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Stochastic Differential Equations Driven by Processes Generated by Divergence Form Operators II: Convergence results

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Abstract: We have seen in a previous article how the theory of “rough paths” allows us to construct solutions of differential equations driven by processes generated by divergence form operators. In this article, we study a convergence criterion which implies that one can interchange the integral with the limit of a family of stochastic processes generated by divergence form operators. As a corollary, we identify stochastic integrals constructed with the theory of rough paths with Stratonovich or Itô integrals already constructed for stochastic processes generated by divergence form operators by using time-reversal techniques.

Keywords: Rough paths, stochastic differential equations, stochastic process generated by divergence form operators, Condition UTD, convergence of stochastic integrals

AMS Classification: 60H10, 60J60

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1 Introduction

In [Lej03b], we have seen how the theory of rough paths developed in [Lyo98] (See also [Lej03a, LQ02]) could be used to prove the pathwise existence of stochastic integrals of type

$$Z_t = z + \sum_{i=1}^N \int_0^t g_i(X_r) dX_r^i \quad (1)$$

and solutions of the Stochastic Differential Equations (SDE) of type

$$Y_t = y + \sum_{i=1}^N \int_0^t f_i(Y_r) dX_r^i \quad (2)$$

provided that f and g are smooth enough, where $X = (X^1, \dots, X^N)$ is the stochastic process generated by a divergence form operator of type

$$L = \sum_{i,j=1}^N \frac{1}{2} \frac{\partial}{\partial x_i} \left(a_{i,j} \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} \quad (3)$$

for a measurable function a taking its values in the space of symmetric $N \times N$ -matrix and a measurable function b from \mathbb{R}^N to \mathbb{R} . Here, a and b are bounded and a is uniformly elliptic, but a and b are not assumed to be continuous.

Let us denote by \mathfrak{K} the map giving Z from X in (1), and \mathfrak{J} the map giving Y from X in (2). Since in our case X has the same regularity properties of the Brownian motion's trajectories, the maps \mathfrak{K} and \mathfrak{J} are not functions of X , but, when X_0 is fixed, functions of a pair $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$, called a *rough path* or a *multiplicative functional*, where $\mathbf{X}_{s,t}^1 = X_t - X_s$ and \mathbf{X}^2 is a function from $\Delta^+ = \{(s, t) \mid 0 \leq s \leq t \leq T\}$ to $\mathbb{R}^N \otimes \mathbb{R}^N$. On the space $\mathbb{R}^N \oplus \mathbb{R}^N \otimes \mathbb{R}^N$ in which \mathbf{X} lives, we introduce the norm

$$|(\mathbf{x}^1, \mathbf{x}^2)| = \max\{|\mathbf{x}^1|, \sqrt{|\mathbf{x}^2|}\}, \quad \mathbf{x}^1 \in \mathbb{R}^N, \quad \mathbf{x}^2 \in \mathbb{R}^N \otimes \mathbb{R}^N.$$

When restricted to the rough paths \mathbf{X} lying above some continuous paths X (*i.e.*, $\mathbf{X}_{s,t} = X_t - X_s$ for all $(s, t) \in \Delta^+$) with a given starting point $X_0 = x$, the maps \mathfrak{K} and \mathfrak{J} are continuous when one uses, for $\alpha \in [2, 3)$, the topology \mathcal{V}^α generated by the norm

$$\|\mathbf{X}\|_{\mathcal{V}^\alpha} = \sup_{(s,t) \in \Delta_+} |\mathbf{X}_{s,t}| + \text{Var}_{\alpha, [0, T]}(\mathbf{X})$$

where

$$\text{Var}_{\alpha, [0, T]}(\mathbf{X}) = \left(\sup_{\text{partition } 0 \leq t_1 < \dots < t_k \leq T} \sum_{i=1}^{k-1} |\mathbf{X}_{t_i, t_{i+1}}|^\alpha \right)^{1/\alpha}. \quad (4)$$

The quantity $\text{Var}_{\alpha,[0,T]} \mathbf{X}$ is the α -variation of \mathbf{X} . The choice of the range of α depends on the regularity of the trajectories of X (a function which is β -Hölder continuous is of finite β^{-1} -variation).

In [Lej03b], we have shown that if (X, \mathbb{P}_x) is the process generated by L given by (3), one can use for $\mathbf{X}_{s,t}$ the process $(X_t^i - X_s^i, K_{s,t}^{i,j}(X))_{i,j=1,\dots,N}$, where $K_{s,t}^{i,j}(X)$ is either

$$K_{s,t}^{i,j}(X) = \int_s^t (X_r^i - X_s^i) \circ dX_r^j \text{ or } K_{s,t}^{i,j}(X) = \int_s^t (X_r^i - X_s^i) dX_r^j.$$

These integrals are defined using the forward-backward martingales decomposition (or Lyons-Zheng decomposition) of X , as constructed by A. Rozkosz [Roz96a] and T. Lyons and L. Stoica [LS99]. In addition to have proved that \mathbf{X} is of finite α -variation under \mathbb{P}_x for any starting point x , we also established a result of type Wong-Zakai.

Let us consider a family $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$ where X^ε is generated by a divergence form operator

$$L^\varepsilon = \sum_{i,j=1}^N \frac{1}{2} \frac{\partial}{\partial x_i} \left(a_{i,j}^\varepsilon \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^N b_i^\varepsilon \frac{\partial}{\partial x_i},$$

for which the uniform ellipticity and boundedness constants of a^ε and b^ε are uniform in ε . If $(X^\varepsilon)_{\varepsilon>0}$ converges uniformly in distribution to X , what can be said on the convergence of $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$ in \mathcal{V}^α ? In particular, does $(K(X^\varepsilon))_{\varepsilon>0}$ converge to $K(X)$? If true, the continuity of the maps \mathfrak{K} and \mathfrak{J} implies the convergence of the stochastic integrals and solutions of SDEs driven by X^ε to the same stochastic integral and solution of SDE where X^ε is replaced by X . As some example coming from the homogenization theory shows us, the answer is not always positive, even if $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$ is tight in \mathcal{V}^α . Yet we give a natural sufficient (but not necessary!) condition to ensure that it is true, which turns out to be *Condition UTD* introduced by F. Coquet, A. Rozkosz and L. Słomiński in [RS98, CS99]. This condition implies that, given family of Dirichlet processes converging to a Dirichlet process, then their martingale parts and their zero-quadratic variation parts also converge jointly the the corresponding parts of the limiting process. In the context of semi-martingales seen as rough paths, the equivalent criteria is Condition UT (see for example [KP96]), as seen in [CL03]. From the heuristic point of view, this may be understood with the notion of (p, q) -rough paths defined in [LV06], although the results in this last article cannot be applied directly in our case. The basic idea is that the limit of $K(X^\varepsilon)$ may not correspond to $K(X)$ when the part of finite or zero-quadratic variation “contributes” to the martingale part of the limit. The homogenization theory

provides us with a nice example of such a phenomenon, as studied in [Lej02, LL06].

In Section 4, we prove the continuity of $\mathfrak{R}(\mathbf{X}) = \int g(X_s) d\mathbf{X}_s$ with respect to g . Although this continuity follows easily from the very construction of \mathfrak{R} , it seems to have never been stated. The convergence of \mathfrak{J} with respect to the function f is given in [CFV05] under a slightly stronger assumption of convergence of the vector fields, but the proof is easily adapted in our case.

In Section 5, we identify the integrals given by the theory of rough paths with the integrals of Itô or Stratonovich type in function of the choice of $K(X)$. This identification is done by smoothing the coefficients of L and using a similar identification for semi-martingales proved in [CL03].

2 Stochastic integral driven by process generated by divergence-form operators

For $0 < \lambda < \Lambda$, let $\Xi^{\text{coeff}}(\lambda, \Lambda)$ denotes the set of all the functions (a, b) such that

$$\begin{aligned} & a \text{ and } b \text{ are measurable,} \\ & a(x) = (a_{i,j}(x))_{i,j=1,\dots,N} \text{ is a symmetric matrix,} \\ & \forall \xi \in \mathbb{R}^N, \forall x \in \mathbb{R}^N, \lambda |\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \\ & b = (b_i)_{i=1,\dots,N} \text{ and } \forall x \in \mathbb{R}^N, \sup_{i=1,\dots,N} |b_i(x)| \leq \Lambda. \end{aligned}$$

For $(a, b) \in \Xi^{\text{coeff}}(\lambda, \Lambda)$, we may define L by (3) as a closed operator on $L^2(\mathbb{R}^N)$. This operator is then the infinitesimal generator of a continuous, strong Markov, stochastic process $(X, (\mathcal{F}_t)_{t>0}, (\mathbb{P}_x)_{x \in \mathbb{R}^N})$: See for example [Str88, Lej00].

We denote by $\Xi(\lambda, \Lambda)$ the class of processes generated by divergence form operators with coefficients in $\Xi^{\text{coeff}}(\lambda, \Lambda)$.

Unless its diffusion coefficient a is smooth enough, $X \in \Xi(\lambda, \Lambda)$ is not a semi-martingale, but belongs to the more general class of Dirichlet processes under \mathbb{P}_x for any starting point x [RS98, Theorem 2.2] (note that there are several possible definitions of a Dirichlet process, with slight variations. For example, the definition of a Dirichlet processes given by the theory of Dirichlet forms [FOT04], which may be used to study the process X , is different).

Definition 1 ([Föl81b]). Let $(\Pi^n)_{n \in \mathbb{N}}$ be a family of partitions of $[0, T]$ whose meshes decrease to 0 as $n \rightarrow \infty$. For a continuous function A , we introduce

$$Q(A, \Pi^n) = \sum_{i=0}^{\ell^n - 1} |A_{t_{i+1}^n} - A_{t_i^n}|^2 \text{ if } \Pi^n = \{t_0^n \leq t_1^n \leq \dots \leq t_{\ell^n}^n\}$$

and we say that a process A is of *zero quadratic variation* (along $(\Pi^n)_{n \in \mathbb{N}}$) if $Q(A, \Pi^n)$ converges to 0 in probability as $n \rightarrow \infty$.

A *Dirichlet process* (along $(\Pi^n)_{n \in \mathbb{N}}$) is a \mathcal{F} -adapted process the sum of a local martingale and a term of zero quadratic variation which are both \mathcal{F} -adapted.

Let $\Upsilon^p(\mathbb{R}^N)$ be the class of functions $g \in W_{\text{loc}}^{1,p}(\mathbb{R}^N)$ for which $\nabla g \in L^\infty(\mathbb{R}^N)$.

