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# IMPLICITIZATION OF SURFACES IN $\mathbb{P}^3$ IN THE PRESENCE OF BASE POINTS

LAURENT BUSÉ, DAVID COX, AND CARLOS D'ANDREA

ABSTRACT. We show that the method of moving quadrics for implicitizing surfaces in  $\mathbb{P}^3$  applies in certain cases where base points are present. However, if the ideal defined by the parametrization is saturated, then this method rarely applies. Instead, we show that when the base points are a local complete intersection, the implicit equation can be computed as the resultant of the first syzygies.

## 1. INTRODUCTION

Let  $x(s, t, u)$ ,  $y(s, t, u)$ ,  $z(s, t, u)$  and  $w(s, t, u)$  be homogeneous polynomials of degree  $n$  such that the parametrization

$$(1) \quad X = \frac{x(s, t, u)}{w(s, t, u)}, \quad Y = \frac{y(s, t, u)}{w(s, t, u)}, \quad Z = \frac{z(s, t, u)}{w(s, t, u)}$$

defines a surface in  $\mathbb{P}^3$ . The implicitization problem consists in the computation of a homogeneous polynomial  $P(X, Y, Z, W)$  whose vanishing defines the projective closure of this surface.

The implicit equation can always be found using Gröbner bases. However, complexity issues mean that in practice, this method is rarely used in geometric modeling, especially in situations where real-time modeling is involved. A more common method for finding the implicit equation is to eliminate  $s, t, u$  by computing the resultant of the three polynomials

$$x(s, t, u) - Xw(s, t, u), \quad y(s, t, u) - Yw(s, t, u), \quad z(s, t, u) - Zw(s, t, u).$$

But in many applications, the resultant vanishes identically due to the presence of base points, which are points  $(s_0 : t_0 : u_0) \in \mathbb{P}^2$  such that

$$x(s_0, t_0, u_0) = y(s_0, t_0, u_0) = z(s_0, t_0, u_0) = w(s_0, t_0, u_0) = 0$$

(see [CGZ] and the references therein).

In [SC], Sederberg and Chen introduced a new technique for finding the implicit equation (1) called the *method of moving quadrics*. This

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method is based in the construction of a matrix  $\mathbb{M}$  whose entries are the coefficients in the monomial basis of certain syzygies of the ideal  $I = \langle x, y, z, w \rangle \subset \mathbb{C}[s, t, u]$ , and syzygies of  $I^2$ . The determinant of this matrix is—under suitable assumptions—the implicit equation. Having a determinantal representation of the implicit equation is useful for geometric modeling because of well-known algorithms for computing symbolic determinants. There is also considerable theoretical interest in knowing when a resultant [SZ, WZ, DD, DE] or an implicit equation [Be] can be represented as a single determinant.

Until now, the method of moving quadrics has been proved valid only in the case where there are no base points (see [CGZ, D, Co]). The motivation for this paper is twofold: first, we wanted to prove the validity of this method in the presence of base points under suitable algebraic conditions on  $I$ , and second, we were curious what tools and concepts from commutative algebra would be required.

Our results provide a positive answer to the first open question given in the last section of [CGZ], in the sense that as the number of base points of the parametrization increases, so does the number of moving planes which occur in the matrix  $\mathbb{M}$ . Moreover, we show that if the number of base points is greater than or equal to the degree of the parametrization, then the implicit equation may be computed as the determinant of a smaller matrix than the one proposed in [CGZ].

One can check also that our method, when applied to the case of no base points, recovers the results of [CGZ]. Hence our method may be regarded as a generalization of [CGZ]. When base points are present, our methods require that they be a local complete intersection. The main theoretical tool used in the proof is the regularity of a homogeneous ideal.

In the second part of the paper, we turn to the case where the ideal  $I = \langle x, y, z, w \rangle$  is saturated. We show in Proposition 4.1 that the method of moving quadrics works only if the degree of the parametrization is 3. So other methods will be needed. Here, the key observation [Co] is that in the saturated case, the syzygy module of  $x, y, z, w$  is a free  $\mathbb{C}[s, t, u]$ -module with 3 generators. If we regard a syzygy  $(A, B, C, D)$  as a polynomial  $AX + BY + CZ + DW$ , then we show in Theorem 4.3 that when  $I$  is a local complete intersection, we can recover the implicit equation by taking the resultant to these syzygies. This allows us to regard a basis of the syzygy module as a generalization of the  $\mu$ -basis for curves given in [CSC] (see also [CZS]).

In general, the search of formulas for implicitization rational surfaces with base points is a very active area of research due to the fact that, in practical industrial design, base points show up quite frequently.

In [MC], a perturbation is applied to resultants in order to obtain a nonzero multiple of the implicit equation. In the recent paper [Bu], a new projection operator called the *residual resultant* (introduced in [BEM]) is developed for computing the implicit equation when the base points locus is a local complete intersection. Recently, Abdallah Al-Amrani informed us that the notes [J] show how to compute the implicit equation as the determinant of the approximation complexes discussed by Vasconcelos in [V].

The paper is organized as follows. In Section 2, we present and discuss the formal structure of the method of moving quadrics in terms of syzygies. We show that there are only two possible sizes for the matrix of moving planes and moving quadrics. Then, in Section 3 we prove that under suitable assumptions on  $I$ , the method actually computes the implicit equation. We illustrate our results with some examples.

In Section 4, we discuss the case where the ideal is a saturated local complete intersection. We prove Theorem 4.3, which asserts that the resultant of a basis of the syzygy module gives the implicit equation raised to a power equal to the degree of the parametrization.

The paper concludes with some open questions in Section 5 suggested by the results of Sections 3 and 4. Appendices A and B give technical results about basepoints and regularity used in Section 3.

## 2. THE FORMAL STRUCTURE OF THE METHOD

Let  $R := \mathbb{C}[s, t, u]$ . A syzygy on  $I = \langle x, y, z, w \rangle \subset R$  is a linear form

$$aX + bY + cZ + dW \in R[X, Y, Z, W]$$

such that  $ax+by+cz+dw = 0$ . This is a *moving plane*. In the same way, we will define a syzygy on  $I^2$  as a quadratic form in  $R[X, Y, Z, W]$  such that it vanishes when the variables are substituted by the polynomials  $x, y, z, w$ . These syzygies are called *moving quadrics*.

The method, as described in [CGZ] for the case where the projective variety  $V(I)$  is empty, consists in fixing a degree  $d$  (in that case  $d = n - 1$ ) and constructing a matrix  $\mathbb{M}$  of size  $\binom{d+2}{2}$ —the number of monomials in three variables of degree  $d$ —having in their rows the coefficients in the monomial basis of a basis of the syzygies of degree  $d$  in the variables  $s, t, u$  on  $I$  and—if there remains space—some linearly independent syzygies on  $I^2$ , also having degree  $d$  in  $s, t, u$ .

It is straightforward to verify that  $\det(\mathbb{M})$  is a homogeneous polynomial in  $\mathbb{C}[X, Y, Z, W]$  such that it vanishes on the surface (1). If this polynomial is not identically zero, then it must be a multiple of

$P(X, Y, Z, W)$ . Under appropriate assumptions on the parametrization, one can show that this determinant gives a non-zero constant times  $P(X, Y, Z, W)$  (see [CGZ, Co]).

Suppose that  $V(I)$  consists of only finitely many points (possibly the empty set). Assume also that the parametrization (1) is proper and that  $V(I)$  is a local complete intersection (possibly empty). Then, it is well-known that the degree of  $P(X, Y, Z, W)$  is equal to  $n^2 - \deg(V(I))$ , where  $\deg(V(I)) = \dim_{\mathbb{C}}(R/I)_k$  for  $k \gg 0$  (see [Co]).

