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# Intersection and self-intersection of surfaces by means of Bezoutian matrices

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## Abstract

The computation of intersection and self-intersection loci of parameterized surfaces is an important task in Computer Aided Geometric Design. Computer algebra tools need to be developed further for computing their implicit equations. We address these problems via four resultants with separated variables. Two of them are specializations of more general ones and the others are determinantal. We give a rigorous study in these cases and provide new and useful formulas via adapted computations of Bezoutians.

*Key words:* Algebraic surfaces, intersection locus, self-intersection locus, resultants, Bezoutians.

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## 1 Introduction

A common representation of surfaces in Solid Modeling and Computer Aided Geometric Design (CAGD) uses parameterized patches, i.e. images of maps

$$\begin{aligned} \Phi : [0, 1] \times [0, 1] &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto \Phi(u, v) = \left( \frac{\Phi_1(u, v)}{\Phi_0(u, v)}, \frac{\Phi_2(u, v)}{\Phi_0(u, v)}, \frac{\Phi_3(u, v)}{\Phi_0(u, v)} \right) \end{aligned} \quad (1)$$

where  $\Phi_0, \Phi_1, \Phi_2, \Phi_3$  are polynomials with real coefficients and bi-degree  $(n, m)$ . They are called Bézier when  $\Phi_0(u, v) = 1$ . These patches are encountered in many applications [9], and spline surfaces are made by gluing them together.

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There are many articles presenting methods and algorithms to intersect two patches (see [23,19,14,22,13]), and also some papers address the computation of self-intersections [11,20,24,26,25]. These operations need still to be improved and computer algebra techniques could help. Among them, generalized (or sparse) resultant techniques have been successfully employed [17,18,7].

The computation of self-intersection of a patch or intersection of two patches are among important problems in CAGD, where computer algebra can help; they were the main topics of the European project GAIA II<sup>1</sup>. Several strategies have been developed to address these problems: either via multivariate resultants [7], or via special cases study [12] or via approximate implicitization [8] and more recently [25], or via numerical methods [11].

Numerical methods algorithms are very efficient, however they often rely on samplings, and have the weakness common to these techniques: if the size of a loop in the (self-)intersection locus is smaller than the step-size of the sampling, it might become invisible and the program is unable to compute it. There are heuristics to choose the step-size, but they do not really overcome this difficulty.

Therefore, symbolic (or semi-numeric) methods are essential in order to solve completely these tasks. For that purpose, the parameterizations should be also algebraic. The most commonly used algebraic representations in CAGD are the so-called NURBS and particularly the Bézier patches.

In this paper, we will present new tools developed for this purpose. Namely specific sparse resultants and corresponding Bezoutian formulas allowing to compute, in each case presented below, a bivariate polynomial which is the equation of a plane projection of the double points locus. This polynomial is represented as the determinant of a small matrix, and thus it is useful for further post-processing.

For this setting, we prepare the equations for the elimination procedure in order to get a special format that we call “with separated variables”. Then we show that the obtained systems have nice properties that we further study and exploit.

We construct an unmixed bivariate resultant  $\text{Res}(P_1, P_2, P_3)$  where the three polynomials have the special form  $P_i(x, y) = f_i(x) - g_i(y)$ , and applied it to the computation of implicit equations of self-intersection and intersection loci of Bézier surfaces which is an important task in Computer Aided Geometric Design and Solid Modeling. In practice, this resultant is computed via a Bezoutian matrix, and it is related to (but different from) the toric resultants studied in [18,17]. In the case of polynomial parameterizations, this resultant is

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<sup>1</sup> <http://www.sintef.no/static/AM/gaiatwo/>

an anisotropic one and in the more general case of rational parameterizations it is a determinantal resultant. For each of these two kinds of resultant we use two different formulations: a dense one to study the intersection problem and a residual one for the self-intersection problem. For each case, we show how to compute the adapted resultant via Bezoutian formulas.

We implemented in Maple our algorithms; illustrations, examples, and comparisons with the Maple package `bires` written by Amit Khetan are presented. We notice that the size of matrices used in our approach to compute the different resultants are significantly smaller than those given in [17].

We begin with the presentation of the methodology (that we will follow below) to obtain an elimination formula adapted to our special situation, by specializing a known resultant designed for a generic situation. Starting from a given polynomial system  $E$ , the first step is to "homogenize" it (i.e. to embed it in an irreducible projective variety) with the condition that the obtained homogeneous system  $E^h$  do not have base-points. Then, by the general theory of resultants, this system can be seen as a particular instance of a general resultant which is not identically zero. Moreover, the particular instance is always geometrically irreducible because, due to the hypothesis that  $E^h$  is base-point free, the incidence variety is a vector bundle. However, it needs not to be algebraically reduced, but this property can be shown through well chosen specializations. This methodology will be applied to the four different cases corresponding to Bézier and rational Bézier parameterizations.

The paper is organized as follows. The following section provides a short overview on the Bezoutian matrix and its main property. In section 3, we precise the application in CAGD and set the equations and the elimination problems for intersection and self-intersection problems. In section 4, we go back to our main CAGD applications and make explicit how our resultants can be used. We sketch also the algorithms and present examples. Sections 5 and 6 are more theoretical and they contain the proofs of our results. In section 5, we prove that in our situation, the specialization process is valid and that resultants (which are anisotropic) can be computed by nice Bezoutian formulas. In section 6, we describe an adapted version of determinantal resultant and prove that resultants can also be computed by simple Bezoutian formulas.

Hereafter, we use the following notations:  $\mathbb{P}^n$  (resp.  ${}^a\mathbb{P}^n$ ) denotes the (resp. anisotropic) projective space of dimension  $n$  over  $\mathbb{C}$ .

## 2 Overview on the Bezoutian matrix

In this short section we recall a well-known matrix construction to eliminate variables from polynomial systems. We will apply it to prove formulas for bivariate resultants that will be used to solve the intersection and self-intersection problems that we address.

**Definition 1** *Let  $f_0, \dots, f_n$  be  $n+1$  polynomials in  $n$  variables  $x_1, \dots, x_n$  with coefficients in a ring  $R$ . The Bezoutian of  $f_0, \dots, f_n$  is the following polynomial in  $2n$  variables  $x_1, \dots, x_n, y_1, \dots, y_n$ :*

$$\text{Bez}_{f_0, \dots, f_n}(x; y) := \begin{vmatrix} f_0(x) & \partial_1 f_0(x, y) & \cdots & \partial_n f_0(x, y) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(x) & \partial_1 f_n(x, y) & \cdots & \partial_n f_n(x, y) \end{vmatrix}$$

where

$$\partial_i f_j(x, y) := \frac{f_j(y_1, \dots, y_{i-1}, x_i, \dots, x_n) - f_j(y_1, \dots, y_i, x_{i+1}, \dots, x_n)}{x_i - y_i}.$$

Set  $\text{Bez}_{f_0, \dots, f_n}(x; y) = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta$  with  $c_{\alpha, \beta} \in R$ . Fixing an order on the monomials, the matrix  $(c_{\alpha, \beta})_{\alpha, \beta}$  whose rows are indexed by the  $y^\beta$  and lines indexed by the  $x^\alpha$  is called the Bezoutian matrix of  $f_0, \dots, f_n$ .

We remark that as  $f_j(x) - f_j(y) = \sum_{i=1}^n (x_i - y_i) \partial_i f_j(x, y)$  we can substitute  $x$  by  $y$  in the first column of the determinant  $\text{Bez}_{f_0, \dots, f_n}(x; y)$ .

**Remark 2** *It is known that if  $\text{Res}_X$  stands for a generalized resultant on a certain projective variety  $X$ , then every maximal minor of the Bezoutian matrix associated to  $f_0(x), \dots, f_n(x)$  (viewed as polynomials in  $x$ ) is a multiple of  $\text{Res}_X$ ; we refer the reader to [5, 2].*

In this paper, we will actually refine this remark by providing new exact formulas in terms of Bezoutian matrices for some special classes of bivariate generalized resultants.

## 3 The main results

In this section we state four theorems addressing two important tasks in CAGD: the intersection and the self-intersection problems for rational surfaces. The proofs of these theorems are rather technical as they require some knowledge in algebraic geometry and rely on the use of specific resultants;

they will be given in sections 5 and 6. Note also that the condition “*sufficiently generic*” that we will require in the following theorems will also be completely described at that time.

### 3.1 Intersection problem

First we describe a particular projection of the pre-image curve (i.e the curve in the parameter space which maps to the intersection locus). We consider the intersection of two patches  $\mathcal{S}_1$  and  $\mathcal{S}_2$  given by the parameterizations  $\Phi(u, v)$  and  $\Psi(s, t)$  with the same features than (1) that we view in  $\mathbb{P}^3$ .