Let $g \in \Upsilon^p(\mathbb{R}^N)$ with $p > \max(N, 2)$ and $T > 0$, as well as $X \in \Xi(\lambda, \Lambda)$. Let us denote by $(\overline{\mathcal{F}}_t)_{t \in [0, T]}$ the backward filtration of X , that is $\overline{\mathcal{F}}_t = \sigma(X_s; s \in [t, T])$. From [LS99, Roz96a, RS98], the process $g(X)$ may be decomposed under \mathbb{P}_x for $x \in \mathbb{R}^N$ as

$$g(X_t) = g(X_0) + \frac{1}{2}M_t^g + \frac{1}{2}(\overline{M}_{T-t}^g - \overline{M}_T^g) + V_t^g, \quad t \in [0, T], \quad (5)$$

where M^g is a \mathcal{F} -martingale, \overline{M}^g is a $\overline{\mathcal{F}}$ -martingale and

$$V_t^g = \int_0^t b(X_r) \nabla g(X_r) dr + \int_0^t \frac{1}{p(r, x, X_r)} a(X_r) \nabla p(r, x, X_r) \nabla g(X_r) dr.$$

Such a decomposition is called a *Lyons-Zheng decomposition*. In addition, V^g is a term of integrable variation under \mathbb{P}_x and M^g and \overline{M}^g are square-integrable martingales with brackets

$$\langle M^g \rangle_t = \int_0^t a \nabla g \cdot \nabla g(X_s) ds \text{ and } \langle \overline{M}^g \rangle_t = \int_0^t a \nabla g \cdot \nabla g(X_{T-s}) ds.$$

Finally, the process $g(X)$ is a Dirichlet process under \mathbb{P}_x for any starting point x (provided we choose a version of g which is continuous around x , whose existence follows from the Morrey theorem [A75, Theorem 5.4, p. 97]) whose term of zero quadratic variation A^g is

$$A_t^g = -\frac{1}{2}M_t^g + \frac{1}{2}(\overline{M}_{T-t}^g - \overline{M}_T^g) + V_t^g,$$

and M^g is its martingale part [Roz96a, RS98]. The backward martingale \overline{M}^g is the martingale part of the reversed process $g(\overline{X}.) = g(X_{T-})$, which is also a Dirichlet process.

If $p > \max(2, N)$, we set for $g, \varphi \in \Upsilon^p$,

$$\mathfrak{L}(t; X, g, \varphi) = \int_0^t g(X_s) \circ d\varphi(X_s) \quad (6)$$

$$\text{or } \mathfrak{L}(t; X, g, \varphi) = \int_0^t g(X_s) d\varphi(X_s) \quad (7)$$

in the way these integrals are defined in [Roz96a] (see also [LS99]). There are two ways to characterize them: In the Stratonovich case, where \mathfrak{L} is defined by (6),

$$\mathfrak{L}(t; X, g, \varphi) = \lim_{n \in \mathbb{N}} \mathfrak{L}^n(t; X, g, \varphi)$$

$$\text{with } \mathfrak{L}^n(t; X, g, \varphi) = \frac{1}{2} \sum_{j=0}^{\ell^n(t)-1} (g(X_{t_{j+1}^n}) + g(X_{t_j^n})) (\varphi(X_{t_{j+1}^n}) - \varphi(X_{t_j^n}))$$

In the Itô case, where \mathfrak{L} is defined by (7),

$$\mathfrak{L}(t; X, g, \varphi) = \lim_{n \in \mathbb{N}} \mathfrak{L}^n(t; X, g, \varphi)$$

$$\text{with } \mathfrak{L}^n(t; X, g, \varphi) = \sum_{j=0}^{\ell^n(t)-1} g(X_{t_j^n}) (\varphi(X_{t_{j+1}^n}) - \varphi(X_{t_j^n})).$$

In both cases, the limits hold in probability under \mathbb{P}_x for any $x \in \mathbb{R}^N$. Of course, the difference between the Itô and the Stratonovich case lies in the quadratic variation, since

$$\begin{aligned} & \frac{1}{2} \sum_{j=0}^{\ell^n(t)-1} (g(X_{t_{j+1}^n}) + g(X_{t_j^n})) (\varphi(X_{t_{j+1}^n}) - \varphi(X_{t_j^n})) \\ &= \frac{1}{2} \sum_{j=0}^{\ell^n(t)-1} (g(X_{t_{j+1}^n}) - g(X_{t_j^n})) (\varphi(X_{t_{j+1}^n}) - \varphi(X_{t_j^n})) \\ & \quad + \sum_{j=0}^{\ell^n(t)-1} g(X_{t_j^n}) (\varphi(X_{t_{j+1}^n}) - \varphi(X_{t_j^n})). \quad (8) \end{aligned}$$

With (5), we also get that in the Stratonovich case, \mathbb{P}_x -almost surely,

$$\begin{aligned} \mathfrak{L}(t; X, g, \varphi) &= \frac{1}{2} \int_0^t (g(X_r) - g(x)) dM_r^\varphi + \frac{1}{2} \int_0^t (g(\bar{X}_r) - g(\bar{X}_0)) d\bar{M}_r^\varphi \\ & \quad + \frac{1}{2} (g(X_t) - g(x)) \bar{M}_t^\varphi + \int_0^t (g(X_r) - g(x)) dV_r^\varphi \\ & \quad - \frac{1}{2} \langle g(\bar{X}), \bar{M}^\varphi \rangle_t + \frac{1}{2} \langle g(X), M^\varphi \rangle_t \end{aligned}$$

with $\bar{X}_t = X_{T-t}$ for $t \in [0, T]$. In the Itô case, \mathbb{P}_x -almost surely,

$$\begin{aligned} \mathfrak{L}(t; X, g, \varphi) &= \frac{1}{2} \int_0^t (g(X_r) - g(x)) \, dM_r^\varphi + \frac{1}{2} \int_0^t (g(\bar{X}_r) - g(\bar{X}_0)) \, d\bar{M}_r^\varphi \\ &\quad + \frac{1}{2} (g(X_t) - g(x)) \bar{M}_t^\varphi + \int_0^t (g(X_r) - g(x)) \, dV_r^\varphi - \frac{1}{2} \langle g(\bar{X}), \bar{M}^\varphi \rangle_t. \end{aligned}$$

We end this section with a convergence result of the integral $\mathfrak{L}(X, g, \varphi)$ with respect to g and φ .

Proposition 1. *Let $(g_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ be sequences of functions in $\Upsilon^p(\mathbb{R}^N)$ and $g_0, \varphi_0 \in \Upsilon^p(\mathbb{R}^N)$. We assume that g_n, φ_n are bounded uniformly for $n \geq 0$ by a constant K . We assume moreover, that g_n (resp. φ_n) and ∇g_n (resp. $\nabla \varphi_n$) converge uniformly to g_0 (resp. φ_0) and ∇g_0 (resp. $\nabla \varphi_0$). Then for any $x \in \mathbb{R}^N$, $t \mapsto \mathfrak{L}(t; X, g_n, \varphi_n)$ converges in probability under \mathbb{P}_x to $t \mapsto \mathfrak{L}(t; X, g, \varphi)$ with respect to the uniform norm. This is true both for the Stratonovich and the Itô case.*

Proof. Let us note first that $\varphi \mapsto M^\varphi$ and $\varphi \mapsto V^\varphi$ are linear maps, as well as $(g, \varphi) \mapsto \mathfrak{L}(X, g, \varphi)$. Hence,

$$\mathfrak{L}(X, g_n, \varphi_n) - \mathfrak{L}(X, g_0, \varphi_0) = \mathfrak{L}(X, g_n - g_0, \varphi_n) + \mathfrak{L}(X, g_n, \varphi_n - \varphi_0).$$

We could then assume that $g_0 = \varphi_0 = 0$.

Hence, for any $C > 0$, the Burkholder-Davis-Gundy inequality (see for example Theorem 3.28 in [KS91, p. 166]),

$$\begin{aligned} \mathbb{P}_x \left[\sup_{t \in [0, T]} \left| \int_0^t g_n(X_r) \, dM_r^{\varphi_n} \right| \geq C \right] \\ \leq \frac{1}{C^2} \mathbb{E}_x \left[\Lambda \int_0^T |\nabla \varphi_n(X_r)|^2 g_n(X_r)^2 \, dr \right] \xrightarrow{n \rightarrow \infty} 0. \quad (9) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{P}_x \left[\sup_{t \in [0, T]} \left| \int_0^t g_n(\bar{X}_r) \, d\bar{M}_r^{\varphi_n} \right| \geq C \right] \\ \leq \frac{1}{C^2} \mathbb{E}_x \left[\Lambda \int_0^T |\nabla \varphi_n(X_r)|^2 g_n(X_r)^2 \, dr \right] \xrightarrow{n \rightarrow \infty} 0. \quad (10) \end{aligned}$$

In addition, since $\mathbb{E}_x \left[\int_0^T |dV_r^{\varphi_n}| \right]$ is bounded by $\|\nabla \varphi_n\|_\infty$ times a constant that depends only on λ, Λ and T (see [Lej03b]),

$$\mathbb{E}_x \left[\sup_{t \in [0, T]} \left| \int_0^t (g_n(X_r) - g_n(x)) \, dV_r^{\varphi_n} \right| \right] \leq 2 \|g_n\|_\infty \mathbb{E}_x \left[\int_0^T |dV_r^{\varphi_n}| \right] \xrightarrow{n \rightarrow \infty} 0. \quad (11)$$

This proves that, in the Stratonovich case,

$$\sup_{t \in [0, T]} |\mathfrak{L}(t; X, g_n, \varphi_n) - \mathfrak{L}(t; X, 0, \varphi_n)| \xrightarrow{n \rightarrow \infty} 0,$$

since $\mathfrak{L}(t; X, 0, \varphi_n) = 0$.

From Remark 2.5 in [Roz96a, p. 107],

$$\begin{aligned} \langle g_n(X), M^{\varphi_n} \rangle_t &= \int_0^t a \nabla g_n \cdot \nabla \varphi_n(X_r) dr \\ \text{and } \langle g_n(\bar{X}), \bar{M}^{\varphi_n} \rangle_t &= \int_{T-t}^T a \nabla g_n \cdot \nabla \varphi_n(X_r) dr. \end{aligned}$$

It follows easily that $\langle g_n(X), M^{\varphi_n} \rangle$ and $\langle g_n(\bar{X}), \bar{M}^{\varphi_n} \rangle$ converges uniformly in $t \in [0, T]$ to 0 as $n \rightarrow \infty$. This proves that, in the Itô case, that $\sup_{t \in [0, T]} |\mathfrak{L}(t; X, g_n, \varphi_n) - \mathfrak{L}(t; X, 0, \varphi_n)|$ converges to 0.

To prove the convergence of $\mathfrak{L}(\cdot, X, g_n, \varphi_n)$ to 0 when $n \rightarrow \infty$ (since $\mathfrak{L}(\cdot, X, g_n, 0) = 0$), it is sufficient to use the previous inequalities (9), (10) and (11) as well as the convergence of $\langle g_n(X), M^{\varphi_n} \rangle$ and $\langle g_n(\bar{X}), \bar{M}^{\varphi_n} \rangle$ to 0. \square

3 Convergence of processes generated by a divergence form operator

Given a process $X \in \Xi(\lambda, \Lambda)$, we have seen in [Lej03b] how to construct a rough path \mathbf{X} lying above X . For this, we have constructed \mathbf{X} with $\mathbf{X}_{s,t}^1 = X_t - X_s$ and $\mathbf{X}_{s,t}^{i,j,2} = K_{s,t}^{i,j}(X)$, where $K_{s,t}^{i,j}(X)$ is either $\int_s^t (X_r^i - X_s^i) \circ dX_r^j$ (Stratonovich case) or $\int_s^t (X_r^i - X_s^i) dX_r^j$ (Itô case).

According to Lemma Lemma 5.4.1 in [LQ02], all the rough paths lying above X may be constructed by adding to \mathbf{X} a function $(s, t) \in \Delta_+ \mapsto \psi_t - \psi_s$ where ψ takes its values in $\mathbb{R}^N \otimes \mathbb{R}^N$ and is of finite $\alpha/2$ -variation. In this article, we consider only the Stratonovich and the Itô cases, for they correspond to natural choices.

