



HAL
open science

Optimality of the parameterization of quadrics and their intersections

Daniel Lazard

► **To cite this version:**

Daniel Lazard. Optimality of the parameterization of quadrics and their intersections. Journées de cloture Visi3D et CoSTIC, 2001, Paris, France. inria-00099402

HAL Id: inria-00099402

<https://inria.hal.science/inria-00099402v1>

Submitted on 26 Sep 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Optimality of the Parameterization of Quadrics and their Intersections

Daniel Lazard

In collaboration with

Sylvain Lazard, Sylvain Petitjean
and Laurent Dupont

December 11, 2001

Optimality

means here optimality in term of

the number of radicals involved

Quadrics of rank 1 or 2

Gauss-Jordan reduction leads to equations :

$$x^2$$

Rank 1 :

Rational parameterization = no radicals

Clearly optimal

$$x^2 - ay^2$$

Rank 2 :

Rational parameterization in $\mathbb{Q}(\sqrt{a})$

Optimal : There are regular rational points iff \sqrt{a} is rational.

Quadratics of rank 3

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2$$

Parameterization :

$$x = 2uv, y = \frac{b}{u^2 - av^2}, z = \frac{\sqrt{bc}}{u^2 + av^2}, t = w$$

\implies At most one square root

There exists a parameterization rational over \mathbb{Q}

\implies The conic $ax^2 + by^2 - cz^2$ has a rational point over \mathbb{Q}

\implies Gauss-Jordan gives equation $x^2 + y^2 - z^2$ when starting from this

point

Example where a **square root** is needed :

$$x^2 + y^2 - 3z^2 \text{ has no rational point (Proof over } \mathbb{Z}/4\mathbb{Z})$$

Quadratics of signature (2,2)

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2 - dt^2$$

Parameterization :

$$x = \frac{a}{uv+av'}, y = \frac{b}{uv-bv'}, z = \frac{\sqrt{ac}}{uv'-av'}, t = \frac{\sqrt{bd}}{uv+bv'}$$

2 square roots

$\delta := abcd$ = discriminant of the quadric.

Invariant **up to a square** by change of coordinates

We claim :

$\sqrt{\delta}$ always needed

Another square root is needed **iff** the quadric has no rational point.

Quadratics of signature (2,2) (continued)

- \mathcal{Q} admits a rational parameterization over \mathbb{K} , linear in one parameter \Leftrightarrow
- \mathcal{Q} contains a line which is rational over \mathbb{K} \Leftrightarrow
- \mathcal{Q} has a rational point over \mathbb{K} and δ is a square in \mathbb{K} \Leftrightarrow
- \mathcal{Q} has equation $x^2 + y^2 - z^2 - t^2$, in some rational frame over \mathbb{K} \Leftrightarrow
- \Leftrightarrow The field of a rational parameterization contains $\mathbb{Q}(\sqrt{\delta})$
- \mathcal{Q} has a rational point over \mathbb{Q} \Leftrightarrow It has a rational point over \mathbb{Q} and equation $x^2 + y^2 - z^2 - \delta t^2$ in some rational frame.

$$x^2 + y^2 - 3z^2 - 11t^2$$

has no rational point over \mathbb{Q}

and no rational parameterization over $\mathbb{Q}(\sqrt{33})$

Proof over $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$

Intersections of quadrics \mathcal{P} and \mathcal{Q} quadrics

The intersection is contained in any quadric of the pencil $\lambda\mathcal{P} + \mu\mathcal{Q}$

There is exactly one quadric of the pencil passing through a point outside of the intersection (**linear equation to solve**)

If the intersection is not empty nor singular

(i.e. $\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has no multiple (projective) root),

the pencil contains a quadric of signature $(2,2)$ with a rational point

Parameterization through this quadric involves $\sqrt{2}$:

A constant one, $\sqrt{\delta}$, the discriminant of the quadric
 One depending on the parameter, $\sqrt{\Delta}$

If the intersection is non singular, $\sqrt{\Delta}$ is needed

Putting the parameterization of the quadric in \mathcal{P} gives an equation

between the parameters.

Isomorphism of the intersection

with a curve in $\mathbb{P}_1 \times \mathbb{P}_1$, the spaces of parameters

De-homogenizing and re-homogenizing :

bi-rational equivalence with a curve in \mathbb{P}_2

of degree 4 with 2 ordinary singular points at infinity.