The set  $\mathcal{S}_1 \cap \mathcal{S}_2$  corresponds to the quintuple of parameters  $(u, v, s, t, \lambda)$ , with  $\lambda \neq 0$ , such that

$$\Phi(u, v) = \lambda\Psi(s, t).$$

This set form a curve  $\mathcal{C}$  in  $\mathbb{R}^5$  that we can describe (i.e. give the topology and witness points on each component) via a well chosen projection on a plane

$$\mathcal{C}_1 = \{(u, t) : \exists (v, s, \lambda) \text{ with } \lambda \neq 0 \text{ and } \Phi(u, v) = \lambda\Psi(s, t)\}.$$

Our geometric task is equivalent to describe the curve  $\mathcal{C}_1$ . Its implicit equation is obtained by eliminating  $v, s, \lambda$  in the system of 4 equations defining  $\mathcal{C}$ .

In the Bézier case (i.e.  $\Phi_0 = \Psi_0 = 1$ ), the question reduces to the elimination of 2 variables in a system of 3 polynomial equations in 4 variables (there is no  $\lambda$ ). We have the following

**Theorem 3 (see subsection 5.1)** *Given two sufficiently generic parameterized Bézier surfaces of bi-degree  $(m, n)$*

$$(u, v) \mapsto (\Phi_1(u, v), \Phi_2(u, v), \Phi_3(u, v)), (s, t) \mapsto (\Psi_1(s, t), \Psi_2(s, t), \Psi_3(s, t)),$$

*the implicit equation of the projection of their intersection locus onto the plane  $(u, t)$  equals the determinant of the  $mn \times mn$  Bezoutian matrix associated to the polynomials*

$$\Phi_1(u, v) - \Psi_1(s, t), \Phi_2(u, v) - \Psi_2(s, t), \Phi_3(u, v) - \Psi_3(s, t)$$

*viewed as polynomials in  $v$  and  $s$ .*

In the rational Bézier case, we have the

**Theorem 4 (see subsection 6.1)** *Given two sufficiently generic parameterized rational Bézier surfaces of bi-degree  $(m, n)$*

$$(u, v) \mapsto \left( \frac{\Phi_1(u, v)}{\Phi_0(u, v)}, \frac{\Phi_2(u, v)}{\Phi_0(u, v)}, \frac{\Phi_3(u, v)}{\Phi_0(u, v)} \right), (s, t) \mapsto \left( \frac{\Psi_1(s, t)}{\Psi_0(s, t)}, \frac{\Psi_2(s, t)}{\Psi_0(s, t)}, \frac{\Psi_3(s, t)}{\Psi_0(s, t)} \right),$$

the implicit equation of the projection of their intersection locus onto the plane  $(u, t)$  equals the determinant of the  $mn \times mn$  matrix  $\mathbb{B}(u, t)$  defined by

$$U\mathbb{B}(u, t)V^t = \frac{1}{(v - v_1)(s - s_1)} \begin{vmatrix} \Phi_0(u, v) & \Phi_0(u, v_1) & \Psi_0(s, t) & \Psi_0(s_1, t) \\ \Phi_1(u, v) & \Phi_1(u, v_1) & \Psi_1(s, t) & \Psi_1(s_1, t) \\ \Phi_2(u, v) & \Phi_2(u, v_1) & \Psi_2(s, t) & \Psi_2(s_1, t) \\ \Phi_3(u, v) & \Phi_3(u, v_1) & \Psi_3(s, t) & \Psi_3(s_1, t) \end{vmatrix}$$

where  $U$  and  $V$  denote respectively the monomial vectors

$$U := [s^i v^j, i = 0 \dots m - 1, j = 0 \dots n - 1]$$

and

$$V := [s_1^i v_1^j, i = 0 \dots m - 1, j = 0 \dots n - 1].$$

It is clear that the previous determinant is a polynomial in  $v, v_1, s, s_1, u, t$ .

### 3.2 Self-intersection problem

We start with a parametrization  $\Phi$  given by (1) that we view in  $\mathbb{P}^3$  and, as in the previous subsection, we will describe a well chosen projection of the pre-image curve of the self-intersection locus.

The set of double points of  $\Phi$  can be characterized by a curve

$$\mathcal{C} = \{(u_1, v_1, u_2, v_2, \lambda) : (u_1, v_1) \neq (u_2, v_2), \lambda \neq 0, \Phi(u_1, v_1) = \lambda\Phi(u_2, v_2)\}$$

in  $\mathbb{R}^5$ . It can also be described in a simpler way by one of its well chosen projection (generically 1 to 1)  $\mathcal{C}_1$  on  $\mathbb{R}^2$ :

$$\mathcal{C}_1 = \{(u, v) : \exists (k, l, \lambda) \text{ such that} \\ (k, l) \neq 0, \lambda \neq 0, \text{ and } \Phi(u, v + k) = \lambda\Phi(u + l, v)\}.$$

Our geometric task is equivalent to describe the curve  $\mathcal{C}_1$ . Its implicit equation is obtained by elimination of  $l, k, \lambda$  in the system of 4 equations defining  $\mathcal{C}_1$ .

If the variables  $u, v$  are fixed, we remark that the point  $(k, l, \lambda) = (0, 0, 1)$  is a base-point (i.e. a solution) of the polynomial system  $\Phi(u, v + k) - \lambda\Phi(u + l, v) = 0$ . We will exploit this observation to develop an adapted resultant to the self-intersection problem (see subsections 5.2 and 6.2).

In the Bézier case we have the

**Theorem 5 (see subsection 5.2)** *Given a sufficiently generic parameterized Bézier surface of bi-degree  $(m, n)$*

$$(u, v) \mapsto (\Phi_1(u, v), \Phi_2(u, v), \Phi_3(u, v)),$$

*the implicit equation of the projection of its self-intersection locus onto the plane  $(u, v)$  equals the determinant of the  $(mn - 1) \times (mn - 1)$  Bezoutian matrix associated to the polynomials*

$$\Phi_1(u, v + k) - \Phi_1(u + l, v), \Phi_2(u, v + k) - \Phi_2(u + l, v), \Phi_3(u, v + k) - \Phi_3(u + l, v)$$

*viewed as polynomials in  $k$  and  $l$ .*

In the rational Bézier case, we have the

**Theorem 6 (see subsection 6.2)** *Given a sufficiently generic parameterized rational Bézier surface of bi-degree  $(m, n)$*

$$(u, v) \mapsto \left( \frac{\Phi_1(u, v)}{\Phi_0(u, v)}, \frac{\Phi_2(u, v)}{\Phi_0(u, v)}, \frac{\Phi_3(u, v)}{\Phi_0(u, v)} \right),$$

*the implicit equation of the projection of its self-intersection locus onto the plane  $(u, v)$  equals the determinant of the  $(mn - 1) \times (mn - 1)$  matrix  $\mathbb{B}(u, v)$  defined by*

$$U\mathbb{B}(u, v)V^t = \frac{1}{(k - k_1)(l - l_1)} \begin{vmatrix} \Phi_0(u, v + k) & \Phi_0(u, v + k_1) & \Phi_0(u + l, v) & \Phi_0(u + l_1, v) \\ \Phi_1(u, v + k) & \Phi_1(u, v + k_1) & \Phi_1(u + l, v) & \Phi_1(u + l_1, v) \\ \Phi_2(u, v + k) & \Phi_2(u, v + k_1) & \Phi_2(u + l, v) & \Phi_2(u + l_1, v) \\ \Phi_3(u, v + k) & \Phi_3(u, v + k_1) & \Phi_3(u + l, v) & \Phi_3(u + l_1, v) \end{vmatrix}$$

*where  $U$  and  $V$  denote respectively the monomial vectors*

$$U := [l^i k^j, (i, j) \in \{0, \dots, m - 1\} \times \{0, \dots, n - 1\} \setminus \{0, 0\}]$$

*and*

$$V := [l_1^i k_1^j, (i, j) \in \{0, \dots, m - 1\} \times \{0, \dots, n - 1\} \setminus \{0, 0\}].$$

It is clear that the determinant in this theorem is polynomial in  $k, k_1, l, l_1, u, v$ .



## 4 Comments and illustrative examples

We now illustrate our approach to the intersection and self-intersection problems of algebraic surfaces. We will actually focus on the self-intersection problem in the Bézier case; we mention however that applications of our results in CAGD follow the same strategy in the other cases.