We have seen in [Lej03b] that for any $\alpha > 2$, any X in $\Xi(\lambda, \Lambda)$, any starting point x and any $\eta > 0$, there exists C large enough such that

$$\begin{cases} \mathbb{P}_x \left[\text{Var}_{\alpha, [0, T]}(\mathbf{X}) > C \right] < \eta, \\ C \text{ depends only on } (\alpha, \lambda, \Lambda, T), \end{cases} \quad (12)$$

and that for any $C > 0$ and any $\eta > 0$, there exists δ small enough such that

$$\begin{cases} \mathbb{P}_x \left[\sup_{|t-s| < \delta} |\mathbf{X}| > C \right] \leq \eta, \\ \delta \text{ depends only on } (\alpha, \lambda, \Lambda, T). \end{cases} \quad (13)$$

3.1 Tightness results

Our first results on sequences of processes in $\Xi(\lambda, \Lambda)$ concern their tightness. For $X \in \Xi(\lambda, \Lambda)$, $K(X)$ denotes either the second-order iterated integrals of X in the Stratonovich or the Itô case.

We now consider a sequence $(X^\varepsilon)_{\varepsilon>0}$ of processes in $\Xi(\lambda, \Lambda)$. We denote by \mathbb{P}_x the distribution of X^ε to denote that $\mathbb{P}_x[X^\varepsilon = x] = 1$.

Proposition 2. *For any $\varepsilon > 0$, let X^ε be in $\Xi(\lambda, \Lambda)$. Under \mathbb{P}_x for any $x \in \mathbb{R}^N$, the sequence $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$ is tight in \mathcal{V}^α for any $\alpha > 2$.*

Proof. With (12) and (13), the proof is an immediate consequence from the tightness criterion presented in [Lej03a, Lej03b].

However, we give a short proof, restricted to the first level for the sake of simplicity, to endow the role of (12) and (13).

For some constant C , let $K(C)$ be the set of continuous functions x from $[0, T]$ to \mathbb{R}^N such that $\text{Var}_{\alpha, [0, T]}(x) \leq C$, where $\text{Var}_{\alpha, [0, T]}(x)$ is defined by (4) with $\mathbf{X}_{s,t}$ replaced by $x_t - x_s$ and $|\cdot|$ is the Euclidean norm on \mathbb{R}^N . Let also K' be a set of continuous functions from $[0, T]$ to \mathbb{R}^N which is relatively compact for the uniform convergence. If $(x^\varepsilon)_{\varepsilon>0}$ is a sequence of functions in $K' \cap K(C)$ that converges to a continuous function x^0 of finite α -variation, then $(x^\varepsilon)_{\varepsilon>0}$ also converges to x^0 in β -variation for any $\beta > \alpha$. This follows from

$$\text{Var}_{\beta, [0, T]}(x^\varepsilon - x^0)^\beta \leq 2^{\alpha-1} \|x^\varepsilon - x^0\|_\infty^{\beta-\alpha} (\text{Var}_{\alpha, [0, T]}(x^\varepsilon)^\alpha + \text{Var}_{\alpha, [0, T]}(x^0)).$$

This also implies that $K' \cap K(C)$ is relatively compact with respect to the space of continuous functions of finite α -variation with the norm $\|\cdot\|_{\mathcal{V}^\beta}$ for all $\beta > \alpha$.

Now, since the choice of C and η in (12) depend only on λ and Λ and $(X^\varepsilon)_{\varepsilon>0}$ is tight for the uniform norm, we deduce that for any $\eta > 0$, there exists a constant C such that $\sup_{\varepsilon>0} \mathbb{P}_x[X^\varepsilon \in K(C) \cap K'] \leq \eta$, where $K' = (X^\varepsilon)_{\varepsilon>0}$, which is relatively compact for the uniform norm according to (13). Thus $K(C) \cap K'$ is relatively compact for the space of continuous functions of finite α -variation with the $\|\cdot\|_{\mathcal{V}^\beta}$ -norm for any $\beta > \alpha > 2$.

The similar reasoning is easily carried to deal with the second level. \square

Hypothesis 1. *The elements of the sequence $(X^\varepsilon)_{\varepsilon>0}$ belong to $\Xi(\lambda, \Lambda)$ and converge in distribution to X^0 in $\Xi(\lambda, \Lambda)$ in $(\mathcal{C}([0, T]; \mathbb{R}^N), \|\cdot\|_\infty)$ under \mathbb{P}_x for any starting point x .*

Remark 1. Under Hypothesis 1 and Proposition 2, it follows immediately from the Prohorov theorem [Bil68] that $(X^\varepsilon)_{\varepsilon>0}$ converges to X^0 in \mathcal{V}^α for

any $\alpha > 2$: As the convergence in \mathcal{V}^α implies the uniform convergence, any possible limit of $(X^\varepsilon)_{\varepsilon>0}$ in \mathcal{V}^α is necessarily X^0 . Note that however, \mathcal{V}^α is not separable and thus the convergence of $(X^\varepsilon)_{\varepsilon>0}$ in \mathcal{V}^α does not necessarily imply its tightness.

Remark 2. Given a sequence $(X^\varepsilon)_{\varepsilon>0}$ of processes in $\Xi(\lambda, \Lambda)$ corresponding to a sequence of coefficients $(a^\varepsilon, b^\varepsilon)$, it is generally true that any cluster point X^0 belongs to $\Xi(\lambda', \Lambda')$ for some $0 < \lambda' < \Lambda'$. At least, this is always true if $b^\varepsilon = 0$ for any $\varepsilon > 0$. For that, one has to combine the results of [Mar79] and [Roz96b] for example. This means that Hypothesis 1 is not stringent at all.

Under Hypothesis 1 and in view of Proposition 2, a natural question is:

$$\text{Is the limit of } (K(X^\varepsilon))_{\varepsilon>0} \text{ equal to } K(X^0)? \quad (\text{Q})$$

This is false in general, as shown in the next section.

Remark 3. Let $L(X^0)$ be the limit of $(K(X^{\varepsilon_k}))_{k \in \mathbb{N}}$ for some subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ decreasing to 0. Then it follows from Lemma 5.4.1 in [LQ02] that there exists a process ψ from $[0, T]$ to $\mathbb{R}^N \otimes \mathbb{R}^N$ such that ψ is almost surely of finite $\alpha/2$ -variation and $L_{s,t}(X^0) = K_{s,t}(X^0) + \psi_t - \psi_s$ for any $(s, t) \in \Delta^+$. Hence, the limits of $\mathfrak{K}(X^{\varepsilon_k}, K(X^{\varepsilon_k}))$ and $\mathfrak{J}(X^{\varepsilon_k}, K(X^{\varepsilon_k}))$ are $\mathfrak{K}(X^0, L(X^0))$ and $\mathfrak{J}(X^0, L(X^0))$. They differ from $\mathfrak{K}(X^0, K(X^0))$ and $\mathfrak{J}(X^0, K(X^0))$ only by a drift (See [Lyo98, Lej03a] for precise results and further comments).

3.2 A counter-example coming from the homogenization theory

Let us consider (a, b) in $\Xi^{\text{coeff}}(\lambda, \Lambda)$ such that $b = 0$ and the diffusion coefficient a is smooth and 1-periodic. Set $a^\varepsilon(x) = a(x/\varepsilon)$ and $b^\varepsilon = 0$. For any $\varepsilon > 0$, $(a^\varepsilon, b^\varepsilon)$ belongs to $\Xi^{\text{coeff}}(\lambda, \Lambda)$ and gives rise to a stochastic process X^ε which is solution to the SDE

$$X_t^\varepsilon = x + \int_0^t \sigma(X_s^\varepsilon/\varepsilon) B_t^\varepsilon + \frac{1}{2\varepsilon} \int_0^t \nabla a(X_s^\varepsilon/\varepsilon) ds, \quad \mathbb{P}_x\text{-almost surely,}$$

where B^ε is a Brownian motion. Using a scaling argument, one gets that X^ε is equal to distribution to $\varepsilon X_{t/\varepsilon}$ under $\mathbb{P}_{x/\varepsilon}$, where X is process generated by $(a, 0)$.

3.2.1 Convergence of the process to a non-standard Brownian motion

The homogenization theory aims to study the asymptotic limit of X^ε . In the case of smooth coefficients, it is standard that X^ε converges to some non-standard Brownian motion, whose diffusion coefficient ‘‘catches’’ the large-scale properties of the behavior of X : see for example the books [BPL78,

JKO94]. The case of processes generated by a divergence form operators, which we present here with smooth coefficients, can be carried without any regularity assumptions, as shown in [Lej00, Lej01].

Let us characterize the constant matrix σ that is such that $(X^\varepsilon)_{\varepsilon>0}$ converges to $\sigma\bar{B}$ for a N -dimensional Brownian motion \bar{B} .

For $i = 1, \dots, N$, let v_i be the variational solution to

$$\sum_{k,j=1,\dots,N} \frac{\partial}{\partial x_k} \left(a_{k,j}(x) \frac{\partial v_i(x)}{\partial x_j} \right) = - \sum_{j=1}^N \frac{a_{i,j}(x)}{\partial x_j}, \quad v_i \text{ periodic.}$$

Thanks to the Fredholm alternative, this problem has a unique solution which is, since a is smooth, a classical solution. The functions v_i are called *correctors*. With the Itô formula applied to $x \mapsto x + v_i(x)$ and an ergodic theorem applied to the projection of X on the torus (whose invariant measure is the Lebesgue measure on $[0, 1]^N$), one gets that for any $t \geq 0$,

$$X_t^\varepsilon + (v_1^\varepsilon(X_t^\varepsilon), \dots, v_N^\varepsilon(X_t^\varepsilon)) = x + (v_1^\varepsilon(x), \dots, v_N^\varepsilon(x)) + R_t^\varepsilon,$$

where, for $i = 1, \dots, N$, $v_i^\varepsilon(x) = \varepsilon v_i(x/\varepsilon)$ (which decreases uniformly to 0) and R^ε is a martingale such that for $i, j = 1, \dots, N$,

$$\begin{aligned} \langle R^{i,\varepsilon}, R^{j,\varepsilon} \rangle_t &\stackrel{\text{dist.}}{\underset{\varepsilon \rightarrow \infty}{\longrightarrow}} \varepsilon^2 \int_0^{t/\varepsilon^2} (1 + \nabla v_i)(1 + \nabla v_j)(X_s) ds \longrightarrow ta^{\text{eff}} \\ \text{with } a^{\text{eff}} &\stackrel{\text{def.}}{=} \sum_{k,\ell=1}^N \int_{[0,1]^N} a_{k,\ell}(x) \left(1 + \frac{\partial v_i}{\partial x_k} \right) \left(1 + \frac{\partial v_j}{\partial x_\ell} \right) (x) dx. \end{aligned}$$

It follows from a Central Limit Theorem on the martingales [EK86] that there exists a N -dimensional Brownian motion \bar{B} such that R^ε converges in distribution to $\sigma\bar{B}$ with $\sigma\sigma^\top = a^{\text{eff}}$. The coefficient a^{eff} is called an *effective coefficient*.

3.2.2 Convergence of the iterated integrals

In [Lej02] and [LL06], we have studied the convergence of $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$, and shown that this sequence converges in distribution in \mathcal{V}^α to $(s, t) \mapsto (\sigma\bar{B}_t - \sigma\bar{B}_s, K_{s,t}(\sigma\bar{B}) + c(t - s))$, where $c = (c_{i,j})_{i,j=1,\dots,N}$ is the $N \times N$ -anti-symmetric matrix defined in our context by (this follows from a short computation performed on the formula given in Proposition 5 in [Lej02] regarding SDEs)

$$c_{i,j} = -\frac{1}{2} \int_{[0,1]^N} \left(\sum_{\substack{k=1,\dots,N, \\ k \neq i,j}} a_{i,k} \frac{\partial u_j}{\partial x_k} - \sum_{\substack{\ell=1,\dots,N, \\ \ell \neq i,j}} a_{j,\ell} \frac{\partial u_i}{\partial x_\ell} \right) dx. \quad (14)$$

Of course, this problem is not relevant in dimension 1, but we remark also that $c = 0$ if $N = 2$ or if a is diagonal. Yet it is possible to construct some cases where $c \neq 0$.