We want to find moving planes and moving quadrics of degree  $d$  such that the determinant of the above matrix  $\mathbb{M}$  equals the implicit equation of the surface. To see what conditions the degree  $d$  must satisfy, consider the exact sequence of  $\mathbb{C}$ -vector spaces:

$$(2) \quad 0 \rightarrow \text{Syz}(I)_d \rightarrow R_d^4 \xrightarrow{B} R_{d+n} \rightarrow (R/I)_{d+n} \rightarrow 0.$$

Here,  $\text{Syz}(I)_d$  is the  $\mathbb{C}$ -vector space of all syzygies of degree  $d$  on  $I$  in the variables  $s, t, u$ , and the map  $B$  is given by  $(x, y, z, w)$ .

Let  $m := \dim \text{Syz}(I)_d$  and  $i := \dim(R/I)_{d+n}$ . Since (2) is exact, we obtain

$$(3) \quad i + 4 \binom{d+2}{2} = m + \binom{d+n+2}{2}.$$

Denote by  $\mathbb{M}$  the matrix of moving planes and moving quadrics as explained at the beginning of this section. It is of size  $\binom{d+2}{2}$ , where  $m$  of the rows are homogeneous of degree one in  $X, Y, Z, W$  and the remaining are of degree 2. We want the determinant of  $\mathbb{M}$  to equal  $P(X, Y, Z, W)$  (up to a nonzero constant). Comparing degrees, we get the equation

$$(4) \quad m + 2 \left( \binom{d+2}{2} - m \right) = n^2 - \deg(V(I)),$$

with the additional condition

$$(5) \quad \binom{d+2}{2} - m \geq 0,$$

which says that the number of syzygies of degree  $d$  on  $I$  is less than or equal to the size of  $\mathbb{M}$ .

However, if we compare  $\deg(V(I)) = \dim_{\mathbb{C}}(R/I)_k$  for  $k \gg 0$  with  $i = \dim_{\mathbb{C}}(R/I)_{d+n}$ , it makes sense to also assume that

$$(6) \quad i = \dim_{\mathbb{C}}(R/I)_{d+n} = \deg(V(I)).$$

Combining this with (4) gives

$$(7) \quad m + 2 \left( \binom{d+2}{2} - m \right) = n^2 - i,$$

and solving equations (3) and (7) in  $d$  and  $m$  leads to the following solutions:

$$(8) \quad \begin{aligned} & d = n - 1, \text{ in which case } m = n + i, \\ & d = n - 2, \text{ in which case } m = i - n. \end{aligned}$$

From this we see that, in the case where there are few base points, the only possibility is  $d = n - 1$ .

**Remark 2.1.** [CGZ] treats the case when the parametrization has no base points. As just noted, this implies  $d = n - 1$ . Furthermore,  $\deg(V(I)) = 0$  in this case, so that (6) is equivalent to the surjectivity of the map  $B$  in (2). In [CGZ], the matrix of  $B$  is denoted  $MP$ , so that the surjectivity of  $B$  means that  $MP$  has maximal rank. This is a part of the hypothesis of Theorem 5.2 of [CGZ].

### 3. EXTENSION OF THE METHOD

In this section, we will extend the method of moving quadrics to the case where base points are present. In order to do this, we impose the following *base point conditions* on the input polynomials:

- BP1:  $x(s, t, u)$ ,  $y(s, t, u)$ ,  $z(s, t, u)$  and  $w(s, t, u)$  are homogeneous of degree  $n$  and linearly independent over  $\mathbb{C}$ .
- BP2:  $V(I)$  consists of a finite number of points and  $\deg(V(I))$  equals the sum of the multiplicities of the distinct points in the locus  $V(I)$ .
- BP3: There is  $d \in \{n-2, n-1\}$  such that  $\dim_{\mathbb{C}}(R/I)_{d+n} = \deg(V(I))$ .
- BP4:  $w \in \text{sat}(x, y, z)$  (where “sat” denotes saturation).
- BP5:  $\dim \text{Syz}(x, y, z)_d = 0$ , where  $d$  is as in BP3.

We can explain these conditions as follows:

1. Condition BP1 is obvious, except possibly for the linear independence. For this, observe that a linear relation among  $x, y, z, w$  implies that the image of the parametrization is a plane. This case is trivial.

2. The finiteness of  $V(I)$  in Condition BP2 is equivalent to assuming that  $x, y, z, w$  have no common factors. Also, the degree formula for the image of the parametrization given in [Co] involves the sum of the multiplicities of the base points. Finally note that  $\deg(V(I))$  equals the sum of the multiplicities of the distinct points in the locus  $V(I)$  if and only if  $V(I)$  is a local complete intersection.

3. Condition BP3 is explained by (6) and (8) from Section 2. The surprise is that BP3 is equivalent to the following regularity condition:

BP3': There is  $d \in \{n - 2, n - 1\}$  such that  $I$  is  $(d + n)$ -regular.

This follows from Theorem B.4 since  $d \in \{n - 2, n - 1\}$  implies that  $d + n \geq 2n - 2$ . Regularity will play an important role in the proof of Theorem 3.4. See [BS] for a discussion of regularity.

4. Since  $V(I)$  is a local complete intersection, Corollary A.2 implies that we can obtain condition BP4 by replacing the input polynomials with generic linear combinations of them.

5. Consider the exact sequence

$$0 \rightarrow \text{Syz}(x, y, z)_d \rightarrow \text{Syz}(x, y, z, w)_d \rightarrow R_d,$$

where the first map sends  $(A, B, C)$  to  $(A, B, C, 0)$  and the second sends  $(A, B, C, D)$  to  $D$ . This shows that BP5 implies the inequality (5) for the given value of  $d$ . Also note that in the case where there are no base points, BP5 is satisfied provided that the homogeneous resultant of  $x, y, z$  is different than zero. This is shown in [CGZ, Lemma 5.1].

In order to see the independence between the conditions, consider the following examples.

**Example 3.1.** Take  $x = s^5$ ,  $y = t^5$ ,  $z = su^4$ , and  $w = st^2u^2$ . Here, we have  $n = 5$  and  $V(I)$  is the local complete intersection consisting of the point  $(0 : 0 : 1)$  of multiplicity 5. Thus Conditions BP1 and BP2 are satisfied. However,  $d = 3$  or  $4$  implies  $d + n = 8$  or  $9$ , yet one can check easily with `Macaulay 2` that regularity of  $I$  is 10. Thus BP3' and its equivalent BP3 fail in this case.

**Example 3.2.** Set  $x = su^2$ ,  $y = t^2(s + u)$ ,  $z = st(s + u)$ , and  $w = tu(s + u)$ . For this parametrization, we have  $n = 3$  and  $V(I)$  is the local complete intersection consisting of the three points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$  of respective multiplicity 2, 3 and 1. The implicit equation is hence a cubic surface. One can compute with `Macaulay 2` that  $I$  is  $2n - 2 = 4$ -regular. It follows easily that Conditions BP1–BP4 are satisfied by taking  $d = 1$ . However, one can prove that  $\dim_{\mathbb{C}}(\text{Syz}(x', y', z')_1) = 1$ , where  $x', y', z'$  are generic linear combinations of  $x, y, z, w$ . Thus BP5 does not hold in this example, even if we use generic linear combinations of  $x, y, z, w$ . We will see later that the method of moving quadrics fails in this case.

**3.1. Construction of the moving plane coefficient matrix.** Let  $x, y, z, w$  satisfy BP1–BP5 and consider the following algorithm:

$$\begin{aligned} I_w &:= \emptyset; \\ \Omega &:= \{s^i t^j u^{d-i-j}(x, y, z), 0 \leq i + j \leq d\}; \\ \Gamma_w &:= \{s^i t^j u^{d-i-j}w, 0 \leq i + j \leq d\}. \end{aligned}$$

While  $\Gamma_w \neq \emptyset$

- Select a column  $s^i t^j u^{d-i-j} w$  from  $\Gamma_w$ , and remove it from  $\Gamma_w$ ;
- If  $s^i t^j u^{d-i-j} w$  is linearly independent from the columns in  $\Omega$ , then add it to  $\Omega$ ; otherwise, add  $(i, j)$  to  $I_w$ .