Consider the following three polynomials

$$P_i(u, v, l, k) := \Phi_i(u, v + k) - \Phi_i(u + l, v) , \quad i = 1 \dots 3 ,$$

that we view as polynomials of bi-degree  $(m, n)$  in the two variables  $(l, k)$  with coefficients polynomials in  $(u, v)$  of bi-degree  $(m, n)$ , and the resultant polynomial (which depends on  $(u, v)$ )

$$R(u, v) := \text{Res}(P_1(u, v, l, k), P_2(u, v, l, k), P_3(u, v, l, k))$$

(the precise definition of the resultant we are using will be given in the sections 5 and 6). In order to draw the portion of the self-intersection curve  $\mathcal{D}$  contained in  $[0, 1]^2$ , we should find a point on each of its connected components. The ones crossing the border of the square are easily found, so the difficulty is essentially concentrated on the internal loops. A common way to find a point on such a loop is to compute all the critical points of the projection of  $\mathcal{D}$  on an axis, e.g. the  $u$ -axis. This amounts to solve the bivariate system:

$$R(u, v) = \frac{\partial R}{\partial u}(u, v) = 0.$$

Once we get a point on each connected component of  $\mathcal{D}$  contained in  $[0, 1]^2$ , the computer algebra part of the process is completed. Then a numerical marching algorithm is called which computes the points  $(u, v + k, u + l, v)$  in  $[0, 1]^4$  needed to draw efficiently the pre-image curve or any of its plane projection. Note that part of this process can be done by exploiting the kernel of the matrix which is used to compute  $R(u, v)$ : for instance, if the couple  $(u, v)$  satisfies  $\det(\mathbb{B}(u, v)) = 0$ , the vector  $U$  of theorem 5 is in the kernel of this matrix, so we can often obtain  $s$  and  $t$  as the quotient of two sub-minors of  $\mathbb{B}(u, v)$  (see [4] for precise results in this spirit).

We compared the new elimination formulas that we propose to the one's given by Amit Khetan in [17]. We implemented the constructions of our matrices in Maple and used the Maple package `bires` written by Khetan for this comparison. Note that Khetan's formulas are designed for bivariate resultants of any three polynomials, whereas our formulas are only available for bivariate resultants of polynomials with separated variables. However, since we took into account this particular structure, it turns out that our formulas com-

pared most of the time favorably to the ones given by Khetan. We observed sometimes that Khetan’s hybrid matrix is not full rank in some cases where the determinant of the matrix we propose gives exactly the projection of the self-intersection curve that we are looking for. Usually, the size of matrices in our approach is significantly smaller than the Khetan’s one, so the construction of the resultant matrices and the computation of determinants are also faster with our method.

**Example 7 (bi-degree (2, 2))** We consider here a *random* rational Bézier surface of bi-degree (2, 2). In this case, the matrix computed by `bi-res` is square of size 11 but not full rank; it has rank 10. The matrix given in theorem 6 is square of size 3 and full rank; its determinant, which is quickly computed, gives a representation of the self-intersection curve of bi-degree (10, 10) and total degree 20.

**Example 8 (bi-degree (3, 2))** We consider here a randomly chosen rational Bézier surface of bi-degree (3, 2) given by the parameterization:

$$\begin{aligned}\phi_1(u, v) &:= 85 + 35u - 79u^2 - 63u^3 - (-55 + 97u + 56u^2 + 57u^3)v - (-37 + 50u + 49u^2 - 59u^3)v^2 \\ \phi_2(u, v) &:= -45 - 92u - 77u^2 + 5u^3 - (-8 + 43u + 66u^2 + 99u^3)v - (-93 - 62u + 54u^2 - 61u^3)v^2 \\ \phi_3(u, v) &:= -50 + 31u + u^2 - 47u^3 + (-12 - 26u - 47u^2 - 61u^3)v + (-18 - 62u - 91u^2 + 41u^3)v^2 \\ \phi_0(u, v) &:= -58 - u - 86u^2 + 19u^3 + (-90 + 94u + 23u^2 - 50u^3)v + (53 + 83u - 84u^2 + 88u^3)v^2.\end{aligned}$$

The implicit equation of a plane projection of the pre-image curve in  $\mathbb{R}^5$ , is given by a polynomial  $Q(u, v)$  of total degree 44, and bi-degree (26, 18). It is computed as a  $5 \times 5$  determinant from the Bezoutian.

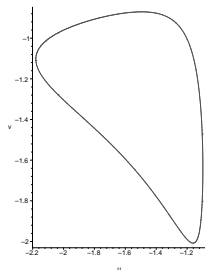


Fig. 1. A view of the double-point locus

We remark an oval, in the double-point locus of this surface (see Figure 1). If we restrict to that oval, we expect self-intersections on the 3D picture, that we show on Figure 2.

The matrix computed by `bi-res` in this case is a full rank  $17 \times 17$  matrix.

**Example 9 (bi-degree (3, 3))** We consider here a *random* bi-cubic rational Bézier surface; it is an important class of surfaces for applications in CAGD. In this case, the two methods gives the correct answer. However, the matrix

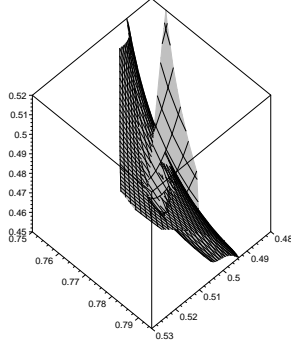


Fig. 2. A patch with self-intersections

generated by `bires` is a full rank  $27 \times 27$  matrix and we did not succeed to compute its determinant in a reasonable time. With our approach we got a full rank  $8 \times 8$  matrix and we were able to compute its determinant in few minutes, obtaining a curve of bi-degree  $(44, 44)$  and total degree 88, as predicted by the theory (see for instance [12]).

From now on we will prove the results given in section 3. For this purpose, we need to use tools from elimination theory and algebraic geometry.

## 5 Bivariate resultants with separated variables

In this section, we suppose given three univariate polynomials of degree  $m \geq 1$

$$f_i(x) := a_{i,0}x^m + a_{i,1}x^{m-1} + \cdots + a_{i,m-1}x + a_{i,m} \in \mathbb{Z}[a_{i,0}, \dots, a_{i,m}][x]$$

and three univariate polynomials of degree  $n \geq 1$

$$g_i(y) := b_{i,0}y^n + b_{i,1}y^{n-1} + \cdots + b_{i,n-1}y + b_{i,n} \in \mathbb{Z}[b_{i,0}, \dots, b_{i,n}][y].$$

Hereafter we will consider resultants of the three following bivariate polynomials with separated variables

$$P_i(x, y) := f_i(x) - g_i(y) \in A[x, y], \quad i = 1, 2, 3,$$

where  $A := \mathbb{Z}[(a_{i,j})_{i=1,2,3;j=0,\dots,m}, (b_{i,j})_{i=1,2,3;j=0,\dots,n}]$  denotes the universal coefficients ring of these polynomials.

### 5.1 The dense case

This case will be applied to solve the intersection problem for Bézier surfaces.

To compute the classical resultant (sometimes called Macaulay's resultant) of the polynomials  $P_1(x, y)$ ,  $P_2(x, y)$  and  $P_3(x, y)$ , we first need to homogenize them to the projective space by introducing a new variable. But it is clear that this resultant is identically zero, since this homogenization produces at least one base-point (i.e. a common root of the homogenizations of  $P_1, P_2, P_3$ ) in  $\mathbb{P}^2$ . To overcome this problem, a natural idea is to change variables and compute the classical resultant of the polynomials  $P_1(x^n, y^m)$ ,  $P_2(x^n, y^m)$  and  $P_3(x^n, y^m)$  which is not identically zero. However, this resultant could be not reduced: it could be a power of an irreducible polynomial. As described in [15], its reduced part corresponds to a resultant which is obtained by a (special) quasi-homogenization of  $P_1(x, y)$ ,  $P_2(x, y)$  and  $P_3(x, y)$  to an anisotropic projective space. We will rely on this construction which has been fully studied in [15,16]. We also refer the reader to [1] for an introduction to anisotropic projective spaces.

Introducing a new variable  $z$ , we homogenize  $P_1(x, y)$ ,  $P_2(x, y)$  and  $P_3(x, y)$  to the anisotropic projective space  ${}^a\mathbb{P}^2$  with homogeneous coordinates  $x, y, z$  of weight  $n, m, 1$  respectively. We get, for  $i = 1, 2, 3$ , the quasi-homogeneous polynomials in  $A[x, y, z]$

$$f_i^h(x, z) := \sum_{j=0}^m a_{i,j} x^{m-j} z^{nj} \quad , \quad g_i^h(y, z) := \sum_{j=0}^n b_{i,j} y^{n-j} z^{mj}$$

and  $P_i^h(x, y, z) := f_i^h(x, z) - g_i^h(y, z)$  of weight  $mn$ . We denote by  $\text{Res}_{{}^a\mathbb{P}^2}$  the corresponding anisotropic resultant of three polynomials in  ${}^a\mathbb{P}^2$ . Hereafter we consider the graded polynomial ring  $A$  with the usual grading: each of its variables has weight 1.