For this, let us assume that $N = 3$ and use for the diffusion coefficient a the matrix

$$a(x_1, x_2, x_3) = \begin{bmatrix} 1 & g(x_2, x_3) & 0 \\ g(x_2, x_3) & 1 & 0 \\ 0 & 0 & f(x_2, x_3) \end{bmatrix},$$

where f and g are two smooth functions which are periodic and such that a is uniformly elliptic. With (14), the coefficient $c_{1,3}$ of the matrix c is equal to

$$\begin{aligned} c_{1,3} &= -\frac{1}{2} \iiint_{[0,1]^3} \left(a_{1,2}(x) \frac{\partial u_3(x)}{\partial x_2} - a_{3,2} \frac{\partial u_1(x)}{\partial x_2} \right) dx \\ &= -\frac{1}{2} \iiint_{[0,1]^3} g(x_2, x_3) \frac{\partial u_3(x)}{\partial x_2} dx. \end{aligned}$$

As f and g depend only on x_2 and x_3 , u_3 depends also only on (x_2, x_3) . Hence, u_3 is solution to

$$\frac{\partial^2 v_3(x)}{\partial x_2^2} + \frac{\partial}{\partial x_3} \left(f(x_2, x_3) \frac{\partial u_3(x)}{\partial x_3} \right) = -\frac{\partial f(x_2, x_3)}{\partial x_3}.$$

In addition, it does not depend on the choice of g . We use for g the function $g(x_2, x_3) = C + \frac{\partial v_3}{\partial x_2}(x_2, x_3)$, where C is chosen large enough to guarantee the uniform ellipticity of a . Since v_3 is periodic,

$$\iiint_{[0,1]^3} C \frac{\partial v_3}{\partial x_2}(x_2, x_3) dx_1 dx_2 dx_3 = 0$$

and

$$\begin{aligned} \iiint_{[0,1]^3} g(x_2, x_3) \frac{\partial v_3}{\partial x_2}(x_2, x_3) dx_1 dx_2 dx_3 \\ = \iint_{[0,1]^2} \left(\frac{\partial v_3}{\partial x_2}(x_2, x_3) \right)^2 dx_2 dx_3 > 0 \end{aligned}$$

unless v_3 is constant (which implies that $f(x_2, x_3)$ does not depend on x_3). Thus, $c_{1,3} \neq 0$ for a good choice of f .

3.3 Condition UTD

Let $(\Pi^n)_{n \in \mathbb{N}}$ be a family of deterministic partitions of $[0, T]$ whose meshes decrease to 0 as $n \rightarrow \infty$.

Definition 2 (Condition UTD [RS98]). A family of Dirichlet processes $(X^\varepsilon, \mathbb{P})$ (along $(\Pi^n)_{n \in \mathbb{N}}$) with decomposition $X_t^\varepsilon = X_0^\varepsilon + M_t^\varepsilon + A_t^\varepsilon$ for $t \in [0, T]$ satisfies Condition UTD if $(\langle M^\varepsilon \rangle_T)_{\varepsilon > 0}$ and $(\sup_{t \in [0, T]} |A_t^\varepsilon|)_{\varepsilon > 0}$ are tight, and for any $C > 0$,

$$\sup_{\varepsilon > 0} \mathbb{P} [Q(A^\varepsilon, \Pi^n) > C] \xrightarrow{n \rightarrow +\infty} 0.$$

Condition UTD is a natural generalization of Condition UCV for semimartingales (See [KP96] for a review of this notion). The following Proposition is a “specialization” of Theorem 1.1, p. 84 in [RS98] to processes generated by divergence form operators.

Proposition 3 ([RS98]). *Let $(X^\varepsilon)_{\varepsilon > 0}$ and X^0 be as in Hypothesis 1. We assume that $(X^\varepsilon)_{\varepsilon > 0}$ satisfies Condition UTD under \mathbb{P}_x . The decomposition of X^ε as a Dirichlet process is written $x + M^\varepsilon + A^\varepsilon$. Then $(M^\varepsilon, A^\varepsilon)_{\varepsilon > 0}$ converges in distribution to (M^0, A^0) , where $X^0 = x + M^0 + A^0$ is the decomposition of X^0 as a Dirichlet process under \mathbb{P}_x .*

Remark 4. It follows from Definition 1 that M^0 and A^0 are both $(\mathcal{F}_t^0)_{t \geq 0}$ -adapted. Condition UTD (that does not necessarily concern processes generated by divergence form operators) was introduced in [CS99] where a similar result is given but without ensuring that M^0 and A^0 are $(\mathcal{F}_t^0)_{t \geq 0}$ -adapted. It is shown in [RS98] that the limit (M^0, A^0) of $(M^\varepsilon, A^\varepsilon)_{\varepsilon > 0}$ is really adapted to the filtration generated by X^0 and may then be identified with the decomposition of X^0 as a Dirichlet process given in Section 2.

Remark 5. Let us note that Condition UTD also implies that $(M^\varepsilon, \langle M^\varepsilon \rangle)_{\varepsilon > 0}$ converges in distribution to $(M^0, \langle M^0 \rangle)$ (see for example [KP96, Theorem 7.12, p. 30]). If the limit of $(\langle M^\varepsilon \rangle)_{\varepsilon > 0}$ is different from $\langle M^0 \rangle$, then $(X^\varepsilon)_{\varepsilon > 0}$ does not satisfy Condition UTD. This can be an easy way to identify the sequences $(X^\varepsilon)_{\varepsilon > 0}$ for which more work has to be done just by studying the convergence of the brackets of their martingales parts.

Following the results in Section 3.2 on homogenization, we remark that the sequence $(X^\varepsilon)_{\varepsilon > 0}$ there does not satisfy Condition UTD. Indeed, the martingale part M^ε of X^ε converges to $\rho \bar{B}'$, where \bar{B}' is a N -dimensional Brownian motion and ρ is a symmetric $N \times N$ -matrix. This is easily proved using arguments similar to the one given in Section 3.2, since the brackets $\langle M^\varepsilon \rangle$ of M^ε are equal in distribution to $t \mapsto \varepsilon^2 \int_0^{t/\varepsilon^2} a(X_s) ds$ and converge

almost surely uniformly on $[0, T]$ to $t \mapsto ta^{\text{mart}}$ with

$$a^{\text{mart}} = \rho \rho^{\text{T}} = \int_{[0,1]^N} a_{i,j}(x) dx.$$

The latter quantity differs in general from a^{eff} , and then $(\langle M^\varepsilon \rangle)_{\varepsilon>0}$ does not necessarily converge to the bracket of $\sigma \bar{B}$ which is equal to $t \mapsto ta^{\text{eff}}$.

Lemma 1. *Let g and φ be some functions in $\Upsilon^p(\mathbb{R}^N)$ and $(X^\varepsilon)_{\varepsilon>0}$ a sequence of processes in $\Xi(\lambda, \Lambda)$. Then for all $x \in \mathbb{R}^N$ and all $\varepsilon > 0$, $\mathfrak{L}^n(t; X^\varepsilon, g, \varphi)$ converges in probability under \mathbb{P}_x to $\mathfrak{L}(t; X^\varepsilon, g, \varphi)$ uniformly in $t \in [0, T]$. If in addition, $(g(X^\varepsilon))_{\varepsilon>0}$ satisfies Condition UTD, then for any $\delta > 0$ and any $C > 0$, there exists n_0 large enough such that*

$$\sup_{\varepsilon>0} \mathbb{P}_x \left[\sup_{t \in [0, T]} |\mathfrak{L}^n(t; X^\varepsilon, g, \varphi) - \mathfrak{L}(t; X^\varepsilon, g, \varphi)| \geq C \right] \leq \delta$$

for any $n \geq n_0$ and any $x \in \mathbb{R}^N$.

These statements are true both for the Stratonovich and the Itô integrals.

Proof. Let X be in $\Xi(\lambda, \Lambda)$. For $R > 0$, let us denote by $\Phi(x, R)$ the event $\{\sup_{t \in [0, T]} |X_t - x| \geq R\}$. One knows from Lemma II.1.12 in [Str88] that there exist some constants K and K' that depend only on λ , Λ , T and the dimension N such that

$$\mathbb{P}_x [\Phi(x, R)] \leq K \exp(-K'R^2). \quad (15)$$

Hence, for any $C > 0$, $t \in [0, T]$ and $R > 0$,

$$\begin{aligned} \mathbb{P}_x \left[\sup_{t \in [0, T]} |\mathfrak{L}^n(t; X, g, \varphi) - \mathfrak{L}(t; X, g, \varphi)| \geq C \right] &\leq \mathbb{P}_x [\Phi(x, R)] \\ &+ \mathbb{P}_x \left[\sup_{t \in [0, T]} |\mathfrak{L}^n(t; X, \tilde{g}, \tilde{\varphi}) - \mathfrak{L}(t; X, \tilde{g}, \tilde{\varphi})| \geq C; \Phi(x, R)^c \right], \end{aligned}$$

where \tilde{g} (resp. $\tilde{\varphi}$) is any continuous function equal to g (resp. φ) on the ball of radius R around x .

Thus, using this localization argument, one may assume that g and φ are bounded on \mathbb{R}^N when x is fixed.

To estimate the speed of convergence of $\mathfrak{L}^n(\cdot; X, g, \varphi)$ to $\mathfrak{L}(\cdot; X, g, \varphi)$, and

in view of (8), we have only to consider the speed of convergence to 0 of

$$\begin{aligned}
I_1^n(t) &= \int_0^t g(X_r) dM_r^\varphi - \sum_{i=1}^{k^n(t)-1} g(X_{t_i^n})(M_{t_{i+1}^n}^\varphi - M_{t_i^n}^\varphi), \\
I_2^n(t) &= \int_0^t g(\bar{X}_r) d\bar{M}_r^\varphi - \sum_{i=1}^{k^n(t)-1} g(\bar{X}_{t-t_i^n})(\bar{M}_{t-t_i}^\varphi - \bar{M}_{t-t_{i+1}^n}^\varphi), \\
I_3^n(t) &= \int_0^t g(X_r) dV_r^\varphi - \sum_{i=1}^{k^n(t)-1} g(X_{t_i^n})(V_{t_{i+1}^n}^\varphi - V_{t_i^n}^\varphi), \\
I_4^n(t) &= \langle g(\bar{X}), \bar{M}^\varphi \rangle_t - \sum_{i=1}^{k^n(t)-1} (g(\bar{X}_{t-t_{i+1}^n}) - g(\bar{X}_{t-t_i^n}))(\bar{M}_{t-t_{i+1}^n}^\varphi - \bar{M}_{t-t_i^n}^\varphi)
\end{aligned}$$

and, in order to deal with the Itô integral,

$$I_5^n(t) = \langle g(X), M^\varphi \rangle_t - \sum_{i=1}^{k^n(t)-1} (g(X_{t_{i+1}^n}) - g(X_{t_i^n}))(M_{t_{i+1}^n}^\varphi - M_{t_i^n}^\varphi),$$

where $0 \leq t_1^n \leq \dots \leq t_{k^n(t)}^n \leq t$ are the points of $\Pi^n \cap [0, t]$.