Observe that at the beginning of the algorithm, the set  $\Omega$  is linearly independent by Condition BP5.

**Remark 3.3.** It is straightforward to check that at the end of the algorithm,  $|I_w| = m =$  the dimension of the syzygies of degree  $d$  on  $x, y, z, w$ .

Also note that in the case where there are no base points, this algorithm constructs the matrix denoted by  $MP_I$  in [CGZ].

Now define  $MP_{I_w}$  to be the coefficient matrix of the polynomials

$$\begin{aligned} & s^i t^j u^{d-i-j}(x, y, z), \quad 0 \leq i + j \leq d, \\ & s^i t^j u^{d-i-j} w, \quad (i, j) \notin I_w. \end{aligned}$$

Observe that  $MP_{I_w}$  is a submatrix of  $MP$  having the same rank as  $MP$  and is maximal with this property. Thus  $MP_{I_w}$  has maximal rank.

**3.2. The moving quadrics coefficient matrix.** Let  $MQ$  be the coefficient matrix of the polynomials

$$s^i t^j u^{d-i-j}(x^2, y^2, z^2, xy, xz, yz, xw, yw, zw, w^2),$$

and let  $MQ_{I_w}$  be the submatrix determined by the polynomials

$$(9) \quad \begin{aligned} & s^i t^j u^{d-i-j}(x^2, y^2, z^2, xy, xz, yz), \quad 0 \leq i + j \leq d, \\ & s^i t^j u^{d-i-j}(xw, yw, zw) \quad (i, j) \notin I_w. \end{aligned}$$

The Theorem 5.1 in [CGZ] may be extended as follows.

**Theorem 3.4.** *Let  $x, y, z, w$  satisfy BP1–BP5 and construct  $MQ_{I_w}$  as above. Then  $MQ_{I_w}$  has maximal rank. Furthermore, the columns of  $MQ_{I_w}$  are a basis of the  $\mathbb{C}$ -vector space  $I_{d+2n}^2$ .*

*Proof.* Suppose that there exist homogeneous polynomials of degree  $d$ , say  $p_1(s, t, u), \dots, p_9(s, t, u)$ , such that

$$(10) \quad p_1 x^2 + p_2 y^2 + p_3 z^2 + p_4 xy + p_5 xz + p_6 yz + p_7 xw + p_8 yw + p_9 zw = 0,$$

where the exponents of the monomials which appear in  $p_7, p_8, p_9$  are of the form  $(i, j, d - i - j)$ ,  $(i, j) \notin I_w$ .

Rewrite equation (10) as

$$(11) \quad (p_1 x + p_4 y + p_5 z + p_7 w)x + (p_2 y + p_6 z + p_8 w)y + (p_3 z + p_9 w)z = 0.$$

Equation (11) is a syzygy on  $x, y, z$ .

Condition BP4 implies that  $V(x, y, z) = V(I)$  as subschemes of  $\mathbb{P}^2$ . It follows that (11) is a syzygy which vanishes at the base point locus



$Z = V(x, y, z)$  in the sense of [CS] (i.e., a syzygy  $(a_1, a_2, a_3)$  on  $x, y, z$  vanishes on  $V(x, y, z)$  if  $a_i \in \text{sat}(x, y, z)$  for all  $i$ ).

As the ideal generated by  $x, y, z$  is a local complete intersection by Condition BP2, Theorem 1.7 of [CS] implies that these syzygies are “Koszul syzygies” in the sense of [Co, CS]. Thus, there are polynomials  $a, b, c$  of degree  $d$  such that

$$\begin{aligned} p_1x + p_4y + p_5z + p_7w &= ay + bz, \\ p_2y + p_6z + p_8w &= -ax + cz, \\ p_3z + p_9w &= -bx - cy. \end{aligned}$$

Since the exponents of the monomials in  $p_9$  are not in  $I_w$ , the third equality tells us that the columns of  $MP_{I_w}$  are linearly dependent unless  $p_3 = p_9 = b = c = 0$ , but if this happens, then the second equality will be  $p_2y + p_6z + p_8w = -ax$ . Since  $MP_{I_w}$  has maximal rank, it follows that  $a = p_2 = p_6 = p_8 = 0$ . But then the first equation implies that  $MP_{I_w}$  doesn't have maximal rank. This contradiction proves that  $MQ_{I_w}$  has maximal rank.

It follows the columns of  $MQ_{I_w}$  are  $\mathbb{C}$ -linearly independent, and they lie in the space of polynomials of degree  $d + 2n$  which belong to  $I^2$ . In order to see that they generate all of  $I_{d+2n}^2$ , we argue as follows.

As  $I^2 \subset \text{sat}(I^2)$ , we clearly have  $I_{d+2n}^2 \subset \text{sat}(I^2)_{d+2n}$ . We will show that the columns of  $MQ_{I_w}$  are actually a basis of  $\text{sat}(I^2)_{d+2n}$ .

Since  $V(I)$  consists of a finite number of points, and  $I$  is  $(d + n)$ -regular by BP3, Corollary 5 of [Ch] implies that  $\text{sat}(I^2)$  is  $(d + 2n)$ -regular. As in Section 2, we let  $i$  denote the dimension of  $(R/I)_{d+n}$ . Using the exact sequence  $0 \rightarrow I/I^2 \rightarrow R/I^2 \rightarrow R/I \rightarrow 0$  and the fact that  $I$  is a local complete intersection of codimension two (see the proof of Theorem 2.4 in [CS]), one can show that the Hilbert polynomial of  $R/I^2$  is equal to  $3i$ .

As  $I^2$  and  $\text{sat}(I^2)$  have the same Hilbert polynomial, and as  $\text{sat}(I^2)$  is  $(d + 2n)$ -regular, we see that  $\dim_{\mathbb{C}}(R/\text{sat}(I^2))_{d+2n} = 3i$ , so the dimension of  $\text{sat}(I^2)_{d+2n}$  is equal to  $\binom{d+2n+2}{2} - 3i$ . Using (9) and Remark 3.3, we see that the rank of  $MQ_{I_w}$  is equal to

$$3 \binom{d+n+2}{2} - 3 \binom{d+2}{2} - 3i.$$

This number equals  $\dim_{\mathbb{C}}(\text{sat}(I^2)_{d+2n})$  when  $d = n - 2$  or  $n - 1$ . So the polynomials in (9) are actually a basis of  $\text{sat}(I^2)_{d+2n}$ . From this the last part of the theorem follows straightforwardly.  $\square$

**Remark 3.5.** The argument of Theorem 3.4 shows that

$$I_{d+2n}^2 = \text{sat}(I^2)_{d+2n}.$$



which by (4) equals

$$n^2 - i.$$

This is the degree of the implicit equation since the parametrization is generically one-to-one.

Checking the diagonal of  $\mathbb{M}$ , we can see that the determinant of  $\mathbb{M}$  has the term  $W^{n^2-i}$ , provided that it does not cancel with other term of the same form. But it is straightforward to see that this is the highest power of  $W$  which appears in the expansion of the determinant. So,  $\det(\mathbb{M}) \neq 0$ , and it is easy to see that it vanishes whenever the point  $(X, Y, Z, W)$  lies on the parametric surface because each row represents a moving plane or quadric that follows the surface.

It now follows easily that  $\det(\mathbb{M})$  is the implicit equation of the surface.  $\square$

**Example 3.7.** Take  $x = st$ ,  $y = u^2$ ,  $z = s^2 + tu$ , and  $w = tu$ . Here, the implicit equation is  $W^2(Z - W) - X^2Y$ , and the zero locus of  $x, y, z, w$  in  $\mathbb{P}^2$  is  $\{(0 : 1 : 0)\}$ , which is a base point of multiplicity 1.

