_1 \downarrow & & \hat{A}_2 \downarrow & & \hat{A}_3 \downarrow & & \hat{A}_4 \downarrow & & \\ M_0 & \xleftarrow{D_0} & M_1 & \xleftarrow{D_1} & M_2 & \xleftarrow{D_2} & M_3 & \xleftarrow{D_3} & M_4 & \xleftarrow{D_4} & \dots \end{array}$$

Each element in the family  $\hat{A}_0, \hat{A}_1, \dots$  adds information to the denotation of  $A$ . This extension is coherent in the sense that the function  $A_p$  can be deduced from  $A_{p+1}$ .

*Remark 2.* This definition is reminiscent of that of [13] where a type constructor  $\delta$  for context extension is introduced. Then, a function symbol of binding arity  $\langle k_1, \dots, k_n \rangle$  is interpreted as a map  $\delta^{k_1} A \times \dots \times \delta^{k_n} A \rightarrow A$ . However our goal here is somewhat different from that of [13], as it is to relate our notion of model to provability.

**Definition 10 (Denotation in a model in binding logic).** *The denotation of a term (resp. a proposition) in a model in binding logic is defined as follows. Let  $x_1, \dots, x_p$  be variables and  $\phi$  be an assignment mapping each free variable of  $t$  but  $x_1, \dots, x_p$  to an element of  $M_0$ .*

- $\llbracket x_k \rrbracket_\phi^{x_1, \dots, x_p} = \mathbf{k}_p,$
- $\llbracket y \rrbracket_\phi^{x_1, \dots, x_p} = \phi(y) \sqcup_{p,0} \langle \rangle$  if  $y$  is not one of the  $x_1, \dots, x_p,$
- $\llbracket f(y_1^1 \dots y_{k_1}^1 t_1, \dots, y_1^n \dots y_{k_n}^n t_n) \rrbracket_\phi^{x_1, \dots, x_p}$   
 $= \hat{f}_p(\llbracket t_1 \rrbracket_\phi^{y_{k_1}^1, \dots, y_1^1, x_1, \dots, x_p}, \dots, \llbracket t_n \rrbracket_\phi^{y_{k_n}^n, \dots, y_1^n, x_1, \dots, x_p}),$
- $\llbracket P(y_1^1 \dots y_{k_1}^1 t_1, \dots, y_1^n \dots y_{k_n}^n t_n) \rrbracket_\phi = \hat{P}(\llbracket t_1 \rrbracket_\phi^{y_{k_1}^1, \dots, y_1^1}, \dots, \llbracket t_n \rrbracket_\phi^{y_{k_n}^n, \dots, y_1^n}),$
- $\llbracket A \Rightarrow B \rrbracket_\phi = \dots,$
- $\llbracket \forall x A \rrbracket_\phi = 1$  if for all  $a$  in  $M_0,$   $\llbracket A \rrbracket_{\phi + \langle x, a \rangle} = 1,$
- $\llbracket \exists x A \rrbracket_\phi = 1$  if for some  $a$  in  $M_0,$   $\llbracket A \rrbracket_{\phi + \langle x, a \rangle} = 1.$

### 3 Examples of models

*Example 6.* As an example, we give a model where the axioms of equality are valid, but not the extensionality scheme. The existence of such a model will allow us to deduce that the extensionality scheme is not a consequence of the axioms of equality. We consider function symbols  $f$  of arity  $\langle 0 \rangle$  and  $A$  of arity  $\langle 1 \rangle$ , a predicate symbol  $=$  of arity  $\langle 0, 0 \rangle$ . We build a model where the axioms

$$\forall x (x = x)$$

$$\begin{aligned}
& \forall x \forall y (x = y \Rightarrow y = x) \\
& \forall x \forall y \forall z (x = y \Rightarrow (y = z \Rightarrow x = z)) \\
& \forall x \forall y (x = y \Rightarrow f(x) = f(y)) \\
& \forall x \forall y (x = y \Rightarrow \Lambda z x = \Lambda z y)
\end{aligned}$$

are valid, but not the instance of the extensionality scheme

$$(\forall x f(x) = x) \Rightarrow \Lambda x f(x) = \Lambda x x$$

The base set  $M_0$  of this model is a pair  $\{k_0, l_0\}$ . The set  $M_n$  contains the projections  $\mathbf{1}_n, \dots, \mathbf{n}_n$ , objects  $\bar{1}_n, \dots, \bar{n}_n$  and two objects  $k_n$  and  $l_n$ . Informally, the objects  $\bar{1}_n, \dots, \bar{n}_n$  can be seen as extensionally equal to, but distinct from, the projections and the objects  $k_n$  and  $l_n$  as the constant functions taking the values  $k_0$  and  $l_0$ .

The model  $\mathcal{M}$  is defined by  $M_n = \{k_n, l_n, \mathbf{1}_n, \dots, \mathbf{n}_n, \bar{1}_n, \dots, \bar{n}_n\}$ . Let  $a$  be in  $M_n$  and  $b_1, \dots, b_n$  be in  $M_p$ , the element  $a \square_{p,n} \langle b_1, \dots, b_n \rangle$  of  $M_p$  is defined as follows.

- if  $a = k_n$  then  $a \square_{p,n} \langle b_1, \dots, b_n \rangle = k_p$ ,
- if  $a = l_n$  then  $a \square_{p,n} \langle b_1, \dots, b_n \rangle = l_p$ ,
- if  $a = \mathbf{i}_n$  then  $a \square_{p,n} \langle b_1, \dots, b_n \rangle = b_i$ ,
- if  $a = \bar{i}_n$  then  $a \square_{p,n} \langle b_1, \dots, b_n \rangle = -b_i$ , where the involution  $-$  is defined by

$$-k_n = k_n \quad -l_n = l_n \quad -\mathbf{i}_n = \bar{i}_n \quad -\bar{i}_n = \mathbf{i}_n$$

**Proposition 1.**  $(-a) \square_{p,n} \langle b_1, \dots, b_n \rangle = -(a \square_{p,n} \langle b_1, \dots, b_n \rangle)$

*Proof.* – if  $a = k_n$ , both sides are equal to  $k_p$ ,  
– if  $a = l_n$ , both sides are equal to  $l_p$ ,  
– if  $a = \mathbf{i}_n$ , both sides are equal to  $-b_i$ ,  
– if  $a = \bar{i}_n$ , both sides are equal to  $b_i$ .