We have seen in [Lej03b, Lemma 3] that for any $\beta < 1/2$, there exists some random variable C_β such that $\mathbb{E}_x [C_\beta]$ is finite and depends only on $(\lambda, \Lambda, \beta)$ and such that X is β -Hölder continuous with constant C_β (and so is \bar{X}). In addition, since this relies on the Kolmogorov Lemma [KS91, Theorem 2.1, p. 25], it is easily obtained that $\mathbb{E}_x [C_\beta^2]$ is also finite and depends only on $(\lambda, \Lambda, \beta)$. Since g is Lipschitz continuous (as $p > N$, this result follows from the Morrey theorem [A75, Theorem 5.4, p. 97]), we get that $g(X)$ and $g(\bar{X})$ are β -Hölder continuous with constant $C_\beta \|\nabla g\|_\infty$. As

$$\langle M^\varphi, M^\varphi \rangle_t = \int_0^t a \nabla \varphi \cdot \nabla \varphi(X_r) dr \leq \Lambda t \|\nabla \varphi\|_\infty^2 \text{ for any } t \geq 0,$$

the Burkholder-Davis-Gundy inequality implies that for some constant K ,

$$\begin{aligned}
\mathbb{E}_x \left[\sup_{t \in [0, T]} I_1^n(t)^2 \right] &\leq K \mathbb{E}_x \left[\int_0^T (g(X_r) - g(X_{t_i^n}))^2 \mathbf{1}_{r \in [t_i^n, t_{i+1}^n)} d\langle M^\varphi, M^\varphi \rangle_r \right] \\
&\leq KT \Lambda \|\nabla \varphi\|_\infty^2 \|\nabla g\|_\infty^2 \mathbb{E}_x [C_\beta^2] (\text{mesh } \Pi^n)^\beta.
\end{aligned}$$

Also $\langle \bar{M}^\varphi, \bar{M}^\varphi \rangle_t \leq \Lambda t \|\nabla \varphi\|_\infty^2$ for any $t \geq 0$ and similarly,

$$\mathbb{E}_x \left[\sup_{t \in [0, T]} I_2^n(t)^2 \right] \leq KT \Lambda \|\nabla \varphi\|_\infty^2 \|\nabla g\|_\infty^2 \mathbb{E}_x [C_\beta^2] (\text{mesh } \Pi^n)^\beta.$$

As seen in [Lej03b], the term V^φ is of finite variation and $\mathbb{E}_x \left[\text{Var}_{1,[0,T]} V^\varphi \right]$ is finite and bounded by a constant depending only on λ , Λ and $\|\nabla\varphi\|_\infty$. Besides, $I_3^n(t) = \int_0^t Z_r^n dV_r^\varphi$ with $Z_r^n = g(X_r) - g(X_{t_i^n})$ when $r \in [t_i^n, t_{i+1}^n)$. Hence, $\mathbb{E}_x \left[\sup_{r \in [0,T]} |Z_r^n| \right] \leq \mathbb{E}_x [C_\beta] \|\nabla g\|_\infty (\text{mesh } \Pi^n)^\beta$. Hence, for any $C > 0$ and any $K > 0$,

$$\begin{aligned} \mathbb{P}_x \left[\sup_{t \in [0,T]} |I_3^n(t)| > C \right] &\leq \mathbb{P}_x \left[\sup_{t \in [0,T]} |Z_r^n| \text{Var}_{1,[0,T]} V^\varphi > C \right] \\ &\leq \mathbb{P}_x \left[\sup_{t \in [0,T]} |Z_r^n| > K \right] + \mathbb{P}_x \left[\text{Var}_{1,[0,T]} V^\varphi > \frac{C}{K} \right] \\ &\leq \|\nabla g\|_\infty \frac{\mathbb{E}_x [C_\beta]}{K} (\text{mesh } \Pi^n)^\beta + \frac{K}{C} \mathbb{E}_x \left[\text{Var}_{1,[0,T]} V^\varphi \right]. \end{aligned}$$

For any $\varepsilon > 0$ and any $C > 0$, one may first choose K small enough such that $KC^{-1}\mathbb{E}_x \left[\text{Var}_{1,[0,T]} V^\varphi \right] < \varepsilon/2$. Then, we choose n_0 large enough such that for any $n \geq n_0$, $\|\nabla g\|_\infty K^{-1}\mathbb{E}_x [C_\beta] (\text{mesh } \Pi^n)^\beta \leq \varepsilon/2$ for any $n \geq n_0$. It yields that $\mathbb{P}_x \left[\sup_{t \in [0,T]} |I_3^n(t)| > C \right] \leq \varepsilon$ for any $n \geq n_0$ and n_0 may be chosen in function of $\text{mesh}(\Pi^n)$, β , $\|\nabla g\|_\infty$, $\|\nabla\varphi\|_\infty$, λ , Λ , and of course, ε and C .

We now estimate of the speed of convergence of I_4^n to 0 in probability.

It is easily seen that for $s \in [0, T]$, $g(\overline{X}_s) = g(\overline{X}_0) + \frac{1}{2}\overline{M}_s^g + \frac{1}{2}(M_{T-s}^g - M_t^g) + V_{T-s}^g - V_T^g$. Since V^g is of integrable variation and $g(X_t)$ is a Dirichlet process,

$$\overline{B}_t^g = g(\overline{X}_0) + V_{T-t}^g - V_T^g - \frac{1}{2}\overline{M}_t^g + \frac{1}{2}M_{T-t}^g - \frac{1}{2}M_T^g$$

is of zero quadratic variation along $(\overline{\Pi}^n)_{n \in \mathbb{N}}$ where $\overline{\Pi}^n = \{T - t \mid t \in \Pi^n\}$. Indeed, it is the term $t \in [0, T] \mapsto M_t^g - \overline{M}_{T-t}^g$ which is of zero-quadratic variation along $(\Pi^n)_{n \in \mathbb{N}}$.

To simplify the notations, we set for two processes X and Y ,

$$Q_t^n(X, Y) = \sum_{i=1}^{k^n(t)-1} (X_{T-t_{i+1}^n} - X_{T-t_i^n})(Y_{T-t_{i+1}^n} - Y_{T-t_i^n}).$$

Using the previous decomposition of $g(\overline{X})$,

$$Q_t^n(g(\overline{X}), \overline{M}^\varphi) = Q_t^n(\overline{M}^g, \overline{M}^\varphi) + Q_t^n(\overline{B}^g, \overline{M}^\varphi),$$

and $\langle g(\overline{X}), \overline{M}^\varphi \rangle = \langle \overline{M}^g, \overline{M}^\varphi \rangle$ almost surely.

As seen in Lemma 3 in Appendix A, the family $(t \mapsto \langle \overline{M}^\varphi \rangle_{t - \sum_{i=1}^{k^n(t)-1} (\overline{M}_{t_{i+1}^n}^\varphi - \overline{M}_{t_i^n}^\varphi)^2})_{n \in \mathbb{N}}$ converges in probability to 0 uniformly in $t \in [0, T]$ at a speed that

depends only mesh Π^n , $\|\nabla\varphi\|_\infty$ and Λ , since $\langle \overline{M^\varphi} \rangle_t - \langle \overline{M^\varphi} \rangle_s \leq \Lambda(t-s)\|\nabla\varphi\|_\infty^2$ for any $0 \leq s \leq t$.

Hence, it remains to study the speed of convergence of $Q_t^n(\overline{B^g}, \overline{M^\varphi})$ to 0.

From the Cauchy-Schwarz inequality and since $t \mapsto Q_t^n(\overline{B^g})$ and $t \mapsto Q_t^n(\overline{M^\varphi})$ are increasing,

$$\sup_{t \in [0, T]} |Q_t^n(\overline{B^g}, \overline{M^\varphi})| \leq Q_T^n(\overline{B^g})^{1/2} Q_T^n(\overline{M^\varphi})^{1/2}$$

with $Q_t^n(\overline{B^g}) = Q_t^n(\overline{B^g}, \overline{B^g})$ and $Q_t^n(\overline{M^\varphi}) = Q_t^n(\overline{M^\varphi}, \overline{M^\varphi})$.

As for any $a, b, \eta > 0$, $\sqrt{ab} \leq \eta a + \eta^{-1}b$, for any $C > 0$,

$$\begin{aligned} \mathbb{P}_x \left[\sup_{t \in [0, T]} Q_t^n(\overline{B^g}, \overline{M^\varphi}) > C \right] &\leq \mathbb{P}_x \left[Q_T^n(\overline{B^g}) \geq C/\eta \right] + \mathbb{P}_x \left[Q_T^n(\overline{M^\varphi}) \geq \eta C \right] \\ &\leq \mathbb{P}_x \left[Q_T^n(\overline{B^g}) \geq C/\eta \right] + \mathbb{P}_x \left[|Q_T^n(\overline{M^\varphi}) - \langle \overline{M^\varphi} \rangle_T| \geq \eta C/2 \right] \\ &\quad + \mathbb{P}_x \left[\langle \overline{M^\varphi} \rangle_T \geq \eta C/2 \right]. \end{aligned}$$

In addition $\mathbb{P}_x \left[\langle \overline{M^\varphi} \rangle_T \geq \eta C/2 \right] \leq \mathbb{P}_x \left[T\Lambda\|\nabla\varphi\|_\infty^2 \geq \eta C/2 \right]$. We choose first η and then n_0 large enough so that $\mathbb{P}_x \left[|Q_T^n(\overline{M^\varphi}) - \langle \overline{M^\varphi} \rangle_T| \geq \eta C/2 \right] \leq \varepsilon/2$ and

$$\mathbb{P}_x \left[Q_T^n(\overline{B^g}) \geq C/\eta \right] \leq \varepsilon/2 \tag{16}$$

for any $n \geq n_0$. The last convergence holds since $\overline{B^g}$ is of zero quadratic variation along $(\overline{\Pi}^n)_{n \in \mathbb{N}}$.

The convergence of $I_5^n(t)$ to 0 is proved using similar computations by replacing $\overline{B^g}$ with $B_t^g = g(x) - \frac{1}{2}M_t^g + \frac{1}{2}(\overline{M}_{T-t}^g - \overline{M}_T^g) + V_t^g$ of zero quadratic variation along $(\Pi^n)_{n \in \mathbb{N}}$.

We deduce that $\mathfrak{L}^n(t; X, g, \varphi)$ converges to $\mathfrak{L}(t; X, g, \varphi)$ in probability uniformly in $t \in [0, T]$. Moreover, except *a priori* for the convergence of $Q_t^n(\overline{B^g})$ and $Q_t^n(B^g)$ to 0, the rate of convergence depends only on λ , Λ , T , $\|\nabla g\|_\infty$, $\|\nabla\varphi\|_\infty$ and $\text{mesh}(\Pi^n)$.

We now apply this result on X^ε . As $(g(X^\varepsilon))_{\varepsilon > 0}$ satisfies Condition UTD, it is easily obtained that $(g(\overline{X^\varepsilon}))_{\varepsilon > 0}$ also satisfies Condition UTD, but along the family of partitions $(\overline{\Pi}^n)_{n \in \mathbb{N}}$, where $\overline{\Pi}^n$ is constructed from Π^n by changing any point t in Π^n into $T - t$. Using Condition UTD in (16) with $\overline{B^g}$ replaced by the zero-quadratic variation term $\overline{B^{\varepsilon, g}}$ of $g(\overline{X^\varepsilon})$ and $B^{\varepsilon, g}$ of $g(X^\varepsilon)$, one sees that the speed of convergence of $\mathfrak{L}^n(\cdot; X^\varepsilon, g, \varphi)$ to $\mathfrak{L}(\cdot; X^\varepsilon, g, \varphi)$ is uniform in ε . \square

3.4 A criterion ensuring the convergence

The goal of this section is to prove the following convergence theorem. Of course, its interest lies in the continuity result of the integrals and the solutions of the differential equations driven by rough paths: See Theorem 5.5.2, p. 143 and Corollary 6.3.2, p. 179 in [LQ02].