All the base point conditions are satisfied here. The degree of the parametrization is  $n = 2$  and the degree of the implicit equation is  $3 = 2^2 - 1$ . The unique value of  $d$  possible here is  $d = n - 1 = 1$ . The matrix  $MP$  in the lexicographic order  $s > t > u$  is

$$MP = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The rows of this matrix correspond to the coefficients in the monomial basis of the polynomials  $(s, t, u)(x, y, z, w)$ . It is straightforward to verify that the last three rows (corresponding to  $(s, t, u)w$ ) are linear combinations of the previous rows, so we may choose  $I_w =$

$\{(1, 0), (0, 1), (0, 0)\}$ . This gives

$$MP_{I_w} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has maximal rank. This matrix gives three linearly independent moving planes of degree 1:

$$\begin{aligned} P_{0,0} &= uW - tY, \\ P_{1,0} &= sW - uX, \\ P_{0,1} &= tW - tZ + sX. \end{aligned}$$

In this case, as  $i = 1$  and  $(n^2 - n)/2 - i = 0$ , so that there are no moving quadrics to consider. Then the matrix of moving planes and moving quadrics is

$$\mathbb{M} = \begin{pmatrix} W & 0 & -X \\ X & W - Z & 0 \\ 0 & Y & -W \end{pmatrix},$$

and the determinant of this matrix gives the implicit equation.

**Example 3.8.** Let  $x = s^3$ ,  $y = t^2u$ ,  $z = s^2t + u^3$ , and  $w = stu$ . One can check that all conditions are satisfied and  $V(I) = \{(0 : 1 : 0)\}$  with multiplicity 2. Again, the only possibility is  $d = n - 1$ . With the aid of `Maple`, we found the following five moving planes of degree 2:

$$\begin{aligned} P_{0,0} &= u^2W + t^2X - stZ, \\ P_{0,1} &= tuW - suY, \\ P_{1,1} &= stW - s^2Y, \\ P_{2,0} &= s^2W - tuX, \\ P_{0,2} &= t^2W - stY, \end{aligned}$$

and the moving quadric  $Q_{1,0} = suW^2 - u^2XY$ . This gives

$$\mathbb{M} = \begin{pmatrix} W & 0 & 0 & 0 & -X & 0 \\ -Y & W & 0 & 0 & 0 & 0 \\ 0 & 0 & W^2 & 0 & 0 & -XY \\ 0 & -Y & 0 & W & 0 & 0 \\ 0 & 0 & -Y & 0 & W & 0 \\ 0 & -Z & 0 & X & 0 & W \end{pmatrix}.$$

One computes that  $\det(\mathbb{M}) = W^7 - X^2Y^3ZW + X^3Y^4$ , which is the implicit equation of the parametric surface.

**Example 3.9.** We present here a case where  $d = n - 2$ . This example is taken from [SC]. Consider the following parametrization of a cubic surface with 6 base points:

$$\begin{aligned} x &= s^2t + 2t^3 + s^2u + 4stu + 4t^2u + 3su^2 + 2tu^2 + 2u^3, \\ y &= -s^3 - 2st^2 - 2s^2u - stu + su^2 - 2tu^2 + 2u^3, \\ z &= -s^3 - 2s^2t - 3st^2 - 3s^2u - 3stu + 2t^2u - 2su^2 - 2tu^2, \\ w &= s^3 + s^2t + t^3 + s^2u + t^2u - su^2 - tu^2 - u^3. \end{aligned}$$

One can check with `Macaulay 2` that  $I$  is saturated, local complete intersection and its regularity is 3. As shown in [SC], we have the following basis of syzygies of degree  $d = n - 2 = 1$ :

$$(12) \quad \begin{aligned} &sX + tY + uZ, \\ &s(Y + W) + t(2Y - Z) + u(Y + 2W), \\ &s(Z - Y) + t(-X + 2W) + u(X - Y). \end{aligned}$$

The first syzygy shows that  $\text{Syz}(x, y, z)_d \neq 0$ , so that Condition BP5 is not verified. But if we consider  $x, y, w$  instead, then it is straightforward to check that all conditions are satisfied, and the method produces the following matrix of moving planes (again there are no moving quadrics to consider here):

$$\mathbb{M} = \begin{pmatrix} Z - Y & -X + 2W & X - Y \\ -Y - W & Z - 2Y & -Y - 2W \\ X & Y & Z \end{pmatrix}.$$

The determinant of this matrix is the determinant of the matrix of syzygies in (12), which has been shown in [SC] to be the implicit equation of the surface.

**Example 3.10.** In this example we focus on Condition BP5 and show that the method of moving surfaces may fail if  $\text{Syz}(x', y', z')_d$  is nonzero when  $x', y', z'$  are generic linear combinations of  $x, y, z, w$ . We take the

parametrization given in Example 3.2, and  $d = 1$ . The following is a basis of syzygies of degree 1:

$$\begin{aligned} sW - uZ, \\ tW - uY, \\ tZ - sY. \end{aligned}$$

Following the method of moving quadrics, this gives the  $3 \times 3$  matrix  $\mathbb{M}$  whose rows are given by these syzygies. However, one easily sees that  $\det(\mathbb{M})$  is identically 0 in this case and hence it is not an implicit equation.

**Remark 3.11.** More generally, suppose that  $x, y, z, w$  is a parametrization which satisfies BP1–BP4 but fails BP5 even after a generic coordinate change, as in the previous example. Then it is easy to see that the method of moving quadrics must fail in one of two ways. To see this, recall that BP5 implies the equality

$$\binom{d+2}{2} - m \geq 0.$$

So when BP5 fails, this inequality may fail, which means that the number of linearly independent moving planes is greater than the number of rows of  $\mathbb{M}$ . But even when the above inequality holds, there are still problems, which we explain as follows. When we replace  $x, y, z, w$  with generic linear combinations, the implicit equation of the surface must contain  $W^{n^2-i}$ . However, since  $\text{Syz}(x', y', z')_d$  is nonzero, it follows that at least one row of  $\mathbb{M}$  will not contain  $W$ , so that  $W$  appears to the power at most  $n^2 - i - 1$  in  $\det(\mathbb{M})$ . This contradiction shows that the method of moving quadrics fails.

#### 4. THE SATURATED CASE

Now we will concentrate on the case where  $I$  is a saturated local complete intersection, with  $V(I)$  consisting in a finite number of points. We will show that the method of moving quadrics rarely applies and that when it does fail, it can often be replaced with a nice resultant.

In this situation, it is well-known (see [Co, Prop. 5.2]) that  $R/I$  is Cohen-Macaulay and the syzygy module  $\text{Syz}(I)$  is a free graded  $\mathbb{C}[s, t, u]$ -module. We also have the following resolution of  $I$  (see [Co]):

$$(13) \quad 0 \rightarrow R(-n - \mu_1) \oplus R(-n - \mu_2) \oplus R(-n - \mu_3) \xrightarrow{A} R(-n)^4 \xrightarrow{B} I \rightarrow 0,$$

where  $\mu_1 + \mu_2 + \mu_3 = n$ , the map  $B$  is given by  $(x, y, z, w)$ , and the columns of  $A$  give three syzygies of degrees  $\mu_1, \mu_2, \mu_3$  respectively which are free generators of  $\text{Syz}(I)$ .

**4.1. Limitations on the method of moving quadrics.** The following proposition shows that in the saturated case, the method described in the previous section can be used only for very low degrees.

**Proposition 4.1.** *If  $I$  is saturated and satisfies Conditions BP1–BP5, then the method of moving quadrics works only for  $d = n - 2$  and  $n \leq 3$ .*

*Proof.* Let us first prove that the method does not work for  $d = n - 1$ . By (8), this implies  $m = n + i$ , and we also have  $i = \frac{1}{2}(n^2 + \mu_1^2 + \mu_2^2 + \mu_3^2)$  by [Co, Proposition 5.3]. Then the inequality (5) becomes

$$\frac{(n+1)n}{2} \geq n + \frac{1}{2}(n^2 + \mu_1^2 + \mu_2^2 + \mu_3^2),$$

which is impossible for positive values of  $n$ .

