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# Optimality of the Parameterization of Quadrics and their Intersections

Daniel Lazard

In collaboration with

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Optimality  
means here optimality in term of  
the number of radicals involved

## Quadratics of rank 1 or 2

Gauss-Jordan reduction leads to equations :

Rank 1 :

$$x^2$$

Rational parameterization = no radicals

Clearly optimal

Rank 2 :

$$x^2 - ay^2$$

Rational parameterization in  $\mathbb{Q}(\sqrt{a})$

Optimal : There are regular rational points iff  $\sqrt{a}$  is rational.

## Quadratics of rank 3

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2$$

Parameterization :

$$x = 2uv, y = \frac{b}{u^2 - av^2}, z = \frac{\sqrt{bc}}{u^2 + av^2}, t = w$$

$\iff$  At most one square root

There exists a parameterization rational over  $\mathbb{Q}$

$\iff$  The conic  $ax^2 + by^2 - cz^2$  has a rational point over  $\mathbb{Q}$

$\iff$  Gauss-Jordan gives equation  $x^2 + y^2 - z^2$  when starting from this

point

Example where a **square root** is needed :

$$x^2 + y^2 - 3z^2 \text{ has no rational point (Proof over } \mathbb{Z}/4\mathbb{Z})$$

## Quadratics of signature (2,2)

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2 - dt^2$$

Parameterization :

$$x = \frac{uv+av'v}{uv-bu'v'}, y = \frac{b}{uv-bu'v'}, z = \frac{uv'-av'v}{uv-bu'v'}, t = \frac{\sqrt{bd}}{uv-bu'v'}$$

2 square roots

$\delta := abcd$  = discriminant of the quadric.

Invariant **up to a square** by change of coordinates

We claim :

$\sqrt{\delta}$  always needed

Another square root is needed **iff** the quadric has no rational point.

## Quadratics of signature (2,2) (continued)

- $\mathcal{Q}$  admits a rational parameterization over  $\mathbb{K}$ , linear in one parameter  $\Leftrightarrow$
- $\mathcal{Q}$  contains a line which is rational over  $\mathbb{K}$   $\Leftrightarrow$
- $\mathcal{Q}$  has a rational point over  $\mathbb{K}$  and  $\delta$  is a square in  $\mathbb{K}$   $\Leftrightarrow$
- $\mathcal{Q}$  has equation  $x^2 + y^2 - z^2 - t^2$ , in some rational frame over  $\mathbb{K}$   $\Leftrightarrow$
- $\Leftrightarrow$  The field of a rational parameterization contains  $\mathbb{Q}(\sqrt{\delta})$
- $\mathcal{Q}$  has a rational point over  $\mathbb{Q}$   $\Leftrightarrow$   $\mathcal{Q}$  has a rational point over  $\mathbb{Q}$  and equation  $x^2 + y^2 - z^2 - \delta t^2$  in some rational frame.

$$x^2 + y^2 - 3z^2 - 11t^2$$

has no rational point over  $\mathbb{Q}$

and no rational parameterization over  $\mathbb{Q}(\sqrt{33})$

Proof over  $\mathbb{Z}/4\mathbb{Z}$  and  $\mathbb{Z}/8\mathbb{Z}$



## Intersections of quadrics $\mathcal{P}$ and $\mathcal{Q}$ quadrics

The intersection is contained in any quadric of the pencil  $\lambda\mathcal{P} + \mu\mathcal{Q}$

There is exactly one quadric of the pencil passing through a point outside of the intersection (**linear equation to solve**)

If the intersection is not empty nor singular

(i.e.  $\det(\lambda\mathcal{P} + \mu\mathcal{Q})$  has no multiple (projective) root), the pencil contains a quadric of signature  $(2,2)$  with a rational point

**Parameterization through this quadric involves  $\sqrt{2}$  :**

A constant one,  $\sqrt{\delta}$ , the discriminant of the quadric  
One depending on the parameter,  $\sqrt{\Delta}$

If the intersection is non singular,  $\sqrt{\Delta}$  is needed

Putting the parameterization of the quadric in  $\mathcal{P}$  gives an equation

between the parameters.

**Isomorphism** of the intersection

with a curve in  $\mathbb{P}_1 \times \mathbb{P}_1$ , the spaces of parameters

De-homogenizing and re-homogenizing :

**bi-rational equivalence** with a curve in  $\mathbb{P}_2$

of degree 4 with 2 ordinary singular points at infinity.