**Proposition 10** *The anisotropic resultant  $\text{Res}_{{}^a\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  is a non-zero irreducible element in  $A$ . Moreover, it is homogeneous of degree  $mn$  with respect to the coefficients of each polynomial  $P_1, P_2$  and  $P_3$ , and hence has total degree  $3mn$ .*

*Proof.* By specializing  $P_1^h$  to  $x^m$ ,  $P_2^h$  to  $y^n$  and  $P_3^h$  to  $z^{mn}$  the anisotropic resultant  $\mathcal{R} := \text{Res}_{{}^a\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  equals  $\text{Res}_{{}^a\mathbb{P}^2}(x^m, y^n, z^{mn}) = 1$  and it is hence non-zero. Moreover, it is known that  $\mathcal{R}$  is homogeneous with respect to the coefficients of each polynomial  $P_1, P_2$  and  $P_3$  of degree  $m^2n^2/mn = mn$ .

To prove the irreducibility of  $\mathcal{R}$  we first prove that  $\mathcal{R}$  is geometrically irreducible (i.e. it is a power of an irreducible polynomial). To do this we consider the incidence variety

$$\mathcal{W} := \{((x : y : z), (a_{i,j}, b_{i,j})) : P_1^h = P_2^h = P_3^h = 0\} \subset {}^a\mathbb{P}^2 \times \mathbb{A}^{3(m+n+2)}$$

and its two canonical projections  $\pi_1$  and  $\pi_2$  onto the first and the second factor respectively. By construction  $\pi_1$  gives  $\mathcal{W}$  a structure of vector bundle over  ${}^a\mathbb{P}^2$

which is an irreducible variety. We deduce that  $\mathcal{W}$  is itself an irreducible variety and therefore that its image by  $\pi_2$  is irreducible in the affine space  $\mathbb{A}^{3(m+n+2)}$  ( $= \text{Spec}(\mathbb{A})$ ). This image has the same support than the zero locus of  $\mathcal{R}$  and hence it follows that  $\mathcal{R}$  is a certain power, say  $p$ , of an irreducible polynomial. Finally, to see that  $p = 1$  we need to prove that the projection of  $\mathcal{W}$  by  $\pi_2$  is birational onto its image. This is done by considering the specialization  $\phi$  defined, for generic  $\alpha_i$ 's and  $\beta_j$ 's, by

$$\phi(P_1^h) = \prod_{i=1}^m (x - \alpha_i z^n), \quad \phi(P_2^h) = \prod_{j=1}^n (y - \beta_j z^m), \quad \phi(P_3^h) = x^m - y^n$$

which sends the anisotropic resultant  $\mathcal{R}$  to

$$\begin{aligned} \phi(\mathcal{R}) &= \text{Res}_{\mathbb{A}^2} \left( \prod_{i=1}^m (x - \alpha_i z^n), \prod_{j=1}^n (y - \beta_j z^m), x^m - y^n \right) = \\ &= \prod_{i,j=1}^{m,n} \text{Res}_{\mathbb{A}^2} (x - \alpha_i z^n, y - \beta_j z^m, x^m - y^n) = \prod_{i,j=1}^{m,n} (\alpha_i^n - \beta_j^m). \end{aligned}$$

(these two equalities rely on properties of anisotropic resultants proved in [15]). Since the above product contains the irreducible and reduced factor  $(\alpha_1 - \beta_1)$ ,  $p$  must be equal to 1.  $\square$

In order to compute directly, for our setting, the above anisotropic resultant we now turn to the Bezoutian (see definition 1) associated to the polynomials  $P_1(x, y)$ ,  $P_2(x, y)$  and  $P_3(x, y)$ . It is easy to check that

$$\text{Bez}_{P_1, P_2, P_3}(x, y; x_1, y_1) = - \begin{vmatrix} f_1(x) - g_1(y) & \frac{f_1(x) - f_1(x_1)}{x - x_1} & \frac{g_1(y) - g_1(y_1)}{y - y_1} \\ f_2(x) - g_2(y) & \frac{f_2(x) - f_2(x_1)}{x - x_1} & \frac{g_2(y) - g_2(y_1)}{y - y_1} \\ f_3(x) - g_3(y) & \frac{f_3(x) - f_3(x_1)}{x - x_1} & \frac{g_3(y) - g_3(y_1)}{y - y_1} \end{vmatrix}$$

and deduce, after manipulation on the columns, that this Bezoutian has a particular symmetry property (which is not true in general), namely

$$\text{Bez}_{P_1, P_2, P_3}(x_1, y_1; x, y) = \text{Bez}_{P_1, P_2, P_3}(x, y; x_1, y_1).$$

The  $(x_1, y_1)$ -monomial support of this Bezoutian (i.e. the set of monomials  $x_1^i y_1^j$  having non-zero coefficients) is

$$\{0, \dots, m-1\} \times \{0, \dots, n-1\}$$

and have hence exactly  $mn$  elements. By the symmetry property, the  $(x, y)$ -monomial support is the same. Therefore the Bezoutian matrix  $\mathbb{B}$  of  $P_1, P_2, P_3$  is, in this case, a square matrix of size  $mn$  and we have:

**Theorem 11** *The determinant of the Bezoutian matrix  $\mathbb{B}$  of  $P_1, P_2, P_3$  equals the anisotropic resultant  $\text{Res}_{\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  up to a sign<sup>2</sup> in  $A$ .*

*Proof.* Using remark 2 we know that  $\text{Res}_{\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  must divide  $\det(\mathbb{B})$ . Moreover, it is easy to see by definition 1 that each entry of  $\mathbb{B}$  is a polynomial in  $A$  which has total degree 3 and is homogeneous of degree 1 with respect to the coefficients of each polynomial  $P_1, P_2$  and  $P_3$ . Therefore,  $\det(\mathbb{B}) \in A$  has total degree  $3mn$  and more precisely is homogeneous with respect to the coefficients of each  $P_1, P_2$  and  $P_3$  of degree  $mn$ . Now, if  $\det(\mathbb{B})$  is non-zero, it equals the anisotropic resultant  $\text{Res}_{\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  up to a sign. Considering the specialization  $\phi$  defined by  $\phi(P_1) = x^m$ ,  $\phi(P_2) = y^n$  and  $\phi(P_3) = 1$ , it is easy to check that

$$\phi\left(\text{Bez}_{P_1, P_2, P_3}(x, y; x_1, y_1)\right) = \frac{x^m - x_1^m}{x - x_1} \times \frac{y^n - y_1^n}{y - y_1},$$

and that  $\phi(\det(\mathbb{B})) = \pm 1$ . So  $\det(\mathbb{B}) \neq 0$  and the theorem is proved.  $\square$

Note that, in more general setting, other formulations of the anisotropic resultant as a determinant of a square matrix has already been established in [16], and also in [17] as a sparse resultant. Theorem 11 is a simple expression in the particular case of polynomial systems with separated variables.

## 5.2 The residual case

This case is adapted to the determination of the self-intersection of Bézier surfaces.

We keep the notations of the subsection 5.1, but now we assume that the polynomials  $f_i(x)$  and  $g_i(x)$ , for all  $i = 1, 2, 3$ , have the same constant term (which is the case in the self-intersection problem), or in other words that

$$P_1(0, 0) = P_2(0, 0) = P_3(0, 0) = 0.$$

Thus we have for  $i = 1, 2, 3$ ,  $a_{i,m} = b_{i,n}$  and we view  $P_1, P_2$  and  $P_3$  as polynomials in  $A'[x, y]$ , where  $A'$  is the universal coefficients ring

$$A' := \mathbb{Z}[(a_{i,j})_{i=1,2,3;j=0,\dots,m}, (b_{i,j})_{i=1,2,3;j=0,\dots,n-1}].$$

---

<sup>2</sup> It is possible to fix the sign in this equality by requiring a precise construction of the Bezoutian matrix  $\mathbb{B}$ , namely an ordering on the  $(x, y)$ -monomial and the  $(x_1, y_1)$ -monomial supports.