Now consider the case  $d = n - 2$ . Here,  $m = i - n$ , and we have the same formula for  $i$ . Thus the inequality (5) becomes

$$\frac{n(n-1)}{2} \geq \frac{1}{2}(n^2 + \mu_1^2 + \mu_2^2 + \mu_3^2) - n,$$

which is equivalent to  $n \geq \mu_1^2 + \mu_2^2 + \mu_3^2$ . As  $\mu_1 + \mu_2 + \mu_3 = n$ , we obtain

$$n \geq \mu_1^2 + \mu_2^2 + \mu_3^2 \geq \frac{n^2}{3}.$$

This shows that  $n$  must be at most 3.

From here, it is now easy to see that the only nontrivial case is  $n = 3$  and  $\mu_1 = \mu_2 = \mu_3 = 1$  (otherwise, the surface will be a plane). In this case, we have  $m = 3$ , which is the number of monomials of degree 1 in 3 variables, and the matrix  $\mathbb{M}$  is the matrix of the basis of syzygies on  $(x, y, z, w)$  of degree one. The determinant of this matrix gives the implicit equation. This can be proved by hand, or regarded as a special case of Theorem 4.3 (see Corollary 4.4).  $\square$

**4.2. The implicit equation as a resultant.** In [CSC], an exact sequence similar to (13) was used to represent the implicit equation of a parametric curve as the resultant of the homogeneous polynomials which were free generators of the syzygy module (see also [Co] for an exposition of this). We will discuss whether these results extend to surfaces in the saturated case.

Consider again the exact sequence (13). Write

$$A = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{pmatrix}.$$

This means that the polynomials

$$\begin{aligned}\mathbf{p} &= p_1X + p_2Y + p_3Z + p_4W, \\ \mathbf{q} &= q_1X + q_2Y + q_3Z + q_4W, \\ \mathbf{r} &= r_1X + r_2Y + r_3Z + r_4W\end{aligned}$$

are syzygies of degrees  $\mu_1, \mu_2, \mu_3$  in the variables  $s, t, u$ , which generate the syzygy module of  $x, y, z, w$ . Let  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\cdot, \cdot, \cdot)$  be the homogeneous resultant of three homogeneous polynomials of degrees  $\mu_1, \mu_2, \mu_3$  as defined in [CLO]. One may ask whether  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})$  computes a power of the implicit equation, as in the case of curves. Unfortunately, the following example shows that this is not always the case.

**Example 4.2.** Let

$$\begin{aligned}x &= st^3 - s^4 - 2s^2t^2 + s^2tu + 4s^3t - 2t^3u, \\ y &= s^2tu - s^3t - 2s^3u + 3st^2u - t^3u, \\ z &= s^3u - st^3 - 4s^2tu + 6t^3u - st^2u, \\ w &= s^3u - 3st^3 - 2st^2u + 6s^2t^2 + t^4 - ts^3.\end{aligned}$$

The ideal generated by these polynomials is saturated, and  $A$  is the following matrix:

$$\begin{pmatrix} s & 2s & tu \\ s & t & s^2 \\ 2t & s & t^2 \\ t & 3t & su \end{pmatrix}.$$

All entries in this matrix vanish under the substitution  $s \mapsto 0, t \mapsto 0$ , so the homogeneous resultant of the first syzygies will be identically zero due to the fact that the polynomials  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  have the common root  $(0 : 0 : 1)$  in projective space.

However, the ideal of Example 4.2 is not a local complete intersection. If we add this hypothesis (which is part of Condition BP2 from Section 3), then we get the following nice result.

**Theorem 4.3.** *Assume that  $I$  is saturated and satisfies Conditions BP1 and BP2. Then*

$$(14) \quad \text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = P(X, Y, Z, W)^h,$$

where  $h$  is the degree of the parametrization and  $P(X, Y, Z, W) = 0$  is an implicit equation of the surface.

*Proof.* Let  $S = V(P) \subset \mathbb{P}^3$  be the Zariski closure of the image of the parametrization (1). To prove the theorem, first suppose that the



resultant vanishes at a point  $(X_0 : Y_0 : Z_0 : W_0) \in \mathbb{P}^3$ . This means that the system of equations in variables  $s, t, u$  given by

$$(15) \quad \begin{aligned} p_1 X_0 + p_2 Y_0 + p_3 Z_0 + p_4 W_0 &= 0, \\ q_1 X_0 + q_2 Y_0 + q_3 Z_0 + q_4 W_0 &= 0, \\ r_1 X_0 + r_2 Y_0 + r_3 Z_0 + r_4 W_0 &= 0 \end{aligned}$$

has a non-trivial solution  $(s_0, t_0, u_0)$ . We will show that

$$(16) \quad (X_0 : Y_0 : Z_0 : W_0) \in S \cup \bigcup_{p \in V(I)} L_p,$$

where  $L_p$  is a line. Since the right-hand side is a proper subvariety of  $\mathbb{P}^3$ , this will prove that  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})$  is a nonzero polynomial and hence has zero locus of pure codimension 1. Since  $L_p$  has codimension 2 and  $V(I)$  is finite, this will prove that the zero locus lies in  $S$ .

Given the solution  $(s_0, t_0, u_0)$  of (15), we can specialize the variables  $s, t, u$  to  $s_0, t_0, u_0$  in the exact sequence (13). This transforms (13) into a complex of vector spaces

$$0 \rightarrow \mathbb{C}^3 \xrightarrow{A_0} \mathbb{C}^4 \xrightarrow{B_0} \mathbb{C} \rightarrow 0,$$

where  $A_0$  is the matrix  $A$  specialized, and  $B_0 = (X_0, Y_0, Z_0, W_0)$ . As  $B_0$  is surjective and—because of (13)— $B_0 A_0 = 0$ , we see that the complex is exact if and only if  $A_0$  is injective. By the Hilbert-Burch Theorem, the maximal minors of  $A_0$  are  $x(s_0, t_0, u_0)$ ,  $y(s_0, t_0, u_0)$ ,  $z(s_0, t_0, u_0)$ ,  $w(s_0, t_0, u_0)$ . So, if we are outside of the zero locus of  $I$ , the complex is exact, and the determinant of the complex is non-zero (see [GKZ, Appendix A] for a definition of the determinant of a complex). Moreover, applying the Cayley formula for computing this determinant with respect to the monomial bases, we get the following:

$$\begin{aligned} x(s_0, t_0, u_0) &= X_0 D, \\ y(s_0, t_0, u_0) &= Y_0 D, \\ z(s_0, t_0, u_0) &= Z_0 D, \\ w(s_0, t_0, u_0) &= W_0 D, \end{aligned}$$

where  $D$  is the determinant of the complex. From here, it is easy to see that the point  $(X_0 : Y_0 : Z_0 : W_0)$  belongs to the surface  $S$ .

However, if  $p = (s_0 : t_0 : u_0) \in V(I)$ , then the above argument fails. To see what happens in this case, we first study the rank of the specialized matrix  $A_0$ . Localizing (13) at  $p$  gives

$$(17) \quad 0 \rightarrow \mathcal{O}_p^3 \xrightarrow{A} \mathcal{O}_p^4 \xrightarrow{B} \mathcal{I}_p \rightarrow 0,$$

where  $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$  and  $\mathcal{I}$  is the ideal sheaf associated to  $I$ . Since  $\mathcal{I}_p \subset \mathcal{O}_p$  is a complete intersection, the minimal resolution of  $\mathcal{I}_p$  is of the form:

$$0 \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_p^2 \rightarrow \mathcal{I}_p \rightarrow 0.$$

This means that (17) is isomorphic to the exact sequence obtained from the minimal resolution by adding the trivial complex

$$0 \rightarrow \mathcal{O}_p^2 = \mathcal{O}_p^2 \rightarrow 0 \rightarrow 0.$$

Hence  $A$  has a  $2 \times 2$  minor which doesn't vanish at  $p$ . In other words, the matrix  $A_0$  has rank  $\geq 2$ . It follows that substituting  $p$  into (15) gives a system of linear equations of rank  $\geq 2$  when regarded as equations in  $X_0, Y_0, Z_0, W_0$ . However, we also know that the rank is  $< 3$  since the  $3 \times 3$  minor of  $A$  are  $x, y, z, w$ , which vanish at  $p \in V(I)$ . Projectively, this means that  $(X_0 : Y_0 : Z_0 : W_0)$  belongs to the line  $L_p \subset \mathbb{P}^3$  defined by substituting  $p$  into (15). This completes the proof of (16).