**Proposition 2.** *The structure  $\mathcal{M}$  is an intensional functional structure.*

*Proof.* First property:

$$\mathbf{i}_n \square_{p,n} \langle a_1, \dots, a_n \rangle = a_i$$

By definition of  $\square$ .

Second property:

$$a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle = a$$

- if  $a = k_n$  then  $a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle = k_n$ ,
- if  $a = l_n$  then  $a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle = l_n$ ,
- if  $a = \mathbf{i}_n$  then  $a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle = \mathbf{i}_n$ ,
- if  $a = \bar{i}_n$  then  $a \square_{n,n} \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle = \bar{i}_n$ .

Third property:

$$\begin{aligned}
(a \square_{p,n} \langle b_1, \dots, b_n \rangle) \square_{q,p} \langle c_1, \dots, c_p \rangle = \\
a \square_{q,n} \langle b_1 \square_{q,p} \langle c_1, \dots, c_p \rangle, \dots, b_n \square_{q,p} \langle c_1, \dots, c_p \rangle \rangle
\end{aligned}$$

- If  $a = k_n$ , both sides are equal to  $k_q$ ,
- if  $a = l_n$ , both sides are equal to  $l_q$ ,
- if  $a = \mathbf{i}_n$ , both sides are equal to  $b_i \square \langle c_1, \dots, c_q \rangle$ ,
- if  $a = \bar{i}_n = -\mathbf{i}_n$ , both sides are equal to  $-(b_i \square \langle c_1, \dots, c_p \rangle)$ .

We define the functions  $\hat{f}_n$  from  $M_n$  to  $M_n$  by  $\hat{f}_n(x) = -x$  and the functions  $\hat{\Lambda}_n$  from  $M_{n+1}$  to  $M_n$  by

$$\begin{aligned} \hat{\Lambda}_n(k_{n+1}) &= k_n & \hat{\Lambda}_n(l_{n+1}) &= l_n \\ \hat{\Lambda}_n(\mathbf{1}_{n+1}) &= k_n & \hat{\Lambda}_n(\bar{1}_{n+1}) &= l_n \\ \hat{\Lambda}_n(\mathbf{i}_{n+1}) &= (\mathbf{i} - \mathbf{1})_n \text{ if } i \geq 2 & \hat{\Lambda}_n(\bar{i}_{n+1}) &= \overline{(i - 1)}_n \text{ if } i \geq 2 \end{aligned}$$

At last, the function  $\hat{\cdot}$  from  $M_0 \times M_0$  to  $\{0, 1\}$  is equality.

**Proposition 3.** *The families  $\hat{f}_i$  and  $\hat{\Lambda}_i$  are coherent.*

*Proof.* For the case of the family  $\hat{f}_i$ , we have to prove that

$$\hat{f}_p(a) \square_{q,p} \langle b_1, \dots, b_p \rangle = \hat{f}_q(a \square_{q,p} \langle b_1, \dots, b_p \rangle)$$

- If  $a = k_p$ , both sides are equal to  $k_q$ ,
- if  $a = l_p$ , both sides are equal to  $l_q$ ,
- if  $a = \mathbf{i}_p$ , both sides are equal to  $-b_i$ ,
- if  $a = \bar{i}_p = -\mathbf{i}_p$ , both sides are equal to  $b_i$ .

For the case of the family  $\hat{\Lambda}_i$ , we have to prove that

$$\hat{\Lambda}_p(a) \square_{q,p} \langle b_1, \dots, b_p \rangle = \hat{\Lambda}_q(a \square_{q+1,p+1} \langle \mathbf{1}_{q+1}, b_1 \square S, \dots, b_p \square S \rangle)$$

where  $S = \langle \mathbf{2}_{q+1}, \dots, (\mathbf{q} + \mathbf{1})_{q+1} \rangle$ .

We first prove that if  $b$  is an element of  $M_q$  then  $\Lambda_q(b \square S) = b$ .

- if  $b = k_q$ , both sides are equal to  $k_q$ ,
- if  $b = l_q$ , both sides are equal to  $l_q$ ,
- if  $b = \mathbf{i}_q$ , both sides are equal to  $\mathbf{i}_q$ ,
- if  $b = \bar{i}_q$ , both sides are equal to  $\bar{i}_q$ .

Then we prove that

$$\hat{\Lambda}_p(a) \square_{q,p} \langle b_1, \dots, b_p \rangle = \hat{\Lambda}_q(a \square_{q+1,p+1} \langle \mathbf{1}_{q+1}, b_1 \square S, \dots, b_p \square S \rangle)$$

- if  $a = k_{p+1}$ , both sides are equal to  $k_q$ ,
- if  $a = l_{p+1}$ , both sides are equal to  $l_q$ ,
- if  $a = \mathbf{1}_{p+1}$ , then  $a \square_{q+1,p+1} \langle \mathbf{1}_{q+1}, b_1 \square S, \dots, b_p \square S \rangle = \mathbf{1}_{q+1}$  and thus both sides are equal to  $k_q$ ,
- if  $a = \bar{1}_{p+1}$ , then  $a \square_{q+1,p+1} \langle \mathbf{1}_{q+1}, b_1 \square S, \dots, b_p \square S \rangle = \bar{1}_{q+1}$  and thus both sides are equal to  $l_q$ ,
- if  $a = \mathbf{i}_{p+1}$  ( $i \geq 2$ ), then using the property above, both sides are equal to  $b_{i-1}$ ,

- if  $a = \bar{i}_{p+1}$  ( $i \geq 2$ ), then then using the property above, both sides are equal to  $-b_{i-1}$ .

**Proposition 4.** *The axioms of equality are valid in the model above.*

*Proof.* This is obvious as equality is interpreted by equality. We give the example of the axiom

$$\forall x \forall y (x = y \Rightarrow \Lambda z x = \Lambda z y)$$

For every  $a$  of  $M_0$ , we have  $\llbracket \Lambda z x \rrbracket_{x/a} = \hat{\Lambda}_0(a \square \langle \rangle) = \llbracket \Lambda z y \rrbracket_{a/y}$ .