Similarly to the previous subsection, we introduce a new variable  $z$  and homogenize the polynomials to the anisotropic projective space  ${}^a\mathbb{P}^2$  with coordinates  $x, y, z$  of weight  $n, m, 1$  respectively. Then we have, for  $i = 1, 2, 3$ ,

$$f_i^h(x, z) := \sum_{j=0}^m a_{i,j} x^{m-j} z^{nj}, \quad g_i^h(y, z) := a_{i,m} z^{mn} + \sum_{j=0}^{n-1} b_{i,j} y^{n-j} z^{mj}$$

and  $P_i^h(x, y, z) := f_i^h(x, z) - g_i^h(x, z)$  are quasi-homogeneous of weight  $mn$ . However, in this new setting, the anisotropic resultant  $\text{Res}_{{}^a\mathbb{P}^2}(P_1^h, P_2^h, P_3^h)$  is degenerate because the polynomials  $P_1^h, P_2^h$  and  $P_3^h$  vanish at the same point  $P := (0 : 0 : 1)$  independently of their coefficients.

In order to deal with this base-point and get a non-identically zero “resultant”, we consider the blow-up

$$\pi_P : {}^a\tilde{\mathbb{P}}^2 \rightarrow {}^a\mathbb{P}^2$$

of the anisotropic projective space along the point  $P$  whose definition ideal is  $I_P := (x, y)$ , with exceptional divisor  $D$ . Following techniques developed in [2,6] (see also [7] for a quick overview), we obtain the so-called *residual resultant*, and denoted by  $\text{Res}_{{}^a\tilde{\mathbb{P}}^2}(P_1^h, P_2^h, P_3^h)$ . It provides a necessary and sufficient condition for the zero locus in  ${}^a\mathbb{P}^2$  of the polynomial system

$$P_1^h(x, y, z) = P_2^h(x, y, z) = P_3^h(x, y, z) = 0$$

to be, scheme-theoretically, strictly bigger than the point  $P$ . For more details, we refer the reader to [2,7,6].

**Proposition 12** *The residual anisotropic resultant  $\text{Res}_{{}^a\tilde{\mathbb{P}}^2}(P_1^h, P_2^h, P_3^h)$  is a non-zero irreducible polynomial in  $A'$ . Moreover, it is homogeneous of degree  $mn - 1$  with respect to the coefficients of each polynomial  $P_1, P_2, P_3$  and of total degree  $3mn - 3$ .*

*Proof.* The proof is similar to that of proposition 10, as it amounts to consider an anisotropic residual resultant instead of a simple anisotropic one. Set  $\tilde{\mathcal{R}} := \text{Res}_{{}^a\tilde{\mathbb{P}}^2}(P_1^h, P_2^h, P_3^h)$ .

The geometric irreducibility of  $\tilde{\mathcal{R}}$  follows from two facts. First, the blow-up  ${}^a\tilde{\mathbb{P}}^2$  is an irreducible projective variety, since  ${}^a\mathbb{P}^2$  is itself a projective irreducible variety. Second, the incidence variety

$$\tilde{\mathcal{W}} = \{(\tilde{t}, (a_{i,j}, b_{i,j})) : \tilde{P}_1^h = \tilde{P}_2^h = \tilde{P}_3^h = 0\} \subset {}^a\tilde{\mathbb{P}}^2 \times \mathbb{A}^{3(m+n+1)},$$

is a vector bundle over  ${}^a\tilde{\mathbb{P}}^2$ . Here  $\tilde{P}_i^h$  denotes the virtual transform of  $P_i^h$  by the blow-up  $\pi_P$  (i.e.  $\pi_P^*(P_i^h)$  divided only one time by  $D$ ).

As  $P$  is a smooth point in  ${}^a\mathbb{P}^2$ , the self-intersection of  $D$  equals  $-1$ . Since each  $P_i^h$  is quasi-homogeneous of degree  $mn$ , the known multi-degree formula for

residual resultants (see [2]) shows that  $\tilde{\mathcal{R}}$  is homogeneous, with respect to the coefficients of each  $P_i$ , of degree

$$\frac{m^2 n^2}{mn} + \int_{a_{\tilde{\mathbb{P}}^2}} D^2 = mn - 1.$$

Finally, to prove that  $\tilde{\mathcal{R}} \neq 0$  and it is not a power of an irreducible polynomial, we exhibit a specialization of  $P_1$ ,  $P_2$  and  $P_3$  such that the corresponding specialization of  $\tilde{\mathcal{R}}$  is non-zero and contains an irreducible and reduced factor, as we did in the proof of proposition 10. For that purpose, we compute

$$\text{Res}_{a_{\tilde{\mathbb{P}}^2}}(x \prod_{i=1}^{m-1} (x - \alpha_i z^n), y \prod_{i=1}^{n-1} (y - \beta_i z^m), x^m - y^n) = \prod_{i,j=1}^{m-1, n-1} (\alpha_i^n - \beta_j^m).$$

□

To compute this anisotropic residual resultant we proceed exactly as in the dense case, by considering the Bezoutian matrix of the polynomials  $P_1, P_2, P_3$ . The only difference is that  $a_{i,m} = b_{i,n}$  for  $i = 1, 2, 3$ , which implies that the  $(x, y)$ -monomial support (resp. the  $(x_1, y_1)$ -monomial support) of this Bezoutian does not contain the monomial  $1 = x^0 y^0$  (resp.  $1 = x_1^0 y_1^0$ ) and hence contains only  $mn - 1$  elements. Therefore, we get a square matrix  $\tilde{\mathbb{B}}$  of size  $mn - 1$  and we have:

**Theorem 13** *The determinant of the Bezoutian matrix  $\tilde{\mathbb{B}}$  of  $P_1, P_2, P_3$  is exactly the residual anisotropic resultant  $\text{Res}_{a_{\tilde{\mathbb{P}}^2}}(P_1^h, P_2^h, P_3^h)$  up to a sign in  $A'$ .*

*Proof.* We already know by remark 2 that  $\text{Res}_{a_{\tilde{\mathbb{P}}^2}}(P_1^h, P_2^h, P_3^h)$  must divide  $\det(\tilde{\mathbb{B}})$  and it is also straightforward to check that each entry of  $\tilde{\mathbb{B}}$  is a polynomial in  $A'$  which has total degree 3 and is homogeneous of degree 1 with respect to the coefficients of each  $P_1, P_2$  and  $P_3$ . Therefore  $\det(\tilde{\mathbb{B}}) \in A'$  has degree  $3mn - 3$  and more precisely is homogeneous with respect to the coefficients of each polynomial  $P_1, P_2$  and  $P_3$  of degree  $mn - 1$ . Now, the theorem is proved if we show that  $\det(\tilde{\mathbb{B}})$  is non-zero. Considering the specialization  $\phi$  defined by  $\phi(P_1) = x^m + x$ ,  $\phi(P_2) = y^n$  and  $\phi(P_3) = x - y$ , we obtain

$$\begin{aligned} \phi(\text{Bez}_{P_1, P_2, P_3}(x, y; x_1, y_1)) = \\ \left( \frac{x^m - x_1^m}{x - x_1} + 1 \right) \left( y^n + (x - y) \frac{y^n - y_1^n}{y - y_1} \right) - (x^m + x) \left( \frac{y^n - y_1^n}{y - y_1} \right) \end{aligned}$$

and this implies  $\phi(\det(\tilde{\mathbb{B}})) = \pm 1$ . □



## 6 Bivariate determinantal resultants with separated variables

To study the intersection and self-intersection problems for more general (i.e. rational Bézier) surfaces, we need an extension of the previous anisotropic resultant to a determinantal one.

With the strategy described in section 3, to compute the intersection locus of two rational Bézier surfaces, we need to eliminate the variables  $x, y, \lambda$  from the polynomial system of the form

$$\phi_i(x) = \lambda\psi_i(y), \quad i = 0 \dots 3 \text{ and } \lambda \neq 0$$

where the  $\phi_i$ 's (resp. the  $\psi_i$ 's) are univariate polynomials in the variable  $x$  (resp.  $y$ ) of given degree  $m \geq 1$  (resp.  $n \geq 1$ ).

Introducing new variables  $\bar{x}$  and  $\bar{y}$ , we can put this system into a (bi)-projective context to first eliminate the variable  $\lambda$ . Then, we eliminate the two couples of homogeneous variables  $x, \bar{x}$  and  $y, \bar{y}$  from the bi-homogeneous polynomial system

$$\begin{aligned} (\phi_0^h(x, \bar{x}) : \phi_1^h(x, \bar{x}) : \phi_2^h(x, \bar{x}) : \phi_3^h(x, \bar{x})) = \\ (\psi_0^h(y, \bar{y}) : \psi_1^h(y, \bar{y}) : \psi_2^h(y, \bar{y}) : \psi_3^h(y, \bar{y})) \in \mathbb{P}^3, \end{aligned}$$

where, for  $i = 0, 1, 2, 3$ ,  $\deg(\phi_i^h) = m \geq 1$  and  $\deg(\psi_i^h) = n \geq 1$ .