The next step is to show that the resultant vanishes on  $V(P)$ , and for this, it is enough to show that it vanishes on a Zariski dense subset. For instance, we can take the image of the parametrization

$$\begin{aligned} \mathbb{P}^2 \setminus V(I) &\rightarrow \mathbb{P}^3, \\ (s : t : u) &\mapsto (x(s, t, u) : y(s, t, u) : z(s, t, u) : w(s, t, u)). \end{aligned}$$

For  $(X_0 : Y_0 : Z_0 : W_0)$  in the image, we can find  $(s_0 : t_0 : u_0) \in \mathbb{P}^2$  in the preimage. It is straightforward to check that the syzygies  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  vanish after the specialization of all the variables. Thus  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})$  vanishes at  $(X_0 : Y_0 : Z_0 : W_0)$ .

Since  $P$  is irreducible, it follows that

$$\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = cP(X, Y, Z, W)^\delta$$

for some  $\delta \in \mathbb{N}$  and a non-zero constant multiplier  $c$ . To see that  $\delta$  is the degree of the parametrization, note that by [Co], the degree of the surface is equal to  $\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$ , which is equal to the degree of  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})$  in the variables  $X, Y, Z, W$ . By the degree formula (see [Co, Appendix]), this number must be  $h$  times the degree of  $P(X, Y, Z, W)$ .  $\square$

**Corollary 4.4.** *If  $n = 3$  and  $\mu_1 = \mu_2 = \mu_3 = 1$ , then the implicit equation is the determinant of the first syzygy module.*

## 5. OPEN QUESTIONS

**Question 5.1.** Most of the base point conditions imposed on the ideal  $I$  in Section 3 were needed in order to prove that matrix  $\mathbb{M}$  has nonzero

determinant. A straightforward computation shows that—for the degrees  $d$  of Section 2—there is a natural map

$$(18) \quad \begin{aligned} \text{Syz}(I)_d^4 &\rightarrow \text{Syz}(I^2)_d, \\ (S_1, S_2, S_3, S_4) &\mapsto S_1X + S_2Y + S_3Z + S_4W \end{aligned}$$

whose cokernel has dimension greater than or equal to  $\binom{d+1}{2} - m$ . Thus, if inequality (5) holds, then we can fill  $\mathbb{M}$  with moving quadrics which do not come from the previous map. It is easy to see that, in order to have  $\det(\mathbb{M}) \neq 0$ , the moving quadrics of  $\mathbb{M}$  must not belong to the image of (18). Is this a sufficient condition? This would make it easier to compute the implicit equation, and we would have a general result with fewer conditions on the base points.

**Question 5.2.** In order to construct matrix  $\mathbb{M}$  we used *all* moving planes of a given degree. Can we make this condition weaker, i.e., can we use matrices which use some but not all moving planes of degree  $d$ ?

**Question 5.3.** In the situation of Theorem 4.3, one can ask how the resultant  $\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})$  relates to the implicit equation  $P = 0$  when  $I$  is not necessarily a local complete intersection. In general, one can show that if  $h$  is the degree of the parametrization, then

$$(19) \quad \begin{aligned} h \deg(P) &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 - \sum_{p \in V(I)} (e_p - d_p), \\ &= \deg(\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})) - \sum_{p \in V(I)} (e_p - d_p), \end{aligned}$$

where

$$\begin{aligned} e_p &= \text{multiplicity of } \mathcal{I}_p \subset \mathcal{O}_p, \\ d_p &= \text{degree of } \mathcal{I}_p \subset \mathcal{O}_p. \end{aligned}$$

Here,  $\mathcal{I}_p$  is the ideal of the local ring  $\mathcal{O}_p$  induced by  $I$ .

To analyze this, let  $A$  be as in (13) and let  $V_i(A) \subset \mathbb{P}^2$  be the subscheme defined by the vanishing of the  $i \times i$  minors of  $A$ . Then

$$V_1(A) \subset V_2(A) \subset V_3(A) = V(I),$$

where the last equality holds by the Hilbert-Burch Theorem. Hence there are three cases to consider:

*Case 1:*  $V_1(A) \neq \emptyset$ . In this situation, it is easy to see that the resultant vanishes identically. This is what happened in Example 4.2.

*Case 2:*  $V_2(A) = \emptyset$ . When  $I$  has a resolution of the form (13), it is easy to show that

$$V_2(A) = \emptyset \Leftrightarrow I \text{ is a local complete intersection.}$$

Hence this case is covered by Theorem 4.3.

*Case 3:*  $V_1(A) = \emptyset$  and  $V_2(A) \neq \emptyset$ . When  $p \in V_2(A)$  is substituted into (15), the resulting system of linear equations has rank 1 and hence defines a plane  $H_p \subset \mathbb{P}^3$ . Then the argument used to prove (16) can be modified so show that

$$(20) \quad V(\text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r})) = S \cup \bigcup_{p \in V_2(A)} H_p.$$

It follows that the resultant has extraneous factors in this case. If  $\ell_p = 0$  is the equation of the plane  $H_p$ , then we have the following conjectural formula for the resultant.

**Conjecture 5.4.** *Let  $I \subset \mathbb{C}[s, t, u]$  be generated by  $x, y, z, w$  of degree  $n$  such that  $V(I)$  is finite. Also assume that:*

- (1)  *$I$  is saturated with free resolution given by (13).*
- (2)  *$V_1(A) = \emptyset$  and  $V_2(A) \neq \emptyset$ .*

*Then, up to a nonzero constant, we have*

$$(21) \quad \text{Res}_{\mu_1, \mu_2, \mu_3}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = P(X, Y, Z, W)^h \prod_{p \in V_2(A)} \ell_p^{e_p - d_p}.$$

This conjecture is compatible with (20) since  $p \in V_2(A) \Rightarrow \mathcal{I}_p$  is not a complete intersection  $\Rightarrow e_p > d_p$ . Furthermore, (19) shows that each side of (21) has the same degree since  $e_p = d_p$  for  $p \in V(I) \setminus V_2(A)$ .

For  $p \in V_2(A)$ , notice that  $e_p - d_p$  measures how far  $V(I)$  is from being a complete intersection at  $p$ . Hence, Conjecture 5.4, if true, would show that resultants are sensitive to subtle features of the base point locus. Also, how does  $e_p - d_p$  relate to the subscheme structure of  $V_2(A) \subset \mathbb{P}^2$  at  $p$ ?

Finally, suppose that  $V_1(A)$  is nonempty (as in Case 1) and is a local complete intersection. Is there a version of Conjecture 5.4 which uses the residual resultant (see [Bu])?

The following example illustrates how extraneous components can arise as predicted by Conjecture 5.4.

**Example 5.5.** Consider the parametrization given by

$$\begin{aligned} x &= st^2 - t^3 - tu^2, \\ y &= t^3 - stu - t^2u + tu^2 + u^3, \\ z &= stu - 2tu^2, \\ w &= t^2u - 2tu^2 + u^3. \end{aligned}$$

Here, we have  $n = 3$ . Using `Macaulay2`, it is easy to compute that  $I = \langle x, y, z, w \rangle$  is saturated with free resolution (13) where the matrix

$A$  is given by:

$$A = \begin{pmatrix} -u & t & 0 \\ -u & -s + t + u & -u \\ t - u & -s - t + 2u & -u \\ u & s - u & t + u \end{pmatrix}.$$

One can also show that  $V(I)$  consists of points  $p = (1 : 0 : 0)$  and  $q = (2 : 1 : 1)$ . The point  $p$  is in  $V_2(A)$ , the point  $q$  is in  $V(I) \setminus V_2(A)$ , and one easily checks that  $V_1(A)$  is empty. Hence the hypothesis of Conjecture 5.4 is satisfied.