**Proposition 5.** *The proposition*

$$(\forall x f(x) = x) \Rightarrow \Lambda x f(x) = \Lambda x x$$

*is not valid in the model above.*

*Proof.* First we check that the proposition  $\forall x f(x) = x$  is valid, i.e. that for every  $a$  we have  $\llbracket f(x) \rrbracket_{a/x} = \llbracket x \rrbracket_{a/x}$ . This is a consequence of the fact that  $\llbracket f(x) \rrbracket_{a/x} = \hat{f}_0(a) = a = \llbracket x \rrbracket_{a/x}$ . Then we check that the proposition  $\Lambda x f(x) = \Lambda x x$  is not valid, i.e. that  $\llbracket \Lambda x f(x) \rrbracket \neq \llbracket \Lambda x x \rrbracket$ . We have  $\llbracket \Lambda x f(x) \rrbracket = \hat{\Lambda}_0(\llbracket f(x) \rrbracket^x) = \hat{\Lambda}_0(\hat{f}_1(\llbracket x \rrbracket^x)) = \hat{\Lambda}_0(\hat{f}_1(\mathbf{1}_1)) = \hat{\Lambda}_0(\bar{1}_1) = l_0$  and  $\llbracket \Lambda x x \rrbracket = \hat{\Lambda}_0(\llbracket x \rrbracket^x) = \hat{\Lambda}_0(\mathbf{1}_1) = k_0$ . Thus  $\llbracket \Lambda x f(x) \rrbracket \neq \llbracket \Lambda x x \rrbracket$ .

**Proposition 6.** *There are instances of the extensionality scheme*

$$(\forall x t = u) \Rightarrow \Lambda x t = \Lambda x u$$

*that are not consequences of the axioms of equality.*

*Example 7.* To the theory of example 2, we add a constant  $a$  of arity  $\langle \rangle$  and we want to prove that the proposition

$$\delta(a, x a, y a) = a$$

is not provable.

We construct the following model. We take  $M_0 = \mathbb{N}$ , and for  $n \geq 1$ ,  $M_n = \mathbb{N}^n \rightarrow \mathbb{N}$ . The functions  $\mathbf{n}_p$  and  $\square_{n,p}$  are defined as in example 4. Thus the model is an intensional functional structure. Then we take

$\hat{a}_0 = 1$  and accordingly  $\hat{a}_n = x_1, \dots, x_n \mapsto 1$ ,

$\hat{i}_0 = x \mapsto 2 * x$  and accordingly  $\hat{i}_n = x_1, \dots, x_n, x \mapsto 2 * x$ ,

$\hat{j}_0 = x \mapsto 2 * x + 1$  and accordingly  $\hat{j}_n = x_1, \dots, x_n, x \mapsto 2 * x + 1$ ,

$\hat{\delta}_0 = d, f, g \mapsto$  if  $d = 0$  or  $d = 1$  then 0 else if  $d$  is even then  $f(d/2)$  else  $g((d-1)/2)$ ,

and accordingly

$\hat{\delta}_n = x_1, \dots, x_n, d, f, g \mapsto \hat{\delta}_0(d(x_1, \dots, x_n), y \mapsto f(x_1, \dots, x_n, y), y \mapsto g(x_1, \dots, x_n, y))$ .

At last equality denotes equality.

The coherence properties are verified. The axioms of the theory are valid.

We have  $\llbracket \delta(a, x a, y a) \rrbracket = 0$  and  $\llbracket a \rrbracket = 1$ . Thus the proposition

$$\delta(a, x a, y a) = a$$

is not valid. Hence it is not provable.

## 4 Deduction modulo

We now want to prove that a sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if for every model of  $\Gamma$ , there is a proposition in  $\Delta$  that is valid. To prove this soundness and completeness theorem, we shall translate binding logic into predicate logic. We will prove that this translation preserves provability. Then we shall relate the models of a theory in binding logic with the models of its translation in predicate logic.

This translation is easier to define if we choose a formulation of predicate logic where some axioms are replaced by computation rules. This is the purpose of Deduction modulo introduced in [10]. In this formalism, the notions of language, term and proposition are that of predicate logic. But, a theory is formed with a set of axioms  $\Gamma$  and a congruence  $\equiv$  defined on propositions. As usual, the set  $\Gamma$  must be decidable. The congruence  $\equiv$  must also be decidable and, in this paper, congruences are defined by confluent and normalizing rewrite systems on terms.

Deduction rules are modified to take this congruence into account. For instance, the right rule of conjunction is not stated as usual:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

As its conclusion need not be exactly  $A \wedge B$  but may be only congruent to this proposition, it is stated:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \text{ if } C \equiv A \wedge B$$

All rules of sequent calculus modulo are defined in a similar way (see figure 2).

Here is a very simple example, where the congruence on propositions is defined by a rewrite system on terms. In arithmetic, we can define a congruence with the following rules:

$$\begin{aligned} 0 + y &\rightarrow y \\ S(x) + y &\rightarrow S(x + y) \\ 0 \times y &\rightarrow 0 \\ S(x) \times y &\rightarrow x \times y + y \end{aligned}$$

In the theory formed with the axiom  $\forall x \, x = x$  and this congruence, we can prove, in sequent calculus modulo, that the number 4 is even:

$$\frac{\frac{\frac{}{4 = 4 \vdash 2 \times 2 = 4} \text{ axiom}}{\forall x \, x = x \vdash 2 \times 2 = 4} (x, x = x, 4) \, \forall\text{-left}}{\forall x \, x = x \vdash \exists x \, 2 \times x = 4} (x, 2 \times x = 4, 2) \, \exists\text{-right}$$

The sequent  $4 = 4 \vdash 2 \times 2 = 4$ , that requires a tedious proof in usual formulations of arithmetic, can simply be proved with the axiom rule here, as  $(4 = 4) \equiv (2 \times 2 = 4)$ .