This elimination problem does not correspond to the projection of a complete intersection incidence variety, as in section 5, then the previous theory of resultants does not apply. However, there is a generalization of the previous approach where the incidence variety is a determinantal variety, and one can define a *determinantal resultant* (see [3]). It turns out that this can be applied in our setting: this elimination problem fits into the class of *determinantal* polynomial systems (which is much larger than the class of complete intersection ones).

We re-formulate the elimination problem as the necessary and sufficient condition on the parameters of the polynomials  $\phi_i^h$  and  $\psi_i^h$ ,  $i = 0 \dots 3$ , so that there exists a point  $((x : \bar{x}), (y : \bar{y})) \in \mathbb{P}^1 \times \mathbb{P}^1$  which satisfies

$$\text{rank} \begin{pmatrix} \phi_0(x, \bar{x}) & \phi_1(x, \bar{x}) & \phi_2(x, \bar{x}) & \phi_3(x, \bar{x}) \\ \psi_0(y, \bar{y}) & \psi_1(y, \bar{y}) & \psi_2(y, \bar{y}) & \psi_3(y, \bar{y}) \end{pmatrix} < 2.$$

Let us fix the notations. From now on, we suppose given four univariate poly-

nomials of degree  $m \geq 1$ ,  $i = 0 \dots 3$ ,

$$\phi_i(x) := a_{i,0}x^m + a_{i,1}x^{m-1} + \dots + a_{i,m-1}x + a_{i,m} \in \mathbb{Z}[a_{i,0}, \dots, a_{i,m}][x]$$

and their corresponding homogenizations  $\phi_i^h(x, \bar{x}) \in \mathbb{Z}[a_{i,0}, \dots, a_{i,m}][x, \bar{x}]$ , as well as four univariate polynomials of degree  $n \geq 1$ ,  $i = 0 \dots 3$ ,

$$\psi_i(y) := b_{i,0}y^n + b_{i,1}y^{n-1} + \dots + b_{i,n-1}y + b_{i,n} \in \mathbb{Z}[b_{i,0}, \dots, b_{i,n}][y]$$

and their corresponding homogenizations  $\psi_i^h(y, \bar{y}) \in \mathbb{Z}[b_{i,0}, \dots, b_{i,n}][y, \bar{y}]$ . We denote by  $A := \mathbb{Z}[(a_{i,j})_{i=0,\dots,3;j=0,\dots,m}, (b_{i,j})_{i=0,\dots,3;j=0,\dots,n}]$  the universal coefficients ring of all these polynomials.

We will use the theory of *determinantal resultants* developed in [3]. In particular, we will provide new formulas to compute determinantal resultants with separated variables using Bezoutian matrices.

### 6.1 The dense case

Let us denote by  $X := \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathcal{O}_X$  the vector bundle of rational functions on  $X$ , and consider two vector bundles  $E := \mathcal{O}_X^4$  and  $F := \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$ . Taking the coefficients of the polynomials  $\phi_i$ 's and  $\psi_i$ 's in  $\mathbb{C}$ , this polynomial system corresponds to a map of vector bundles on  $X$ ,  $\rho : E \rightarrow F$ , whose matrix is given by

$$\mathbb{M}((x, \bar{x}), (y, \bar{y})) := \begin{pmatrix} \phi_0(x, \bar{x}) & \phi_1(x, \bar{x}) & \phi_2(x, \bar{x}) & \phi_3(x, \bar{x}) \\ \psi_0(y, \bar{y}) & \psi_1(y, \bar{y}) & \psi_2(y, \bar{y}) & \psi_3(y, \bar{y}) \end{pmatrix}.$$

Following [3], the determinantal resultant associated to this morphism exists and corresponds to the necessary and sufficient condition on the  $\phi_i$ 's and  $\psi_i$ 's coefficients so that there exists a point  $((x_0 : \bar{x}_0), (y_0 : \bar{y}_0)) \in \mathbb{P}^1 \times \mathbb{P}^1$  satisfying

$$\text{rank } \mathbb{M}((x_0 : \bar{x}_0), (y_0 : \bar{y}_0)) \leq 1.$$

We denote this determinantal resultant, which is an element in  $A$  defined up to a sign, by  $\text{Res}_X(\mathbb{M})$ . In this case, we have the following proposition:

**Proposition 14** *The determinantal resultant  $\text{Res}_X(\mathbb{M})$  is a non-zero irreducible polynomial in  $A$ . Moreover, for each  $j \in \{0, \dots, 3\}$ , it is a homogeneous polynomial with respect to the coefficients of  $\phi_j$  and  $\psi_j$  of degree  $mn$ , and it is hence an homogeneous element in  $A$  of degree  $4mn$ .*

*Proof.* First, we will justify the existence of  $\text{Res}_X(\mathbb{M})$ . Set  $H = \mathcal{H}om(E, F)$ . Although the vector bundle  $H$  is not very ample on  $X$ , the proof of [3] theorem

1 applies. In this proof, the very ampleness hypothesis is used to show that the projection from the incidence variety  $W$  to the projectivized parameter space  $\mathbb{P}(H)$

$$W = \{(x, \varphi) \in X \times \mathbb{P}(H) : \text{rank}(\phi(x)) \leq 1\} \rightarrow \mathbb{P}(H)$$

is bi-rational onto its image (which is called the resultant variety). The argument is the following: given a zero-dimensional sub-scheme  $z$  of  $\mathbb{P}^1 \times \mathbb{P}^1$  of degree two (that is two distinct points or a double point), the locus of matrices  $\varphi$  in  $\mathbb{P}(H)$  of rank lower or equal to 1 on  $z$  is of co-dimension twice the co-dimension of matrices of rank lower or equal to 1 on only one smooth point (which is here 3). In our particular situation, this property remains true, even if  $H$  is not very ample.

To obtain the claimed partial degrees we use the intersection theory as developed in [10]: given an integer  $j \in \{0, \dots, 3\}$  we know from [3] that the degree of  $\text{Res}_X(\mathbb{M})$  with respect to the coefficients of  $\phi_j$  and  $\psi_j$  equals the coefficient of the monomial  $\alpha_j$  in the coefficient of  $t^3$  in the expansion of the series

$$-\frac{\prod_{i=1}^4 (1 - \alpha_i t)}{c_t(F)} = -\frac{1}{c_t(F)} (1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)t + \dots)$$

where  $c_t(F)$  denotes the Chern polynomial of  $F$ . We deduce that this degree is exactly the degree of the coefficient of  $t^2$  in the series  $1/c_t(F)$ . Denoting by  $H_1$  (resp.  $H_2$ ) the class of the generic point in the first (resp. second) factor  $\mathbb{P}^1$  of  $X$ , we have

$$\begin{aligned} c_t(F) &= c_t(\mathcal{O}_X(m; 0))c_t(\mathcal{O}_X(0; n)) = (1 + mH_1)(1 + nH_2) \\ &= 1 + mH_1 + nH_2 + mnH_1H_2. \end{aligned}$$

It follows that

$$\frac{1}{c_t(F)} = 1 - c_1(F)t + (c_1(F)^2 - c_2(F))t^2 + \dots$$

from which we obtain that the degree of  $\text{Res}_X(\mathbb{M})$  with respect to the coefficients of  $\phi_j$  and  $\psi_j$  equals

$$\begin{aligned} \int_X c_1(F)^2 - c_2(F) &= \int_X (mH_1 + nH_2)^2 - mnH_1H_2 \\ &= \int_X 2mnH_1H_2 - mnH_1H_2 = \int_X mnH_1H_2 = mn. \end{aligned}$$

□

**Remark 15** We recall from [3] that  $\text{Res}_X(\mathbb{M})$  is, for all  $(p, q) \in \mathbb{N}^2$  such that  $p \geq 3m - 1$  and  $q \geq 3n - 1$ , the determinant of the following graded part of the Eagon-Northcott complex

$$\begin{array}{ccc}
S[-3, -1]_{(p-3m; q-n)} & & \\
\oplus & & S[-2, -1]_{(p-2m; q-n)}^4 \\
S[-2, -2]_{(p-2m; q-2n)} & \xrightarrow{\partial_2} & \oplus \\
\oplus & & S[-1, -2]_{(p-m; q-2n)}^4 \\
S[-1, -3]_{(p-m; p-3n)} & & \\
& \xrightarrow{\partial_1} & S[-1, -1]_{(p-m; q-n)}^6 \xrightarrow{\wedge^2 \rho} S[0, 0]_{(p; q)}
\end{array}$$

where  $S$  is the ring

$$S := (\mathbb{Z}[(a_{i,j})_{i=0,\dots,3,j=0,\dots,m}]) \otimes_{\mathbb{Z}} \mathbb{Z}[(b_{i,j})_{i=0,\dots,3,j=0,\dots,n}] \otimes_{\mathbb{Z}} (\mathbb{Z}[x, \bar{x}] \otimes_{\mathbb{Z}} \mathbb{Z}[y, \bar{y}])$$

which is equipped (via tensor products) with two (bi-graded) gradings that we denote by  $S[-, -]_{(-, -)}$ . In particular,  $\text{Res}_X(\mathbb{M})$  vanishes if and only if the rank of the  $9mn \times 24mn$  matrix

$$S[-1, -1]_{(2m-1; 2n-1)}^6 \xrightarrow{\wedge^2 \rho} S[0, 0]_{(3m-1; 3n-1)}$$

drops. Moreover, from this result we can deduce that  $\text{Res}_X(\mathbb{M}) \in A$  is homogeneous with respect to the coefficients of  $(\phi_i)_{i=0,\dots,3}$  (resp.  $(\psi_i)_{i=0,\dots,3}$ ) of degree  $2mn$ .