Non singular intersection has **genus 1**

⇐ No rational parameterization over \mathbb{C}

$\sqrt{\delta}$ is not needed
iff

a surface of degree 8 has a rational point

If $\sqrt{\delta}$ is not needed,

the intersection contains pair of conjugate points of degree 2
the quadric passing through a point on a rational line
defined by such a pair has a square discriminant

\Leftrightarrow The surface

$$z^2 = \det(\mathcal{Q}(x, y, 1, 0) \mathcal{P} - \mathcal{P}(x, y, 1, 0) \mathcal{Q})$$

has a rational point.

This surface has no rational point in the case :

$$\mathcal{P} = 5y^2 + 6xy + 2z^2 - t^2 + 6zt;$$

$$\mathcal{Q} = 3x^2 + y^2 - z^2 - t^2$$

Singular intersections

0 is a multiple root of $\det(\mathcal{F} + \lambda \mathcal{Q})$

$\Leftrightarrow \mathcal{F}$ has rank ≤ 2 (pair of planes) **or**

\mathcal{F} is a cone and all quadrics of the pencil are tangent at its vertex

\Leftrightarrow The intersection has a **singular (complex) point** which is also a

singular point of \mathcal{F}

Up to a real projective transformation,

there are **28 singular intersections** corresponding to **46 pencils**.

Algorithm sketched by Sylvain Lazard

managed through the number of multiple roots of $\det(\lambda P + \mu Q)$



If the intersection does not consist in 4 lines

at most 2 \surd for parameterizing all components

at most 1 \surd may be unnecessary

it may be removed by finding a rational point on a conic

If the intersection consists in 4 non coplanar lines (skew quadrilateral)

at most 2 \surd for each line

possibility of 3 \surd for all lines together

always optimal for each line and for all lines together.

In the case of 4 concurrent lines

general degree 4 equation

Singular quartic

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has a single multiple root

Parameterize the cone corresponding to this root : 1 or 0 \checkmark

Plug it in the equation of \mathcal{P}

\Leftarrow equation which is linear in one of the parameter.

One \checkmark , zero if one finds a rational point on the cone

Cubic an line

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has 2 double roots, real or not

Parameterize through a (2, 2) quartic : no \checkmark

\Leftarrow Equation in the parameters factors (GCD computation) in

a univariate factor (the line)

and a factor linear in that variable (the cubic)

Totally rational parameterization

TWO CONICS

$\det(\lambda P + \mu Q)$ has 1 (rational) double root

The corresponding quadric is a pair of planes

Compute these planes : 1 or 0 $\sqrt{\quad}$

Parameterize through a (2,2) quartic and plug the parameters in the equations of the planes : 1 or 0 $\sqrt{\quad}$

\iff Linear equations in the parameters

First $\sqrt{\quad}$ is needed if the planes are not rational

Second $\sqrt{\quad}$ may be avoided for each conic with a rational point on the field of the planes

Conic and 2 lines

$\det(\lambda P + \mu Q)$ has 2 (rational) double roots

The corresponding quadrics are a cone and a pair of planes

Parameterize through a (2, 2) quadric : one $\sqrt{\quad}$ iff the lines are irrational

\implies Equation in the parameters factors (GCD computation) in

2 univariate factors (the lines)

and a bilinear one (the conic)

At most 1 $\sqrt{\quad}$

It may be avoided for the conic iff it has a rational point

4 lines

$\det(\lambda P + \mu Q)$ has 2 double roots

The corresponding quadrics are pairs of planes

If the roots are rational

Compute the planes and their intersection : 0, 1 or 2 $\sqrt{\delta}$, unavoidable

Else

Parameterize through a $(2, 2)$ quadric : one $\sqrt{\delta}$

\implies Equation in the parameters factors (GCD computation)

in 2 univariate factors of degree 2.

Each linear factor parameterizes one line with at most 2 $\sqrt{\delta}$

(all together 3 $\sqrt{\delta}$)

If the roots of $\det(\lambda P + \mu Q)$ generates the same field as $\sqrt{\delta}$

Plug the parameterization of the quadric in each plan of a pair :

2 $\sqrt{\delta}$ all together

Always optimal