Since  $\mu_1 = \mu_2 = \mu_3 = 1$ , the degree of  $I$  is  $\frac{1}{2}(n^2 + \mu_1^2 + \mu_2^2 + \mu_3^2) = 6$  by [Co]. Thus  $d_p = d_q = 1$ . The resultant of  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  is the determinant

$$\det \begin{pmatrix} 0 & -Y - Z + W & 0 \\ Z & X + Y - Z & W \\ -X - Y - Z + W & Y + 2Z - W & -Y - Z + W \end{pmatrix},$$

which can be factored into

$$(-YZ - Z^2 + XW + YW + 2ZW - W^2)(Y + Z - W).$$

One can check that the first factor gives the implicit equation, which hence has degree 2, and that the parametrization has degree 1. From this, we know that  $3^2 - e_p - e_q = 2$ . Moreover, since  $q \in V(I) \setminus V_2(A)$ ,  $\mathcal{I}_q$  is a local complete intersection and hence  $d_q = e_q$ . Furthermore, since  $p \in V_2(A)$ ,  $\mathcal{I}_p$  is not a local complete intersection and hence  $e_p > d_p$ . This implies that  $e_p - d_p = 1$ , so that Conjecture 5.4 predicts that the resultant has a single extraneous component of multiplicity 1. This is confirmed by the above factorization.

**Question 5.6.** The proof of Theorem 4.3 also shows that each base point  $p \in V(I)$  blows up to a line  $L_p \subset S$ . What happens if we drop the hypothesis that  $V(I)$  is a local complete intersection from the theorem? For example, suppose that  $V_1(A) = \emptyset$  and  $p \in V_2(A)$ . Then the resultant has an extraneous factor  $\ell_p$  as in Conjecture 5.4. If  $H_p \subset \mathbb{P}^3$  is the plane defined by  $\ell_p = 0$ , then presumably  $p \in V_2(A)$  blows up to the plane curve  $S \cap H_p \subset H_p$ . But then what happens if  $p \in V_1(A)$ ? Does  $p$  blow up to a space curve which doesn't lie any plane? All of this indicates an interesting relation between the geometry of a parametrization and the structure of various subschemes of its base point locus.

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### APPENDIX A. A THEOREM ABOUT BASEPOINTS

We begin with the following general result.

**Theorem A.1.** *Let  $X$  be a Cohen-Macaulay variety of dimension  $d$  and let  $\mathcal{L}$  be a line bundle on  $X$ . Also assume that  $L \subset H^0(X, \mathcal{L})$  is a subspace such that  $V(L) \subset X$  is a 0-dimensional subscheme. If  $s_0, \dots, s_d \in L$  are generic, then:*

- (1)  $V(s_0, \dots, s_d) = V(L)$  as sets.
- (2)  $V(s_0, \dots, s_d)$  and  $V(L)$  have the same multiplicity at all points.

*Furthermore, if  $V(L)$  is a local complete intersection, then we have  $V(s_0, \dots, s_d) = V(L)$  as subschemes.*

*Proof.* Let  $m = \dim_{\mathbb{C}}(L)$ . If  $m \leq d + 1$ , then  $s_0, \dots, s_d$  span  $L$  and it follows that  $V(s_0, \dots, s_d) = V(L)$  as subschemes of  $X$ . Hence we may assume that  $m > d + 1$ .

From  $L$  we get the morphism  $\varphi : X \setminus V(L) \rightarrow \mathbb{P}(L^*) \simeq \mathbb{P}^{m-1}$ . The image of  $\varphi$  is a constructible set of dimension at most  $d$ . Hence we can find a linear subvariety  $P$  of codimension  $d + 1$  which is disjoint from the image. We can write  $P$  as the intersection of  $d + 1$  generic hyperplanes. However, hyperplanes in  $\mathbb{P}(L^*)$  are defined by elements of  $L$ . Thus  $P$  is defined by  $s_0, \dots, s_d \in L$ . Furthermore,  $P$  being disjoint from the image of  $\varphi$  implies that  $s_0, \dots, s_d$  don't vanish simultaneously on  $X \setminus V(L)$ , i.e.,  $V(s_0, \dots, s_d) \subset V(L)$ . The other inclusion is obvious, which completes the proof of part (a) of the theorem.

For part (b), fix  $p \in V(L)$  and let  $I_p \subset \mathcal{O}_p$  be the ideal generated by  $L$  in the local ring  $\mathcal{O}_p$  of  $X$  at  $p$ . Corollary 4.5.10 of Bruns and Herzog [BH] implies that  $\mathcal{O}_p$  has a system of parameters which generates a reduction ideal  $J_p$  for  $I_p$ . Note that this system of parameters is a regular sequence since  $\mathcal{O}_p$  is Cohen-Macaulay (Theorem 2.12 of [BH]). Furthermore, the proofs of Proposition 4.5.8, Theorem 1.5.17 and Proposition 1.5.12 of [BH] show that the system of parameters can be chosen to be generic linear combinations of generators of  $I_p$ . Since we can use a basis of  $L$  as generators of  $I_p$ , it follows that the system of parameters can be

chosen to be generic elements of  $L$ . This system has  $d$  elements since  $\mathcal{O}_p$  has dimension  $d$ .

It follows that  $s_0, \dots, s_{d-1}$  can be assumed to be a regular sequence which generates a reduction ideal  $J_p$  for  $I_p$ . Furthermore, since this is true for generic elements of  $L$  and  $V(L)$  is finite, we can assume that this holds for all  $p \in V(L)$ .

Now let  $\tilde{I}_p$  be the ideal of  $\mathcal{O}_p$  generated by  $s_0, \dots, s_d$ . Then we have the obvious inclusions

$$J_p \subset \tilde{I}_p \subset I_p,$$

which gives the inequalities

$$e(J_p) \geq e(\tilde{I}_p) \geq e(I_p).$$

However, the first and third terms are equal since  $J_p$  is a reduction ideal for  $I_p$ . This proves the desired equality of multiplicities.

Finally, if  $I_p$  is a complete intersection, then it coincides with all of its reduction ideals (this is easy to prove). Thus  $J_p = I_p$ , which by the above inclusions implies  $\tilde{I}_p = I_p$ . This shows that  $V(s_0, \dots, s_d)$  and  $V(L)$  have the same scheme structure at  $p$ . When  $V(L)$  is a local complete intersection, this is true for all of its points, and it follows that  $V(s_0, \dots, s_d) = V(L)$  as schemes.  $\square$

**Corollary A.2.** *Suppose  $x, y, z, w \in \mathbb{C}[s, t, u]$  are homogeneous of degree  $n$  with no common factor. If we replace  $x, y, z$  with generic linear combinations of  $x, y, z, w$ , then*

- (1)  $V(x, y, z) = V(x, y, z, w)$  as sets.
- (2)  $V(x, y, z)$  and  $V(x, y, z, w)$  have the same multiplicity at each point.

Furthermore, if  $V(x, y, z, w)$  is a local complete intersection, then we have  $V(x, y, z) = V(x, y, z, w)$  as subschemes and  $w \in \text{sat}(x, y, z)$ .

*Proof.* Apply Theorem A.1 to  $L = \text{Span}(x, y, z, w) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ . For the final assertion, note that  $V(x, y, z) = V(x, y, z, w)$  as subschemes  $\Leftrightarrow \text{sat}(x, y, z) = \text{sat}(x, y, z, w) \Leftrightarrow w \in \text{sat}(x, y, z)$ .  $\square$

## APPENDIX B. A THEOREM ABOUT REGULARITY

We begin with a lemma in two variables.