$$\begin{array}{c}
\frac{}{A \vdash B} \text{ axiom if } A \equiv B \\
\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash \Delta} \text{ cut if } A \equiv B \\
\frac{\Gamma, B_1, B_2 \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left if } A \equiv B_1 \equiv B_2 \\
\frac{\Gamma \vdash B_1, B_2, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right if } A \equiv B_1 \equiv B_2 \\
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left} \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right} \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta} \Rightarrow\text{-left if } C \equiv (A \Rightarrow B) \\
\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash C, \Delta} \Rightarrow\text{-right if } C \equiv (A \Rightarrow B) \\
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, C \vdash \Delta} \wedge\text{-left if } C \equiv (A \wedge B) \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \wedge\text{-right if } C \equiv (A \wedge B) \\
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta} \vee\text{-left if } C \equiv (A \vee B) \\
\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash C, \Delta} \vee\text{-right if } C \equiv (A \vee B) \\
\\
\frac{}{\Gamma, A \vdash \Delta} \perp\text{-left if } A \equiv \perp \\
\frac{\Gamma, C \vdash \Delta}{\Gamma, B \vdash \Delta} (x, A, t) \forall\text{-left if } B \equiv (\forall x A) \text{ and } C \equiv (t/x)A \\
\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} (x, A) \forall\text{-right if } B \equiv (\forall x A) \text{ and } x \notin FV(\Gamma \Delta) \\
\frac{\Gamma, A \vdash \Delta}{\Gamma, B \vdash \Delta} (x, A) \exists\text{-left if } B \equiv (\exists x A) \text{ and } x \notin FV(\Gamma \Delta) \\
\frac{\Gamma \vdash C, \Delta}{\Gamma \vdash B, \Delta} (x, A, t) \exists\text{-right if } B \equiv (\exists x A) \text{ and } C \equiv (t/x)A
\end{array}$$

**Fig. 2.** Sequent calculus modulo

**Proposition 7.** *If the sequent  $A_1, \dots, A_n \vdash B_1, \dots, B_p$  has a proof and  $A_1 \equiv A'_1, \dots, A_n \equiv A'_n, B_1 \equiv B'_1, \dots, B_p \equiv B'_p$  then the sequent  $A'_1, \dots, A'_n \vdash B'_1, \dots, B'_p$  has a proof of the same length.*

*Proof.* By induction over proof structure.

**Definition 11 (Model of a theory modulo).** *A model of a congruence is a model such that two congruent terms or propositions have the same denotation. A model of a theory modulo is a model of its axioms and of its congruence.*

Gödel's completeness theorem generalizes to Deduction modulo.

**Proposition 8 (Soundness and completeness for Deduction modulo).** *([8]) A sequent  $\Gamma \vdash \Delta$  is provable modulo a congruence  $\equiv$  if and only if for every model of  $\langle \Gamma, \equiv \rangle$  there is a proposition of  $\Delta$  that is valid.*

## 5 Translation of binding logic into Deduction modulo

**Definition 12 (Translation of a language).** *Let  $\mathcal{L}$  be a language in binding logic. We consider a language  $\mathcal{L}'$  in many-sorted predicate logic, with sorts of the form  $n$  where  $n$  is a natural number and  $\langle n, p \rangle$  where  $n$  and  $p$  are natural numbers. The sorts  $n$  are those of terms appearing in a context where  $n$  variables are bound. The sort  $\langle n, p \rangle$  are those of explicit substitutions mapping  $p$  variables to  $p$  terms, these terms being of sort  $n$ .*

*To every function symbol  $f$  of binding arity  $\langle k_1, \dots, k_n \rangle$  we associate an infinite number of function symbols  $f_0, \dots, f_p, \dots$  of rank  $\langle k_1 + p, \dots, k_n + p \rangle p$ . To every predicate symbol  $P$  of binding arity  $\langle k_1, \dots, k_n \rangle$  we associate a predicate symbol also written  $P$  of rank  $\langle k_1, \dots, k_n \rangle$ . Then we introduce more constants and function symbols:*

- for all natural numbers  $n$ ,  $n$  constants  $1_n, \dots, n_n$  of sort  $n$ ,
- for all natural numbers  $n$  and  $p$ , a function symbol  $\llbracket_{n,p}$  of rank  $\langle p, \langle n, p \rangle \rangle n$ ,
- for all natural numbers  $n$ , a constant  $id_n$  of sort  $\langle n, n \rangle$ ,
- for all natural numbers  $n$  and  $p$ , a function symbol  $\cdot_{n,p}$  (read cons) of rank  $\langle n, \langle n, p \rangle \rangle \langle n, p + 1 \rangle$ ,
- for all natural numbers  $n$ , a constant  $\uparrow_n$  of sort  $\langle n + 1, n \rangle$ ,
- for all natural numbers  $n, p$  and  $q$ , a function symbol  $\circ_{n,p,q}$  of rank  $\langle \langle p, n \rangle, \langle q, p \rangle \rangle \langle q, n \rangle$ .

Notice that these symbols are those of the calculus of explicit substitutions [1, 6] with de Bruijn indices [4]. Calculi of explicit substitutions have first been introduced to decompose the substitutions initiated by  $\beta$ -reduction in  $\lambda$ -calculus. It has then be noticed that having explicit renaming/relocating operators permitted to enforce scope constraints in  $\lambda$ -calculus and then in other languages with bound variables [9, 11, 19, 20, 22]. In this paper, we are not particularly interested in  $\beta$ -reduction and explicit substitutions are only used to enforce scope constraints.

We give now the translation from  $\mathcal{L}$  to  $\mathcal{L}'$ . This translation uses an auxiliary list of variables, which is written  $l$ . The length of this list  $l$  is written  $|l|$  and, as usual,  $\epsilon$  is the empty list,  $x.l$  is the list  $l$  prefixed by the term  $x$  and  $l_p$  is the  $p$ -th element of the list  $l$ .

**Definition 13 (Pre-cooking).** Pre-cooking is a function from  $\mathcal{L}$  to  $\mathcal{L}'$  defined as follows:

- $F(x, l) = n_{|l|}$  if  $l_n = x$  and  $l_p \neq x$  for  $p < n$ ,
- $F(x, l) = x[\uparrow_0 \circ \uparrow_1 \cdots \circ \uparrow_{|l|-1}]$  if  $l_p \neq x$  for all  $p$ . The variable  $x$  is given the sort 0 in the language  $\mathcal{L}'$ . This term is simply written  $x[\uparrow^{|l|}]$  if there is no ambiguity,
- $F(f(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n), l) = f_{|l|}(F(t_1, x_{k_1}^1 \cdots x_1^1.l), \dots, F(t_n, x_{k_n}^n \cdots x_1^n.l)),$
- $F(P(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n), \epsilon) = P(F(t_1, x_{k_1}^1 \cdots x_1^1), \dots, F(t_n, x_{k_n}^n \cdots x_1^n)),$
- $F(A \Rightarrow B, \epsilon) = F(A, \epsilon) \Rightarrow F(B, \epsilon),$   
 $F(A \wedge B, \epsilon) = F(A, \epsilon) \wedge F(B, \epsilon),$   
 $F(A \vee B, \epsilon) = F(A, \epsilon) \vee F(B, \epsilon),$   
 $F(\perp, \epsilon) = \perp,$   
 $F(\forall x A, \epsilon) = \forall x F(A, \epsilon),$   
 $F(\exists x A, \epsilon) = \exists x F(A, \epsilon).$

We write  $t'$  for  $F(t, \epsilon)$  and  $A'$  for  $F(A, \epsilon)$ .