Theorem 1. *If $(X^\varepsilon)_{\varepsilon>0}$ satisfies Hypothesis 1 and Condition UTD, then $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon>0}$ converges in distribution under \mathbb{P}_x to $(X^0, K(X^0))$ in \mathcal{V}^α for any $\alpha > 2$ for any starting point x .*

Remark 6. Still using our example on homogenization in Section 3.2 and Remark 5, Condition UTD is sufficient but not necessary, since the sequence $(X^\varepsilon)_{\varepsilon>0}$ we constructed there does not satisfy Condition UTD, but may converge to some process X^0 such that $(K(X^\varepsilon))_{\varepsilon>0}$ also converges to $K(X^0)$. Of course, this is the case in dimension 1, but also for some cases in dimension 2 or more as discussed in Section 3.2.2.

Before proving this theorem, let us state a useful application.

Corollary 1. *Let $(a^0, b^0) \in \Xi^{\text{coeff}}(\lambda, \Lambda)$ and $(a^\varepsilon, b^\varepsilon) \in \Xi^{\text{coeff}}(\lambda, \Lambda)$ such that $(a^\varepsilon, b^\varepsilon)$ converges almost everywhere to (a, b) . For $\varepsilon \geq 0$, let X^ε be the process corresponding to $(a^\varepsilon, b^\varepsilon)$. Then $(X^\varepsilon, K(X^\varepsilon))$ converges in distribution under \mathbb{P}_x to $(X^0, K(X^0))$ in \mathcal{V}^α for any $\alpha \geq 2$ for any starting point x .*

Proof. It is well known that the convergence of $(a^\varepsilon, b^\varepsilon)$ to (a^0, b^0) ensures the convergence in distribution of X^ε to X^0 under \mathbb{P}_x (see [Str88, Roz96b] for example).

If φ is a smooth function with compact support, then the core of the proof of Theorem 2.2 in [RS98] is to establish that $(\varphi(X^\varepsilon))_{\varepsilon>0}$ satisfies Condition UTD (see in particular (2.33) in [RS98, p. 103]).

With a localization argument, this proves that $(X^\varepsilon)_{\varepsilon>0}$ also satisfies Condition UTD.

For this, let us note first that, if $M^\varepsilon = (M^{1,\varepsilon}, \dots, M^{N,\varepsilon})$ is the martingale part of X^ε , then $\langle M^{i,\varepsilon} \rangle_T \leq \Lambda T$ for $i = 1, \dots, N$ and then $(\langle M^\varepsilon \rangle_T)_{\varepsilon>0}$ is tight.

Now, as in the proof of Lemma 1, let us set $\Phi^\varepsilon(R) = \{\sup_{t \in [0, T]} |X_t^\varepsilon - x| \geq R\}$ for $R > 0$, and choose $\varphi(x)$ such that $\varphi(x) = x_i$ on the ball $\{y \in \mathbb{R}^N \mid |y - x| \leq R\}$. Let us denote $A^{R,\varepsilon}$ (resp. A^ε) the term of zero-quadratic variation of $\varphi(X^\varepsilon)$ (resp. X^ε). Then

$$\mathbb{P}_x \left[\sup_{t \in [0, T]} A^\varepsilon > C \right] \leq \mathbb{P}_x \left[\sup_{t \in [0, T]} A^\varepsilon > C; \Phi^\varepsilon(R)^c \right] + \mathbb{P}_x [\Phi^\varepsilon(R)]$$

On $\Phi^\varepsilon(R)$, $A^\varepsilon = A^{R,\varepsilon}$, so that

$$\mathbb{P}_x \left[\sup_{t \in [0, T]} A^\varepsilon > C \right] \leq \mathbb{P}_x \left[\sup_{t \in [0, T]} A^{R,\varepsilon} > C \right] + \mathbb{P}_x [\Phi^\varepsilon(R)]$$

With (15) (which is uniform in ε), it is easily deduced that $(\sup_{t \in [0, T]} A^\varepsilon)_{\varepsilon > 0}$ is tight. The same computation with $\sup_{t \in [0, T]} A^\varepsilon$ replaced by $Q(A^\varepsilon, \Pi^n)$ proves that $(X^\varepsilon)_{\varepsilon > 0}$ satisfies Condition UTD.

Theorem 1 allows us to conclude. \square

To prove Theorem 1, let us remark first that in the Stratonovich or the Itô case, for $i, j = 1, \dots, N$,

$$K_{0,t}^{i,j}(X^\varepsilon) = \mathfrak{L}(t; X^\varepsilon, \chi_i, \chi_j) \text{ with } \chi_i(x) = x_i.$$

In addition, for all $\varepsilon \geq 0$,

$$K_{s,t}^{i,j}(X^\varepsilon) = K_{0,t}^{i,j}(X^\varepsilon) - K_{0,s}^{i,j}(X^\varepsilon) - (X_s^i - X_0^i)(X_t^j - X_s^j), \quad \forall (s, t) \in \Delta_+. \quad (17)$$

Theorem 1 is proved using Lemma 1 and the following Lemma.

Lemma 2. *If for $x \in \mathbb{R}^N$, any $t \in [0, T]$, $i, j = 1, \dots, N$ and any constant $C > 0$,*

$$\limsup_{n \rightarrow \infty} \sup_{\varepsilon > 0} \mathbb{P}_x [|\mathfrak{L}^n(t; X^\varepsilon, \chi_i, \chi_j) - \mathfrak{L}(t; X^\varepsilon, \chi_i, \chi_j)| \geq C] = 0 \quad (18)$$

then $(X^\varepsilon, K(X^\varepsilon))$ converges in distribution under \mathbb{P}_x to $(X^0, K(X^0))$ as $\varepsilon \rightarrow 0$ in \mathcal{V}^α for any $\alpha > 2$.

Proof. As for each integer n and each $t \in [0, T]$, $(X^\varepsilon, \mathfrak{L}^n(t; X^\varepsilon, \chi_i, \chi_j))$ converges in distribution to $(X^0, \mathfrak{L}^n(t; X^0, \chi_i, \chi_j))$, Condition (18) together with Theorem 4.2 in [Bil68, p. 25] allow us to assert that $(X^\varepsilon, K_{0,t}^{i,j}(X^\varepsilon))$ with $K_{0,t}^{i,j}(X^\varepsilon) = \mathfrak{L}(t; X^\varepsilon, \chi_i, \chi_j)$ converges in distribution to $(X^0, K_{0,t}^{i,j}(X^0))$. With (17), $(X^\varepsilon, K_{s,t}^{i,j}(X^\varepsilon))$ converges in distribution to $(X^0, K_{s,t}^{i,j}(X^0))$ for all $(s, t) \in \Delta_+$. The tightness of $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon > 0}$ in \mathcal{V}^α (see Proposition 2) and the continuity of $(s, t) \mapsto K_{s,t}(X^0)$ allows us to uniquely identify any possible limit of $(X^\varepsilon, K(X^\varepsilon))_{\varepsilon > 0}$ with $(X^0, K(X^0))$. \square

4 Continuity of integrals along rough paths

We give now a general result of convergence that allows us to ensure that the rough path integral $\int f_\delta(X_s) d\mathbf{X}_s$ converges to $\int f(X_s) d\mathbf{X}_s$ when f_δ converges

to f in an appropriate sense. This result will be used in the next section to identify the integrals constructed in [Lej03b] using the rough paths theory with the one constructed using time-reversal techniques presented in Section 2.

We denote by $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$ a rough path of finite α -variation with $\alpha \in [2, 3)$. We assume that \mathbf{X} lies above X with $X_0 = x$ for a fixed x , which means that $\mathbf{X}_{s,t}^1 = X_t - X_s$.

For $\gamma \in (0, 1)$ such that $2 + \gamma > \alpha$, we denote by $\text{Lip}(\gamma)$ the set of continuous, bounded functions $g : \mathbb{R}^N \rightarrow \mathbb{R}^m$ with a bounded first derivative which is γ -Hölder continuous. On this space, we define the norm

$$\|g\|_{\text{Lip}} = \|g\|_{\infty} + \sum_{i=1}^N \|\partial_{x_i} g\|_{\infty} + \sum_{i=1}^N \sup_{x, y \in \mathbb{R}^N, x \neq y} \frac{|\partial_{x_i} g(x) - \partial_{x_i} g(y)|}{|x - y|^{\gamma}}.$$

The theory of rough path allows us to construct a new rough path $\mathbf{Z}(f) = (\mathbf{Z}^1(f), \mathbf{Z}^2(f))$ from $(f_1, \dots, f_N) \in \text{Lip}(\gamma)^N$ and \mathbf{X} that corresponds to the integral of f — identified as a differential form — along \mathbf{X} . We set

$$\mathbf{Z}_{0,t}(f) = \int_0^t f(X_s) d\mathbf{X}_s.$$

For such a function $f = (f_1, \dots, f_N)$, we set $\|f\|_{\text{Lip}} = \max_{i=1, \dots, N} \|f_i\|_{\text{Lip}}$.

Proposition 4. *Let $f \in \text{Lip}(\gamma)^N$ and $(f_{\delta})_{\delta > 0}$ a sequence of functions in $\text{Lip}(\gamma)^N$ such that $\|f_{\delta}\|_{\text{Lip}}$ remains bounded in δ , f_{δ} converges uniformly to f and ∇f_{δ} converges uniformly to ∇f as $\delta \rightarrow 0$.*

Then $\mathbf{Z}(f_{\delta})$ converges to $\mathbf{Z}(f)$ in α -variation.

Proof. For $(s, t) \in \Delta_+$, set

$$\mathbf{Y}_{s,t}^1 = \sum_{i=1}^N f(X_s) \mathbf{X}_{s,t}^{i,1} + \sum_{i,j=1}^N \frac{\partial f_i}{\partial x_j}(X_s) \mathbf{X}_{s,t}^{i,j,2}$$

and

$$\mathbf{Y}_{s,t}^2(f) = \sum_{i,j=1}^N f_i(X_s) f_j(X_s) \mathbf{X}_{s,t}^{i,j,2}.$$

We also assume that \mathbf{X} is controlled by some function $\omega : \Delta_+ \rightarrow \mathbb{R}$, which means that

$$|\mathbf{X}_{s,t}| \leq \omega(s, t)^{1/\alpha} \text{ with } \omega(s, r) + \omega(r, t) \leq \omega(s, t)$$

for all $0 \leq s \leq r \leq t \leq T$ and ω is continuous near its diagonal. Then, $\mathbf{Y}(f)$ is an *almost rough path*, in the sense that

$$|\mathbf{Y}_{s,t}(f) - \mathbf{Y}_{s,r}(f) \otimes \mathbf{Y}_{r,t}(f)| \leq C\omega(s, t)^{\theta}, \quad \forall 0 \leq s \leq r \leq t \leq T,$$

for $\theta = (\gamma + 1)/\alpha > 1$ and a constant C that depends only on $\omega(0, T)$, α , γ and $\|f\|_{\text{Lip}}$.