**Lemma B.1.** *Let  $\bar{I} \subset \mathbb{C}[s, t]$  have  $r$  minimal homogeneous generators of degree  $n$ . If  $V(\bar{I}) = \emptyset$  in  $\mathbb{P}^1$ , then  $\bar{I}$  is  $m$ -regular for all  $m \geq 2n - r + 1$ .*

*Proof.* Let  $S = \mathbb{C}[s, t]$ . The Hilbert syzygy theorem, together with a Hilbert polynomial calculation and the fact that  $\bar{I}_k = S_k$  for  $k \gg 0$ ,

imply that  $\bar{I}$  has a minimal graded free resolution

$$0 \rightarrow \bigoplus_{i=1}^{r-1} S(-n - \mu_i) \rightarrow S(-n)^r \rightarrow \bar{I} \rightarrow 0,$$

where  $\mu_i \geq 1$  for all  $i$  and  $\sum_{i=1}^{r-1} \mu_i = n$ .

Since  $S(-n)$  and  $S(-n - \mu_i)$  have generators of degrees  $n$  and  $n + \mu_i$  respectively, Definition 3.2(c) of [BM] implies that  $\bar{I}$  is  $m$ -regular whenever  $m \geq \max\{n, n + \mu_i - 1\} = n + \max\{\mu_i\} - 1$ . However,  $\mu_i \geq 1$  and  $\sum_{i=1}^{r-1} \mu_i = n$  imply that for each  $i$ , we have  $n \geq \mu_i + r - 2$ . This implies  $2n - r + 1 \geq n + \max\{\mu_i\} - 1$ , and the lemma follows.  $\square$

For the rest of this appendix, we will study the regularity of certain homogeneous ideals  $I \subset R = \mathbb{C}[s, t, u]$  using the inductive method found on page 34 of [BM]. We begin with the following result.

**Lemma B.2.** *Let  $I \subset R = \mathbb{C}[s, t, u]$  have  $r \geq 4$  minimal homogeneous generators, all of degree  $n$ , and assume that  $V(I) \subset \mathbb{P}^2$  is finite and the rational map from  $\mathbb{P}^2$  to  $\mathbb{P}^{r-1}$  given by the minimal generators is generically finite. Given a generic element of  $\ell \in R_1$ , let  $I_\ell$  be the image of  $I$  in the quotient ring  $R/\langle \ell \rangle$ . Then  $I_\ell$  has at least 3 minimal generators.*

*Proof.* Let  $p_1, \dots, p_r$  be minimal homogeneous generators of  $I$ , where each  $p_i$  has degree  $n$ . Then let  $Z \subset \mathbb{P}^{r-1} \times \mathbb{P}(R_1) = \mathbb{P}^{r-1} \times \mathbb{P}^2$  be defined by

$$Z = \{([a_1, \dots, a_r], [\ell]) : \ell|a_1p_1 + \dots + a_rp_r\},$$

and let  $\pi_1 : Z \rightarrow \mathbb{P}^{r-1}$  and  $\pi_2 : Z \rightarrow \mathbb{P}^2$  be the natural projections. Note that our hypothesis implies  $n > 1$ .

Since  $V(I) \subset \mathbb{P}^2$  is finite, we know that  $p_1, \dots, p_r$  have no common factors. Thus the linear system of divisors given by  $a_1p_1 + \dots + a_rp_r = 0$  is reduced in the sense of [I, p. 130], and the image of the rational map it determines has dimension 2 by hypothesis. It follows by the Bertini theorem [I, Thm. 7.19] that the general member of the linear system is irreducible. Since  $n > 1$ , we conclude that  $\pi_1^{-1}(p) = \emptyset$  for a general point  $\mathbf{a} \in \mathbb{P}^{r-1}$ . Furthermore, if  $\pi_1^{-1}(\mathbf{a}) \neq \emptyset$ , then  $\pi_1^{-1}(\mathbf{a})$  is finite since  $a_1p_1 + \dots + a_rp_r$  is divisible by at most  $n$  linear forms. Standard arguments then imply that  $Z$  has dimension  $\leq r - 2$ .

Now consider a generic  $\ell \in \mathbb{P}^2$  and let  $(p_i)_\ell$  denote the image of  $p_i$  in  $R/\langle \ell \rangle$ . If  $\pi_2^{-1}(\ell) = \emptyset$ , then we are done since the  $(p_i)_\ell$  are linearly independent in this case. On the other hand, suppose that  $\pi_2^{-1}(\ell) \neq \emptyset$  when  $\ell$  is generic. Since  $Z$  has dimension  $\leq r - 2$ , it follows that

$$\pi_2^{-1}(\ell) \text{ has dimension at most } r - 4$$



for generic  $\ell$ . However,

$$\pi_2^{-1}(\ell) = \mathbb{P}(\text{space of linear relations among the } (p_i)_\ell),$$

so that for generic  $\ell$ , the space of linear relations among the  $(p_i)_\ell$  has dimension  $\leq r - 3$ . This implies that at least 3 of  $(p_i)_\ell$  are linearly independent for generic  $\ell$ .  $\square$

We now state our first main result.

**Theorem B.3.** *Let  $I \subset R = \mathbb{C}[s, t, u]$  have  $r \geq 4$  minimal homogeneous generators, all of degree  $n$ , and assume that  $V(I) \subset \mathbb{P}^2$  is finite and the rational map from  $\mathbb{P}^2$  to  $\mathbb{P}^{r-1}$  given by the minimal generators is generically finite. If  $\mathcal{I}$  is the associated sheaf on  $\mathbb{P}^2$ , then:*

- (1)  $H^2(\mathcal{I}(k)) = 0$  for all  $k \geq 0$ .
- (2)  $H^1(\mathcal{I}(k)) = 0$  for all  $k \geq 2n - 3$ .

*Proof.* Let  $Z = V(I) \subset \mathbb{P}^2$ . The statement for  $H^2(\mathcal{I}(k))$  is then a trivial consequence of

$$(22) \quad 0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_Z \rightarrow 0$$

and the vanishing of the higher cohomology of  $\mathcal{O}_Z$ .

To prove the second statement, let  $\ell$  be a generic element of  $R_1$ . Since  $Z$  is finite, we may assume that  $Z \cap V(\ell) = \emptyset$ . By Lemma B.2, we may also assume that

$$\bar{I} = I_\ell \subset S = R/\langle \ell \rangle$$

has at least 3 minimal homogeneous generators, all of degree  $n$ . Then Lemma B.1 implies that  $\bar{I}$  is  $m$ -regular for  $m \geq 2n - 3 + 1 = 2n - 2$ . If  $\bar{\mathcal{I}}$  is the sheaf associated to  $\bar{I}$ , then by Definition 3.2(b) of [BM], we have

$$(23) \quad \bar{I}_k \rightarrow H^0(\bar{\mathcal{I}}(k)) \text{ is an isomorphism for } k \geq 2n - 2,$$

$$(24) \quad H^1(\bar{\mathcal{I}}(k)) = 0 \text{ for } k \geq 2n - 3.$$

We now use the argument of [BM, p. 34]. Tensoring

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$$

with  $\mathcal{I}(k)$  gives the exact sequence

$$\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{I}(k), \mathcal{O}_{\mathbb{P}^1}) \rightarrow \mathcal{I}(k-1) \rightarrow \mathcal{I}(k) \rightarrow \bar{\mathcal{I}}(k) \rightarrow 0.$$

However,  $\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{I}(k), \mathcal{O}_{\mathbb{P}^1})$  is supported on  $V(\ell) \simeq \mathbb{P}^1$ , and  $\mathcal{I}(k)$  is locally free on  $\mathbb{P}^2 - V(I)$ . Then  $\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{I}(k), \mathcal{O}_{\mathbb{P}^1}) = 0$  follows from  $V(I) \cap V(\ell) = \emptyset$ .

Thus we have an exact sequence

$$0 \rightarrow \mathcal{I}(k-1) \rightarrow \mathcal{I}(k) \rightarrow \bar{\mathcal{I}}(k) \rightarrow 0,$$



Combining this with  $\dim(R/I)_m = \deg(Z)$  immediately implies that  $\dim H^0(\mathcal{I}(m)) = \dim I_m$ , and the theorem is proved.  $\square$

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