**Proposition 9.** If  $t$  is a term of  $\mathcal{L}$  then  $t'$  is a term of sort 0.

*Proof.* We prove, by induction on terms, the more general proposition: if  $t$  is a term of  $\mathcal{L}$  then  $F(t, l)$  is a term of sort  $|l|$ .

- If  $t$  is a variable of  $l$ , then  $F(t, l) = n_{|l|}$  and thus it has sort  $|l|$ .
- If  $t$  is a variable  $x$  not belonging to  $l$ , then  $F(t, l) = x[\uparrow^{|l|}]$ . The term  $\uparrow^{|l|} = \uparrow_0 \circ \uparrow_1 \cdots \circ \uparrow_{|l|-1}$  has sort  $\langle |l|, 0 \rangle$ . The variable  $x$  has sort 0. So,  $x[\uparrow^{|l|}]$  has sort  $|l|$ .
- If  $t = f(x_1^1 \cdots x_{k_1}^1 t_1, \dots, x_1^n \cdots x_{k_n}^n t_n)$  then  $F(t, l) = f_{|l|}(F(t_1, x_{k_1}^1 \cdots x_1^1.l), \dots, F(t_n, x_{k_n}^n \cdots x_1^n.l))$ . By induction hypothesis,  $F(t_i, x_{k_i}^i \cdots x_1^i.l)$  have sort  $k_i + |l|$  and the application of the symbol  $f_{|l|}$  yields a term of sort  $|l|$ .

*Example 8.* Let  $f$  be a function symbol of binding arity  $\langle 0, 0 \rangle$  and  $A$  a function symbol of binding arity  $\langle 1 \rangle$  and  $=$  a predicate symbol of binding arity  $\langle 0, 0 \rangle$ . The proposition:

$$\forall x \forall y (f(x, y) = \lambda z. f(x, z))$$

is translated as:

$$\forall x \forall y (f_0(x, y) = \lambda_0(f_1(x[\uparrow], 1_1)))$$

The variables  $x$  and  $y$ , that are bound by quantifiers, are kept as named variables, the variable  $z$ , that is bound by a binding symbol is replaced by a de Bruijn index, the first occurrence of  $f$  is translated as  $f_0$  and the second as  $f_1$



as it is under one binder, the variable  $x$  under the binder  $\Lambda$  is protected by a substitution  $\uparrow$  to avoid captures. The terms  $x[\uparrow]$ ,  $1_1$  and  $f_1(x[\uparrow], 1_1)$  have sort 1 while the term  $\Lambda_0(f_1(x[\uparrow], 1_1))$  has sort 0.

**Definition 14 (Translation of a theory).**

Let  $\Gamma$  be a theory in binding logic. The translation of  $\Gamma$  is a theory modulo having as axioms the translations of those of  $\Gamma$  and as congruence that defined by the following rewrite system  $\sigma$ :

$$\begin{aligned}
n+1 &\rightarrow 1[\uparrow^n] \\
1[t.s] &\rightarrow t \\
t[id] &\rightarrow t \\
(t[s])[s'] &\rightarrow t[s \circ s'] \\
id \circ s &\rightarrow s \\
\uparrow \circ (t.s) &\rightarrow s \\
(s_1 \circ s_2) \circ s_3 &\rightarrow s_1 \circ (s_2 \circ s_3) \\
(t.s) \circ s' &\rightarrow t[s'].(s \circ s') \\
s \circ id &\rightarrow s \\
1.\uparrow &\rightarrow id \\
1[s].(\uparrow \circ s) &\rightarrow s \\
f_p(t_1, \dots, t_n)[s] &\rightarrow f_q(t_1[1.1[\uparrow]. \dots 1[\uparrow^{k_1-1}].s \circ \uparrow^{k_1}], \\
&\quad \dots, \\
&\quad t_n[1.1[\uparrow]. \dots 1[\uparrow^{k_n-1}].s \circ \uparrow^{k_n}])
\end{aligned}$$

where, in the last rule,  $f$  has binding arity  $\langle k_1, \dots, k_n \rangle$  and  $s$  has sort  $\langle q, p \rangle$ .

The system  $\sigma$  is locally confluent as all critical pairs converge. To obtain termination, it suffices to note that  $\sigma$  shares all the rules but the first and the last ones with the system  $\sigma$  of [1, 6], called  $\sigma_e$  here, which is strongly normalizing. Here we use a version of  $\sigma_e$  with pairs, which is also strongly normalizing [6]. For the first and the last rule, we translate  $n+1$  onto  $1[\uparrow^n][id]$  and  $f_p(t_1, \dots, t_n)$  onto  $\langle \dots \langle \lambda^{k_1} t_1, \lambda^{k_2} t_2 \rangle, \dots, \lambda^{k_n} t_n \rangle$ . The right member of the last rule translates onto the term  $\langle \dots \langle \lambda^{k_1} t_1[1.1[\uparrow]. \dots 1[\uparrow^{k_1-1}].s \circ \uparrow^{k_1}], \dots \rangle, \dots, \lambda^{k_n} t_n[1.1[\uparrow]. \dots 1[\uparrow^{k_n-1}].s \circ \uparrow^{k_n}] \rangle$ , which is the reduct by rules of  $\sigma_e$  of the translation of the left member. Thus, every rewriting step of  $\sigma$  is translated as a non empty sequence of rewriting steps of  $\sigma_e$ . As  $\sigma_e$  is strongly normalizing, this is also the case for  $\sigma$ .

If  $t$  is a term or a proposition, we write  $t \downarrow$  for its normal form.