We now turn to the computation of this determinantal resultant in our setting. We consider the four polynomials

$$P_i(x, y, \lambda) := \phi_i(x) - \lambda \psi_i(y), \quad i = 0 \dots 3,$$

and compute their associated Bezoutian  $\text{Bez}_{P_0, \dots, P_3}(x, y, \lambda, x_1, y_1, \lambda_1)$  which equals

$$-\lambda \begin{vmatrix} \phi_0(x) & \frac{\phi_0(x) - \phi_0(x_1)}{x - x_1} & \psi_0(y) & \frac{\psi_0(y) - \psi_0(y_1)}{y - y_1} \\ \phi_1(x) & \frac{\phi_1(x) - \phi_1(x_1)}{x - x_1} & \psi_1(y) & \frac{\psi_1(y) - \psi_1(y_1)}{y - y_1} \\ \phi_2(x) & \frac{\phi_2(x) - \phi_2(x_1)}{x - x_1} & \psi_2(y) & \frac{\psi_2(y) - \psi_2(y_1)}{y - y_1} \\ \phi_3(x) & \frac{\phi_3(x) - \phi_3(x_1)}{x - x_1} & \psi_3(y) & \frac{\psi_3(y) - \psi_3(y_1)}{y - y_1} \end{vmatrix}.$$

This Bezoutian does not depend on the variable  $\lambda_1$ , and we can eliminate  $\lambda$  from it by defining

$$\text{Bez}'_{P_0, \dots, P_3}(x, y, x_1, y_1) := \frac{\text{Bez}_{P_0, \dots, P_3}(x, y, \lambda, x_1, y_1, \lambda_1)}{\lambda}.$$

It is clear that  $\text{Bez}'_{P_0, \dots, P_3}$  has the symmetry property. Its  $(x_1, y_1)$ -monomial support (and hence also its  $(x, y)$ -monomial support) corresponds to the monomials  $x_1^i y_1^j$  with  $(i, j) \in \{0 \dots m-1\} \times \{0 \dots n-1\}$ . Therefore we can construct a square matrix  $\mathbb{B}'$  of size  $mn$  from this Bezoutian and we have:

**Theorem 16** *The determinant of the matrix  $\mathbb{B}'$  equals to the determinantal resultant  $\text{Res}_X(\mathbb{M})$  in  $A$  up to a sign.*

*Proof.* We already know by remark 2 that  $\text{Res}_X(\mathbb{M})$  must divide  $\det(\mathbb{B}')$ . Since they have the same degree as homogeneous elements in  $A$ , we deduce that the claimed result will be proved if we show that  $\det(\mathbb{B}')$  is non-identically zero. For this, we consider the specialization  $\rho$  defined by

$$\begin{aligned} \rho(\phi_0) &= x^m, \quad \rho(\phi_1) = 0, \quad \rho(\phi_2) = 0, \quad \rho(\phi_3) = 1, \\ \rho(\psi_0) &= 0, \quad \rho(\psi_1) = y^n, \quad \rho(\psi_2) = 1, \quad \rho(\psi_3) = 0. \end{aligned}$$

and it is straightforward to check that  $\rho(\det(\mathbb{B}')) = \pm 1$  (the sign depends on the ordering chosen on monomials to construct  $\mathbb{B}'$ ).  $\square$

## 6.2 The residual case

With the same notations as in the subsection 6.1 and we now assume that for  $i = 0 \dots 3$ , the two polynomials  $\phi_i(x)$  and  $\psi_i(y)$  have the same constant term (which is the case for the self-intersection problem); that is

$$\phi_i(0) = a_{i,m} = b_{i,n} = \psi_i(0) \text{ for } i = 0, 1, 2, 3.$$

The ring  $A' := \mathbb{Z}[(a_{i,j})_{i=0,1,2,3;j=0,\dots,m}, (b_{i,j})_{i=0,1,2,3;j=0,\dots,n-1}]$  denotes the universal coefficients ring.

We remark that the determinantal resultant  $\text{Res}_X(\mathbb{M}) \in A'$  of proposition 14 is identically zero in this case, because the matrix  $\mathbb{M}$  is always of rank less or equal to 1 at

$$P = ((0 : 1), (0 : 1)) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

So we are no longer considering all the morphisms  $E \rightarrow F$  but only those having rank 1 at this point  $P$ .

Similarly to what we did in subsection 5.2, we can overtake this difficulty by considering the blow-up

$$\pi_P : \tilde{X} \rightarrow X = \mathbb{P}^1 \times \mathbb{P}^1$$

of  $X$  along  $P$ . Then the morphism  $\pi_P^*(E) \rightarrow \pi_P^*(F)$  have generically rank 2 outside the exceptional divisor  $D$  and rank 1 on  $D$ . Therefore, for such a general morphism, the co-kernel  $Q$  fits in the exact sequence

$$\pi_P^*(E)|_D \rightarrow \pi_P^*(F)|_D \rightarrow Q \rightarrow 0$$

and is a vector bundle of rank 1 over  $D$ . It follows that the kernel  $\tilde{F}$  of the canonical surjective morphism  $\pi_P^*(F) \rightarrow Q \rightarrow 0$  is also a vector bundle<sup>3</sup> over  $\tilde{X}$  and has the same rank than  $F$ . We have the exact sequence

$$0 \rightarrow \tilde{F} \rightarrow \pi_P^*(F) \rightarrow Q \rightarrow 0. \quad (2)$$

It turns out that the morphism  $\pi_P^*(E) \rightarrow \pi_P^*(F)$  factors through  $\tilde{F}$  and the residual determinantal resultant we are looking for is the determinantal resultant of vector bundles  $\tilde{E} := \pi_P^*(E)$  and  $\tilde{F}$  over  $\tilde{X}$ . We will denote it by  $\text{Res}_{\tilde{X}}(\mathbb{M})$ . It gives a necessary and sufficient condition so that the determinantal locus in  $X$  corresponding to the condition

$$\text{rank} \begin{pmatrix} \phi_0(x, \bar{x}) & \phi_1(x, \bar{x}) & \phi_2(x, \bar{x}) & \phi_3(x, \bar{x}) \\ \psi_0(y, \bar{y}) & \psi_1(y, \bar{y}) & \psi_2(y, \bar{y}) & \psi_3(y, \bar{y}) \end{pmatrix} < 2$$

is, scheme-theoretically, strictly bigger than the point  $P$ .