Given a family of partitions $(\Pi^n)_{n \in \mathbb{N}}$ whose meshes decreases to 0, we set

$$\mathbf{Y}_{s,t}^{\Pi^n}(f) = \mathbf{Y}_{s,t_{\ell(s)}^n}(f) \otimes \mathbf{Y}_{t_{\ell(s)}^n, t_{\ell(s)^{n+1}}^n}(f) \otimes \cdots \otimes \mathbf{Y}_{t_{\ell(t)}^n-1, t_{\ell(t)}^n}(f) \otimes \mathbf{Y}_{t_{\ell(t)}^n, t}(f)$$

where $\Pi^n = \{0 = t_0^n \leq \cdots \leq t_{\ell^n}^n = T\}$ and $\ell(s)^n$ and $\ell(t)^n$ are such that $t_{\ell(s)^n-1}^n < s \leq t_{\ell(s)}^n$ and $t_{\ell(t)}^n \leq t < t_{\ell(t)+1}^n$. The rough path $\mathbf{Z}(f)$ is constructed from $\mathbf{Y}(f)$ as the limit of $\mathbf{Y}^{\Pi^n}(f)$. Indeed, by construction (see Theorem 3.2.1 in [LQ02, p. 41]), one gets that there exists a sequence $(K_n)_{n \in \mathbb{N}}$ decreasing to 0 such that

$$|\mathbf{Z}_{s,t}(f) - \mathbf{Y}_{s,t}^{\Pi^n}(f)| \leq K_n \omega(s, t)^\theta, \quad \forall (s, t) \in \Delta_+. \quad (19)$$

These constants K_n depend on C and n , and thus on $\|f\|_{\text{Lip}}$, $\omega(0, T)$, α and γ .

For each integer n , $\mathbf{Y}_{s,t}^{\Pi^n}(f_\delta)$ converges to $\mathbf{Y}_{s,t}^{\Pi^n}(f)$ as $\delta \rightarrow 0$. In addition, for each $\delta > 0$, $\mathbf{Y}_{s,t}^{\Pi^n}(f_\delta)$ converges to $\mathbf{Z}(f_\delta)$ at a speed that does not depend on δ , since $(\|f_\delta\|_{\text{Lip}})_{\delta > 0}$ is bounded. This is sufficient to ensure that $\mathbf{Z}_{s,t}(f_\delta)$ converges to $\mathbf{Z}_{s,t}(f)$ for each $(s, t) \in \Delta_+$.

Moreover, it is easily checked that

$$|\mathbf{Y}_{s,t}^{\Pi^n}(f_\delta) - \mathbf{Y}_{s,t}^{\Pi^n}(f)| \leq C \max\{\sup_{\delta > 0} \|f_\delta - f\|_\infty, \sup_{\delta > 0} \|\nabla f_\delta - \nabla f\|_\infty\} \omega(s, t)^{1/\alpha}$$

for some constant C that depends only on $\sup_{\delta > 0} \|f_\delta\|_{\text{Lip}}$ and $\|f\|_{\text{Lip}}$. With (19), one obtains that for any $\varepsilon > 0$, there exists δ_0 small enough such that for all $\delta < \delta_0$, $|\mathbf{Z}_{s,t}(f_\delta) - \mathbf{Z}_{s,t}(f)| \leq \varepsilon \omega(s, t)^{1/\alpha}$ for all $(s, t) \in \Delta_+$.

The result is then proved. \square

5 Identification of the integrals

Let us note that a function in $\text{Lip}(\gamma)$ also belongs to \mathcal{Y}^p . Given $g = (g_1, \dots, g_N) \in \text{Lip}(\gamma)^N$, we are given two ways in considering the integrals of type $\sum_{i=1}^N \int_0^t g_i(X_s) \circ dX_s^i$ and $\sum_{i=1}^N \int_0^t g_i(X_s) dX_s^i$ for $X \in \Xi(\lambda, \Lambda)$: either by using the Lyons-Zheng decomposition (5) or by using the rough paths $(X, K(X))$, where $K(X)$ is either the Stratonovich or the Itô integral, as constructed in [Lej03b].

We have seen in [CL03] that the integrals given by the proper choice of $K(X)$ are the same as the Itô or Stratonovich integral when X is a semimartingale. The identification relies on the Wong-Zakai theorem for semimartingales.

In our case, we will approximate a process X by semi-martingales obtained by smoothing the coefficients of its infinitesimal generator, and then pass to the limit.

We denote by $\mathfrak{K}_s(t; X, g)$ the integral $\mathbf{Z}_{0,t}^1(g)$ with $\mathbf{Z} = \int g(X_s) d\mathbf{X}_s$, where \mathbf{X} is the rough path $\mathbf{X} = (X, K(X))$ with $K(X)$ defined as a Stratonovich integral. Similarly, if we use an Itô integral for $K(X)$, then we denote $\mathbf{Z}_{0,t}^1(g)$ by $\mathfrak{K}_i(t; X, g)$.

We also set $\mathfrak{L}_s(t; X, g) = \sum_{i=1}^N \mathfrak{L}(t; X, g_i, \chi_i)$, when \mathfrak{L} is defined in the Stratonovich, and $\mathfrak{L}_i(t; X, g) = \sum_{i=1}^N \mathfrak{L}(t; X, g_i, \chi_i)$ when \mathfrak{L} is defined in the Itô sense.

Remark 7. Here, we consider only the “first level”, and not the iterated integrals of $\int g(X) dX$ against itself since it is useless for our result.

Theorem 2. *For any $X \in \Xi(\lambda, \Lambda)$ and $g \in \text{Lip}(\gamma)^N$ with $\gamma \in (0, 1)$. Then $\mathfrak{K}_s(X, g) = \mathfrak{L}_s(X, g)$ and $\mathfrak{K}_i(X, g) = \mathfrak{L}_i(X, g)$ under \mathbb{P}_x for any starting point x .*

We already know that this result is true for semi-martingales.

Proposition 5 ([CL03]). *Let (X, \mathbb{P}) be a continuous semi-martingale and $g \in \text{Lip}(\gamma)^N$ with $\gamma \in (0, 1)$. Then \mathbb{P} -almost surely, $\mathfrak{K}_s(X, g) = \mathfrak{L}_s(X, g)$ and $\mathfrak{K}_i(X, g) = \mathfrak{L}_i(X, g)$.*

Proof of Theorem 2. We drop any references to the subscripts s and i in \mathfrak{L} and \mathfrak{K} , since the proof is the same in both cases.

Let $(a^\varepsilon, b^\varepsilon)$ a sequence of elements in $\Xi^{\text{coeff}}(\lambda, \Lambda)$ such that a^ε and b^ε are smooth and converge almost everywhere to a and b . The corresponding process X^ε is then a semi-martingale, since $L^\varepsilon = \frac{1}{2}\nabla(a^\varepsilon\nabla\cdot) + b^\varepsilon\nabla$ may be transformed into a non-divergence form operator.

In addition, let us assume in a first time that g is smooth. From the proof of Theorem 2.2 in [Roz96a] (see also the proof of Corollary 1 on the localization argument), $(g(X^\varepsilon))_{\varepsilon>0}$ satisfies Conditions UTD. From Lemma 1, one knows that for $i = 1, \dots, N$, $\mathfrak{L}^n(t; X^\varepsilon, g, \chi_i)$ converges to $\mathfrak{L}(t; X^\varepsilon, g, \chi_i)$ in probability at a speed that is uniform in ε . As in the proof of Lemma 2, it is then easily deduced from Theorem 4.2 in [Bil68] that $\mathfrak{L}(t; X^\varepsilon, g)$ converges $\mathfrak{L}(t; X, g)$ in distribution. Yet we have seen in Corollary 1 that $(X^\varepsilon, K(X^\varepsilon))$ converges in \mathcal{V}^α to $(X, K(X))$. From the continuity of \mathfrak{K} in \mathcal{V}^α , we deduce that $\mathfrak{K}(X^\varepsilon, g)$ converges in distribution to $\mathfrak{K}(X, g)$. Necessarily, $\mathfrak{L}(X, g) = \mathfrak{K}(X, g)$ under the assumption that g is smooth.

Now, if g only belongs to $\text{Lip}(\gamma)^N$, let us introduce a sequence of mollifiers $(\varphi_\delta)_{\delta>0}$. Then the convolution $g \star \varphi_\delta$ of g with φ_δ is a smooth functions

satisfies $\|g \star \varphi_\delta\|_{\text{Lip}} \leq \|g\|_{\text{Lip}}$. In addition, $g \star \varphi_\delta$ and $\nabla g \star \varphi_\delta$ converge uniformly to g and ∇g .

The conclusion follows from Propositions 1 and 4. \square

A On the convergence of the sum of squares of a martingale to its bracket

The following Lemma is used in the proof of Lemma 1. It corresponds to a classical result, but we write it explicitly to show the dependence of the rate of convergence with respect to the Lipschitz constant of the brackets.

Lemma 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$. Let M be a square integrable martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ such that for some constant K , $\langle M \rangle_t - \langle M \rangle_s \leq K(t - s)$ for any $0 \leq s \leq t \leq T$ for a fixed $T > 0$. For each $n \geq 1$, let $\{t_i^n\}_{i=1, \dots, n}$ be a partition of $[0, T]$ such that $\sup_{i=1, \dots, n-1} (t_{i+1}^n - t_i^n)$ decreases to 0 as $n \rightarrow \infty$. Let*

$$Q_t^n = \sum_{i=1}^{\ell^n(t)-1} (M_{t_{i+1}^n} - M_{t_i^n})^2,$$

where $\ell(t)$ is the integer such that $t_{\ell(t)}^n \leq t < t_{\ell(t)+1}^n$.

Then, for any $\kappa > 0$ and any $\varepsilon > 0$, there exists an integer n_0 such that

$$\mathbb{P} \left[\sup_{t \in [0, T]} |Q_t^n - \langle M \rangle_t| \geq \kappa \right] \leq \varepsilon \text{ for all } n \geq n_0$$

and the choice of n_0 depends only on K, κ, T and the mesh $\sup_{i=1, \dots, n-1} (t_{i+1}^n - t_i^n)$ of $\{t_i^n\}_{i=1, \dots, n}$.

Proof. The convergence in probability of Q_t^n to $\langle M \rangle_t$ for any $t \geq 0$ is classical (See for example [KS91, Theorem 1.5.8, p. 33]). However, we write it to show that the rate of convergence depends only on K . Let N be the martingale defined by $N = M^2 - \langle M \rangle$ (note that with our condition on the growth of $\langle M \rangle$, N is really a martingale and not a local martingale).

Let us set $\Delta_i^n M = M_{t_{i+1}^n} - M_{t_i^n}$, $\Delta_i^n N = N_{t_{i+1}^n} - N_{t_i^n}$, $\Delta_i^n \langle M \rangle = \langle M \rangle_{t_{i+1}^n} - \langle M \rangle_{t_i^n}$.