We now want to prove that the sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if  $\Gamma' \vdash \Delta'$  is provable in Deduction modulo  $\sigma$ . First, we show that substitution (with renaming) in  $\mathcal{L}$  collapses to grafting (without renaming) in  $\mathcal{L}'$ .

**Proposition 10.** *Let  $t$  and  $u$  be terms of  $\mathcal{L}$ . Then:*

$$((t/x)u)' \equiv \langle t'/x \rangle u'$$

*Proof.* To ease reading, we leave implicit the sorts of the symbols. We prove by induction on the structure of  $u$  the more general statement: for all lists  $l$ , if the

free variables of  $t$  do not appear in  $l$ , then:

$$F((t/x)u, l) = \langle t'/x \rangle F(u, l)$$

The result follows for  $l = \epsilon$ . The non trivial case is when  $u$  is  $x$ , in this case we need:

$$F(t, l) = \langle t'/x \rangle F(x, l)$$

i.e.:

$$F(t, l) = t'[\uparrow^{|l|}] = F(t, \epsilon)[\uparrow^{|l|}]$$

To prove this, we show by induction on the structure of  $t$ , that for all lists  $l$  and  $l'$  such that the free variables of  $t$  do not appear in  $l$ , we have:

$$F(t, l' l) = F(t, l')[1. \dots |l'|. \uparrow^{|l|+|l'|}]$$

where  $l' l$  is the concatenation of the two lists.

- If  $t$  is a variable of  $l'$ , then  $t$  is the  $k^{th}$  variable of  $l'$  for some  $k$  and both terms reduce to  $k$ .
- If  $t$  is a variable  $x$  that does not belong to  $l'$  then it does not belong to  $l' l$  and both terms reduce to  $x[\uparrow^{|l|+|l'|}]$ .
- If  $t = f(x_1^1 \dots x_{k_1}^1 t_1, \dots, x_1^n \dots x_{k_n}^n t_n)$  then, expliciting only the first argument of  $f$  for the sake of simplicity, we get:

$$F(t, l' l) = f_{|l'|+|l|}(F(t_1, x_{k_1}^1 \dots x_1^1 l' l), \dots)$$

and by induction hypothesis:

$$F(t, l' l) = f_{|l'|+|l|}(F(t_1, x_{k_1}^1 \dots x_1^1 l')[1. \dots |l'| + k_1. \uparrow^{|l|+|l'|+k_1}], \dots)$$

Then, we have:

$$F(t, l')[1. \dots |l'|. \uparrow^{|l|+|l'|}] = f_{|l'|}(F(t_1, x_{k_1}^1 \dots x_1^1 l'), \dots)[1. \dots |l'|. \uparrow^{|l|+|l'|}]$$

and with the last rule of  $\sigma$ , we get:

$$F(t, l')[1. \dots |l'|. \uparrow^{|l|+|l'|}] = f_{|l'|+|l|}(F(t_1, x_{k_1}^1 \dots x_1^1 l')[1. \dots |l'| + k_1. \uparrow^{|l|+|l'|+k_1}], \dots)$$

Thus:

$$F(t, l' l) = F(t, l')[1. \dots |l'|. \uparrow^{|l|+|l'|}]$$

**Proposition 11.** *Let  $t$  and  $A$  be a term and a proposition of  $\mathcal{L}$ . Then:*

$$((t/x)A)' \equiv (t'/x)A'$$

*Proof.* By induction over the structure of  $A$ . If  $A$  is atomic then we apply the proposition 10. Notice that if  $u$  is a term without bound variables, then  $(t/x)u = \langle t/x \rangle u$ . If  $A$  has the form  $B \Rightarrow C$ ,  $B \wedge C$ ,  $B \vee C$  or  $\perp$ , we apply the induction hypothesis. If  $A$  has the form  $\forall y B$ , then let  $z$  be a fresh variable, we have:

$$(t/x)A = (t/x)(\forall y B) = \forall z ((t/x)\langle z/y \rangle B)$$

Thus  $((t/x)A)' = \forall z ((t/x)\langle z/y \rangle B)' \equiv \forall z (t'/x)(\langle z/y \rangle B)' = (t'/x)\forall z (\langle z/y \rangle B)' = (t'/x)\forall y B' = (t'/x)A'$ . If  $A$  has the form  $\exists y B$ , then the proof is similar.

The purpose of the following definition and proposition is to characterize the image of pre-cooking.

**Definition 15 (F-term).** *A F-term (resp. a F-proposition) is a  $\sigma$ -normal term (resp. proposition) of  $\mathcal{L}'$  containing only variables of sort 0.*

**Proposition 12.** *If  $t$  is a F-term of sort 0 (resp. a F-proposition), then there is a term (resp. a proposition)  $u$  in  $\mathcal{L}$  such that  $t = u'$ .*

*Proof.* We first prove by induction on the structure of  $t$  that if  $t$  is a F-term of sort  $n$  then there is a term  $u$  and a sequence  $l$  of  $n$  variables such that  $t = F(u, l)$ . The non trivial case is when  $t = x[s]$ . This term has sort  $n$  thus  $s$  has sort  $\langle n, 0 \rangle$  and it is normal, thus  $s = \uparrow^n$ .

Then, we prove by induction on the structure of  $A$  that if  $A$  is a F-proposition then there is a proposition  $B$  such that  $A = B'$ .

**Proposition 13.** *Let  $\Gamma \vdash \Delta$  be a sequent containing only F-propositions. If the sequent  $\Gamma \vdash \Delta$  has a proof, then it also has a proof where all propositions are F-propositions and all the witnesses F-terms of sort 0.*

*Proof.* First, as  $\sigma$  is a term rewrite system, Deduction modulo  $\sigma$  has the cut elimination property [12] and thus  $\Gamma \vdash \Delta$  has a cut free proof. Then we reason by induction on the size of such a cut free proof.

If the last rule is an axiom then the result is obvious. Otherwise, we apply the induction hypothesis to the sub-proofs. Consider for instance the case of the right rule of the existential quantifier:

$$\frac{\pi}{\frac{\Gamma \vdash C, \Delta}{\Gamma \vdash B, \Delta} (x, A, t) \exists\text{-right}}$$

With  $B \equiv \exists x A$  and  $C \equiv (t/x)A$ .