Non singular intersection has **genus 1**

⇐ No rational parameterization over  $\mathbb{C}$

$\sqrt{\delta}$  is not needed  
iff

a surface of degree 8 has a rational point

If  $\sqrt{\delta}$  is not needed,

the intersection contains pair of conjugate points of degree 2  
the quadric passing through a point on a rational line  
defined by such a pair has a square discriminant

$\Leftrightarrow$  The surface

$$z^2 = \det(\mathcal{Q}(x, y, 1, 0) \mathcal{P} - \mathcal{P}(x, y, 1, 0) \mathcal{Q})$$

has a rational point.

This surface has no rational point in the case :

$$\mathcal{P} = 5y^2 + 6xy + 2z^2 - t^2 + 6zt;$$

$$\mathcal{Q} = 3x^2 + y^2 - z^2 - t^2$$

## Singular intersections

$0$  is a multiple root of  $\det(\mathcal{F} + \lambda \mathcal{Q})$

$\Leftrightarrow \mathcal{F}$  has rank  $\leq 2$  (pair of planes) **or**

$\mathcal{F}$  is a cone and all quadrics of the pencil are tangent at its vertex

$\Leftrightarrow$  The intersection has a **singular (complex) point** which is also a

singular point of  $\mathcal{F}$

Up to a real projective transformation,

there are **28 singular intersections** corresponding to **46 pencils**.

Algorithm sketched by Sylvain Lazard

managed through the number of multiple roots of  $\det(\lambda P + \mu Q)$



If the intersection does not consist in 4 lines

at most 2  $\surd$  for parameterizing all components

at most 1  $\surd$  may be unnecessary

it may be removed by finding a rational point on a conic

If the intersection consists in 4 non coplanar lines (skew quadrilateral)

at most 2  $\surd$  for each line

possibility of 3  $\surd$  for all lines together

always optimal for each line and for all lines together.

In the case of 4 concurrent lines

general degree 4 equation

## Singular quartic

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$  has a single multiple root

Parameterize the cone corresponding to this root : 1 or 0  $\checkmark$

Plug it in the equation of  $\mathcal{P}$

$\Leftarrow$  equation which is linear in one of the parameter.

One  $\checkmark$ , zero if one finds a rational point on the cone

## Cubic an line

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$  has 2 double roots, real or not

Parameterize through a (2, 2) quartic : no  $\checkmark$

$\Leftarrow$  Equation in the parameters factors (GCD computation) in

a univariate factor (the line)

and a factor linear in that variable (the cubic)

Totally rational parameterization

## TWO CONICS

$\det(\lambda P + \mu Q)$  has 1 (rational) double root

The corresponding quadric is a pair of planes

Compute these planes : 1 or 0  $\sqrt{\quad}$

Parameterize through a (2, 2) quartic and plug the parameters in the equations of the planes : 1 or 0  $\sqrt{\quad}$

$\iff$  Linear equations in the parameters

First  $\sqrt{\quad}$  is needed if the planes are not rational

Second  $\sqrt{\quad}$  may be avoided for each conic with a rational point on the field of the planes

## Conic and 2 lines

$\det(\lambda P + \mu Q)$  has 2 (rational) double roots

The corresponding quadrics are a cone and a pair of planes

Parameterize through a (2, 2) quadric : one  $\sqrt{\quad}$  iff the lines are irrational

$\Leftrightarrow$  Equation in the parameters factors (GCD computation) in

2 univariate factors (the lines)

and a bilinear one (the conic)

At most 1  $\sqrt{\quad}$

It may be avoided for the conic iff it has a rational point



## 4 lines

$\det(\lambda P + \mu Q)$  has 2 double roots

The corresponding quadrics are pairs of planes

If the roots are rational

Compute the planes and their intersection : 0, 1 or 2  $\sqrt{\delta}$ , unavoidable

Else

Parameterize through a  $(2, 2)$  quadric : one  $\sqrt{\delta}$

$\implies$  Equation in the parameters factors (GCD computation)

in 2 univariate factors of degree 2.

Each linear factor parameterizes one line with at most 2  $\sqrt{\delta}$

(all together 3  $\sqrt{\delta}$ )

If the roots of  $\det(\lambda P + \mu Q)$  generates the same field as  $\sqrt{\delta}$

Plug the parameterization of the quadric in each plan of a pair :

2  $\sqrt{\delta}$  all together

Always optimal