**Proposition 17** *The residual determinantal resultant  $\text{Res}_{\tilde{X}}(\mathbb{M})$  is a non-zero irreducible polynomial in  $A'$ . Moreover, for each integer  $j \in \{0, \dots, 3\}$ , it is homogeneous with respect to the coefficients of  $\phi_j$  and  $\psi_j$  of degree  $mn - 1$ , and hence it is an homogeneous element in  $A'$  of degree  $4mn - 4$ .*

*Proof.* The proof of the irreducibility of  $\text{Res}_{\tilde{X}}(\mathbb{M})$  is exactly the same than the one given in proposition 14 using the general theory developed in [3]. Given an integer  $j \in \{0, \dots, 3\}$ , the degree of  $\text{Res}_{\tilde{X}}(\mathbb{M})$  with respect to the coefficients of  $\phi_j$  and  $\psi_j$  equals the degree of the coefficient of  $t^2$  in the series  $1/c_t(\tilde{F})$ . From the exact sequence (2) we deduce that  $c_t(\tilde{F})c_t(Q) = c_t(F)$ . Moreover, using the definition of  $\mathbb{M}$  and  $Q$  we have  $Q \simeq \mathcal{O}_D$ , so we deduce that

$$\begin{aligned} \frac{1}{c_t(\tilde{F})} &= \frac{c_t(\mathcal{O}_D)}{c_t(F)} = \frac{1}{c_t(F)c_t(\mathcal{O}_{\tilde{X}}(-D))} = \frac{1}{c_t(F)} \times \frac{1}{1 - Dt} \\ &= \left(1 - c_1(F)t + (c_1(F)^2 - c_2(F))t^2 + \dots\right) \left(1 + Dt + D^2t^2 + \dots\right) \end{aligned}$$

and hence that the degree of  $\text{Res}_{\tilde{X}}(\mathbb{M})$  with respect to the coefficients of  $\phi_j$  and  $\psi_j$  equals

<sup>3</sup> this is a well-known consequence of the fact that  $Q$  has rank 1. The construction of  $\tilde{F}$  is known as an *elementary transformation* of the vector bundle  $F$  (see [21] for more details).

$$\begin{aligned} \int_{\tilde{X}} D^2 - c_1(F)D + c_1(F)^2 - c_2(F) &= \int_{\tilde{X}} c_1(F)^2 - c_2(F) + \int_{\tilde{X}} D^2 \\ &= mn - 1. \end{aligned}$$

As a consequence, the residual determinantal resultant  $\text{Res}_{\tilde{X}}(\mathbb{M})$  is an homogeneous element in  $A'$  of degree  $4(mn - 1) = 4mn - 4$ .  $\square$

To compute this residual determinantal resultant we use, as before, Bezoutian matrices. With the same notations and computations as in the subsection 6.1, we build the Bezoutian matrix  $\mathbb{B}'$  of size  $mn \times mn$ . However in this more particular setting, the couple  $(0, 0)$  (associated to the monomial 1) is not in the  $(x, y)$ -monomial support and not also in the  $(x_1, y_1)$ -monomial support. In this way, we consider the  $mn - 1 \times mn - 1$  sub-matrix  $\tilde{\mathbb{B}}'$  of  $\mathbb{B}'$  corresponding to erase the row and the column which defines the monomial 1.

**Theorem 18** *The determinant of the matrix  $\tilde{\mathbb{B}}'$  equals the residual determinantal resultant  $\text{Res}_{\tilde{X}}(\mathbb{M})$  in  $A'$  up to a sign.*

*Proof.* Using remark 2,  $\text{Res}_{\tilde{X}}(\mathbb{M})$  divides  $\det(\tilde{\mathbb{B}}')$ . Since these two polynomials have the same degree as homogeneous elements in  $A'$  we deduce that the claimed result is proved if we show that  $\det(\tilde{\mathbb{B}}')$  is non-identically zero. For this, we consider the specialization  $\rho$  defined by

$$\begin{aligned} \rho(\phi_0) &= x^m + x, \quad \rho(\phi_1) = 0, \quad \rho(\phi_2) = x, \quad \rho(\phi_3) = 1, \\ \rho(\psi_0) &= 0, \quad \rho(\psi_1) = y^n, \quad \rho(\psi_2) = -y, \quad \rho(\psi_3) = 1. \end{aligned}$$

It is straightforward to check that  $\rho(\det(\tilde{\mathbb{B}}')) = \pm 1$  (the sign depends on the ordering chosen on monomials to construct the matrix  $\tilde{\mathbb{B}}'$ ).  $\square$

## 7 Conclusion

In this paper we presented a re-formulation of computer algebra problems related to the determination of the self-intersection and intersection loci of Bézier surfaces patch and in the more general case of rational patches used for NURBS. We gave simple and efficient formulas to compute the implicit equation of a certain projection of these loci. The next step is to study a numeric adaptation of the original approach described in this paper.

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## References

- [1] M. Beltrametti and L. Robbiano. Introduction to the theory of weighted projective spaces. *Exposition. Math.*, 4(2):111–162, 1986.
- [2] L. Busé. *Etude du résultant sur une variété algébrique*. PhD thesis, Université de Nice Sophia-Antipolis, 2001.
- [3] L. Busé. Resultants of determinantal varieties. *J. Pure Appl. Algebra*, 193(1-3):71–97, 2004.
- [4] L. Busé and C. D’Andrea. Inversion of parameterized hypersurfaces by means of subresultants. In *ISSAC 2004*, pages 65–71. ACM, New York, 2004.
- [5] L. Busé, M. Elkadi, and B. Mourrain. Generalized resultants over unirational algebraic varieties. *J. Symbolic Comput.*, 29(4-5):515–526, 2000. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).
- [6] L. Busé, M. Elkadi, and B. Mourrain. Resultant over the residual of a complete intersection. *J. Pure Appl. Algebra*, 164(1-2):35–57, 2001. Effective methods in algebraic geometry (Bath, 2000).
- [7] L. Busé, M. Elkadi, and B. Mourrain. Using projection operators in computer aided geometric design. In *Topics in algebraic geometry and geometric modeling*, volume 334 of *Contemp. Math.*, pages 321–342. Amer. Math. Soc., Providence, RI, 2003.
- [8] T. Dokken. Approximate implicitization. In *Mathematical methods for curves and surfaces (Oslo, 2000)*, *Innov. Appl. Math.*, pages 81–102. Vanderbilt Univ. Press, Nashville, TN, 2001.
- [9] G. Farin. *Curves and surfaces for computer aided geometric design*. Computer Science and Scientific Computing. Academic Press Inc., Boston, MA, third edition, 1993. A practical guide, With 1 IBM-PC floppy disk (5.25 inch; DD).
- [10] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.
- [11] A. Galligo and J.-P. Pavone. A sampling algorithm for parametric surface self-intersection. Preprint, 2006.



- [12] A. Galligo and M. Stillman. The geometry of bicubic surfaces and splines. To appear in *Journal of Symbolic Computation*, 2006.
- [13] R. N. Goldman. The method of resolvents: a technique for the implicitization, inversion, and intersection of nonplanar parametric, rational cubic curves. *Comput. Aided Geom. Design*, 2(4):237–255, 1985.
- [14] M. Hosaka. *Modeling of Curves and Surfaces in CAD/CAM*. Springer-Verlag, New York, NY, 1992.
- [15] J.-P. Jouanolou. Le formalisme du résultant. *Adv. Math.*, 90(2):117–263, 1991.
- [16] J.-P. Jouanolou. Résultant anisotrope, compléments et applications. *Electron. J. Combin.*, 3(2):Research Paper 2, approx. 91 pp. (electronic), 1996. The Foata Festschrift.
- [17] A. Khetan. The resultant of an unmixed bivariate system. *J. Symbolic Comput.*, 36(3-4):425–442, 2003. International Symposium on Symbolic and Algebraic Computation (ISSAC’2002) (Lille).
- [18] A. Khetan, N. Song, and R. Goldman. Sylvester  $A$ -resultants for bivariate polynomials with planar Newton polygons (extended abstract). In *ISSAC 2004*, pages 205–212. ACM, New York, 2004.
- [19] S. Krishnan and D. Manocha. An efficient surface intersection algorithm based on lower-dimensional formulation. In *ACM Transactions on Graphics*, volume 16, pages 74–106, 1997.
- [20] D. Lasser. Self-intersections of parametric surfaces. In *Proceedings of the International Conference on Engineering Graphics and Descriptive Geometry, Vienna Volume 1 pages 322-321*, 1988.
- [21] M. Maruyama. Elementary transformations in the theory of algebraic vector bundles. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 241–266. Springer, Berlin, 1982.
- [22] N. Patrikalakis. Surface-to-surface intersections. In *IEEE Computer Graphics and Applications*, volume 13, pages 89–95, 1993.
- [23] T. W. Sederberg and R. J. Meyers. Loop detection in surface patch intersections. *Comput. Aided Geom. Design*, 5(2):161–171, 1988.
- [24] W. C. T. Maekawa and N. M. Patrikalakis. Computation of self-intersections of offsets of bezier surface patches. *Journal of Mechanical Design, ASME Transactions*, 119(2):275-283, 1997.
- [25] J. B. Thomassen. Self-intersection problems and approximate implicitization. In *Computational methods for algebraic spline surfaces*, pages 155–170. Springer, Berlin, 2005.
- [26] P. Volino and N. M. Thalmann. Collision and self-collision detection: Efficient and robust solutions for highly deformable surfaces. In D. Terzopoulos and D. Thalmann, editors, *Computer Animation and Simulation’95*, pages 55–65. Springer-Verlag, 1995.