Call  $\theta$  the substitution mapping each variable  $y$  of  $t$  of sort  $n$  ( $n \neq 0$ ) to the term  $y'[\uparrow^n]$  where  $y'$  is a fresh variable of sort 0. By induction on the structure of  $\pi$ , the proof  $\theta\pi$  is a proof of  $\Gamma \vdash \theta C, \Delta$  we have  $\theta C \equiv \theta(t/x)A = (\theta t/x)A \equiv ((\theta t/x)A) \downarrow$ . Thus, by proposition 7, there is a proof  $\pi'$  of the same size as  $\pi$  of the sequent  $\Gamma \vdash ((\theta t/y)A) \downarrow, \Delta$ .

The sequent  $\Gamma \vdash ((\theta t/y)A) \downarrow, \Delta$  contains only F-propositions. Applying the induction hypothesis to the proof  $\pi'$ , we get that there exists a proof  $\pi''$  of  $\Gamma' \vdash ((\theta t/y)A) \downarrow, \Delta$  where all propositions are F-propositions and all the witnesses F-terms of sort 0. We build the proof:

$$\frac{\pi''}{\frac{\Gamma \vdash ((\theta t/y)A) \downarrow, \Delta}{\Gamma \vdash B, \Delta} (x, A, \theta t) \exists\text{-right}}$$

We can now state and prove the theorem:

**Theorem 1.** *The sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if  $\Gamma' \vdash \Delta'$  is provable modulo  $\sigma$ .*

*Proof.* The direct sense is an easy induction on the structure of the proof in binding logic. As an example, we give the case of the right rule of the existential quantifier. The proof has the form:

$$\frac{\pi}{\frac{\Gamma \vdash (t/x)A, \Delta}{\Gamma \vdash \exists x A} \exists\text{-right}}$$

By induction hypothesis, there is a proof  $\pi'$  of  $\Gamma' \vdash ((t/x)A)', \Delta'$ , and, by proposition 11, we have  $(t'/x)A' \equiv ((t/x)A)'$ . We build the proof in Deduction modulo:

$$\frac{\pi'}{\frac{\Gamma' \vdash ((t/x)A)', \Delta'}{\Gamma' \vdash \exists x A', \Delta'} (x, A', t') \exists\text{-right}}$$

Conversely, by the proposition 13, we can build a proof in Deduction modulo of  $\Gamma' \vdash \Delta'$  where all the propositions are  $F$ -propositions and all the witnesses  $F$ -terms of sort 0. By proposition 12, all the propositions and witnesses in this proof are the pre-cooking of some propositions or terms in binding logic. By induction on the structure of this proof we can build a proof of  $\Gamma \vdash \Delta$  in binding logic. As an example, we give the case of the right rule of the existential quantifier. The proof has the form:

$$\frac{\pi}{\frac{\Gamma' \vdash C', \Delta'}{\Gamma' \vdash B', \Delta'} (x, t', A') \exists\text{-right}}$$

where  $B' \equiv \exists x A'$  and  $C' \equiv (t'/x)A'$ .

By proposition 11, we have  $(t'/x)A' \equiv ((t/x)A)'$  and thus  $C' \equiv ((t/x)A)'$ . As  $F$ -terms are  $\sigma$ -normal we have  $B' \equiv \exists x A'$  and  $C' \equiv ((t/x)A)'$ . As pre-cooking is injective, we get  $B \equiv \exists x A$  and  $C \equiv (t/x)A$ .

By induction hypothesis, there exists a proof  $\pi'$  in binding logic of  $\Gamma \vdash (t/x)A, \Delta$ . We build the proof in binding logic:

$$\frac{\pi'}{\frac{\Gamma \vdash (t/x)A, \Delta}{\Gamma \vdash \exists x A, \Delta} \exists\text{-right}}$$

Thus, binding logic can be coded back into predicate logic modulo. We could pursue this by coding back predicate logic modulo into predicate logic (introducing the rules of  $\sigma$  as axioms). However, this is not necessary to define the notion of model, because we can use directly the notion of model of Deduction modulo and the related soundness and completeness result.

*Remark 3.* An axiom scheme, containing schematic variables for terms such as the scheme  $\beta$

$$\forall x (\alpha(\lambda x t, x) = t)$$

is translated as an axiom scheme

$$\forall x (\alpha(\lambda F(t, x.\epsilon), x) = F(t, \epsilon))$$

Using the fact that  $F(t, x.\epsilon)[x.id] = F(t, \epsilon)$  we can replace this scheme by the equivalent

$$\forall x (\alpha(\lambda F(t, x.\epsilon), x) = F(t, 1)[x.\epsilon])$$

i.e. by

$$\forall x (\alpha(\lambda u, x) = u[x.\epsilon])$$

where  $u$  is a schematic variable ranging over terms of the form  $F(t, x.\epsilon)$ .

However, we may also imagine to translate it as a single axiom

$$\forall Y \forall x (\alpha(\lambda Y, x) = Y[x.id])$$

Notice that the variable  $Y$  has sort 1 here, while  $x$  has sort 0.

This remark shows that our method for proving completeness could be adapted for extensions of binding logic with quantification over variables ranging in  $M_i$  for  $i \geq 1$ . We leave this for future work.

## 6 Relating models

**Proposition 14.** *The sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if for every model of  $\langle \Gamma', \equiv \rangle$ , there is a proposition in  $\Delta'$  that is valid.*

*Proof.* The sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if the sequent  $\Gamma' \vdash_{\equiv} \Delta'$  is provable in Deduction modulo. By the soundness and completeness theorem of Deduction modulo this is equivalent to the fact that for every model of  $\langle \Gamma', \equiv \rangle$ , there is a proposition in  $\Delta'$  that is valid.

Now, we want to prove that for every model of  $\Gamma$  there is a proposition in  $\Delta$  that is valid if and only if for every model of  $\langle \Gamma', \equiv \rangle$  there is a proposition in  $\Delta'$  that is valid. This is the purpose of the two next propositions that form a collapse lemma, showing that interpreting  $\mathcal{L}'$  we can restrict to models where explicit substitutions are interpreted like sequences of terms.

**Proposition 15.** *If for every model of  $\langle \Gamma', \equiv \rangle$ , there is a proposition in  $\Delta'$  that is valid, then for every model of  $\Gamma$ , there is a proposition in  $\Delta$  that is valid.*

*Proof.* Let  $\mathcal{M}$  be a model of  $\Gamma$  (in binding logic). We want to prove that some proposition of  $\Delta$  is valid in  $\mathcal{M}$ .