Then

$$\begin{aligned}
\left(\sum_{i=1}^{\ell^n(t)-1} \Delta_i^n M - \langle M \rangle_t \right)^2 &\leq 2 \left(\sum_{i=1}^{\ell^n(t)-1} \Delta_i^n M - \langle M \rangle_{t_{\ell^n(t)}} \right)^2 + 2(\langle M \rangle_t - \langle M \rangle_{t_{\ell^n(t)}})^2 \\
&= 2 \left(\sum_{i=1}^{\ell^n(t)-1} (\Delta_i^n M - \Delta_i^n \langle M \rangle) \right)^2 + 2(\langle M \rangle_t - \langle M \rangle_{t_{\ell^n(t)}})^2 \\
&= 2 \sum_{i=1}^{\ell^n(t)-1} \Delta_i^n N^2 + 2 \sum_{\substack{i,j=1,\dots,\ell^n(t)-1 \\ i \neq j}} \Delta_j^n N \Delta_i^n N + 2(\langle M \rangle_t - \langle M \rangle_{t_{\ell^n(t)}})^2.
\end{aligned}$$

If $j < i$, since N is a martingale,

$$\mathbb{E} \left[\Delta_j^n N \Delta_i^n N \right] = \mathbb{E} \left[\Delta_j^n N \mathbb{E} \left[\Delta_i^n N \mid \mathcal{F}_{t_i^N} \right] \right] = 0.$$

Thus,

$$\mathbb{E} \left[\left(\sum_{i=1}^{\ell^n(t)-1} \Delta_i^n M - \langle M \rangle_t \right)^2 \right] \leq 2 \sum_{i=1}^{\ell^n(t)-1} \mathbb{E} \left[\Delta_i^n N^2 \right] + 2(\langle M \rangle_t - \langle M \rangle_{t_{\ell^n(t)}})^2.$$

With the Burkholder-Davis-Gundy inequality, there exists a constant C' such that for $i = 1, \dots, \ell^n(t) - 1$,

$$\begin{aligned}
\mathbb{E} \left[\Delta_i^n N^2 \right] &= \mathbb{E} \left[\Delta_i^n M^4 \right] - 2\mathbb{E} \left[\Delta_i^n M^2 \Delta_i^n \langle M \rangle^2 \right] + \mathbb{E} \left[\Delta_i^n \langle M \rangle^2 \right] \\
&\leq C' \mathbb{E} \left[\Delta_i^n \langle M \rangle^2 \right] \leq KC' (t_{i+1}^n - t_i^n)^2,
\end{aligned}$$

so that

$$\mathbb{E} \left[\left(\sum_{i=1}^{\ell^n(t)-1} \Delta_i^n M - \langle M \rangle_t \right)^2 \right] \leq 2K(C'T + 1) \sup_{i=1,\dots,n-1} (t_{i+1}^n - t_i^n).$$

This proves the $L^2(\mathbb{P})$ -convergence of the one-dimensional marginals of Q^n to $\langle M \rangle$.

To now turn to prove the uniform convergence of Q^n to $\langle M \rangle$. For this, let us choose two integers $1 \leq k < \ell \leq n$ and let us note that

$$\left(\sum_{i=k}^{\ell-1} \Delta_i^n M^2 \right)^2 = \sum_{i=k}^{\ell-1} \Delta_i^n M^4 + 2 \sum_{\substack{i,j=k,\dots,\ell-1 \\ i \neq j}} \Delta_i^n M^2 \Delta_j^n M^2.$$

With the Burkholder-Davis-Gundy inequality, there exists some constant C' such that

$$\mathbb{E} \left[\sum_{i=k}^{\ell-1} \Delta_i^n M^4 \right] \leq C' \mathbb{E} \left[\sum_{i=k}^{\ell-1} \Delta_i^n \langle M \rangle^2 \right] \leq C' K(t_\ell - t_k) \sup_{i=k, \dots, \ell-1} (t_{i+1}^n - t_i^n). \quad (20)$$

On the other hand,

$$\begin{aligned} \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n M^2 \Delta_j^n M^2 &= \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n N \Delta_j^n \langle M \rangle + \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n \langle M \rangle \Delta_j^n N \\ &+ \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n N \Delta_j^n N + \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n \langle M \rangle \Delta_j^n \langle M \rangle. \end{aligned}$$

As N is martingale, one gets that

$$\mathbb{E} \left[\sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} a_i \Delta_j^n N \right] = \mathbb{E} \left[\sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} a_i \mathbb{E} \left[\Delta_j^n N \mid \mathcal{F}_{r_j} \right] \right] = 0$$

for $a_i = \Delta_i^n N$ or $a_i = \Delta_i^n \langle M \rangle$. In addition, since $\Delta_i^n N^2 = (\Delta_i^n M^2 - \Delta_i^n \langle M \rangle)^2 \leq \Delta_i^n M^4 + \Delta_i^n \langle M \rangle^2$,

$$\begin{aligned} \sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n N \Delta_j^n \langle M \rangle &= \sum_{i=k}^{\ell-1} \Delta_i^n N (\langle M \rangle_{t_\ell^n} - \langle M \rangle_{t_{i+1}^n}) \\ &\leq (\langle M \rangle_{t_\ell^n} - \langle M \rangle_{t_k^n})^2 \sum_{i=k}^{\ell-1} \Delta_i^n N^2 \\ &\leq (\langle M \rangle_{t_\ell^n} - \langle M \rangle_{t_k^n})^2 \left(\sum_{i=k}^{\ell-1} \Delta_i^n M^4 + \sum_{i=k}^{\ell-1} \Delta_i^n \langle M \rangle^2 \right). \end{aligned}$$

With (20), $\mathbb{E} \left[\sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n N \Delta_j^n \langle M \rangle \right] \leq C(t_\ell^n - t_k^n)^4$ for some constant C that depends only on T and K . On the other hand, still using the fact that $\langle M \rangle$ is increasing, $\sum_{\substack{i,j=k, \dots, \ell-1 \\ i \neq j}} \Delta_i^n \langle M \rangle \Delta_j^n \langle M \rangle \leq (\langle M \rangle_{t_\ell^n} - \langle M \rangle_{t_k^n})^2 \leq K^2(t_\ell^n - t_k^n)^2$.

Hence, we finally get that

$$\mathbb{E} \left[\left(\sum_{i=k}^{\ell-1} \Delta_i^n M^2 \right)^2 \right] \leq C(t_\ell^n - t_k^n)^2 \quad (21)$$

for some constant C that depends only on K and T .

Now, for a partition $\{s_i^m\}_{i=1,\dots,m}$ of $[0, T]$,

$$\begin{aligned} & \sup_{t \in [0, T]} |Q_t^n - \langle M \rangle_t| \\ & \leq \sup_{i=1,\dots,m-1} \sup_{t \in [t_i^m, t_{i+1}^m]} \left(|Q_t^n - Q_{s_i^m}^n| + |Q_{s_i^m}^n - \langle M \rangle_{s_i^m}| + |\langle M \rangle_t - \langle M \rangle_{s_i^m}| \right). \end{aligned} \quad (22)$$

Let us note that, since $t \mapsto Q_t^n$ is increasing, for $t \in [s_i^m, s_{i+1}^m]$

$$\begin{aligned} \sup_{t \in [s_i^m, s_{i+1}^m]} |Q_t^n - Q_{s_i^m}^n| & \leq \sum_{i=\ell^-(i,n)}^{\ell^+(i,n)-1} (M_{t_{i+1}^n} - M_{t_i^n})^2 \\ & \quad - (M_{t_{\ell^-(i,n)-1}^n} - M_{t_{\ell^-(i,n)}^n})^2, \end{aligned} \quad (23)$$

where $\ell^-(i, n)$ is such that $t_{\ell^-(i,n)-1}^n < s_i^m \leq t_{\ell^-(i,n)}^n$ and $t_{\ell^+(i,n)}^n < s_{i+1}^m \leq t_{\ell^+(i,n)+1}^n$.

With (22) and (23),

$$\begin{aligned} \mathbb{P} \left[\sup_{t \in [0, T]} |Q_t^n - \langle M \rangle_t| > 4\kappa \right] & \leq \mathbb{P} \left[\sup_{i=1,\dots,m} \sup_{t \in [s_i^m, s_{i+1}^m]} |\langle M \rangle_t - \langle M \rangle_{s_i^m}| > \kappa \right] \\ & \quad + \mathbb{P} \left[\sup_{i=1,\dots,m} \sum_{i=\ell^-(i,n)}^{\ell^+(i,n)-1} (M_{t_{i+1}^n} - M_{t_i^n})^2 > \kappa \right] \\ & \quad + \mathbb{P} \left[\sup_{i=1,\dots,m-1} |Q_{s_i^m}^n - \langle M \rangle_{s_i^m}| > \kappa \right] \\ & \quad + \mathbb{P} \left[\sup_{i=1,\dots,m-1} (M_{t_{\ell^-(i,n)-1}^n} - M_{t_{\ell^-(i,n)}^n})^2 > \kappa \right]. \end{aligned} \quad (24)$$

For any $\kappa > 0$, it follows from (21) that

$$\begin{aligned} \mathbb{P} \left[\sup_{i=1,\dots,m-1} \sum_{j=\ell^-(i,n)}^{\ell^+(i,n)-1} \Delta_i M^2 \geq \kappa \right] & \leq \frac{1}{\kappa^2} \sum_{i=1}^m \mathbb{E} \left[\left(\sum_{j=\ell^-(i,n)}^{\ell^+(i,n)-1} \Delta_i M^2 \right)^2 \right] \\ & \leq \frac{C}{\kappa^2} \sum_{i=1}^{m-1} (s_{i+1}^m - s_i^m)^2 \leq \frac{CT}{\kappa^2} \delta \end{aligned} \quad (25)$$

with $\delta = \sup_{i=1,\dots,m-1} (s_{i+1}^m - s_i^m)$. Moreover,

$$\mathbb{P} \left[\sup_i \sup_{t \in [s_i^m, s_{i+1}^m]} |\langle M \rangle_t - \langle M \rangle_{s_i^m}| > \kappa \right] \leq \mathbb{P} [\text{osc}(\langle M \rangle, \delta) > \kappa] \leq \frac{K\delta}{\kappa}, \quad (26)$$

where $\text{osc}(f, \delta) = \sup_{s,t \in [0,T], |t-s| < \delta} |f(t) - f(s)|$ is the modulus of continuity of a continuous function f . With the Burkholder-Davis-Gundy, there exists a constant C' such that

$$\begin{aligned} & \mathbb{P} \left[\sup_{i=1, \dots, m-1} (M_{t_{\ell^{-(i,n)}-1}}^n - M_{t_{\ell^{-(i,n)}}}^n)^2 > \kappa \right] \\ & \leq \sum_{i=1}^{m-1} \frac{1}{\kappa^2} \mathbb{E} \left[(M_{t_{\ell^{-(i,n)}-1}}^n - M_{t_{\ell^{-(i,n)}}}^n)^2 \right] \leq \frac{mC'K}{\kappa^2} \sup_{j=1, \dots, n-1} (t_{j+1}^n - t_j^n). \end{aligned} \quad (27)$$

Finally,

$$\mathbb{P} \left[\sup_{i=1, \dots, m-1} |Q_{s_i^m}^n - \langle M \rangle_{s_i^m}| > \kappa \right] \leq \sum_{i=1}^{m-1} \mathbb{P} \left[|Q_{s_i^m}^n - \langle M \rangle_{s_i^m}| > \kappa \right]. \quad (28)$$

Given $\kappa, \varepsilon > 0$, the idea is now to fix first the mesh δ of the partition $\{s_i^m\}_{i=1, \dots, m}$, — which is arbitrary chosen — so that the quantities in (25) and (26) are smaller than $\varepsilon/4$. This fix m . We choose then n_0 large enough such that for any $n \geq n_0$, the quantities in (27) and (28) are also smaller than $\varepsilon/4$. We then obtain that with (24)

$$\mathbb{P} \left[\sup_{t \in [0,T]} |Q_t^n - \langle M \rangle_t| > 4\kappa \right] \leq \varepsilon \text{ for all } n \geq n_0$$

and we remark that the choice of δ depends only on K, T and κ , as well as the choice of n_0 , as we claimed. \square

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