We build a model  $\mathcal{N}$  of  $\mathcal{L}'$  in Deduction modulo. We take  $N_n = M_n$ ,  $N_{n,p} = M_n^p$  (the Cartesian product of  $M_n$ ),

- $\hat{p}_n = \mathbf{p}_n$ ,
- $a \hat{\sqcap}_{n,p} \langle b_1, \dots, b_p \rangle = a \square_{n,p} \langle b_1, \dots, b_p \rangle$ ,
- $\hat{id}_n = \langle \mathbf{1}_n, \dots, \mathbf{n}_n \rangle$ ,
- $\hat{\cdot}_{n,p}(a, \langle b_1, \dots, b_p \rangle) = \langle a, b_1, \dots, b_p \rangle$ ,

- $\hat{\uparrow}_n = \langle \mathbf{2}_{n+1}, \dots, (\mathbf{n} + \mathbf{1})_{n+1} \rangle$
- $\hat{\circ}_{n,p,q}(\langle a_1, \dots, a_p \rangle, \langle b_1, \dots, b_q \rangle) = \langle a_1 \sqcap_{n,q} \langle b_1, \dots, b_q \rangle, \dots, a_p \sqcap_{n,q} \langle b_1, \dots, b_q \rangle \rangle$

and we keep the same denotation for the function symbols and the predicate symbols.

Then we verify that the model  $\mathcal{N}$  is a model of the rewrite rules of  $\sigma$ . This is a consequence of the definition of the denotation of the symbols of  $\mathcal{L}'$  and of the fact that  $\mathcal{M}$  is a model in binding logic (hence it verifies the three properties of intensional functional structures and the coherence property for function symbols).

Then, a simple proof by induction over the structure of terms (resp. propositions) shows that the denotation of  $t$  in  $\mathcal{M}$  is the denotation of  $t'$  in  $\mathcal{N}$ .

Hence, the model  $\mathcal{N}$  is a model of  $\Gamma'$ . As it is a model of  $\Gamma'$  and  $\equiv$ , there is a proposition in  $\Delta'$  that is valid in  $\mathcal{N}$ . Thus, there is a proposition of  $\Delta$  that is valid in  $\mathcal{M}$ .

**Proposition 16.** *If for every model of  $\Gamma$ , there is a proposition in  $\Delta$  that is valid, then for every model of  $\langle \Gamma', \equiv \rangle$  there is a proposition in  $\Delta'$  that is valid.*

*Proof.* Let  $\mathcal{N}$  be a model of  $\langle \Gamma', \equiv \rangle$  (in deduction modulo). We want to prove that some proposition of  $\Delta'$  is valid in  $\mathcal{N}$ .

The model  $\mathcal{N}$  is formed with a family of sets  $N_n$ , a family of sets  $N_{n,p}$  and functions  $\hat{p}_n, \hat{\sqcap}_{n,p}, \hat{id}, \hat{\cdot}_{n,p}, \hat{\uparrow}_p, \hat{\circ}_{n,p,q}$ .

We define a model  $\mathcal{M}$  of  $\mathcal{L}$  (in binding logic) by taking  $M_n = N_n$ ,

- $\mathbf{p}_n = \hat{p}_n$ ,
- $a \sqcap_{n,p} \langle b_1, \dots, b_p \rangle = a \hat{\sqcap}_{n,p} (b_1 \hat{\cdot} \dots \hat{\cdot} b_p \cdot \llbracket \uparrow^n \rrbracket)$ .

and we keep the same denotation for the function symbols and the predicate symbols.

As  $\mathcal{N}$  is a model of the rewrite rules of  $\sigma$ ,  $\mathcal{M}$  is an intensional functional structure and verifies the coherence property. Hence, it is a model of  $\mathcal{L}$ .

Then, we prove by induction over the structure of terms (resp. proposition) that if  $t$  is a term in binding logic, then the denotation of  $t$  in  $\mathcal{M}$  is the denotation of  $t'$  in  $\mathcal{N}$ .

The only non trivial case is when  $t$  is a variable. In this case,  $t' = x[\uparrow^n]$ , where  $n$  is the sort of  $t$ . The denotation of  $t$  in  $\mathcal{N}$  is  $\phi(x) \hat{\sqcap}_{n,0} \llbracket \uparrow^n \rrbracket$ . Its denotation in  $\mathcal{M}$  is  $\phi(x) \sqcap_{n,0} \langle \rangle = \phi(x) \hat{\sqcap}_{n,0} \llbracket \uparrow^n \rrbracket$ .

Thus the model  $\mathcal{M}$  is a model of  $\Gamma$  and thus there is a proposition of  $\Delta$  that is valid in  $\mathcal{M}$ . Thus there is a proposition  $\Delta'$  is valid in  $\mathcal{N}$ .

**Theorem 2 (Soundness and Completeness).** *A sequent  $\Gamma \vdash \Delta$  is provable in binding logic if and only if for each model of  $\Gamma$ , there is a proposition of  $\Delta$  that is valid.*

*Proof.* From propositions 14, 15 and 16.

*Remark 4.* Intensional functional structures can be seen as intensional versions of Henkin's models [14, 2]. Hence, we believe that our completeness theorem could also be proved by translating binding logic not to predicate logic using de Bruijn indices and explicit substitutions, but to higher-order logic using higher-order abstract syntax.

## Conclusion

We have defined a logic with binders, that is a simple extension of predicate logic: proof rules are the same and no extra axiom is added. Many theories can be expressed in this framework. The expression of a theory is close to its expression in the informal language of mathematics. Some theories, however, require axiom schemes.

We have given a notion of model for this logic and proved a soundness and completeness theorem. The first naive idea of extensional model is not sufficient because some non provable propositions are always valid in such models. Our models are algebraic, or categorical, structures that permit to speak about intensional functions both in intensional and in extensional mathematics. We have used this notion of model to prove that the extensionality scheme is independent.

Along the way, we have defined a translation of binding logic into predicate logic. This translation may also be interesting for itself.

We conjecture that if for each binder we have an extensionality scheme then we can restrict to models where  $M_n$  is the set of functions of  $A^n$  to  $A$  for some set  $A$ . Other aspects of binding logic such as unification, skolemization and proof search remain to be studied.

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